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► To cite this version:

Hoang Nam Nguyen, Abdel Lisser, Vikas Vikram Singh. Random games under normal mean-variance mixture distributed independent linear joint chance constraints. 2023. <hal-04373176>

HAL Id: hal-04373176

<https://hal.science/hal-04373176v1>

Preprint submitted on 4 Jan 2024

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Random games under normal mean-variance mixture distributed independent linear joint chance constraints

Hoang Nam Nguyen¹, Abdel Lisser^{1,*}, Vikas Vikram Singh²

Abstract

In this paper, we study an n player game where the payoffs as well as the strategy sets are defined using random variables. The payoff function of each player is defined using expected value function and his/her strategy set is defined using a linear joint chance constraint. The random constraint vectors defining the joint chance constraint are independent and follow normal mean-variance mixture distributions. For each player, we reformulate the joint chance constraint in order to prove the existence of a Nash equilibrium using the Kakutani fixed-point theorem under mild assumptions. We illustrate our theoretical results by considering a game between two competing firms in financial market.

Keywords: Chance-constrained game, Normal mean-variance mixture, Nash equilibrium

2000 MSC: 90C15, 90C25, 90C59

1. Introduction

In 1950, Nash (Nash Jr, 1950) showed that there exists a mixed strategy equilibrium point in any finite strategic game, which is called Nash equilibrium nowadays. In (Nash Jr, 1950), the players' payoffs and strategy sets are deterministic, which are not suitable in real life applications. We can deal with the randomness in the players' payoffs in at least two different ways. First, the expectation value of the random payoffs of the players (Ravat and Shanbhag, 2011) often used for risk neutral games. Second, the risk measure CVaR (Kannan et al., 2013) and the chance constraint optimization framework (Singh et al., 2016; Singh and Lisser, 2018) used as an alternative payoff criterion. A natural way to deal with random constraints in the strategy sets is to replace them with chance constraints which is quite popular in real life applications, e.g., risk constraints in portfolio optimization (Ji and Lejeune, 2018), resource constraints in stochastic shortest path problem (Cheng and Lisser, 2012), renewable energy aggregators in the local market (Li et al., 2022). The games involving chance constraint either in payoffs or in strategy sets are called chance-constrained games (CCGs). Several studies exist in the literature on chance-constrained games, e.g., (Singh et al., 2016) showed the existence of Nash equilibrium for the case where the payoff vector is elliptically distributed. Also, (Singh and Lisser, 2018) showed the equivalence between Nash equilibrium of a CCG and the global optimal solution of a mathematical program. The CCGs where strategy sets are defined using individual/joint chance constraint have been widely studied in the literature (Peng et al., 2018, 2021; Singh and Lisser, 2019; Singh et al., 2021; Nguyen et al., 2022).

To the best of our knowledge, the random constraint vectors in the CCGs are often assumed to be elliptically distributed or follow a mixture of elliptical distributions. These families include many known distributions, e.g., Normal distributions, t-distributions, Laplace distributions, Kotz-type distributions, Pearson distributions. However, it is interesting to consider different distributions for the constraint vectors since these families of distributions are not suitable for some practical situations. In power system scheduling problems, both wind power forecast errors and load

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forecasting errors can be better fitted by generalized hyperbolic (GH) distributions (Hodge et al., 2012), which are neither elliptically distributed nor a mixture of elliptical distributions. This family belongs to a more general family of distributions, namely normal mean-variance mixture, which is often used to model many financial applications (Eberlein and Prause, 2002; Bauer, 2000; Rachev, 2003; Protasov, 2004). In this paper, we study an n player CCG where the players' payoffs are defined using the expectation value and the strategy set of each player is defined by a joint chance constraint. We assume that the random constraint vectors are independent and follow normal mean-variance mixture distributions which generalize the family of GH distributions (Barndorff-Nielsen et al., 1982). We derive a new reformulation of the joint chance constraint and show that there exists a Nash equilibrium under mild assumptions, by proving the convexity of the feasible strategy set of each player. In (Nguyen and Lisser, 2021), the authors give some sufficient conditions under which joint chance constraint is convex when the constraint vectors are independent and follow normal mean-variance mixture distribution. In this paper, we relax those conditions and provide a lower bound on the probability level vector for which the joint chance constraint is convex. As an application of these games we study a competition between two firms in a financial market and compute its Nash equilibria using a best response algorithm.

The remainder of this paper is organized as follows. In Section 2, we introduce the game model and some basic concepts used in the paper. In Section 3, we show the existence of a Nash equilibrium. In section 4, we illustrate our theoretical results using a game between two firms from a financial market. We conclude the paper in Section 5.

2. The Game Model and Basic Concepts

2.1. Game Model

In this section, we define a chance-constrained game model. Let $H = \{1, 2, \dots, n\}$ be the set of players and S^i be the strategy set of player $i \in H$ which is assumed to be a non-empty, convex and compact subset of \mathbb{R}^{d_i} . The set $S = \prod_{i=1}^n S^i$ is called the set of strategy profiles of the game. For any player $i \in H$, a strategy profile $x = (x^1, x^2, \dots, x^n) \in S$ can be written as (x^i, x^{-i}) , where $x^i \in S^i$ and $x^{-i} \in S^{-i}$. Here, S^{-i} denote the set of vectors of strategies of all players, except player i . We consider the case where the strategies of each player i further satisfy the following joint chance constraint

$$\mathbb{P}(V^i x^i \leq D^i) \geq \alpha_i, \quad (1)$$

where α_i is a real number defined in $[0, 1]$, $D^i = (D^{i,1}, \dots, D^{i,K_i})^\top \in \mathbb{R}^{K_i}$ is a real vector and $V^i = [V^{i,1}, \dots, V^{i,K_i}]^\top$ is a $K_i \times d_i$ random matrix, $V^{i,k}$ is the k^{th} row of V^i and \top denotes the transposition. Let $J^i = \{1, 2, \dots, K_i\}$ be the index set of i^{th} player's constraints. Then, for any $i \in H$, the feasible strategy set $S_{\alpha_i}^i$ of player i is the set of all $x^i \in S^i$ such that x^i satisfies the chance constraint (1). We assume that for each $i \in H$, $S_{\alpha_i}^i$ is a non-empty set. Let $\alpha = (\alpha_i)_{i \in H}$ and $S_\alpha = \prod_{i=1}^n S_{\alpha_i}^i$ be the set of all feasible strategy profiles. For each feasible strategy profile $x \in S_\alpha$, we assume that player i gains a random payoff given by $k_i(x, \zeta)$, where ζ is an m -dimensional random vector. We define the payoff function of player i by taking the expected value of the random payoff given by

$$p_i(x) = \mathbb{E}[k_i(x, \zeta)], \quad \forall x \in S_\alpha.$$

The tuple $(S_{\alpha_i}^i, p_i)_{i \in H}$ is called a CCG. We assume that the CCG is of complete information, i.e., the vector α , the payoff function p_i and the feasible strategy set $S_{\alpha_i}^i$ for all $i \in H$, are known to all the players. Nash equilibrium is commonly used stability notion for non-cooperative games. For the CCG, a strategy profile z^* is said to be a Nash equilibrium at given confidence level vector α if for any player $i \in H$, the following inequality holds

$$p_i(z^{i*}, z^{-i*}) \geq p_i(x^i, z^{-i*}), \quad \forall x^i \in S_{\alpha_i}^i.$$

The proof of the existence of a Nash equilibrium for a non-cooperative game is mostly done in the literature using fixed point theorems. In this paper, we use the Kakutani fixed-point theorem, which requires the payoff function of each player i to be a continuous function with respect to the strategies of all the players and a concave function with respect to their strategies, where the strategies of other players are fixed, for any player $i \in H$. (Peng et al., 2021) show that under the following assumption, the abovementioned properties hold.

Assumption 1. For each player $i \in H$, the following conditions hold: (i) $k_i(\cdot, x^{-i}, \zeta)$ is concave of x^i , for every $(x^{-i}, \zeta) \in S^{-i} \times \mathbb{R}^m$, (ii) $k_i(\cdot)$ is continuous, (iii) $p_i(x) < \infty$, for every $x \in S$.

2.2. Basic Concepts

In this section, we present some basic definitions, which are useful in the paper.

