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Preferential attachment hypergraph with vertex deactivation

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Abstract—In the field of complex networks, hypergraph models have so far received significantly less attention than graphs. However, many real-life networks feature multiary relations (co-authorship, protein reactions) may therefore be modeled way better by hypergraphs. Also, a recent study by Broido and Clauset suggests that a power-law degree distribution is not as ubiquitous in the natural systems as it was thought so far. They experimentally confirm that a majority of networks (56% of around 1000 networks that undergone the test) favor a power-law with an exponential cutoff over other distributions. We address the two above observations by introducing a preferential attachment hypergraph model which allows for vertex deactivations. The phenomenon of vertex deactivations is rare in existing theoretical models and omnipresent in real-life scenarios (social network accounts which are not maintained forever, collaboration networks in which people retire, technological networks in which devices break down). We prove that the degree distribution of the proposed model follows a power-law with an exponential cutoff. We also check experimentally that a Scopus collaboration network has the same characteristic. We believe that our model will predict well the behavior of systems from a variety of domains.

Index Terms—complex network, hypergraph, preferential attachment, power-law, exponential cutoff

I. INTRODUCTION

The notion of *complex networks* relates to the mathematical structures modeling large real-life systems. Their omnipresence across different life domains is remarkable. Complex networks model biological networks (e.g., protein or gene interactions schemes, maps of neural connections in the brain), social networks (Facebook, Twitter, Snapchat, collaboration networks), technological networks (power grids, transportation networks), the World Wide Web, etc. They allow to predict the behavior of the systems, serve as the benchmarks for testing algorithms that are used later in the real networks, and, in general, allow to understand better the underlying mechanisms that create those systems in nature. Roughly, since 1999, one observes a dynamical growth in experimental and theoretical research on complex networks in computer science, mathematical, and physical societies. It was the year when Barabási and Albert introduced the seminal model of a *preferential attachment* random graph [3]. This model is based on two mechanisms: growth (the graph is growing over time, gaining a new vertex and a bunch of edges at each time step)

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and preferential attachment (an arriving vertex is more likely to attach to other vertices with high degrees rather than with low degrees). It captures the *small world* (small diameter) and the *rich get richer* (leading to a heavy tailed degree distribution) phenomena commonly observed in nature.

Since then, a number of theoretical models were presented, e.g., [10], [11], [15], [26], [30]. These were mostly graph models concentrated on reflecting three phenomena: a small diameter, a high clustering coefficient, and a *power-law* degree distribution. It was thought for a long time that a power-law degree distribution is the most commonly present in nature [7]. However, this statement was recently questioned by Broido and Clauset [9]. They performed statistical tests on almost 1000 social, biological, technological, transportation, and information networks and observed that a “majority of networks (56%) favor the power-law with cutoff model over other distributions”. The cutoff observed in the tail of a distribution may be caused by a finite-size character of the dynamic network, i.e., when the elements deactivate after some time [9]. The phenomenon of vertex deactivations is rare in known theoretical models and omnipresent in real-life scenarios as the extinction events are fundamental in the world surrounding us. Think of social networks (Facebook, Twitter, Instagram, etc.) where users unsubscribe or simply stop using them, collaboration networks in which deactivated nodes represent people who retired, died or stopped working in the given domain, technological networks where a vertex deactivation is interpreted as a breakdown of the device or the web network in which web pages are not maintained forever. Even though some theoretical models featuring deletions or deactivations of vertices were introduced [14], [27], just a few of them lead to a degree distribution following a power-law with an exponential cutoff. One of the widely cited is a balls and bins scheme introduced by Fenner et al. [17], [18].

In the model from [17], [18], information about the degree of each element of the network is kept but information about who is connected with whom is lost. Working with graphs instead of bins and balls allows to keep this information. Nevertheless, graphs have another clear limitation. They reflect only binary relations while in practice we encounter many higher order relations (groups of interest, protein reactions, co-authorship, interactions between biological cells, GitHub users committing to the same repository). Nowadays they are often modeled in graphs by cliques which may lead to a profound information loss [4]. E.g., if there are three researchers in

