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Convexity of linear joint chance constrained optimization with elliptically distributed dependent rows

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ABSTRACT

In this paper, we study the convexity of the linear joint chance constraints. We assume that the constraint row vectors are elliptically distributed. Further, the dependence of the rows is modeled by a family of Archimedean copulas, namely, the Gumbel–Hougaard copulas. Under mild assumptions, we prove the eventual convexity of the feasibility set.

1. Introduction

We consider the following linear optimization with joint chance constraints

$$\begin{aligned} \min \quad & c^{\top} x \\ \text{subject to} \quad & \mathbb{P}\{Vx \leq D\} \geq p \\ & x \in Q. \end{aligned} \tag{1}$$

where Q is a closed convex subset of $\mathbb{R}^n$ such that $\delta_{\mathrm{l}} \leq \|x\| \leq \delta_{\mathrm{u}}$, for any $x \in Q$, $\|\cdot\|$ denotes the Euclidean norm, δ_{l} and δ_{u} are strictly positive real numbers, $D = [D_1, \ldots, D_K]^{\top} \in \mathbb{R}^K$ is a deterministic vector, $V = [v_1, \ldots, v_K]^{\top}$ is a random matrix with size $K \times n$, where v_k is a random vector in $\mathbb{R}^n$, for any $k = 1, 2, \ldots, K$ and $p \in (0, 1)$. We denote $S(p)$ the feasibility set of (1). Let $J = \{1, 2, \ldots, K\}$.

The convexity of chance constraints as well as the analytical properties of the probability function play an important role in convex optimization which are difficult issues and scarcely studied in the literature. This problem was first introduced by Prékopa [1]. He considers the following chance constraints

$$\mathbb{P}(h(x, \xi) \geq 0) \geq p, \tag{2}$$

where $x \in \mathbb{R}^n$ is a decision vector, $\xi : \Omega \to \mathbb{R}^m$ is a random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$ and $p \in [0, 1]$ is a given probability threshold. Theorem 10.2.1 in [1] states that the feasibility set of (2) is convex if $\mathbb{P} \circ \xi^{-1}$ of ξ is a log-concave probability measure on $\mathbb{R}^m$ and the components of h are quasi-concave, where $\mathbb{P} \circ \xi^{-1}$ is the pushforward measure (or

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image measure) of the probability measure $\mathbb{P}$ in $\mathbb{R}^m$ induced by the random vector ξ, denoted by $\mathbb{P}\circ\ \xi^{-1}$. Mathematically, given a Lebesgue measure set $B \subset \mathbb{R}^m$, the pushforward measure $\mathbb{P}\circ\ \xi^{-1}$ of B is given by

$$\mathbb{P} \circ \xi^{-1}(B) = \mathbb{P}(\omega \in \Omega \mid \xi(\omega) \in B).$$

Henrion and Strugarek [2] studied a particular form of (2) by taking $h(x, \xi) = g(x) - \xi$, where $g : \mathbb{R}^m \to \mathbb{R}^n$. Hence, they consider the following form

$$\mathbb{P}(\xi \leq g(x)) \geq p, \tag{3}$$

where they suppose that the components of ξ are independent. They prove that if the cumulative distribution functions of the components of ξ have $r-$ decreasing densities and the components of g are $r-$ concave, the feasibility set of (3) is convex. Henrion and Strugarek [3] generalized this result to the case where the components of ξ are dependent. He uses the theory of copulas to model the dependence of the components of ξ. Marti [4] studied the differentiation of probability functions by an integral transformation method. The derivatives of the probability function can be obtained by applying an integral transformation to its integral representation. Some basic results on the differentiability of a probability function were studied by Kibzun et al. [5]. They proposed new formulations of the gradient of probability functions in different forms, i.e., integral over the surface, volume, or sum of surface and volume integrals. Lobo et al. [6] studied some applications of second-order cone program leading to a new approach for solving chance constraints. A more developed direction was initialized by Henrion and Strugarek [2] which gave a full description of the structure (not only the convexity) of a one-row linear optimization with a chance constraint by introducing a new notion of r-decreasing function. Henrion and Strugarek [2] studied the convexity in the case where the constraints are independent. To deal with the dependent case, Henrion and Strugarek [3], Cheng et al. [7] and Van Ackooij [8,9] used the theory of copulas to model the dependence of the constraints. They supposed that the distribution of the constraint row vectors are elliptically distributed. Under high probability threshold p, they prove the convexity of $S(p)$. Hong et al. [10] proposed to solve joint chance-constrained programs by sequential convex approximations. They proved that the solutions of the sequence of approximations converge to a Karush–Kuhn–Tucker (KKT) point of the original problem. Farshbaf-Shaker et al. [11] proved some properties of chance constraints in infinite dimensions. They supposed that the feasibility set belongs to a Banach space. Under mild conditions, they proved regularity properties of the probability function with an application to PDE constrained optimization. Wim van Ackooij and Malick [12] studied the convexity of the feasibility set in a general framework by using the radial representation of elliptical distributions.

The convexity of chance constraints has been used to show the existence of a Nash equilibrium in chance-constrained games [13–15]. Nguyen et al. [13] assume that the random constraint vectors follow elliptical distributions and show that there exists a Nash equilibrium of the chance-constrained game. Peng et al. [14,15] show a similar result by assuming that the random constraint vectors follow either Normal distributions or mixture of elliptical distributions.

The convexity of chance constraints could significantly impact the fields of operations research, logistics, supply chain management, financial risk management, and many others, by providing more effective ways to handle uncertainties. Here are some of the advantages and potential real-world applications.

- **Efficient Solution Methods:** If chance constraints are shown to be convex, more efficient solution methods (such as interior-point methods) can be applied to solve the problem, saving time and computational resources. This is particularly useful in large-scale stochastic optimization problems where computational efficiency is critical.
- **Quality of Solution:** Convex problems have the property that any local optimum is also a global optimum. Therefore, if the chance constraints are convex, we have the guarantee that if a local optimum is also a global optimum. This leads to better decision making in practical situations.
- **Robustness and Certainty:** Convexity of chance constraints allows for a higher level of robustness and certainty in the solutions obtained. Convex problems are less susceptible to changes in the data. This is especially beneficial in real-world scenarios where data can often change.
- **Modeling Flexibility:** The convexity of chance constraints could increase the modeling flexibility, allowing researchers and practitioners to model complex, real-world situations more accurately.

Chance constraints can be used in operations research to handle the uncertainty in decision-making processes. Here are a few concrete examples

- In supply chain optimization problems, there can be uncertainty in demand, transportation costs, and delivery times. Using chance constraints, a supply chain manager could design a distribution network so that the probability of meeting customer demand is at least a certain percentage. This ensures that the supply chain is robust enough to meet demand under varying conditions, which could reduce costs associated with stockouts or overstocking.
- In portfolio optimization, an investor may wish to ensure that the probability of the portfolio's return falling below a certain level is minimized. This can be formulated as a chance constraint optimization problem, where the objective is to maximize the expected return subject to a chance constraint on the portfolio's return.
- In renewable energy planning, power production from sources like wind and solar is uncertain. Chance constraints can be used to ensure that a certain level of power demand is met with a high probability, given the uncertainty in renewable energy production. For example, an energy planner might need to decide how much backup capacity to maintain, given the uncertainty in wind power production.

- In hospital resource allocation, chance constraints can be used to make sure there is a high probability of having enough resources (beds, doctors, nurses, medical supplies) to meet patient demand. For example, a hospital could use chance constraints to decide how many operating rooms to keep open, given the uncertainty in the number of surgeries that will need to be performed.

In this paper, we study the convexity of the feasible set $S(p)$ when the row vectors v_i follow an elliptical distribution and the dependence of the random constraint vectors is captured by a Gumbel–Hougaard copula. We derive a new reformulation of the joint chance constraints and prove the convexity of $S(p)$ under mild conditions. Our main contributions can be summarized as follows

- Compared with [3,8], we consider the chance constraints with random matrix.
- Compared with the convexity results in [2,12], we consider the dependent rows with copula.
- Cheng et al. [7] consider the joint chance constraints defined in (1). They assume that the dependence of the row vectors $v_i^\top$, $i \in J$ follows a copula independently of x (cf. Assumption 2.21 [7]). In our paper, we prove the convexity of $S(p)$ without this strong assumption.