Definition 1. The function $f : \mathbb{R} \rightarrow \mathbb{R}$, is said to be an r -decreasing function for some real number $r \in \mathbb{R}$, if f is continuous on $(0, +\infty)$ and there exists some strictly positive real number t^* such that the function $t \mapsto t^r f(t)$ is strictly decreasing on $(t^*, +\infty)$.

Definition 2. Let Q be a convex subset of \mathbb{R}^s and $f : Q \rightarrow (0, +\infty)$. For a real number $r \in (-\infty, +\infty)$, f is said to be r -concave on Q if for any $x, y \in Q$ and $\alpha \in [0, 1]$,

$$\begin{aligned} f(\alpha x + (1 - \alpha)y) &\geq [\alpha f(x)^r + (1 - \alpha)f(y)^r]^{\frac{1}{r}}, \text{ if } r \neq 0, \\ f(\alpha x + (1 - \alpha)y) &\geq f(x)^\alpha f(y)^{1-\alpha}, \text{ otherwise.} \end{aligned}$$

Definition 3. An n -dimensional random vector U follows a normal mean-variance mixture distribution with parameters (μ, γ, Σ, W) , i.e., $U \sim \text{NMVM}(\mu, \gamma, \Sigma, W)$, if we have the following representation:

$$U \stackrel{d}{=} \mu + \gamma W + \sqrt{W} \Sigma^{\frac{1}{2}} Z,$$

where: (i) Z is an n -dimension standard Gaussian distribution with mean vector $\mu_Z = 0$ and covariance matrix $\Sigma_Z = \mathbf{I}_n$, where \mathbf{I}_n is the $n \times n$ identity matrix.

(ii) W is a positive random variable with a density function independent of Z .

(iii) $\Sigma \in \mathbb{R}^{n \times n}$ is an $n \times n$ positive definite matrix and $\Sigma^{\frac{1}{2}} \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix such that $\Sigma^{\frac{1}{2}} (\Sigma^{\frac{1}{2}})^\top = \Sigma$.

(iv) μ and γ are n -dimensional real vectors and $\stackrel{d}{=}$ implies that the both sides have the same distribution.

Normal mean-variance mixture distribution can be used to model chance-constraints due to its flexibility, robustness, scalability, optimization capabilities, and simulation abilities (Naderi et al., 2023; Lee and McLachlan, 2021). In the following section, we study the existence of a Nash equilibrium of the CCG where the random constraint vectors follow normal mean-variance mixture distributions.

3. Existence of Nash Equilibrium

For each player $i \in H$, we assume that the row vectors of V^i follow normal mean-variance mixture distributions, i.e., $V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$ for all $k \in J^i$. Moreover, the row vectors $\{V^{i,k} \mid k \in J^i\}$ are mutually independent and the support of $W^{i,k}$ is an open interval $(w_l^{i,k}, w_u^{i,k})$, where $0 < w_l^{i,k} \leq w_u^{i,k} < \infty$. Let $\tilde{S}_{\alpha_i}^i = S_{\alpha_i}^i \setminus \{0\}$, then for $x^i \in \tilde{S}_{\alpha_i}^i$, let

$$g^{i,k}(x^i, W^{i,k}) = \frac{-(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}}, \quad \xi^{i,k}(x^i) = \frac{Z^\top ((\Sigma^{i,k})^{\frac{1}{2}})^\top x^i}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}}, \quad (2)$$

where Z refers to Definition 3. It is well known that $\xi^{i,k}(x^i)$ follows 1-dimensional standard Gaussian distribution (Fang et al., 2018), for any $i \in H$ and $k \in J^i$. Using the independence of the row vectors $V^{i,k}$, the constraint (1) can be written as $\prod_{k \in J^i} \mathbb{P} \{ (V^{i,k})^\top x^i \leq D^{i,k} \} \geq \alpha_i$, which implies that

$$\prod_{k \in J^i} \mathbb{P} \left(\frac{W^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} (x^i)^\top \gamma^{i,k} + \sqrt{W^{i,k}} \frac{Z^\top ((\Sigma^{i,k})^{\frac{1}{2}})^\top x^i}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \leq \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \geq \alpha_i. \quad (3)$$

Using the notations in (2), we rewrite (3) as follows

$$\prod_{k \in J^i} \mathbb{P} \left(\xi^{i,k}(x^i) \leq g^{i,k}(x^i, W^{i,k}) \right) \geq \alpha_i. \quad (4)$$

Using the independence between $V^{i,k}$, the constraint (4) can be reformulated as

$$\prod_{k \in J^i} \mathbb{E}_{W^{i,k}} \left(\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right) \geq \alpha_i, \quad (5)$$

or equivalently

$$\sum_{k \in J^i} \log \left(\mathbb{E}_{W^{i,k}} \left(\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right) \right) \geq \log(\alpha_i), \quad (6)$$

where Φ be the cumulative distribution function of an 1-dimensional standard Gaussian distribution. The main idea for the proof of the existence of Nash equilibrium is based on the Kakutani fixed-point theorem, which requires the convexity of the feasible strategy set $S_{\alpha_i}^i$. We show that there exists a real number $\alpha_i^* \in [0, 1]$ such that $S_{\alpha_i}^i$ is convex, for all $\alpha_i \in (\alpha_i^*, 1]$, where α_i^* is defined in the following assumption.

Assumption 2. For each $i \in H$, let $\alpha_i \in (\alpha_i^*, 1]$, where $\alpha_i^* = \max \left(\alpha_i^{(1)}, \alpha_i^{(2)}, \alpha_i^{(3)} \right)$. The scalars $\alpha_i^{(j)}$, $j = 1, 2, 3$, are given by

$$\begin{aligned} \alpha_i^{(1)} &= \max_{k \in J^i} \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right). \\ \alpha_i^{(2)} &= \max_{k \in J^i} \Phi \left(\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right). \\ \alpha_i^{(3)} &= \max_{k \in J^i} \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right), \end{aligned}$$

where $\|\cdot\|_2$ denotes the Euclidean norm and $\lambda_{i,k,\min}$ is the smallest eigenvalue of the positive definite matrix $\Sigma^{i,k}$.

Lemma 1. Let Assumption 2 hold. Assume that $x^i \in \tilde{S}_{\alpha_i}^i$, for all $i \in H$. Then, $D^{i,k} > (\mu^{i,k})^\top x^i$, for all $i \in H$ and $k \in J^i$.

Proof. For each $i \in H$, let $x^i \in \tilde{S}_{\alpha_i}^i$. Since $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] \in [0, 1]$ and $\alpha_i > \alpha_i^{(1)}$, for any $k \in J^i$, the constraint (5) implies that

$$\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] > \alpha_i^{(1)}.$$

It follows from the definition of $g^{i,k}$ in (2) that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{-(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \right] > \alpha_i^{(1)}. \quad (7)$$

Due to the following three inequalities

$$(i) |(x^i)^\top \gamma^{i,k}| \leq \|x^i\|_2 \|\gamma^{i,k}\|_2, \quad (ii) \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \geq \sqrt{\lambda_{i,k,\min}} \|x^i\|_2, \quad (iii) \sqrt{W^{i,k}} \leq \sqrt{w_u^{i,k}}, \quad (8)$$

we deduce that

$$\left| \frac{-(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \sqrt{W^{i,k}} \right| \leq \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}}. \quad (9)$$

From (7) and (9), by applying the increasing monotonicity of Φ , we get

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \right] > \alpha_i^{(1)}.$$

It is clear from the definition of $\alpha_i^{(1)}$ in Assumption 2 that the following condition holds

$$\alpha_i^{(1)} \geq \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right),$$

which in turn implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \right) \right] > \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right). \quad (10)$$

If $D^{i,k} - (x^i)^\top \mu^{i,k} \leq 0$, using the increasing monotonicity of Φ and (10), we get

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right) \right] > \Phi \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} \right),$$

which gives a contradiction. Therefore, $D^{i,k} - (x^i)^\top \mu^{i,k} > 0$. \square

Lemma 2. For $i \in H$, let $x^i \in \tilde{S}_{\alpha_i}^i$. Then, for any $k \in J^i$ and $z^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$, we have