a triangle in a collaboration graph, one cannot tell whether they published one paper together or three independent papers, each per pair of researchers. Higher order relations can be captured by hypergraphs, that is, a generalization of graphs in which each (hyper)edge possibly links together more than two nodes. Sometimes keeping information about hyperedge may have a profound impact on analyzing the model and drawing conclusions. Consider any example in which a big hyperedge strongly indicates belonging to the same community (e.g., an email sent to a group of people should evidence the existence of a community rather than be treated as a set of bilateral emails). So far hypergraph models have received significantly less attention than graphs in the area of complex networks. Wang et al. introduced a preferential attachment hypergraph model but restricted to a specific subfamily of uniform acyclic hypergraphs (the analogue of trees within graphs) [29]. The first rigorously studied non-uniform hypergraph preferential attachment model was proposed only in 2019 by Avin et al. [2] and featured a power-law degree distribution. Another dynamic hypergraph model with a clear community structure was presented in [20]. Note that there exists an analogy between the hypergraphs and the random intersection graphs [5], [6]. The algorithms and software tools for working with hypergraph networks, even the definitions of some features and measures started appearing only recently [1], [21]–[23].

Results. We propose a preferential attachment hypergraph model in which vertices may become inactive after some time. The hyperedges model multiary and not necessarily uniform relations, we allow for different cardinalities of hyperedges (e.g., articles may have different numbers of co-authors). We prove that the degree distribution of our model follows a power-law with an exponential cutoff and compare it with a real-life example, a Scopus research collaboration network. We believe that our model will be the next step towards developing the hypergraph chapter in the complex networks area and that will serve as a useful tool predicting well the behavior of the systems from a variety of domains.

Paper organization. Sec. II contains basic definitions and notation. In Sec. III, we introduce the hypergraph model with vertex deactivation and prove that its degree distribution follows a power-law with an exponential cutoff using a master equation approach. Due to the presence of hyperedges and to the possibility of vertex deactivation, we had to modify the classical approach (e.g. treat active and inactive vertices separately) and take advantage of some tools that were not used in this context before (e.g. the Stolz-Cesaro Theorem). In Sec. IV, we estimate one of the parameters that appears in the formula for the degree distribution of our model. It is defined as a limit, existence of which we assume (Asm. (4) formulated in Sec. III.B) to prove the main result (Thm. 2). Such an assumption was already present in the literature on models with degree distribution following a power-law with an exponential cutoff, [17], [18]. Even though we also did not manage to prove the existence of this limit directly, the technical novelty is that we give a formally rigid indication on how to estimate its value (using Gaussian hypergeometric functions

and the Banach Fixed Point Theorem) and an experimental justification for its existence in Sec. V. Sec. V also includes the experimental results on real data and the simulations of the model. Further works are discussed in Sec. VI.

II. BASIC DEFINITIONS AND NOTATION

We define a *hypergraph* H as a pair $H = (V, E)$, where V is a set of vertices and E is a multiset of hyperedges, i.e., non-empty, unordered multisets of V . We allow for a multiple appearance of a vertex in a hyperedge (self-loops) as well as a multiple appearance of a hyperedge in E . The degree of a vertex v in a hyperedge e , denoted by $d(v, e)$, is the number of times v appears in e . The cardinality of a hyperedge e is $|e| = \sum_{v \in e} d(v, e)$. The degree of a vertex $v \in V$ in H is understood as the number of times it appears in all hyperedges, i.e., $\deg(v) = \sum_{e \in E} d(v, e)$. If $|e| = k$ for all $e \in E$, H is said to be *k-uniform*.

We consider hypergraphs that grow by adding vertices and/or hyperedges at discrete time steps $t = 0, 1, 2, \dots$ according to some rules involving randomness. The random hypergraph obtained at time t will be denoted by $H_t = (V_t, E_t)$ and the degree of $u \in V_t$ in H_t by $\deg_t(u)$. During this building process some of the vertices may become deactivated. Therefore the set V_t splits into \mathcal{A}_t , the set of vertices active at time t (denote its cardinality by A_t), and \mathcal{I}_t , the set of vertices that are not active at time t (denote its cardinality by I_t); thus $|V_t| = A_t + I_t$. By D_t we denote the sum of degrees of vertices active at time t , i.e., $D_t = \sum_{u \in \mathcal{A}_t} \deg_t(u)$. Moreover, we write Θ_t for the degree of a vertex chosen for deactivation at time t (the description of a deactivation procedure is given within the formal definition of the model in the next section).