This paper is organized as follows. In Section 2, we recall some basic concepts and preliminary results. We propose a new reformulation of the joint chance constraints in Section 2.1. Next, we present some theoretical results in Section 2.2 and Section 3 which are useful to prove the convexity of the feasibility set $S(p)$ in Section 4 and show some numerical results. We conclude the paper in Section 5.

2. Basic concepts and preliminary results

In this section, we recall some useful definitions and propositions for our subsequent analysis.

Definition 1. An n-dimensional random vector X follows a spherical distribution if there exists a function $\Psi : \mathbb{R} \to \mathbb{R}$ such that the characteristic function $\phi_X(t)$ of X is given by

$$\phi_X(t) = \mathbb{E}(e^{it^\top X}) = \Psi(t^\top t).$$

The function Ψ is called a characteristic generator of the spherical distribution.

Definition 2. An n-dimensional random vector U follows an elliptical distribution with location parameter μ, positive definite scale matrix Σ and characteristic generator Ψ (in short $U \sim \text{Ellip}(\mu, \Sigma, \Psi)$), if we have the following representation

$$U \hat{=} \mu + AX,$$

where X follows a spherical distribution with a characteristic generator Ψ, $A \in \mathbb{R}^{n\times n}$ such that $AA^\top = \Sigma$ and $\mu \in \mathbb{R}^n$; $\hat{=}$ implies that the both sides have the same distribution.

The probability density function of all the distributions from elliptical family does not always exist. Whenever it exists, it is of the form

$$f_U(z) = \frac{c}{\sqrt{\det(\Sigma)}} g_{\text{den}}\left(\sqrt{(z-\mu)^\top \Sigma^{-1}(z-\mu)}\right),$$

where g_{den} is a nonnegative function called *radial density* and $c > 0$ is a normalization factor which makes f_U a probability density function.

Definition 3. A function $f : Q \to (0, +\infty)$ is r-concave on a set $Q \subset \mathbb{R}^s$ for a given $r \in (-\infty, +\infty)$ if for any $x, y \in Q$ and $y \in [0, 1]$,

$$f(yx + (1-y)y) \geq [yf(x)^r + (1-y)f(y)^r]^{\frac{1}{r}}, \text{ when } r \neq 0,$$
$$f(yx + (1-y)y) \geq f(x)^y f(y)^{1-y}, \text{ otherwise.}$$

Definition 4. A real function $f : \mathbb{R} \to \mathbb{R}$ is r-decreasing for some real number $r \in \mathbb{R}$, if f is continuous on $(0, +\infty)$ and there exists some strictly positive real number t^* such that the function $t \mapsto t^r f(t)$ is strictly decreasing on $(t^*, +\infty)$.

Table 1 presents some 1-dimensional spherical distributions with r-decreasing densities for some values of r and their thresholds t^* [13].

Definition 5. A function $C : [0, 1]^K \to [0, 1]$ is a K-dimensional copula if C is a joint CDF of a K-dimensional random vector, on the unit cube $[0, 1]^K$, whose marginals are uniformly distributed on $[0, 1]$.

Proposition 1 (*Sklar's Theorem*). *Let $F : \mathbb{R}^K \to [0, 1]$ be a joint CDF of a K-dimensional random vector and $F_1, \ldots, F_K$ are the marginal CDFs. Then, there exists a K-dimensional copula C such that*

$$F(z) = C\left(F_1(z_1), \ldots, F_K(z_K)\right).$$

Table 1
Typical 1-dimensional spherical distributions with r-decreasing densities and their thresholds t^*.

Distribution	Radial density	r	t^*
Normal	$e^{-\frac{1}{2}u^2}$	$r > 0$	$\sqrt{r}$
t	$\left(1+\frac{1}{\nu}u^2\right)^{-(1+\nu)/2}$, $\nu > 0, \nu$ integer	$0 < r < \nu + 1$	$\sqrt{\frac{r\nu}{\nu+1-r}}$
Laplace	$e^{-\lvert u\rvert}$	$r > 0$	$\frac{r}{\sqrt{2}}$
Kotz type	$u^{2(N-1)}e^{-qu^{2s}}$, $q, s > 0, N > \frac{1}{2}$	$r > 2(1-N)$	$\sqrt[2s]{\frac{2(N-1)+r}{2qs}}$
Pearson type VII	$\left(1+\frac{u^2}{m}\right)^{-N}$, $m > 0, N > \frac{1}{2}$	$0 < r < 2N$	$\sqrt{\frac{rm}{2N-r}}$

Table 2
Selected types of strictly Archimedean copulas.

Type of copula	Parameter θ	Generator $\psi_\theta(t)$
Independent	–	-log(t)
Gumbel–Hougaard	$\theta \geq 1$	$[-\log(t)]^\theta$
Frank	$\theta > 0$	$-\log\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$
Clayton	$\theta > 0$	$\frac{1}{\theta}(t^\theta - 1)$
Joe	$\theta \geq 1$	$-\log[1-(1-t)^\theta]$

Moreover, if F_i is continuous for any $i = 1, \ldots, K$, then C is uniquely given by

$$C(u) = F\left(F_1^{(-1)}(u_1), \ldots, F_K^{(-1)}(u_K)\right).$$

Proposition 2 (*Fréchet–Hoeffding Upper Bound*). *For any $K-$dimensional copula C and $u = [u_1, \ldots, u_K]^\top \in [0,1]^K$, we have*

$$C(u) \leq C_M(u) = \min_{k=1,\ldots,K} u_k.$$

Definition 6. A K-dimensional copula C is strictly Archimedean if there exists a continuous and strictly decreasing function $\psi : (0,1] \to [0,+\infty)$, such that $\psi(1) = 0$, $\lim_{t\to 0}\psi(t) = +\infty$, and for any K-dimensional vector $u = (u_1, \ldots, u_K) \in [0,1]^K$, we have

$$C(u) = \psi^{(-1)}\left(\sum_{i=1}^{K}\psi(u_i)\right).$$

The function ψ is called a generator of the copula C.

Table 2 presents a selection of some strictly Archimedean copulas with their generators [13].

Definition 7. A function $f : \mathbb{R} \to \mathbb{R}$ is K-monotonic on an open interval $I \subseteq \mathbb{R}$ for some positive integer $K \geq 2$, if the following three conditions hold:

1. f is differentiable up to the order $(K-2)$ on I,
2. The derivatives of f satisfy

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad 0 \leq k \leq K-2,$$

for any $t \in I$,
3. The function $t \mapsto (-1)^{K-2}\frac{d^{K-2}}{dt^{K-2}} f(t)$ is nonincreasing and convex on I.

Proposition 3 (*Theorem 2.2, [16]*). *Let $\psi : (0,1] \to [0,+\infty)$ be a strictly decreasing function such that $\psi(1) = 0$ and $\lim_{t\to 0}\psi(t) = +\infty$. Then, ψ is the generator of a K-dimensional strictly Archimedean copula if and only if the inverse function $\psi^{(-1)}$ is $K-$monotonic on $(0,+\infty)$ and continuous on $[0,+\infty)$.*

2.1. Reformulation of the probability function

Assume that the random vectors $v_i \sim \text{Ellip}(\mu_i, \Sigma_i, \Psi_i)$, for any $i \in J$. Let

$$\xi_i(x) := \frac{v_i^\top x - \mu_i^\top x}{\sqrt{x^\top \Sigma_i x}}, \qquad g_i(x) := \frac{D_i - \mu_i^\top x}{\sqrt{x^\top \Sigma_i x}}. \tag{4}$$