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] \geq \alpha_i.$$

Proof. For $i \in H$ and $k \in J^i$, let $x^i \in \tilde{S}_{\alpha_i}^i$ and $z^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$. It is clear from the definition of $g^{i,k}$ in (2) that the following condition holds for any $W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$

$$g^{i,k}(x^i, W^{i,k}) = \frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \left(\frac{z^{i,k} - W^{i,k}}{\sqrt{W^{i,k}}} \right).$$

By applying the inequalities (i) – (iii) of (8) and the fact that $|z^{i,k} - W^{i,k}| \leq w_u^{i,k} - w_1^{i,k}$, we deduce the following inequality

$$\frac{(x^i)^\top \gamma^{i,k}}{\sqrt{(x^i)^\top \Sigma^{i,k} x^i}} \left(\frac{z^{i,k} - W^{i,k}}{\sqrt{W^{i,k}}} \right) \leq \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}}$$

which in turn implies that

$$g^{i,k}(x^i, W^{i,k}) \leq \frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}}, \forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \quad (11)$$

Taking the expectation value $\mathbb{E}_{W^{i,k}} (\Phi(\cdot))$ on both sides of (11), we get

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \leq \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right].$$

Since $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(x^i, W^{i,k}))] \in [0, 1]$, for any $k \in J^i$, the constraint (5) implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(g^{i,k}(x^i, W^{i,k}) \right) \right] \geq \alpha_i,$$

which in turn implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] \geq \alpha_i.$$

□

Lemma 3. *Let Assumption 2 hold. Then, for all $i \in H$*

$$\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in J^i} \Omega^{i,k},$$

where Conv represents the convex hull and

$$\Omega^{i,k} = \left\{ x^i \in S^i \left| \begin{array}{l} \frac{-(x^i)^\top \gamma^{i,k} \sqrt{W^{i,k}} + \frac{D^{i,k} - (x^i)^\top \mu^{i,k}}{\sqrt{W^{i,k}}}}{\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_1^{i,k}}}} > 4 \sqrt{(x^i)^\top \Sigma^{i,k} x^i}, \\ \forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \end{array} \right. \right\}, \quad (12)$$

Proof. For $i \in H$, let $x^i \in \tilde{S}_{\alpha_i}^i$. It follows from Lemma 2 that for any $z^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] \geq \alpha_i. \quad (13)$$

Since $\alpha_i > \alpha_i^{(2)}$, the constraint (13) implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] > \alpha_i^{(2)}.$$

It follows from the definition of $\alpha_i^{(2)}$ in Assumption 2 that the following condition holds

$$\alpha_i^{(2)} \geq \Phi \left[\frac{4 \sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right].$$

which in turn implies that

$$\begin{aligned} & \mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}} g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] \\ & > \Phi \left[\frac{4 \sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right]. \end{aligned} \quad (14)$$

If $g^{i,k}(x^i, z^{i,k}) \leq 0$, the constraint (14) implies that

$$\Phi\left(\frac{\|\gamma^{i,k}\|_2(w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right) > \Phi\left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}}\left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}\sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right) + \frac{\|\gamma^{i,k}\|_2(w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right],$$

which derives a contradiction due to the increasing monotonicity of Φ . Therefore, $g^{i,k}(x^i, z^{i,k}) > 0$. Then, the following inequality holds for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$

$$\frac{\sqrt{z^{i,k}}}{\sqrt{W^{i,k}}}g^{i,k}(x^i, z^{i,k}) \leq \frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}}g^{i,k}(x^i, z^{i,k}). \quad (15)$$

It follows from (14) and (15) that

$$\begin{aligned} & \Phi\left(\frac{\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}}g^{i,k}(x^i, z^{i,k}) + \frac{\|\gamma^{i,k}\|_2(w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right) \\ & > \Phi\left[\frac{4\sqrt{w_u^{i,k}}}{\sqrt{w_l^{i,k}}}\left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}\sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right) + \frac{\|\gamma^{i,k}\|_2(w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right]. \end{aligned} \quad (16)$$

By applying the increasing monotonicity of Φ , (16) is equivalent to the following inequality

$$g^{i,k}(x^i, z^{i,k}) > 4\left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}\sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}\sqrt{w_l^{i,k}}}\right). \quad (17)$$

Since (17) holds for any $z^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$ and $x^i \in \tilde{S}_{\alpha_i}^i$, we get

$$\tilde{S}_{\alpha_i}^i \subset \bigcap_{k \in J^i} \Omega^{i,k}. \quad (18)$$

Note that $\Omega^{i,k}$ is a convex set. Then, by taking the convex hull on both side of (18),

$$\text{Conv}(\tilde{S}_{\alpha_i}^i) \subset \bigcap_{k \in J^i} \Omega^{i,k}.$$

□

Lemma 4. *Let Assumption 2 hold. Then, for any convex subset Q^i of $\bigcap_{k \in J^i} \Omega^{i,k}$ such that $0 \notin Q^i$, $g^{i,k}(x^i, W^{i,k})$ is defined and (-2) -concave with respect to x^i on Q^i , for all $i \in H$, $k \in J^i$ and $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$.*

Proof. For $x^i \in Q^i$ and $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, let $f^{i,k}(x^i, W^{i,k}) = \frac{1}{(g^{i,k}(x^i, W^{i,k}))^2}$. It follows from the definition of $g^{i,k}$ in (2) that

$$f^{i,k}(x^i, W^{i,k}) = ((x^i)^\top \Sigma^{i,k} x^i) \cdot M_{i,k}(x^i, W^{i,k})^{-1},$$

where $M_{i,k}(x^i, W^{i,k}) = W^{i,k}[(x^i)^\top \gamma^{i,k}]^2 + \frac{1}{W^{i,k}}[D^{i,k} - (x^i)^\top \mu^{i,k}]^2 + 2[(x^i)^\top \gamma^{i,k}][(x^i)^\top \mu^{i,k} - D^{i,k}]$. In order to simplify the notation, for the rest of the proof, we write $M_{i,k}$ (resp. $f^{i,k}$) instead of $M_{i,k}(x^i, W^{i,k})$ (resp. $f^{i,k}(x^i, W^{i,k})$). It is clear that the (-2) -concavity of $g^{i,k}$ is equivalent to the convexity of $f^{i,k}$ on Q^i . In order to prove the convexity of $f^{i,k}$, we prove that the Hessian matrix of $f^{i,k}$ with respect to x^i is positive semidefinite on Q^i ,

for any $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$. Let $\nabla_{x^i} f^{i,k}$ be the gradient vector of $f^{i,k}$ with respect to x^i and $H_{x^i} f^{i,k}$ be the Hessian matrix of $f^{i,k}$ with respect to x^i . Let $h(x^i) = (x^i)^\top \Sigma^{i,k} x^i$. The gradient vector of h is given as follows

$$\nabla_{x^i} h(x^i) = 2\Sigma^{i,k} x^i,$$

and the gradient vector of $M_{i,k}$ with respect to x^i is given as follows

$$\nabla_{x^i} M_{i,k} = 2[W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}]. \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right).$$

Since $f^{i,k} = h(x^i) \cdot M_{i,k}^{-1}$, the gradient vector of $f^{i,k}$ with respect to x^i can be written as follows

$$\nabla_{x^i} f^{i,k} = \nabla_{x^i} h(x^i) \cdot M_{i,k}^{-1} + h(x^i) \cdot \nabla_{x^i} (M_{i,k}^{-1}).$$

We can write $M_{i,k}^{-1} = u \circ M_{i,k}$, where $u(x) = \frac{1}{x}$. By the chain rule of composite function in calculus, the gradient vector of $M_{i,k}^{-1}$ is given by

$$\nabla_{x^i} (M_{i,k}^{-1}) = u'(M_{i,k}) \cdot \nabla_{x^i} M_{i,k} = \frac{-1}{M_{i,k}^2} \cdot 2[W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}]. \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right),$$

which in turn implies that, for any $(x^i, W^{i,k}) \in \mathcal{Q}^i \times [w_l^{i,k}, w_u^{i,k}]$

$$\nabla_{x^i} f^{i,k} = 2M_{i,k}^{-1} \Sigma^{i,k} x^i - 2M_{i,k}^{-2} \cdot h(x^i) \cdot [W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}]. \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right). \quad (19)$$