$N_{k,t}$ stands for the number of vertices in H_t of degree k . Thus $\sum_{k \geq 1} N_{k,t} = |V_t|$. Similarly, $A_{k,t}$ is the number of active vertices of degree k at time t and $I_{k,t}$ the number of inactive vertices of degree k at time t (denote the corresponding sets by $\mathcal{A}_{k,t}$ and $\mathcal{I}_{k,t}$, respectively); $\sum_{k \geq 1} A_{k,t} = A_t$, $\sum_{k \geq 1} I_{k,t} = I_t$ and $N_{k,t} = A_{k,t} + I_{k,t}$. We write $f(k) \sim g(k)$ if $f(k)/g(k) \xrightarrow{k \rightarrow \infty} 1$. We say that the degree distribution of a random hypergraph follows a *power-law* if the expected fraction of vertices of degree k is proportional to $k^{-\beta}$ for some exponent $\beta > 1$. Formally, we interpret it as $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta}$ for some positive constants c and $\beta > 1$. Similarly, we say that the degree distribution of H_t follows a *power-law with an exponential cutoff* if $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k$, where $\gamma \in (0, 1)$.

We say that an event A occurs *with high probability* (whp) if the probability $\mathbb{P}[A]$ depends on a certain number t and tends to 1 as t tends to infinity.

III. PREFERENTIAL ATTACHMENT HYPERGRAPH WITH VERTEX DEACTIVATION

The model introduced in this section may be seen as a generalization of a hypergraph model presented by Avin et al. in [2]. The model from [2] allows for two different actions at a single time step - attaching a new vertex by a hyperedge to

the existing structure or creating a new hyperedge on already existing vertices. We add another possibility - deactivation of a vertex. Once a vertex is chosen for deactivation, it stays deactivated forever, i.e., it remains in the hypergraph but it can not be chosen to the new hyperedges - its degree freezes and the hyperedges incident with it remain in the hypergraph. Avin et al. proved that the degree distribution of their model follows a power-law. We prove that adding the possibility of deactivation of vertices generates an exponential cutoff in the degree distribution.

A. Model $\mathbf{H}(\mathbf{H}_0, \mathbf{p}_v, \mathbf{p}_e, \mathbf{Y})$

The hypergraph model H is characterized by the following parameters:

- 1) H_0 - the initial hypergraph, seen at $t = 0$;
- 2) $p_v, p_e, p_d = 1 - p_e - p_v$ - the probabilities indicating, what are the chances that a particular type of event occurs at a single time step;
- 3) $Y = (Y_0, Y_1, \dots, Y_t, \dots)$ - independent random variables, giving the cardinalities of the hyperedges that are added at a single time step.

Here is how the structure of $H = H(H_0, p_v, p_e, Y)$ is being built. We start with some non-empty hypergraph H_0 at $t = 0$. We assume for simplicity that H_0 consists of a hyperedge of cardinality 1 over a single vertex. Nevertheless, all the proofs may be generalized to any initial H_0 having constant number of vertices and constant number of hyperedges with constant cardinalities. ‘Vertices chosen from \mathcal{A}_t in proportion to degrees’ means that active vertices are chosen independently (possibly with repetitions) and the probability that any u from \mathcal{A}_t is chosen is

$$\mathbb{P}[u \text{ is chosen}] = \frac{\deg_t(u)}{\sum_{v \in \mathcal{A}_t} \deg_t(v)} = \frac{\deg_t(u)}{D_t}$$

($\deg_t(u)$ and $\deg_t(v)$ refer to the degrees of u and v in the whole H_t). For $t \geq 0$ we form H_{t+1} from H_t choosing only one of the following events according to p_v, p_e, p_d .

- With probability p_v : Add one vertex v . Draw a value y being a realization of Y_t . Then select $y - 1$ vertices from \mathcal{A}_t in proportion to degrees; add a new hyperedge consisting of v and the $y - 1$ selected vertices.
- With probability p_e : Draw a value y being a realization of Y_t . Then select y vertices from \mathcal{A}_t in proportion to degrees; add a new hyperedge consisting of the y selected vertices.
- With probability p_d : Choose one vertex from \mathcal{A}_t in proportion to degrees. Deactivate it, i.e., $\mathcal{A}_{t+1} = \mathcal{A}_t \setminus \{v\}$ and $\mathcal{I}_{t+1} = \mathcal{I}_t \cup \{v\}$.