Using the notations in (4), the chance constraint in (1) can be rewritten as follows

$$\mathbb{P}\left\{\xi_i(x) \leq g_i(x),\ i \in J\right\} \geq p.$$

It is well known that $\xi_i(x)$ follows 1−dimensional spherical distribution with characteristic generator Ψ_i [17]. Our aim is to reformulate this function in order to study the convexity of $S(p)$. By Sklar's Theorem, we have that, there exists a copula C_x such that

$$C_x[F_1(g_1(x)), \dots, F_K(g_K(x))] \geq p, \tag{5}$$

where C_x is the $K-$ dimensional copula of the K-dimensional random vector $\xi(x) = [\xi_1(x), \dots, \xi_K(x)]^\top$ and F_i is the cumulative distribution function of $\xi_i(x)$, for $i = 1, \dots, K$. In [7], the authors assume that C_x does not depend on x, i.e., there exists a copula C such that $C_x = C$, for any $x \in Q$ (cf. Assumption 2.21 [7]). We study the general case, where C_x is a copula, which depends on x. Assume that for any $x \in Q$, C_x is a strictly Archimedean copula with generator ψ_x. Then, the constraint (5) can be rewritten as follows

$$\psi_x^{(-1)}\left(\sum_{i=1}^{K} \psi_x(F_i(g_i(x)))\right) \geq p. \tag{6}$$

Using the decreasing monotonicity of ψ_x, (6) is equivalent to

$$\sum_{i=1}^{K} \psi_x(F_i(g_i(x))) \leq \psi_x(p) \tag{7}$$

By adding auxiliary variables $\{y_i \geq 0, \ i \in J\}$, we reformulate (7) into individual chance constraints [7,18]. Since ψ_x is positive, (7) is equivalent to the following constraints

$$\begin{cases} \text{(i) } \psi_x(F_i(g_i(x))) \leq y_i \psi_x(p), \quad i \in J, \\ \text{(ii) } y_i \geq 0, \quad i \in J, \\ \text{(iii) } \sum_{i=1}^{K} y_i = 1. \end{cases} \tag{8}$$

This means that if $x^* \in S(p)$ then there exists $y^* = [y_1^*, \dots, y_K^*]^\top \in \mathbb{R}^K$ such that (x^*, y^*) satisfies constraints (8). On the other hand, if (x^*, y^*) is a feasible solution for constraints (8) and $x^* \in Q$, then $x^* \in S(p)$. Moreover, for $x^* \in S(p)$, we can choose y^* in order to satisfy constraints (8) as

$$y_i^* = \frac{\psi_{x^*}(F_i(g_i(x^*)))}{\sum_{j=1}^{K} \psi_{x^*}(F_j(g_j(x^*)))}, \quad \forall\, i \in J. \tag{9}$$

Using the decreasing monotonicity of the generator ψ_x, constraints (8) can be written as follows

$$\begin{cases} \text{(i) } F_i(g_i(x)) \geq \psi_x^{(-1)}(y_i \psi_x(p)), \quad i \in J, \\ \text{(ii) } y_i \geq 0, \quad i = 1, \dots, K, \\ \text{(iii) } \sum_{i=1}^{K} y_i = 1. \end{cases} \tag{10}$$

In the rest of the paper, we assume that the following assumption holds.

Assumption 1. C_x is a Gumbel–Hougaard copula, for any $x \in Q$, i.e., the generator ψ_x is given by

$$\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}, \tag{11}$$

for any $(x, t) \in Q \times (0, 1]$, where $\kappa(x) : Q \to (0, 1]$ is a strictly positive function.

Remark 1. Our aim is to show the concavity of $F_i(g_i)$ w.r.t x and the joint convexity of $\psi_x^{(-1)}(y_i \psi_x(p))$ w.r.t (y_i, x).

2.2. Concavity of $F_i(g_i(\cdot))$

In this section, we will show our main result in Lemma 2 that under Assumption 2, $F_i(g_i(\cdot))$ is a concave function on $S(p)$. Define an index set $I^\mu \subset J$ such that $\mu_i \neq 0$ for any $i \in I^\mu$ and $\mu_i = 0$ otherwise. Define a set of real numbers $\{r_i \mid i \in J\}$ such that

$$\begin{cases} r_i > 1, \text{ if } i \in I^\mu, \\ r_i = 1, \text{ if } i \notin I^\mu. \end{cases}$$

Assumption 2 (*i*)**.** The cumulative distribution function F_i has $(r_i + 1)-$ decreasing densities with the thresholds t_i^*, for any $i \in J$. (*ii*) $p > p^*$, where

$$p^* = \max\left\{\frac{1}{2}, \max_{j \in I^\mu} F_i\left(\frac{r_i + 1}{r_i - 1} \lambda_{i,min}^{-\frac{1}{2}} \|\mu_i\|\right), \max_{i \in J} F_i[t_i^*(r_i + 1)]\right\}, \tag{12}$$

where $\lambda_{i,min}$ is the smallest eigenvalue of the positive definite matrix Σ_i, for any $i \in J$.

Lemma 1. *If Assumption 2 holds, then*

$$\mathrm{Conv}(S(p)) \subset \bigcap_{j\in I^{\mu}} \Omega^j,$$

where

$$\Omega^j = \left\{ x \in Q \mid D_j - \mu_j^{\top} x > \frac{r_j+1}{r_j-1} \lambda_{j,min}^{-\frac{1}{2}} \|\mu_j\| \sqrt{x^{\top}\Sigma_j x} \right\},$$

and $Conv(S(p))$ is the convex hull of $S(p)$. Moreover, for any $i = 1, 2, \ldots, K$, $g_i > 0$ and g_i is $(-r_i)-$ concave on any convex subset Q^i of $\bigcap_{j\in I^{\mu}} \Omega^j$.

Proof. The proof is given in Appendix A. □

Using Lemma 1, we prove the following lemma.

Lemma 2. *If Assumption 2 holds, then $F_i(g_i(\cdot))$ is concave on $Conv(S(p))$, for any $i \in J$.*

Proof. Using Lemma 1, g_i is $(-r_i)-$ concave and $g_i > 0$ on $\mathrm{Conv}(S(p))$, for any $i \in J$. Hence, for any $a \in [0,1]$ and $x_1, x_2 \in S(p)$, we have

$$g_i(ax_1 + (1-a)x_2) \geq [a g_i^{-r_i}(x_1) + (1-a) g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}. \tag{13}$$

As $x_1 \in S(p)$ and $p > p^*$, the constraint (5) implies that

$$C_{x_1}[F_1(g_1(x_1)), \ldots, F_K(g_K(x_1))] > p^*,$$

which in turn implies from Proposition 2 and Assumption 2 that

$$F_i(g_i(x_1)) > p^* \geq F_i[t_j^*(r_i+1)], \ \forall \ \ i \in J.$$

Since, $F_i(\cdot)$ is monotonically increasing, we get

$$g_i(x_1) > t_i^*(r_i+1) > 0,$$

which implies that $0 < g_i(x_1)^{-r_i} < (t_i^*(r_i+1))^{-r_i}$, for any $i \in J$. Similarly, we obtain the same inequality for x_2. By taking F_i on both sides of (13),

$$F_i(g_i(ax_1 + (1-a)x_2)) \geq F_i([a g_i^{-r_i}(x_1) + (1-a) g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}). \tag{14}$$

Since $F_i(\cdot)$ has $(r_i+1)-$decreasing density, from Lemma 3.1 of [2], the function $t \mapsto F_i\left(t^{-\frac{1}{r_i}}\right)$ is concave on $(0, (t_i^*)^{-r_i})$. Therefore, we can write

$$F_i([a g_i^{-r_i}(x_1) + (1-a) g_i^{-r_i}(x_2)]^{-\frac{1}{r_i}}) \geq a F_i(g_i(x_1)) + (1-a) F_i(g_i(x_2)). \tag{15}$$

From (14) and (15), we deduce that

$$F_i(g_i(ax_1 + (1-a)x_2)) \geq a\left(F_i(g_i(x_1))\right) + (1-a)\left(F_i(g_i(x_2))\right).$$

Therefore, $F_i(g_i(\cdot))$ is concave on $\mathrm{Conv}(S(p))$. □

3. Convexity of $\psi_x^{(-1)}(y_i \psi_x(p))$

The main result of this section is Lemma 7 which shows that under Assumption 3, U is jointly convex. In this section, we assume that the feasibility set $S(p)$ is non empty. Let c_1 be a real number such that $0 < c_1 \leq 1$. Define an index set $I^D \subset J$ such that $D_i > 0$ for any $i \in I^D$ and $D_i \leq 0$ otherwise. For any $i \in J$, let

$$G_i = \begin{cases} \left(\dfrac{\log\left(F_i\left(\dfrac{D_i}{\sqrt{\lambda_{i,\min}}\,\delta_1} + \dfrac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right)\right)}{\log p} \right)^{\frac{1}{c_1}}, & \text{if } i \in I^D, \\ \left(\dfrac{\log\left(F_i\left(\dfrac{D_i}{\sqrt{\lambda_{i,\max}}\,\delta_u} + \dfrac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}} \right)\right)}{\log p} \right)^{\frac{1}{c_1}}, & \text{if } i \in J \backslash I^D, \end{cases}$$

where $\lambda_{i,\max}$ is the largest eigenvalue of the positive definite matrix Σ_i, for any $i \in J$. Let $h_1 := \min_{1\leq j\leq K}(G_j)$ and $h_u := 1-(n-1)\,h_1$.