Note that the Hessian matrix of $f^{i,k}$ with respect to x^i can be written equivalently as follows

$$H_{x^i} f^{i,k} = \nabla_{x^i} \left[\left(\nabla_{x^i} f^{i,k} \right)^\top \right].$$

Then, it suffices to derive the term on the right of (19) by x^i . Let $v(x) = \frac{1}{x^2}$, $r(x^i) = \Sigma^{i,k} x^i$, $s(x^i) = W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}$, then the gradient vector of $f^{i,k}$ with respect to x^i is given by

$$\nabla_{x^i} f^{i,k} = 2 \cdot M_{i,k}^{-1} \cdot r(x^i) - 2 \cdot v \circ M_{i,k} \cdot h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right). \quad (20)$$

Using (20), we get the Hessian matrix as

$$\begin{aligned} H_{x^i} f^{i,k} &= 2 \cdot \nabla_{x^i} (M_{i,k}^{-1}) \cdot [r(x^i)]^\top + 2 \cdot M_{i,k}^{-1} \cdot \nabla_{x^i} r(x^i) - 2 \cdot v \circ M_{i,k} \cdot h(x^i) \cdot \nabla_{x^i} s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top \\ &\quad - 2 \cdot \nabla_{x^i} (v \circ M_{i,k}) \cdot h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top - 2 \cdot v \circ M_{i,k} \cdot \nabla_{x^i} h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top \\ &= A + B + C + D + E, \end{aligned}$$

where

$$\begin{aligned} A &= 2 \cdot \nabla_{x^i} (M_{i,k}^{-1}) \cdot [r(x^i)]^\top, \quad B = 2 \cdot M_{i,k}^{-1} \cdot \nabla_{x^i} r(x^i), \quad C = -2 \cdot v \circ M_{i,k} \cdot h(x^i) \cdot \nabla_{x^i} s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \\ D &= -2 \cdot \nabla_{x^i} (v \circ M_{i,k}) \cdot h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \quad E = -2 \cdot v \circ M_{i,k} \cdot \nabla_{x^i} h(x^i) \cdot s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top. \end{aligned}$$

We can verify that the following equations hold

$$\begin{aligned}
\text{(i)} \quad \nabla_{x^i}(M_{i,k}^{-1}) &= \frac{-1}{M_{i,k}^2} \cdot 2s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right), \quad \nabla_{x^i} r(x^i) = \Sigma^{i,k}, \\
\text{(ii)} \quad \nabla_{x^i} s(x^i) &= W^{i,k} \gamma^{i,k} + \mu^{i,k}, \quad \nabla_{x^i} h(x^i) = 2\Sigma^{i,k} x^i \\
\text{(iii)} \quad \nabla_{x^i}(v \circ M_{i,k}) &= v'(M_{i,k}) \cdot \nabla_{x^i} M_{i,k} = \frac{-2}{M_{i,k}^3} \cdot 2[W^{i,k}(x^i)^\top \gamma^{i,k} + (x^i)^\top \mu^{i,k} - D^{i,k}] \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right) \\
&= \frac{-2}{M_{i,k}^3} \cdot 2s(x^i) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right),
\end{aligned}$$

which in turn imply that

$$\begin{aligned}
A &= \frac{-4s(x^i)}{M_{i,k}^2} \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right) (x^i)^\top \Sigma^{i,k}, \quad B = \frac{2}{M_{i,k}} \Sigma^{i,k}, \quad C = \frac{-2h(x^i)}{M_{i,k}^2} \cdot (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \\
D &= \frac{8h(x^i)s^2(x^i)}{M_{i,k}^3} \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right) \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top, \quad E = \frac{-4s(x^i)}{M_{i,k}^2} \Sigma^{i,k} x^i \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top.
\end{aligned}$$

Note that $s^2(x^i) = W^{i,k} M_{i,k}$. Then, $D = \frac{8h(x^i)}{M_{i,k}^2} \cdot (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top$. Therefore, the Hessian matrix of $f^{i,k}$ with respect to x^i can be rewritten as

$$\begin{aligned}
H_{x^i} f^{i,k} &= A + B + C + D + E \\
&= \frac{-4s(x^i)}{M_{i,k}^2} \left[\frac{1}{W^{i,k}} \Sigma^{i,k} x^i (\mu^{i,k})^\top + \frac{1}{W^{i,k}} \mu^{i,k} (x^i)^\top \Sigma^{i,k} + \Sigma^{i,k} x^i (\gamma^{i,k})^\top + \gamma^{i,k} (x^i)^\top \Sigma^{i,k} \right] \\
&\quad + \frac{2}{M_{i,k}} \Sigma^{i,k} + \frac{6h(x^i)}{M_{i,k}^2} (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \cdot \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top.
\end{aligned}$$

By multiplying the above equation by $\frac{M_{i,k}^2}{2}$, we get

$$\begin{aligned}
\frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} &= M_{i,k} \Sigma^{i,k} + 3h(x^i) (W^{i,k} \gamma^{i,k} + \mu^{i,k}) \left(\gamma^{i,k} + \frac{\mu^{i,k}}{W^{i,k}} \right)^\top \\
&\quad - 2s(x^i) \left[\frac{1}{W^{i,k}} \Sigma^{i,k} x^i (\mu^{i,k})^\top + \frac{1}{W^{i,k}} \mu^{i,k} (x^i)^\top \Sigma^{i,k} + \Sigma^{i,k} x^i (\gamma^{i,k})^\top + \gamma^{i,k} (x^i)^\top \Sigma^{i,k} \right]. \tag{21}
\end{aligned}$$

We prove that the quadratic form of $H_{x^i} f^{i,k}$ at $z^i \in \mathbb{R}^{d_i}$ is positive, for any $z^i \in \mathbb{R}^{d_i}$ and $(x^i, W^{i,k}) \in Q^i \times [w_1^{i,k}, w_u^{i,k}]$, i.e.,

$$(z^i)^\top H_{x^i} f^{i,k} z^i \geq 0. \tag{22}$$

In fact, by taking the quadratic form of (21) at z^i , we have

$$\begin{aligned}
(z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i &= M_{i,k} (z^i)^\top \Sigma^{i,k} z^i + 3h(x^i) \left(\sqrt{W^{i,k}} ((z^i)^\top \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} ((z^i)^\top \mu^{i,k}) \right)^2 \\
&\quad - 4s(x^i) ((z^i)^\top \Sigma^{i,k} x^i) \left((\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i \right). \tag{23}
\end{aligned}$$

Note that the following inequalities hold

$$\begin{aligned}
\text{(i)} \quad &-4s(x^i) ((z^i)^\top \Sigma^{i,k} x^i) \left((\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i \right) \geq -4|s(x^i)| \cdot |(z^i)^\top \Sigma^{i,k} x^i| \cdot |(\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i|, \\
\text{(ii)} \quad &|(z^i)^\top \Sigma^{i,k} x^i| \leq \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i}, \quad \text{(iii)} \quad |(\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i| \leq |(\gamma^{i,k})^\top z^i| + \frac{1}{W^{i,k}} |(\mu^{i,k})^\top z^i|, \tag{24}
\end{aligned}$$

where (i) and (iii) are trivial. In order to prove (ii), let $e_1 = (z^i)^\top (\Sigma^{i,k})^{\frac{1}{2}}$, $e_2 = (x^i)^\top (\Sigma^{i,k})^{\frac{1}{2}}$, then by Cauchy-Schwarz inequality, (ii) is rewritten as

$$|e_1(e_2)^\top| \leq \|e_1\|_2 \|e_2\|_2.$$

It follows from (i) – (iii) of (24) that

$$\begin{aligned} & -4s(x^i)((z^i)^\top \Sigma^{i,k} x^i) \left((\gamma^{i,k})^\top z^i + \frac{1}{W^{i,k}} (\mu^{i,k})^\top z^i \right) \\ & \geq -4|s(x^i)| \cdot \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \cdot \left(|(\gamma^{i,k})^\top z^i| + \frac{1}{W^{i,k}} |(\mu^{i,k})^\top z^i| \right). \end{aligned} \quad (25)$$