Remark 1. Note that this model can be simplified to many known models by choosing the appropriate set of parameters:

- 1) setting $p_v = 1, p_e = p_d = 0$ and $Y_t = 2$ (all the hyperedges are of size 2 thus one simply builds a graph) one gets the Barabási-Albert tree [3];
- 2) setting $p_d = 0$ and $Y_t = 2$ one gets the preferential attachment scheme for graphs with vertex- and edge-step [12], Chapter 3;

- 3) setting $p_d = 0$ one gets the hypergraph model presented by Avin et al. in [2].

Remark 2. Vertices in this model are deleted in proportion to their degrees, which may reflect aging, retirement, etc. The choice of other deletion procedures appearing in some scenarios (e.g., deleting vertices at random or inversely to the degree) would influence significantly our calculations and might even lead to a different limiting degree distribution. We leave it as a future work.

Remark 3. As the hypergraph gets large, the probability of creating a self-loop can be well bounded and is quite small provided that the sizes of hyperedges are reasonably bounded.

Note that if we want a process to continue then it is reasonable to demand that, on average, we add more vertices to the system than we deactivate. Therefore we always assume $p_v > p_d$. Then the probability that the process will not terminate (i.e., that we never arrive at the moment in which all vertices are deactivated) is positive and equals $1 - (p_d/p_v)^i$, where i is the number of active vertices at time $t = 0$, in our case $i = 1$ (compare with the probability that the gambler’s fortune will increase forever, [16]). We concentrate only on the case when the process does not terminate.

B. Degree distribution of $\mathbf{H}(\mathbf{H}_0, \mathbf{p}_v, \mathbf{p}_e, \mathbf{Y})$

In this section we prove that the degree distribution of $H = H(H_0, p_v, p_e, Y)$ follows a power-law with an exponential cutoff under four assumptions.

First two of them address the distributions of the cardinalities of hyperedges (Y_t) added step by step. We assume that their expectation is constant and their variance sublinear in t , which, we feel, is in accordance with many real-life systems (in particular, with the scientific collaboration network we are working with experimentally in Sec. V).

The third assumption tells that we will restrict ourselves to only such distributions of Y_t for which the distribution of D_t (the sum of degrees of active vertices at time t) remains concentrated. Similar assumption one finds in other papers on complex network models, e.g. in [2] by Avin et al. (presenting a model of a preferential attachment hypergraph with the degree distribution following a power-law) or in [24], [25] by Krapivsky et al. (where the models in which the arriving vertex attaches to the existing node w with probability proportional to $(\deg w)^r$ with $r < 1$ is studied).

The fourth assumption refers to the average sum of degrees of vertices chosen for deactivation, $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$. In Sec. IV we prove that its order is $\Theta(t)$. However, we additionally assume that the limit $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ exists and equals some $\theta \in \mathbb{R}_{>0}$. Such assumption was also already present in the literature on models with degree distribution following a power-law with an exponential cutoff [17], [18]. Since we were not able to (just as the authors of [17] or [18]) theoretically justify the existence of the stated limit we support it by simulations in Sec. V. We also explain in Sec. IV how the limiting value may be obtained, assuming that the limit exists.

Assumptions

- 1) $\mathbb{E}[Y_t] = \mu \in \mathbb{R}_{>0}$ for all $t > 0$.
- 2) $\text{Var}[Y_t] = o(t)$.
- 3) $\mathbb{P}[D_t \neq \mathbb{E}[D_t] + o(t)] = o(1/t)$.
- 4) $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \theta \in \mathbb{R}_{>0}$.

Before we formally state and prove the main theorem we introduce several technical lemmas and theorems that will be helpful later on.

Theorem 1 (Stolz-Cesàro). *Let $(a_t)_{t \geq 1}$ and $(b_t)_{t \geq 1}$ be the sequences of real numbers. Assume $(b_t)_{t \geq 1}$ is strictly monotone and divergent. If $\lim_{t \rightarrow \infty} \frac{a_{t+1} - a_t}{b_{t+1} - b_t} = g$ then $\lim_{t \rightarrow \infty} \frac{a_t}{b_t} = g$.*

Lemma 1 ([12], Chapter 3.3). *Let $(a_t)_{t \geq 1}$, $(b_t)_{t \geq 1}$ and $(c_t)_{t \geq 1}$ be the sequences of real numbers, where $b_t \xrightarrow{t \rightarrow \infty} b > 0$, $c_t \xrightarrow{t \rightarrow \infty} c$ and a_t satisfies the recursive relation $a_{t+1} = (1 - \frac{b_t}{t})a_t + c_t$. Then $\lim_{t \rightarrow \infty} \frac{a_t}{t} = \frac{c}{1+b}$.*

The proofs of Lemmas 2, 3, and 4 can be found in the extended version of this paper [19].