Lemma 3. $0 < h_l \leq h_u < 1$. *Moreover, given $x \in S(p)$ and y_i refers to (9). Hence, $h_l \leq y_i \leq h_u$, for any $i \in J$.*

Proof. The proof is given in Appendix B. □

It follows from Lemma 3 that h_l, h_u belong to $(0,1)$. Therefore, we can define

$$\begin{aligned}
\varphi_1^* &:= c_l\ (\log h_u)\ (\log p)\ h_l.\\
\varphi_2^* &:= \frac{(\log h_l)^2}{4c_l} + \max\left([1+\log h_l\ (1+(\log p)\ h_l)]^2, [1+\log h_l\ c_l\ (1+\log p)]^2\right).\\
\omega &:= \frac{\varphi_2^*}{\varphi_1^*}.
\end{aligned} \tag{16}$$

Remark 2. Since $\varphi_1^* > 0$, then ω is well-defined. Moreover, ω does not depend on (x, y_i).

In order to show the convexity of $U(x,y_i) := \psi_x^{(-1)}(y_i\psi_x(p))$, we first show that the Hessian matrix of U is positive semidefinite. The following lemma is a reformulation of the positive semidefiniteness of the Hessian matrix of U.

Lemma 4. *The positive semidefiniteness of the Hessian matrix of U on the convex set $Q\times[h_l,h_u]$ is equivalent to the positive semidefiniteness of the following n-dimensional symmetric matrix*

$$N(x,y_i) = \frac{\partial^2}{\partial y_i^2}U(x,y_i)\times H_xU(x,y_i) - \left(\bigtriangledown_x \frac{\partial}{\partial y_i}U(x,y_i)\right)\left(\bigtriangledown_x \frac{\partial}{\partial y_i}U(x,y_i)\right)^{\top}, \tag{17}$$

for any $(x,y_i)\in Q\times[h_l,h_u]$, where

$$H_xU = \begin{bmatrix} \frac{\partial^2 U}{\partial x_1^2} & \frac{\partial^2 U}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2 U}{\partial x_1\partial x_n}\\ \frac{\partial^2 U}{\partial x_2\partial x_1} & \frac{\partial^2 U}{\partial x_2^2} & \cdots & \frac{\partial^2 U}{\partial x_2\partial x_n}\\ . & . & \cdots & .\\ . & . & \cdots & .\\ \frac{\partial^2 U}{\partial x_n\partial x_1} & \frac{\partial^2 U}{\partial x_n\partial x_2} & \cdots & \frac{\partial^2 U}{\partial x_n^2} \end{bmatrix},\quad \bigtriangledown_x = \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)^{\top}.$$

Proof. The proof is given in Appendix C. □

Assumption 3. We assume that

(*i*) $p \ge e^{-1}$.

(*ii*) $0 < c_l \le \kappa(x) \le 1$, for any $x \in Q$.

(*iii*) $H_\kappa(x) - \omega\bigtriangledown_x\kappa(x)(\bigtriangledown_x\kappa(x))^{\top}$ is a positive semidefinite matrix for any $x \in Q$, where $H_\kappa(x)$ is the Hessian matrix of $\kappa(x)$; $\bigtriangledown_x\kappa(x)$ is the gradient vector of $\kappa(x)$.

Lemma 5. *Let*

$$\varphi_1(x,y_i) := \kappa(x)\ (\log y_i)\ \left[\kappa(x) - 1 + \kappa(x)(\log p)\ y_i^{\kappa(x)}\right].$$

If Assumption 3 holds, then $\varphi_1(x,y_i) \ge \varphi_1^* > 0$, for any $(x,y_i)\in Q\times[h_l,h_u]$, where φ_1^* refers to (16).

Proof. Since $0 < y_i \le h_u < 1$, then

$$(-\log y_i) \ge (-\log h_u) > 0. \tag{18}$$

As $0 < h_l \le y_i < 1$ and $0 < \kappa(x) \le 1$, we deduce that $1 \ge y_i^{\kappa(x)} \ge y_i \ge h_l > 0$. Note that $1 \ge -\log p > 0$ (because $e^{-1} \le p < 1$ by (i) of Assumption 3). Then, $1 \ge -(\log p)\ y_i^{\kappa(x)} \ge -(\log p)\ h_l > 0$. Since $0 < \kappa(x) \le 1$ and $1 + (\log p)\ y_i^{\kappa(x)} \ge 0$, we have

$$\begin{aligned}
&1 - \kappa(x) - \kappa(x)\log(p).y_i^{\kappa(x)} = 1 - \kappa(x)(1+\log p.y_i^{\kappa(x)})\\
&\ge 1 - (1+\log p.y_i^{\kappa(x)}) = -(\log p)\ y_i^{\kappa(x)} \ge -(\log p)\ h_l > 0,
\end{aligned} \tag{19}$$

for any $(x,y_i)\in Q\times[h_l,h_u]$. Moreover, by (ii) of Assumption 3, we have

$$\kappa(x) \ge c_l > 0,\quad \forall\ x \in Q. \tag{20}$$

Therefore, it suffices to multiply term by term the inequalities (18)–(20) to complete the proof. □

Lemma 6. *Let* $\varphi_2(x,y_i) := \kappa(x)\log(y_i)^2(1+(\log p)\ y_i^{\kappa(x)})\left[1-\kappa(x)-\kappa(x)\ (\log p)\ y_i^{\kappa(x)}\right]$

$+\left(1+\kappa(x)\ (\log y_i)+(\log p)\ (\log y_i)\ y_i^{\kappa(x)}\ \kappa(x)\right)^2$. *If Assumption 3 holds, $0 < \varphi_2(x,y_i) \le \varphi_2^*$, for any $(x,y_i)\in Q\times[h_l,h_u]$, where φ_2^* refers to (16).*

Proof. Since $p \geq e^{-1}$, then $0 > \log p \geq -1$. Using $0 < y_i^{\kappa(x)} < 1$, we have $0 < 1 + (\log p)\ y_i^{\kappa(x)} < 1$. Moreover, as $0 < \kappa(x) \leq 1$, then $0 < \kappa(x)\ (1 + (\log p)\ y_i^{\kappa(x)}) < 1$. Let $s := \kappa(x)\ (1 + (\log p)\ y_i^{\kappa(x)})$. By applying the Cauchy–Schwarz inequality, we deduce that $s(1-s) \leq \frac{1}{4}$ (the equality holds if and only if $s = \frac{1}{2}$). Hence,

$$\kappa(x)(1 + (\log p)\ y_i^{\kappa(x)}) \left[1 - \kappa(x)\ (1 + (\log p)\ y_i^{\kappa(x)})\right] \leq \frac{1}{4},$$

which in turn implies that

$$0 < (1 + (\log p)\ y_i^{\kappa(x)}) \left[1 - \kappa(x)\ (1 + (\log p)\ y_i^{\kappa(x)})\right] \leq \frac{1}{4\ \kappa(x)} \leq \frac{1}{4\ c_1}. \tag{21}$$

Since $1 > y_i \geq h_1 > 0$, we have $0 > \log y_i \geq \log h_1$. Then,

$$0 < (\log y_i)^2 \leq (\log h_1)^2. \tag{22}$$

Moreover,

$$0 < \kappa(x) \leq 1. \tag{23}$$

By multiplying term by term the inequalities (21)–(23), we get

$$0 < \kappa(x) \log(y_i)^2 (1 + (\log p)\ y_i^{\kappa(x)}) \left[1 - \kappa(x) - \kappa(x)\ (\log p)\ y_i^{\kappa(x)}\right] \leq \frac{(\log h_1)^2}{4\ c_1}, \tag{24}$$

for any $(x, y_i) \in Q \times [h_1, h_u]$. By Assumption 3, we have

$$\begin{aligned} &\text{(i) } 0 < c_1 \leq \kappa(x) \leq 1, \ \text{(ii) } 0 < -\log h_u \leq -\log y_i \leq -\log h_1, \\ &\text{(iii) } 0 \leq 1 + \log p \leq 1 + (\log p)\ y_i^{\kappa(x)} \leq 1 + (\log p)\ h_1. \end{aligned} \tag{25}$$