Then, from (23) and (25), we get

$$\begin{aligned} & (z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i \geq M_{i,k} (z^i)^\top \Sigma^{i,k} z^i + 3h(x^i) \left(\sqrt{W^{i,k}} ((z^i)^\top \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} ((z^i)^\top \mu^{i,k}) \right)^2 \\ & - 4|s(x^i)| \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \left(|(\gamma^{i,k})^\top z^i| + \frac{1}{W^{i,k}} |(\mu^{i,k})^\top z^i| \right). \end{aligned}$$

Note that $|s(x^i)| = \sqrt{W^{i,k} M_{i,k}}$ and $3h(x^i) \left(\sqrt{W^{i,k}} ((z^i)^\top \gamma^{i,k}) + \frac{1}{\sqrt{W^{i,k}}} ((z^i)^\top \mu^{i,k}) \right)^2 \geq 0$, which in turn implies that

$$\begin{aligned} & (z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i \\ & \geq M_{i,k} (z^i)^\top \Sigma^{i,k} z^i - 4\sqrt{M_{i,k}} \sqrt{(z^i)^\top \Sigma^{i,k} z^i} \sqrt{(x^i)^\top \Sigma^{i,k} x^i} \left(\sqrt{W^{i,k}} |(\gamma^{i,k})^\top z^i| + \frac{1}{\sqrt{W^{i,k}}} |(\mu^{i,k})^\top z^i| \right). \end{aligned} \quad (26)$$

Assume that $z^i \neq 0$. Dividing both sides of (26) by $((z^i)^\top \Sigma^{i,k} z^i)((x^i)^\top \Sigma^{i,k} x^i)$, we have

$$\frac{(z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i}{((z^i)^\top \Sigma^{i,k} z^i)((x^i)^\top \Sigma^{i,k} x^i)} \geq \frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i} - 4\sqrt{\frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i}} \left(\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \right). \quad (27)$$

Note that $\frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i} = [g^{i,k}(x^i, W^{i,k})]^2$. Since $x^i \in Q^i$ and Q^i is a subset of $\bigcap_{k \in J^i} \Omega^{i,k}$, then $x^i \in \Omega^{i,k}$, for any $k \in J^i$. Then, for any $W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$, it follows from the definition of $\Omega^{i,k}$ in (12) that

$$g^{i,k}(x^i, W^{i,k}) > 0.$$

Therefore, we can write $\sqrt{\frac{M_{i,k}}{(x^i)^\top \Sigma^{i,k} x^i}} = g^{i,k}(x^i, W^{i,k})$ and (27) is equivalent to the following inequality

$$\frac{(z^i)^\top \frac{(M_{i,k})^2}{2} \cdot H_{x^i} f^{i,k} z^i}{((z^i)^\top \Sigma^{i,k} z^i)[(x^i)^\top \Sigma^{i,k} x^i]} \geq [g^{i,k}(x^i, W^{i,k})]^2 - 4g^{i,k}(x^i, W^{i,k}) \left(\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \right).$$

It is easy to see that the following inequalities hold

$$\begin{aligned} \text{(i)} \quad & \left| \frac{(\gamma^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \leq \frac{\|\gamma^{i,k}\|_2 \cdot \|z^i\|_2}{\sqrt{\lambda_{i,k,\min}} \cdot \|z^i\|_2} = \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}, \quad \text{(ii)} \quad \left| \frac{(\mu^{i,k})^\top z^i}{\sqrt{(z^i)^\top \Sigma^{i,k} z^i}} \right| \leq \frac{\|\mu^{i,k}\|_2 \cdot \|z^i\|_2}{\sqrt{\lambda_{i,k,\min}} \cdot \|z^i\|_2} = \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}}, \\ \text{(iii)} \quad & \sqrt{W^{i,k}} \leq \sqrt{w_u^{i,k}}, \quad \frac{1}{\sqrt{W^{i,k}}} \leq \frac{1}{\sqrt{w_1^{i,k}}}, \end{aligned}$$

which in turn implies that

$$\sqrt{W^{i,k}} \left| \frac{(\gamma^{i,k})^\top z}{\sqrt{z^\top \Sigma^{i,k} z}} \right| + \frac{1}{\sqrt{W^{i,k}}} \left| \frac{(\mu^{i,k})^\top z}{\sqrt{z^\top \Sigma^{i,k} z}} \right| \leq \frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_l^{i,k}}}. \quad (28)$$

Then, from (27) and (28), we get

$$\frac{z^\top \frac{(M_{i,k})^2}{2} H_{x^i} f^{i,k} z}{(z^\top \Sigma^{i,k} z) [(x^i)^\top \Sigma^{i,k} x^i]} \geq [g^{i,k}(x^i, W^{i,k})]^2 - 4g^{i,k}(x^i, W^{i,k}) \left(\frac{\|\gamma^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \sqrt{w_u^{i,k}} + \frac{\|\mu^{i,k}\|_2}{\sqrt{\lambda_{i,k,\min}}} \frac{1}{\sqrt{w_l^{i,k}}} \right). \quad (29)$$

Since $x^i \in \Omega^{i,k}$, for any $k \in J^i$, the term on the right-hand side of (29) is positive, which implies that $(z^i)^\top H_{x^i} f^{i,k} z^i \geq 0$. \square

Using the abovementioned lemmas, we prove the following convexity result.

Lemma 5. *Let Assumption 2 hold. Then, $S_{\alpha_i}^i$ is a convex set, for all $i \in H$.*

Proof. Let $\alpha_i \in (\alpha_i^*, 1]$, $\lambda \in (0, 1)$ and $y_1, y_2 \in S_{\alpha_i}^i$. We will show that $\lambda y_1 + (1 - \lambda)y_2 \in S_{\alpha_i}^i$. Consider 4 cases as follows

Case 1: Let $y_1 = 0$ or $y_2 = 0$. Without loss of generality, we assume that $y_2 = 0$. Since $y_0 = 0 \in S_{\alpha_i}^i$, then $\mathbb{P}(0 \leq D^i) \geq \alpha_i$. Since $\alpha_i > 0$ and D^i is a real vector, we have that $D^i \geq 0$, which implies that

$$\mathbb{P}(V^i y_1 \leq D^i) \geq \mathbb{P}(V^i y_1 \leq D^i) \geq \alpha_i.$$

Therefore, $\lambda y_1 + (1 - \lambda)y_2 \in S_{\alpha_i}^i$.

Case 2: Let $y_1 \neq 0$, $y_2 \neq 0$ and $\lambda y_1 + (1 - \lambda)y_2 = 0$. In this case, $y_2 = \frac{-\lambda}{1-\lambda} y_1 \in \tilde{S}_{\alpha_i}^i$ and $y_1 \in \tilde{S}_{\alpha_i}^i$. It follows from Lemma 1 that

$$(\mu^{i,k})^\top y_1 > \frac{\lambda - 1}{\lambda} D^{i,k}, \quad (\mu^{i,k})^\top y_1 < D^{i,k}, \quad \forall k \in J^i.$$

This implies that $D^{i,k} \geq 0$ for all $k \in J^i$. Hence, $\lambda y_1 + (1 - \lambda)y_2 = 0 \in S_{\alpha_i}^i$.

Case 3: Let $y_1 \neq 0$, $y_2 \neq 0$ and $0 \in \text{Seg}(y_1, y_2)$, where

$$\text{Seg}(y_1, y_2) = \{y_1 + l(y_2 - y_1), 0 \leq l \leq 1\}.$$

Then, for any point $x \in \text{Seg}(y_1, y_2)$, either $x \in \text{Seg}(y_1, 0)$ or $x \in \text{Seg}(0, y_2)$. It follows from Case 1 that $\text{Seg}(y_1, 0)$ and $\text{Seg}(0, y_2)$ are subset of $S_{\alpha_i}^i$. Therefore, $\lambda y_1 + (1 - \lambda)y_2 \in S_{\alpha_i}^i$ for all $\lambda \in [0, 1]$.