Note that the condition (iii)(25) holds since $\log p < 0$ and $1 \geq y_i^{\kappa(x)} \geq h_1$. By multiplying (i) – (iii) of (25) term by term, we get

$$-\log h_1\ (1 + (\log p)\ h_1) \geq -(\log y_i)\ \kappa(x)(1 + (\log p)\ y_i^{\kappa(x)}) \geq -(\log h_u)\ c_1\ (1 + \log p),$$

which is equivalent to

$$1 + (\log h_1)\ (1 + (\log p)\ h_1) \leq 1 + (\log y_i)\ \kappa(x)(1 + (\log p)\ y_i^{\kappa(x)}) \leq 1 + (\log h_u)\ c_1(1 + \log p),$$

which in turn implies that

$$\begin{aligned} &0 \leq [1 + (\log y_i)\ \kappa(x)(1 + (\log p)\ y_i^{\kappa(x)})]^2 \\ &\leq \max\left((1 + (\log h_1)\ (1 + (\log p)\ h_1))^2, (1 + (\log h_u)\ c_1\ (1 + \log p))^2\right). \end{aligned} \tag{26}$$

Adding (24) and (26) together, completes the proof. □

Lemma 7. *If Assumption 3 holds, then U is jointly convex on $Q \times [h_l, h_u]$.*

Proof. Using Lemma 4, it suffices to show the positive semidefiniteness of $N(x, y_i)$ in (17) for any $(x, y_i) \in Q \times [h_1, h_u]$. Since $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$, we have $\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}$. Then, we can write

$$U(x, y_i) = e^{-\left\{ y_i(-\log p)^{\frac{1}{\kappa(x)}} \right\}^{\kappa(x)}} = p^{y_i^{\kappa(x)}}. \tag{27}$$

We calculate explicitly the partial derivatives of U as follows

$$\begin{aligned} &\text{(i) } \frac{\partial}{\partial y_i} U(x, y_i) = (\log p)\ p^{y_i^{\kappa(x)}}\ \kappa(x)\ y_i^{\kappa(x)-1}. \\ &\text{(ii) } \frac{\partial^2}{\partial y_i^2} U(x, y_i) = \kappa(x)\ (\log p)\ y_i^{\kappa(x)-2}\ p^{y_i^{\kappa(x)}} [\kappa(x) - 1 + \kappa(x)\ \log(p)\ y_i^{\kappa(x)}]. \\ &\text{(iii) } \bigtriangledown_x U(x, y_i) = (\log p)\ p^{y_i^{\kappa(x)}}\ (\log y_i)\ y_i^{\kappa(x)}\ \bigtriangledown_x \kappa(x). \\ &\text{(iv) } \bigtriangledown_x \frac{\partial}{\partial y_i} U(x, y_i) = \\ &(\log p)\ y_i^{\kappa(x)-1}\ p^{y_i^{\kappa(x)}} [1 + \kappa(x)\ (\log y_i) + (\log p)\ (\log y_i)\ y_i^{\kappa(x)} \kappa(x)]\ \bigtriangledown_x \kappa(x). \\ &\text{(v) } H_x U(x, y_i) = \\ &p^{y_i^{\kappa(x)}}\ y_i^{\kappa(x)}\ (\log p)\ (\log y_i)\ [H_\kappa(x) + (\log y_i + (\log y_i)\ (\log p)\ y_i^{\kappa(x)})\ \bigtriangledown_x \kappa(x) (\bigtriangledown_x \kappa(x))^\top]. \end{aligned}$$

Hence, we obtain the following formulations

$$\text{(i) } \frac{\partial^2}{\partial y_i^2} U(x, y_i) \times H_x U(x, y_i) = \kappa(x)(\log p)^2 y_i^{2\ \kappa(x)-2}\ (\log y_i)\ p^{2\ y_i^{\kappa(x)}} \times$$

$$\left[\kappa(x) - 1 + \kappa(x)\ (\log p)\ y_i^{\kappa(x)}\right]\ \left[H_\kappa(x) + \nabla_x\kappa(x)(\nabla_x\kappa(x))^\top(\log y_i + (\log y_i)\ (\log p)\ y_i^{\kappa(x)})\right].$$

$$\text{(ii)}\ \left(\nabla_x \frac{\partial}{\partial y_i}U(x,y_i)\right)\left(\nabla_x \frac{\partial}{\partial y_i}U(x,y_i)\right)^\top = (\log p)^2\ y_i^{2\ \kappa(x)-2}\ p^{2\ y_i^{\kappa(x)}} \times$$

$$\left(1 + \kappa(x)\ (\log y_i) + (\log p)\ (\log y_i)\ y_i^{\kappa(x)}\kappa(x)\right)^2 \nabla_x\kappa(x)(\nabla_x\kappa(x))^\top. \tag{28}$$

Note that $(\log p)^2\ y_i^{2\kappa(x)-2}\ p^{2\ y_i^{\kappa(x)}}$ is a positive common factor of $\frac{\partial^2}{\partial y_i^2}U(x,y_i) \times H_x U(x,y_i)$ and $\left(\nabla_x \frac{\partial}{\partial y_i}U(x,y_i)\right)\left(\nabla_x \frac{\partial}{\partial y_i}U(x,y_i)\right)^\top$. Then, it follows from (17) and (28) that the positive semidefiniteness of $N(x,y_i)$ is equivalent to the positive semidefiniteness of the following matrix

$$M(x,y_i) =$$
$$\left[\kappa(x) - 1 + \kappa(x)\ (\log p)\ y_i^{\kappa(x)}\right]\ \left[H_\kappa(x) + \nabla_x\kappa(x)(\nabla_x\kappa(x))^\top(\log y_i + (\log y_i)\ (\log p)\ y_i^{\kappa(x)})\right]$$
$$\times\ \kappa(x)\ (\log y_i) - \left(1 + \kappa(x)\ (\log y_i) + (\log p)\ (\log y_i)\ y_i^{\kappa(x)}\kappa(x)\right)^2 \nabla_x\kappa(x)(\nabla_x\kappa(x))^\top.$$

Note that $M(x,y_i)$ can be rewritten as follows

$$M(x,y_i) = \varphi_1(x,y_i)\ H_\kappa(x) - \varphi_2(x,y_i)\ \nabla_x\kappa(x)(\nabla_x\kappa(x))^\top,$$

where $\varphi_1(x,y_i)$ and $\varphi_2(x,y_i)$ refer to Lemmas 5 and 6. By (iii) of Assumption 3, we have $H_\kappa(x) - \omega\nabla_x\kappa(x)(\nabla_x\kappa(x))^\top$ is a positive semidefinite matrix for any $x \in Q$, where ω refers to (16). Since $\varphi_1^*, \varphi_2^* > 0$, it is clear that $\omega > 0$. Moreover, $\nabla_x\kappa(x)(\nabla_x\kappa(x))^\top$ is a positive semidefinite matrix. Hence, $H_\kappa(x)$ is also a positive semidefinite matrix. By Lemmas 5 and 6, we have that $\varphi_1(x,y_i) \geq \varphi_1^* > 0$ and $\varphi_2(x,y_i) \leq \varphi_2^*$, for any $(x,y_i) \in Q \times [h_1, h_u]$. Then,

$$M(x,y_i) \geq \varphi_1^* H_\kappa(x) - \varphi_2^*\nabla_x\kappa(x)(\nabla_x\kappa(x))^\top \geq 0,$$

which completes the proof. □

4. Convexity of the feasibility set $S(p)$

We showed that if Assumption 2 holds, then $F_i(g_i(\cdot))$ is concave on $\mathrm{Conv}(S(p))$, for any $i \in J$. Moreover, given arbitrarily a real number c_l such that $0 < c_l \leq 1$, if Assumption 3 holds, then U is jointly convex on $Q \times [h_l, h_u]$. We will apply these results to prove the convexity of the feasibility set $S(p)$.

Theorem 1. *If Assumptions 2 and 3 hold and the feasibility set $S(p)$ is non empty, then $S(p)$ is a convex set.*

Proof. For any $x_1, x_2 \in S(p)$ and $\beta \in [0,1]$, we show that $x^* := \beta x_1 + (1-\beta)x_2 \in S(p)$. In fact, let $y^1 := (y_1^1, \ldots, y_K^1)$ and $y^2 := (y_1^2, \ldots, y_K^2)$, where y_i^1 and y_i^2, $i \in J$, are the corresponding values of y_i^* defined in (9) w.r.t x_1 and x_2, respectively. It follows from Lemma 3 that $h_l \leq y_i^1, y_i^2 \leq h_u$, for any $i \in J$. Moreover, by Lemmas 2 and 7, we have $F_i(g_i(\cdot))$ is concave on $\mathrm{Conv}(S(p))$ and $U(\cdot,\cdot)$ is jointly convex on $Q \times [h_l, h_u]$. Then, for any $i \in J$, we have

$$F_i(g_i(x^*)) \geq \beta F_i(g_i(x_1)) + (1-\beta)F_i(g_i(x_2))$$
$$\geq \beta\ \psi_{x_1}^{(-1)}\left(y_i^1\psi_{x^*}(p)\right) + (1-\beta)\ \psi_{x_2}^{(-1)}\left(y_i^2\psi_{x^*}(p)\right) = \beta U(x_1, y_i^1) + (1-\beta)U(x_2, y_i^2)$$
$$\geq U(x^*, \beta y_i^1 + (1-\beta)y_i^2) = \psi_{x^*}^{(-1)}\left((\beta y_i^1 + (1-\beta)y_i^2)\psi_{x^*}(p)\right),$$

which in turn implies that $(x^*, \beta y^1 + (1-\beta)y^2)$ satisfies (10). Then, $x^* \in S(p)$. □

We are interested in finding an example which fits all the Assumptions in Theorem 1. In the following, we will study a specific case which fits Assumptions 3 and 2.

4.1. An example of the function κ

In this section, we give an example of κ which satisfies all the conditions in Assumption 3.

Remark 3. It is not necessary to verify the condition (i) of Assumption 3 since $e^{-1} \approx 0.37$ and we consider high value of the probability threshold p.

Lemma 8. *Let d be a real number such that $d < c_l$ and $q : Q \to \mathbb{R}$ be a real-valued function which satisfies the two following conditions.*

1. *q is twice continuously differentiable and convex on Q.*
2. *$\log(c_l - d) \leq q(x) \leq \log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right]$, for any $x \in Q$, where ω refers to (16).*

Then, $\kappa(x) := e^{q(x)} + d$ is a function which satisfies Assumption 3.

Table 3
Selected 1-dimensional elliptical distributions with r-decreasing densities and the value of the threshold p^* respectively.

Distribution	p^*
Normal	0.92
Student's t with 2− degrees of freedom	0.84
Student's t with 3− degrees of freedom	0.87
Student's t with 4− degrees of freedom	0.88
Laplace	0.88

Proof. First we verify the condition (ii) of Assumption 3, which can be implied by the assumption $\log(c_1 - d) \le q(x) \le \log(1-d)$. Since $q(x) \le \log(\frac{1}{\omega})$, we have $e^{q(x)} \le \frac{1}{\omega}$, for any $x \in Q$. Let $\kappa^*(x) := e^{q(x)}$. We have the following formulation

$$H_x \log(\kappa^*(x)) = \frac{\kappa^*(x) H_x \kappa^*(x) - \bigtriangledown_x \kappa^*(x)(\bigtriangledown_x \kappa^*(x))^\top}{\kappa^*(x)^2}, \tag{29}$$

where $H_x \log(\kappa^*(\cdot))$ is the Hessian matrix of the function $\log(\kappa^*(\cdot))$. As q is a convex function on Q and $\log(\kappa^*(.)) = q(.)$, we deduce that $H_x(\log(\kappa^*(x)))$ is a positive semidefinite matrix for any $x \in Q$. It follows from (29) that $\kappa^*(x) H_x \kappa^*(x) - \bigtriangledown_x \kappa^*(x)(\bigtriangledown_x \kappa^*(x))^\top$ is a positive semidefinite matrix. Since $0 < \kappa^*(x)$ and $\bigtriangledown_x \kappa^*(x)(\bigtriangledown_x \kappa^*(x))^\top$ is positive semidefinite, we deduce that $H_x \kappa^*(x)$ is positive semidefinite, for any $x \in Q$. Moreover, $\kappa^*(x) \le \frac{1}{\omega}$, which implies that

$$\frac{1}{\omega} H_x \kappa^*(x) - \bigtriangledown_x \kappa^*(x)(\bigtriangledown_x \kappa^*(x))^\top$$

is a positive semidefinite matrix. On the other hand, since $\kappa(x) = \kappa^*(x) + d$, then $H_x \kappa^*(x) = H_x \kappa(x)$ and $\bigtriangledown_x \kappa^*(x) = \bigtriangledown_x \kappa(x), \forall\, x \in Q$. Therefore, the condition (iii) of Assumption 3 holds. □

Next, we take an example function q which meets the two conditions in Lemma 8.

Lemma 9. *Let $q(x) = \frac{\|x\|^2}{L} + z$ where $L > 0, z \in \mathbb{R}$ are real numbers such that*

$$\log(c_l - d) \le \frac{1}{L}\delta_l^2 + z \le \frac{1}{L}\delta_u^2 + z \le \log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right], \tag{30}$$

where δ_l, δ_l are defined in (1). Then, $q(x)$ satisfies the two conditions in Lemma 8.

Proof. The first condition is trivial. Since $\frac{\delta_l^2}{L} + z = \min_{x\in Q} q(x)$ and $\frac{\delta_u^2}{L} + z = \max_{x\in Q} q(x)$, it is clear that the second condition holds. □

By Lemmas 8 and 9, the function $\kappa(x) = e^{\frac{\|x\|^2}{L} + z} + d$ which satisfies (30) fits all conditions in Assumption 3. It suffices to choose appropriate parameters L, z and d. In fact, (30) is a mild condition. Let d be an arbitrary real number in $(c_1 - \frac{1}{\omega}, c_1)$. As $d < c_1$ and $c_1 \le 1$, We have that $\log(c_1 - d)$ is well defined and

$$\log(c_1 - d) \le \log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right].$$

Let L be an arbitrary real number in $\left[\frac{\delta_u^2 - \delta_l^2}{\log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right] - \log(c_1 - d)}, +\infty\right)$. We deduce from this condition that

$$\log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right] - \log(c_1 - d) \ge \left(\frac{1}{L}\delta_u^2 + z\right) - \left(\frac{1}{L}\delta_l^2 + z\right) > 0.$$

Let z be an arbitrary real number in $\left[\log(c_1 - d) - \frac{\delta_l^2}{L}, \log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right] - \frac{\delta_u^2}{L}\right]$. We deduce from this condition that

$$\begin{cases} \log(c_1 - d) \le \frac{1}{L}\delta_1^2 + z. \\ \frac{1}{L}\delta_u^2 + z \le \log\left[\min\left(\frac{1}{\omega}, 1-d\right)\right]. \end{cases}$$

Therefore, we can verify that this set of $q(x), d, L, z$ satisfy (30).

4.2. Numerical experiments

To verify the $r-$decreasing property of a differentiable density $f(t)$, we check whether the derivative of $t^r f(t)$ is strictly negative for $t > t^*(r) > 0$, which is equivalent to $r.f(t) + t.f'(t) < 0$, for any $t > t^*(r)$. The thresholds for some typical 1−dimensional spherical distributions are given in Table 1.