Case 4: Let $y_1 \neq 0$, $y_2 \neq 0$ such that $0 \notin \text{Seg}(y_1, y_2)$. It is clear that $\text{Seg}(y_1, y_2) \subset \text{Conv}(\tilde{S}_{\alpha_i}^i)$. From Lemmas 3 and 4, $g^{i,k}(\cdot, W^{i,k})$ is defined and (-2) -concave on $\text{Seg}(y_1, y_2)$, for all $W^{i,k} \in [w_l^{i,k}, w_u^{i,k}]$, which implies that

$$g^{i,k}(\lambda y_1 + (1 - \lambda)y_2, W^{i,k}) \geq \left(\lambda (g^{i,k}(y_1, W^{i,k}))^{-2} + (1 - \lambda) (g^{i,k}(y_2, W^{i,k}))^{-2} \right)^{-\frac{1}{2}}. \quad (30)$$

Since, $y_1 \in \tilde{S}_{\alpha_i}^i$ and $\alpha_i > \alpha_i^{(3)}$, using Lemma 2, for any $z \in [w_l^{i,k}, w_u^{i,k}]$, the following condition holds

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(y_1, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right) \right] > \alpha_i^{(3)}.$$

Moreover, it follows from the definition of $\alpha_i^{(3)}$ in Assumption 2 that

$$\alpha_i^{(3)} \geq \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_l^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_l^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_l^{i,k}}} \right),$$

which in turn implies that

$$\mathbb{E}_{W^{i,k}} \left[\Phi \left(\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(y_1, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) \right] > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_1^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right). \quad (31)$$

If $g^{i,k}(z_1, z) \leq 0$, (31) implies that

$$\Phi \left(\frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_1^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right),$$

which gives a contradiction by the increasing monotonicity of Φ . Therefore, $g^{i,k}(y_1, z) \geq 0$. For any $W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$, we have

$$\frac{\sqrt{z}}{\sqrt{W^{i,k}}} g^{i,k}(y_1, z) \leq \frac{\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} g^{i,k}(y_1, z). \quad (32)$$

From (31) and (32), for any $z \in [w_1^{i,k}, w_u^{i,k}]$, we get

$$\Phi \left(\frac{\sqrt{w_u^{i,k}}}{\sqrt{w_1^{i,k}}} g^{i,k}(y_1, z) + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right) > \Phi \left(\frac{\sqrt{3w_u^{i,k}}}{\sqrt{w_1^{i,k}}} + \frac{\|\gamma^{i,k}\|_2 (w_u^{i,k} - w_1^{i,k})}{\sqrt{\lambda_{i,k,\min}} \sqrt{w_1^{i,k}}} \right),$$

which in turn implies that

$$0 < g^{i,k}(y_1, W^{i,k})^{-2} < \frac{1}{3}, \forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}].$$

Similarly,

$$0 < g^{i,k}(y_2, W^{i,k})^{-2} < \frac{1}{3}, \forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}].$$

By applying the non-decreasing function $\Phi(\cdot)$ on both side of (30), we can write

$$\Phi \left(g^{i,k} \left(\lambda y_1 + (1 - \lambda) y_2, W^{i,k} \right) \right) \geq \Phi \left(\left(\lambda (g^{i,k}(y_1, W^{i,k}))^{-2} + (1 - \lambda) (g^{i,k}(y_2, W^{i,k}))^{-2} \right)^{-\frac{1}{2}} \right), \forall W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]. \quad (33)$$

Since, $\Phi(\cdot)$ is the cumulative distribution function of a standard Normal distribution, it follows from Proposition 4.1 of (Henrion and Strugarek, 2008) that $\Phi(\cdot)$ has r -decreasing density, for any $r > 0$ and $t^* = \sqrt{r}$, where t^* refers to Definition 1. By choosing $r = 3$, it follows from Lemma 3.1 of (Henrion and Strugarek, 2008), the function $t \mapsto \Phi \left(t^{-\frac{1}{2}} \right)$ is concave on $(0, \frac{1}{3})$. Therefore, for any $W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$, we obtain

$$\Phi \left(\left(\lambda (g^{i,k}(y_1, W^{i,k}))^{-2} + (1 - \lambda) (g^{i,k}(y_2, W^{i,k}))^{-2} \right)^{-\frac{1}{2}} \right) \geq \lambda \left(\Phi \left(g^{i,k}(y_1, W^{i,k}) \right) \right) + (1 - \lambda) \left(\Phi \left(g^{i,k}(y_2, W^{i,k}) \right) \right). \quad (34)$$

For any $W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$, it follows from (33) and (34) that

$$\Phi \left(g^{i,k}(\lambda y_1 + (1 - \lambda) y_2, W^{i,k}) \right) \geq \lambda \left(\Phi \left(g^{i,k}(y_1, W^{i,k}) \right) \right) + (1 - \lambda) \left(\Phi \left(g^{i,k}(y_2, W^{i,k}) \right) \right), \quad (35)$$

which implies that $\Phi(g^{i,k}(\cdot, W^{i,k}))$ is a concave function on $\text{Seg}(y_1, y_2)$, for any $W^{i,k} \in [w_1^{i,k}, w_u^{i,k}]$. By taking the expectation value $\mathbb{E}_{W^{i,k}}$ on both sides of (35), we deduce that

$$\mathbb{E}_{W^{i,k}} \left[\Phi(g^{i,k}(\lambda y_1 + (1-\lambda)y_2, W^{i,k})) \right] \geq \lambda \left(\mathbb{E}_{W^{i,k}} \left[\Phi(g^{i,k}(y_1, W^{i,k})) \right] \right) + (1-\lambda) \left(\mathbb{E}_{W^{i,k}} \left[\Phi(g^{i,k}(y_2, W^{i,k})) \right] \right) \quad (36)$$

which in turn implies that the function $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(\cdot, W^{i,k}))]$ is a concave function on $\text{Seg}(y_1, y_2)$. It follows from the discussion in Definition 2.1 of (Henrion and Strugarek, 2008) that log-concavity is a weaker property than concavity. Therefore, $\mathbb{E}_{W^{i,k}} [\Phi(g^{i,k}(\cdot, W^{i,k}))]$ is also a log-concave function on $\text{Seg}(y_1, y_2)$. Then, $\lambda y_1 + (1-\lambda)y_2 \in S_{\alpha_i}^i$. \square

The convexity of $S_{\alpha_i}^i$ under normal mean-variance mixture distribution was shown in Theorem 2.3 of (Nguyen and Lisser, 2021) when x^i satisfy certain conditions (e.g. condition 3 and 4 of Theorem 2.3 (Nguyen and Lisser, 2021)). In the numerical section of (Nguyen and Lisser, 2021), it is shown that these assumptions hold under some strict conditions, e.g., $D^{i,k} \geq N \times d_i$, for any $i \in H$, $k \in J^i$, where d_i is the dimension of x^i and N is a large number ($N \geq 200$ in general). In Lemma 5, we relax the strong conditions on x^i used in Theorem 2.3 of (Nguyen and Lisser, 2021) and give the lower bound α_i^* , defined in Assumption 2, on the probability level α_i for which $S_{\alpha_i}^i$ is a convex set. The lower bound α_i^* can be calculated using the parameters of normal mean-variance mixture distribution. If we set $\|\gamma^{i,k}\|_2$ and $\|\mu^{i,k}\|_2$ small compared to $\sqrt{\lambda_{i,k,\min}}$ (which is relevant as shown in Section 4 due to high value of eigenvalue of covariance matrix to model risky decision) and $w_u^{i,k} \approx w_1^{i,k}$, the lower bound of α_i^* tends to $\Phi(\sqrt{3}) \approx 0.958$ because $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ tend to $\Phi(0)$ and $\alpha_i^{(3)}$ tends to $\Phi(\sqrt{3})$. Therefore, the best possible value of the lower bound on probability level is around 0.96. The earlier work (Nguyen and Lisser, 2021) does not give any theoretical lower bound of α_i . The authors presented few examples where the lower bound can go up to 0.99. In practice, a high probability level value (≥ 0.95) for the chance constraints is desirable as it ensures the reliability of the solution. In the next section, we consider a competition between two financial firms whose loss vectors follow normal mean-variance mixture distributions for which $\alpha_i^* = 0.96$, i.e., it attains the best possible theoretical lower bound.

The closeness of $S_{\alpha_i}^i$ is a consequence of Proposition 3.1 in (Römisch and Schultz, 1991). Since $S_{\alpha_i}^i$ is a subset of S^i , which is a compact set, we deduce that $S_{\alpha_i}^i$ is a compact set. We show that there exists a Nash equilibrium of the CCG by the following theorem.