In this section, we study values of p^* as defined in (12). All the numerical results are performed using Python 3.8.8 on a PC with Intel i5 CPU (2.4 GHz), RAM 16G, 512G SSD. Assume that for any $j \in J$, F_j has the same density. For the sake of illustration,

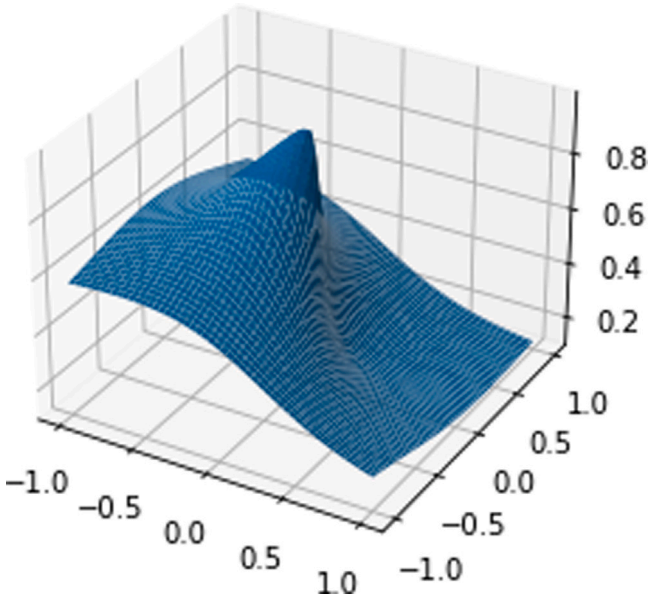

Fig. 1. Surface plot of the probability function f_{proba} on $[-1,1]^2$.

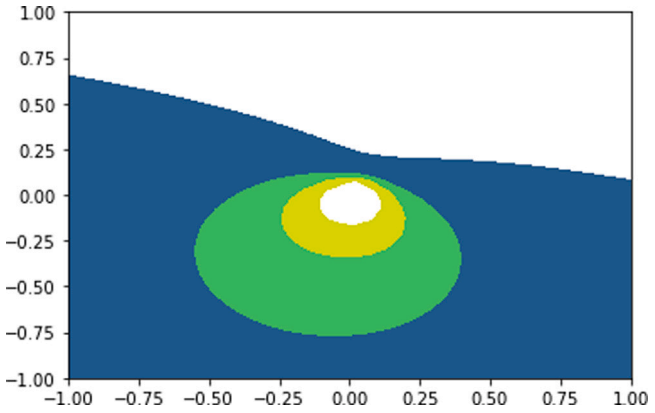

Fig. 2. Contour line of the probability function f_{proba} with 4 levels $(0.6, 0.7, 0.8, 0.9)$.

we set $\lambda_{j,\min}^{-\frac{1}{2}}$ small enough such that $F_i\left(\frac{r_i+1}{r_i-1}\lambda_{i,min}^{-\frac{1}{2}}\|\mu_i\|\right) \leq F_i[t_i^*(r_i+1)]$ and take $\max\left(\frac{1}{2}, F_i[t_i^*(r_i+1)]\right)$ as an upper bound of p^*. We show in Table 3 the bounds of p^* for some typical elliptical distributions. We consider the case with $n=2$ and $K=2$, i.e. we have 2 constraints and 2-dimensional decision variable. The parameters are taken as follows: $p=0.95, c_{\text{l}}=0.9, D_1=1, D_2=0.85, \mu_1=(1,20)^\top, \mu_2=(7,2)^\top, \delta_{\text{l}}=0.2, \delta_{\text{u}}=1.5, \Sigma_1=3\times\mathbb{I}_2$, and $\Sigma_2=30\times\mathbb{I}_2$, where $\mathbb{I}_2$ is the 2×2-identity matrix. h_{l} and h_{u} are calculated as in Section 3. φ_1^*, φ_2^* and ω are calculated by (16). Let $d=c_{\text{l}}-\frac{1}{2\omega}$, $L=\frac{\delta_{\text{u}}^2-\delta_{\text{l}}^2}{\log\left(\min\left(\frac{1}{\omega},1-d\right)\right)-\log(c_{\text{l}}-d)}$ and $z=\log\left(\min\left(\frac{1}{\omega},1-d\right)\right)-\frac{\delta_{\text{u}}^2}{L}$. It is easy to see that this set of parameters fits the conditions in Lemmas 8 and 9.

In our simulations, we consider the probability function

$$f_{\text{proba}}(x)=\psi_x^{(-1)}\left(\sum_{i=1}^{2}\psi_x(F_i(g_i(x)))\right),$$

where ψ_x is defined in (11), F_1 is the cumulative distribution function of a 1− dimensional standard Student's t distribution with 3−degrees of freedom and F_2 is the cumulative distribution function of a 1− dimensional standard Student's t distribution with 4−degrees of freedom, $x=(x_1,x_2)^\top$ where $x_1, x_2 \in [-1,1]$. Fig. 1 shows the curve of function $z=f_{\text{proba}}(x)$ on the domain $[-1,1]$ and Fig. 2 shows its contour lines with four different levels $0.6, 0.7, 0.8$ and 0.9, respectively.

5. Conclusion

In this paper, we studied the convexity of joint chance constraints in the case of elliptical distributions. Further, we modeled the dependence of random variables in different rows by a Gumbel–Hougaard copula. We come up with new convexity results of the feasibility set. We simulated our theoretical result by showing the surface plot of the probability function with its contour lines. It is very clear that the feasibility set is eventually a convex set under high value of the probability level. Further research can be dedicated to other families of copulas.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix A. Proof of Lemma 1

Let $i \in I$ and $x_0 \in S(p)$. It follows from the constraint (5) that

$$C_{x_0}[F_1(g_1(x_0)), \dots, F_K(g_K(x_0))] \geq p, \ \forall \ \ i \in J. \tag{A.1}$$

By applying Proposition 2, we get $F_i(g_i(x_0)) \geq p, \ \forall i \in J$, which in turn implies by Assumption 2 that

$$F_i(g_i(x_0)) > p^* \geq F_i\left(\frac{r_i+1}{r_i-1}\lambda_{i,\min}^{-\frac{1}{2}}\|\mu_i\|\right), \ \forall i \in J$$

Since $F_i(\cdot)$ is an increasing function, we have

$$g_i(x_0) > \frac{r_i+1}{r_i-1}\lambda_{i,\min}^{-\frac{1}{2}}\|\mu_i\|,$$

which implies that

$$D_i - \mu_i^\top x_0 > \frac{r_i+1}{r_i-1}\lambda_{i,\min}^{-\frac{1}{2}}\|\mu_i\|\sqrt{x_0^\top \Sigma_i x_0}.$$

Therefore, $S(p) \subset \bigcap_{j\in I} \Omega^j$. For each $j \in I$, Ω^j is a convex set which implies that $\text{Conv}(S(p)) \subset \bigcap_{j\in I} \Omega^j$. We prove the second part of Lemma 1 by considering the following two cases:

Case 1: Let $i \notin I^\mu$, then $\mu_i = 0$. By Assumption 2, we have $p > \frac{1}{2}$. Let $x_0 \in S(p)$. By applying Proposition 2 on (A.1), we have

$$F_i(g_i(x_0)) \geq p > \frac{1}{2}. \tag{A.2}$$

Since, F_i is the CDF of a 1-dimensional real-valued random variable which is symmetric at 0, $F_i(0) = \frac{1}{2}$. From (A.2) we get $g_i(x_0) > 0$ which in turn implies that $D_i - (\mu_i)^\top x_0 > 0$. Since $\mu_i = 0$, we get $D_i > 0$. In conclusion, the proof follows directly from Lemma 3 of [7].

Case 2: Let $i \in I^\mu$. It follows from Lemma 2 of [7] that the function

$$f_i(x) = \left(\frac{\sqrt{(x)^\top \Sigma_i x}}{D_i - (\mu_i)^\top x}\right)^{r_i}$$

is a convex function on $\bigcap_{j\in I} \Omega^j$. Therefore, for any $y, z \in Q^i \subseteq \bigcap_{j\in I^\mu} \Omega^j$ and $\lambda \in [0,1]$, we have

$$f_i[\lambda y + (1-\lambda)z] \leq \lambda f_i(y) + (1-\lambda) f_i(z). \tag{A.3}$$

Note that $g_i(x) = \left(f_i(x)\right)^{\frac{-1}{r_i}}$ on Q^i. From (A.3), we can write

$$g_i[\lambda y + (1-\lambda)z] \geq \left(\lambda(g_i(y))^{-r_i} + (1-\lambda)(g_i(z))^{-r_i}\right)^{\frac{-1}{r_i}}.$$

It is clear that $g_i > 0$ on $\bigcap_{j\in I} \Omega^j$. Hence, g_i is $(-r_i)$-concave on Q^i.