Theorem 1. Consider an n -player CCG defined in Section 1, where

1. Assumptions 1 and 2 hold.
2. For each $i \in H$, we assume that $V^{i,k} \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$ and the vectors $V^{i,k}$ are mutually independent, $k \in J^i$.

Then, there exists a Nash equilibrium of the CCG for any $\alpha \in (\alpha_1^*, 1] \times \dots \times (\alpha_n^*, 1]$, where $\{\alpha_i^*\}_{i \in H}$ refers to Assumption 2.

Proof. Given $\alpha \in (\alpha_1^*, 1] \times \dots \times (\alpha_n^*, 1]$, and subject to Assumption 1, the payoff function $p_i(x^i, x^{-i})$ is concave in terms of x^i for any $x^{-i} \in S^{-i}$ and continuous with respect to x . By applying Lemma 5, it can be deduced that the feasible strategy set $S_{\alpha_i}^i$, where $i \in H$, is a convex set for all $\alpha_i \in (\alpha_i^*, 1]$. Additionally, each $S_{\alpha_i}^i$ is a compact set. As a consequence, the existence of a Nash equilibrium in the context of the CCG follows from Theorem 4 presented in (Fan, 1966). \square

4. Competition of two firms in financial market

We consider a competition model of two firms in a same financial market. Let $J = \{1, \dots, K\}$ be the set of portfolios and \mathcal{A}_k be the set of assets in portfolio k , for $k \in J$. We assume that both firms invest in the same set of portfolios from J and the portfolios are pairwise disjoint. Each firm i invests its money in the assets. Let x_{kj}^i be the amount of money that firm i invests in asset j of portfolio k . Let $x_k^i = (x_{kj}^i)_{j \in \mathcal{A}_k}$ be the investment vector of firm i in

portfolio k and $x^i = (x_k^i)_{k \in J}$ be the strategy vector of firm i . The strategy set of firm i (or the set of investments) is defined as

$$S^i = \left\{ x^i \mid \sum_{j \in \mathcal{A}_k, k \in J} x_{kj}^i \leq B^i, x_{kj}^i \geq \epsilon_{kj}^i, \text{ for any } k \in J, j \in \mathcal{A}_k \right\},$$

where B^i is the budget of firm i and $\epsilon_{kj}^i > 0$ is the minimal amount firm i must invest in asset j of portfolio k . It is clear that S^i is a convex and compact set with strictly positive components, for any $i = 1, 2$. The vector $(x^1, x^2) \in S^1 \times S^2$ represents a strategy profile (or an investment profile) of both firms. Let $L_k^i = (L_{kj}^i)_{j \in \mathcal{A}_k}$ be a random loss vector of firm i from portfolio k . Then, for a given investment vector x_k^i , the random loss incurred by firm i from portfolio k is $(L_k^i)^\top x_k^i$. Let D_k^i be the maximal loss level of firm i from portfolio k and firm i wants to keep its random loss below this level at probability level α_i , i.e.,

$$\mathbb{P} \{ (x_k^i)^\top L_k^i \leq D_k^i, k \in J \} \geq \alpha_i, \forall i = 1, 2.$$

Hence, the feasible strategy set of firm i is defined as

$$S_{\alpha_i}^i = \{ x^i \in S^i \mid \mathbb{P} \{ (x_k^i)^\top L_k^i \leq D_k^i, k \in J \} \geq \alpha_i \}$$

We assume that for any $i = 1, 2$ and $k \in J$, the random loss vector L_k^i follows a normal mean-variance mixture distribution. Let $R_k^i = -L_k^i$ be the random return vector of firm i from portfolio k . We consider the case where each firm wants to minimize their transaction cost which is incurred due to trades from multiple firms. The transaction cost of a firm usually also depends on the investment of other firms (Lampariello et al., 2021). Therefore, for a given strategy profile (x^1, x^2) , we consider the quadratic transaction cost as follows

$$\text{TC}^i(x^1, x^2) = \sum_{k \in J} (x_k^1 + x_k^2)^\top \Omega_k^i (x_k^1 + x_k^2),$$

where $\text{TC}^i(x^1, x^2)$ is the transaction cost of firm i at strategy profile (x^1, x^2) , the positive semidefinite matrix Ω_k^i represents the market impact of portfolio k on firm i whose entry at position (r, s) is the impact of the liquidity of asset r on the liquidity of asset s . The same formulation of transaction cost has been considered in (Lampariello et al., 2021). The payoff function of firm i is defined as follows

$$u^i(x^1, x^2) = \left(\sum_{k \in J} \mathbb{E}(R_k^i)^\top x_k^i \right) - \text{TC}^i(x^1, x^2),$$

where $\sum_{k \in J} \mathbb{E}(R_k^i)^\top x_k^i$ is the expected return of firm i at strategy profile (x^1, x^2) . It is clear that the payoff function of firm i is a continuous function of the strategy profile and concave with respect to its strategy vector, for every fixed strategy vector of the other firm.

In our case study, we consider two firms with three portfolios where each portfolio consists of three assets, i.e., $J = \{1, 2, 3\}$ and $\mathcal{A}_k = \{1, 2, 3\}$, for any $k \in J$. We assume that the random loss vectors follow normal mean-variance mixture distributions, i.e., for any $i = 1, 2$ and $k \in J$, $L_k^i \sim \text{NMVM}(\mu^{i,k}, \gamma^{i,k}, \Sigma^{i,k}, W^{i,k})$, where $\mu^{i,k}$ is a 3×1 vector taken on $[-0.35, 0]^3$ and $\gamma^{i,k}$ is a 3×1 vectors taken on $[0, 0.02]^3$. $\Sigma^{i,k}$ is a 3×3 positive definite matrix with all eigen values belong to $[5, 10]$ and high values on the main diagonal and it is given by

$$\Sigma^{i,k} = \frac{AA^\top}{3 \times \max(1, \lambda_{\max})} + \beta \times I_3,$$

where A is a 3×3 random matrix whose all the entries are real numbers in $[0, 1]$, λ_{\max} is the largest eigenvalue of the semidefinite positive matrix $\frac{AA^\top}{3}$, β is a real number taken on $[5, 9]$ and I_3 is 3×3 identity matrix. We take negative values of location parameters $\mu^{i,k}$ and high values on the main diagonal of covariance matrix $\Sigma^{i,k}$ because the firms gain positive return in expectation if they decide to invest but they have to make risky decision. $W^{i,k}$ follows

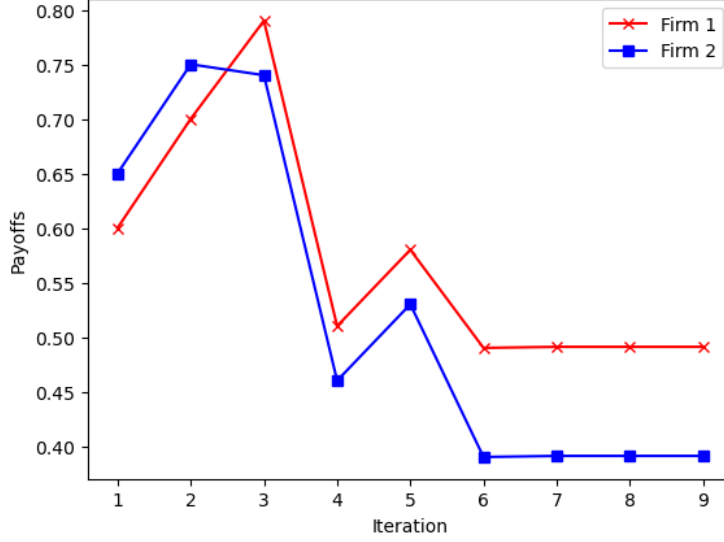


Figure 1: Convergence of payoffs to a Nash equilibrium.

an uniform distribution with support in $[w_l^{i,k}, w_u^{i,k}]$, where $w_l^{i,k}$ is taken on $[0.9, 1]$ and $w_u^{i,k}$ is taken on $[1, 1.1]$. We randomly generate an instance of the above framework and compute the threshold levels $(\alpha_i^*)_{i=1,2}$, which is equal to 0.958. Then, we take the probability levels $\alpha_1 = \alpha_2 = 0.96$ such that Assumption 2 holds. The other parameters are given by $B^1 = B^2 = 10$, $\epsilon_{kj}^1 = \epsilon_{kj}^2 = 0.1$, $D_k^1 = 1$, $D_k^2 = 1.5$, Ω_k^i is a 3×3 positive definite matrix, generated by $\Omega_k^i = \frac{BB^\top}{3}$, where B is a 3×3 random matrix whose all the entries are real numbers belonging to $[0, 1]$. Then, the payoff function of two firms are calculated by

$$u^i(x^1, x^2) = - \left[\sum_{k \in J} (\mu^{i,k})^\top x_k^i \right] - \sum_{k \in J} (x_k^1 + x_k^2)^\top \Omega_k^i (x_k^1 + x_k^2).$$

The numerical results below are performed using Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD. We compute a Nash equilibrium using the well-known best response algorithm. The algorithm converges to a Nash equilibrium point (x^{1*}, x^{2*}) given by

$$\begin{aligned} x^{1*} &= [(0.48, 0.51, 0.63), (0.47, 0.31, 0.55), (0.67, 0.25, 0.4)], \\ x^{2*} &= [(0.52, 0.31, 0.73), (0.57, 0.54, 0.57), (0.47, 0.55, 0.3)], \end{aligned}$$

Figure 1 shows that best response algorithm converges after few iterations.

5. Conclusion

We show that there exists a Nash equilibrium of an n player CCG when the random constraint vectors of the joint chance constraint are independent and follow normal mean-variance mixture distributions. The convexity of joint chance constraint, which is crucial for the existence of Nash equilibrium, does not require two strong conditions used in an earlier work (Nguyen and Lisser, 2021). It only needs a theoretical lower bound on the probability level which is defined in terms of the parameter of the normal mean-variance mixture distribution.

Data availability statement

No data was used for the research described in the article.

Acknowledgments

This research was supported by French government under the France 2030 program, reference ANR-11-IDEX-0003 within the OI H-Code.

References

- O. Barndorff-Nielsen, J. Kent, and M. Sørensen. Normal variance-mean mixtures and z distributions. *International Statistical Review*, 50(2): 145–159, 1982.
- Christian Bauer. Value at risk using hyperbolic distributions. *Journal of Economics and Business*, 52(5):455–467, 2000.
- Jianqiang Cheng and Abdel Lisser. A second-order cone programming approach for linear programs with joint probabilistic constraints. *Operations Research Letters*, 40(5):325–328, 2012. URL <https://doi.org/10.1016/j.orl.2012.06.008>.
- Antoine Augustin Cournot. *Researches into the Mathematical Principles of the Theory of Wealth*. Macmillan Company, New York, 1897.
- Ernst Eberlein and Karsten Prause. The generalized hyperbolic model: Financial derivatives and risk measures. In Geman, Hélyette and Madan, Dilip and Pliska, Stanley R. and Vorst, Ton. (eds), *Mathematical Finance — Bachelier Congress 2000: Selected Papers from the First World Congress of the Bachelier Finance Society, Paris, June 29–July 1, 2000*, pages 245–267. Springer, Berlin, Heidelberg, 2002.
- Ky Fan. Applications of a theorem concerning sets with convex sections. *Mathematische Annalen*, 163(3):189–203, 1966.
- Kai-Tai Fang, Samuel Kotz, and Kai Wang Ng. *Symmetric multivariate and related distributions*. Chapman and Hall/CRC, 2018.
- R. Henrion and C. Strugarek. Convexity of chance constraints with independent random variables. *Computational Optimization and Applications*, 41(2):263–276, 2008.
- Bri-Mathias Hodge, Anthony Florita, Kirsten Orwig, Debra Lew, and Michael Milligan. Comparison of wind power and load forecasting error distributions. Technical report, National Renewable Energy Lab.(NREL), Golden, CO (United States), 2012.
- Ran Ji and Miguel A Lejeune. Risk-budgeting multi-portfolio optimization with portfolio and marginal risk constraints. *Annals of Operations Research*, 262(2):547–578, 2018.
- Aswin Kannan, Uday V Shanbhag, and Harrison M Kim. Addressing supply-side risk in uncertain power markets: stochastic Nash models, scalable algorithms and error analysis. *Optimization Methods and Software*, 28(5):1095–1138, 2013. URL <https://doi.org/10.1080/10556788.2012.676756>.
- Lorenzo Lampariello, Christoph Neumann, Jacopo M. Ricci, Simone Sagratella, and Oliver Stein. Equilibrium selection for multi-portfolio optimization. *European Journal of Operational Research*, 295(1):363–373, 2021. ISSN 0377-2217. URL <https://doi.org/10.1016/j.ejor.2021.02.033>.
- Sharon X. Lee and Geoffrey J. McLachlan. On mean and/or variance mixtures of normal distributions. In Simona Balzano, Giovanni C. Porzio, Renato Salvatore, Domenico Vistocco, and Maurizio Vichi, editors, *Statistical Learning and Modeling in Data Analysis*, pages 117–127, Cham, 2021. Springer International Publishing.
- Xiangyu Li, Chaojie Li, Guo Chen, and Zhao Yang Dong. A risk-averse energy sharing market game for renewable energy microgrid aggregators. *IEEE Transactions on Power Systems*, 37(5):3528–3539, 2022.
- Xun Li and Duan Li. Mean-variance portfolio optimization with a normal mean-variance mixture model. *Journal of Industrial & Management Optimization*, 12(3):1129–1151, 2016.
- Alexander J McNeil, Rüdiger Frey, and Paul Embrechts. *Quantitative risk management: concepts, techniques and tools-revised edition*. Princeton university press, 2015.
- Mehrdad Naderi, Elham Mirfarah, Wan-Lun Wang, and Tsung-I Lin. Robust mixture regression modeling based on the normal mean-variance mixture distributions. *Computational Statistics and Data Analysis*, 180:107661, 2023. ISSN 0167-9473.
- John Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951. URL <https://doi.org/10.2307/1969529>.
- John F Nash Jr. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950.
- Hoang Nam Nguyen and Abdel Lisser. Convexity conditions for normal mean-variance mixture distribution in joint probabilistic constraints. 2021.
- Hoang Nam Nguyen, Abdel Lisser, and Vikas Vikram Singh. Random games under elliptically distributed dependent joint chance constraints. *Journal of Optimization Theory and Applications*, 195(1):249–264, 2022.
- Shen Peng, Vikas Vikram Singh, and Abdel Lisser. General sum games with joint chance constraints. *Operations Research Letters*, 46(5):482–486, 2018.
- Shen Peng, Navnit Yadav, Abdel Lisser, and Vikas Vikram Singh. Chance-constrained games with mixture distributions. *Mathematical Methods of Operations Research*, 2021.
- Rostislav S Protasov. Em-based maximum likelihood parameter estimation for multivariate generalized hyperbolic distributions with fixed λ . *Statistics and Computing*, 14(1):67–77, 2004.
- Svetlozar Todorov Rachev. *Handbook of Heavy Tailed Distributions in Finance: Handbooks in Finance, Book 1*. Elsevier, 2003.
- Uma Ravat and Uday V Shanbhag. On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM Journal on Optimization*, 21(3):1168–1199, 2011. URL <https://doi.org/10.1137/100792644>.
- Werner Römisch and Rüdiger Schultz. Stability analysis for stochastic programs. *Annals of Operations Research*, 30(1):241–266, 1991.
- Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on stochastic programming: modeling and theory*. SIAM, 2021.
- Vikas Vikram Singh and Abdel Lisser. A characterization of Nash equilibrium for the games with random payoffs. *Journal of Optimization Theory and Applications*, 178(3):998–1013, 2018.
- Vikas Vikram Singh and Abdel Lisser. A second-order cone programming formulation for two player zero-sum games with chance constraints. *European Journal of Operational Research*, 275(3):839–845, 2019.
- Vikas Vikram Singh, Oualid Jouini, and Abdel Lisser. Existence of Nash equilibrium for chance-constrained games. *Operations Research Letters*, 44(5):640–644, 2016.

Vikas Vikram Singh, Abdel Lisser, and Monika Arora. An equivalent mathematical program for games with random constraints. *Statistics & Probability Letters*, 174:109092, 2021.