Appendix B. Proof of Lemma 3

Let $x \in S(p)$ and

$$y_i = \frac{\psi_x[F_i(g_i(x))]}{\sum_{j=1}^{K} \psi_x[F_j(g_j(x))]}, \quad \forall\, i \in J, \tag{B.4}$$

where $\psi_x(.)$ refers to (11). It is easy to see that $\sum_{i\in J} y_i = 1$ and $y_i \geq 0$, for any $i \in J$. We prove that $y_i \geq G_i$, for any $i \in J$. Consider two cases as follows:

Case 1: $i \in I^D$. It follows from the Cauchy–Schwarz inequality that

$$|-\mu_i^\top x| \leq \|\mu_i\|\ \|x\|, \ \forall\, i \in J. \tag{B.5}$$

By the property of maximal/minimal eigenvalue, we have

$$\sqrt{\lambda_{i,\max}}\|x\| \geq \sqrt{x^\top \Sigma_i x} \geq \sqrt{\lambda_{i,\min}}\|x\|. \tag{B.6}$$

Since $x \in S(p)$, it follows from the constraint (7) that

$$0 < \sum_{j=1}^{K} \psi_x[F_j(g_j(x))] \leq \psi_x(p). \tag{B.7}$$

From (4), (B.5)–(B.6) and the fact that $\|x\| \geq \delta_{\mathrm{l}}$, we get

$$g_i(x) \leq \frac{D_i}{\sqrt{x^\top \Sigma_i x}} + \frac{|-\mu_i^\top x|}{\sqrt{x^\top \Sigma_i x}} \leq \frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}.$$

As F_i is increasing and ψ_x is decreasing, we get

$$\psi_x(F_i(g_i(x))) \geq \psi_x\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}\right)\right). \tag{B.8}$$

From (B.4), (B.7) and (B.8), we have

$$y_i \geq \frac{\psi_x\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}\right)\right)}{\psi_x(p)} = \left(\frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}\right)\right)}{\log p}\right)^{\frac{1}{\kappa(x)}}. \tag{B.9}$$

Since $0 \leq y_i \leq 1$, the following condition holds

$$0 < \frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}\right)\right)}{\log p} \leq 1,$$

which in turn implies by (ii) of Assumption 3 that

$$\left(\frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}\right)\right)}{\log p}\right)^{\frac{1}{\kappa(x)}} \geq \left(\frac{\log\left(F_i\left(\frac{D_i}{\sqrt{\lambda_{i,\min}}\delta_{\mathrm{l}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}\right)\right)}{\log p}\right)^{\frac{1}{c_{\mathrm{l}}}} = G_i. \tag{B.10}$$

Then, if follows from (B.9) and (B.10) that $y_i \geq G_i$.

Case 2: $i \in J \backslash I^D$. In this case, $D_i \leq 0$. Then, by (B.5), (B.6) and $x \leq \delta_{\mathrm{u}}$, we have

$$g_i(x) \leq \frac{D_i}{\sqrt{\lambda_{i,\max}}\delta_{\mathrm{u}}} + \frac{\|\mu_i\|}{\sqrt{\lambda_{i,\min}}}.$$

It follows the similar proof procedure as Case 1 that $y_i \geq G_i$.

Therefore, combining the results in Case 1 and Case 2, we have $y_i \geq h_{\mathrm{l}} > 0$, for any $i \in J$. Since, $\sum_{i\in J} y_i = 1$, we get

$$y_i = 1 - \sum_{j\in J, j\neq i} y_j \leq 1 - (n-1)\ h_{\mathrm{l}} = h_{\mathrm{u}} < 1. \tag{B.11}$$

Hence, $0 < h_{\mathrm{l}} \leq y_i \leq h_{\mathrm{u}} < 1$, for any $i \in J$.

Appendix C. Proof of Lemma 4

The Hessian matrix of U at a point (x, y_i) is an $(n+1)$-dimensional symmetric matrix which has the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A = H_x U(x, y_i)$, $B = \bigtriangledown_x \frac{\partial}{\partial y_i} U(x, y_i)$, $C = B^\top$, $D = \frac{\partial^2}{\partial y_i^2} U(x, y_i)$. The main idea of the proof is based on the Schur's complement. It suffices to show that $\frac{\partial^2}{\partial y_i^2} U(x, y_i) > 0$, for any $(x, y_i) \in Q \times [h_{\mathrm{l}}, h_{\mathrm{u}}]$. In fact, for $U(x, y_i) = \psi_x^{(-1)}(y_i \psi_x(p))$, we have

$$\frac{\partial^2}{\partial y_i^2} U(x, y_i) = [\psi_x(p)]^2 (\psi_x^{(-1)})''(y_i \psi_x(p)).$$

Since $\psi_x(t) = (-\log t)^{\frac{1}{\kappa(x)}}$, we deduce that

$$\psi_x^{(-1)}(t) = e^{-t^{\kappa(x)}}, \quad (\psi_x^{(-1)})''(t) = e^{-t^{\kappa(x)}} t^{\kappa(x)-2} \kappa(x) \left[\kappa(x) t^{\kappa(x)} - \kappa(x) + 1\right].$$

Using the above formulations, $\psi_x(p) > 0$ and $(\psi_x^{(-1)})''(t) > 0$, for any $t > 0$. Hence, $\frac{\partial^2}{\partial y_i^2} U(x, y_i) > 0$, for any $(x, y_i) \in Q \times [h_l, h_u]$.

References

[1] Prékopa A. Stochastic programming, vol. 324. Springer Science & Business Media; 2013.
[2] Henrion R, Strugarek C. Convexity of chance constraints with independent random variables. Comput Optim Appl 2008;41(2):263–76.
[3] Henrion R, Strugarek C. Convexity of chance constraints with dependent random variables: the use of copulae. In: Stochastic optimization methods in finance and Energy. Springer; 2011, p. 427–39.
[4] Marti K. Differentiation of probability functions: The transformation method. Comput Math Appl 1995;30(3–6):361–82.
[5] Kibzun A, Uryasev S. Differentiability of probability function. Stoch Anal Appl 1998;16(6):1101–28.
[6] Lobo MS, Vandenberghe L, Boyd S, Lebret H. Applications of second-order cone programming. Linear Algebr Appl 1998;284(1–3):193–228.
[7] Cheng J, Houda M, Lisser A. Second-order cone programming approach for elliptically distributed joint probabilistic constraints with dependent Rows. Technical report, 2014, Optimization.
[8] van Ackooij W. Eventual convexity of chance constrained feasible sets. Optimization 2015;64(5):1263–84.
[9] van Ackooij W. Convexity statements for linear probability constraints with gaussian technology matrices and copulæ correlated rows. ResearchGate; 2017, [ResearchGate - Working paper], https://www.researchgate.net/publication/318360771_Convexity_Statements_for_Linear_Probability_Constraints_with_Gaussian_Technology_Matrices_and_Copulae_Correlated_Rows.
[10] Hong LJ, Yang Y, Zhang L. Sequential convex approximations to joint chance constrained programs: A Monte Carlo approach. Oper Res 2011;59(3):617–30.
[11] Farshbaf-Shaker MH, Henrion R, Hömberg D. Properties of chance constraints in infinite dimensions with an application to pde constrained optimization. Set-Valued Var Anal 2018;26(4):821–41.
[12] van Ackooij W, Malick J. Eventual convexity of probability constraints with elliptical distributions. Math Program 2019;175(1):1–27.
[13] Nguyen HN, Lisser A, Singh VV. Random games under elliptically distributed dependent joint chance constraints. J Optim Theory Appl 2022;195(1):249–64.
[14] Peng S, Singh VV, Lisser A. General sum games with joint chance constraints. Oper Res Lett 2018;46(5):482–6.
[15] Peng S, Yadav N, Lisser A, Singh VV. Chance-constrained games with mixture distributions. Math Methods Oper Res 2021;94:71–97.
[16] McNeil AJ, Nešlehová J. Multivariate archimedean copulas, d-monotone functions and L1-norm symmetric distributions. Ann Statist 2009;37(5B):3059–97.
[17] Fang K-T, Kotz S, Ng KW. Symmetric multivariate and related distributions. 1st ed. New York: Chapman and Hall/CRC; 1990.
[18] Cheng J, Lisser A. A second-order cone programming approach for linear programs with joint probabilistic constraints. Oper Res Lett 2012;40(5):325–8.