Lemma 2. *If $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta\right)$ for some positive constants c, β, γ, δ then $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim \frac{c}{p_v} k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta\right)$. (Here “ \sim ” refers to the limit by $k \rightarrow \infty$).*

Lemma 3. *Let $\mathbb{E}[Y_t] = \mu$ for all $t > 0$ and $\text{Var}[Y_t] = o(t)$. Then $\mathbb{E} \left[\frac{Y_t^2}{D_{t-1}^2} \right] = o\left(\frac{1}{t}\right)$.*

Lemma 4. *Let $D_t = \mathbb{E}[D_t] + o(t)$ whp. Then $\mathbb{E} \left[\frac{A_{k,t}}{D_t} \right] = \frac{\mathbb{E}[A_{k,t}]}{\mathbb{E}[D_t]} + o(1)$ for each $k \geq 1$.*

Lemma 5. *Let $\mathbb{E}[Y_t] = \mu$ for all $t > 0$. Assume that $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \theta \in \mathbb{R}_{>0}$. Then $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[D_t]}{t} = (p_v + p_e)\mu - p_d\theta$.*

Proof. The initial hypergraph H_0 consists of a single vertex of degree 1. Since at time $t \geq 1$ we add a hyperedge of cardinality Y_t with probability $p_v + p_e$ and we deactivate a vertex of degree Θ_t with probability p_d we get

$$\mathbb{E}[D_t] = 1 + (p_v + p_e) \sum_{\tau=1}^t \mathbb{E}[Y_\tau] - p_d \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]. \quad (1)$$

The conclusion follows. \square

Theorem 2. *Consider a hypergraph $H = H(H_0, p_e, p_v, Y)$ for any $t > 0$. By Asm. (1-4) the degree distribution of H follows a power-law with an exponential cutoff, i.e.,*

$$\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta\right)$$

$$\text{for } \beta = \frac{\mu(p_v + p_e) - p_d\theta}{p_v(\mu - 1) + p_e\mu + p_d}, \gamma = \frac{p_v(\mu - 1) + p_e\mu}{p_v(\mu - 1) + p_e\mu + p_d},$$

$$\delta = \frac{p_d}{\mu(p_v + p_e) - p_d\theta}, \quad c = \frac{\beta \cdot \Gamma(1 + \beta)}{\gamma},$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$.

Remark 4. *The theorem and its proof presented below remain true if we relax Asm. (3) just to $D_t = \mathbb{E}[D_t] + o(t)$ whp.*

Nevertheless, we leave the stronger version of (3) on the list of assumptions as it will be needed in Sec. IV for estimating θ .

Remark 5. *Setting $p_d = 0$ in the above theorem (i.e., considering the process without deactivation) results in the power-law degree distribution, namely $\mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-(\beta+1)}$, where $\beta = \frac{\mu}{\mu - p_v}$ and $c = \beta \cdot \Gamma(1 + \beta)$. This is in accordance with the result obtained in [2].*

See [19] for the full proof without shortcuts in calculations.

Proof. We take a standard master equation approach that can be found e.g. in Chung and Lu book [12] about complex networks. However, we apply it separately to the number of active vertices and the number of deactivated vertices.

Recall that $N_{k,t}$ denotes the number of vertices of degree k at time t . We need to show that $\lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{N_{k,t}}{|V_t|} \right] \sim c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta\right)$ for the proper constants c, β, γ and δ . However, by Lemma 2 we know that it suffices to show that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} \sim p_v \cdot c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta\right).$$

Recall that $N_{k,t} = A_{k,t} + I_{k,t}$. First, let us evaluate $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t}$ using the mathematical induction on k . In this part we follow closely the lines of the proof that can be found in [2]. Consider the case $k = 1$. Since H_0 consists of a single hyperedge of cardinality 1 over a single vertex, we have $A_{1,0} = 1$. To formulate a master equation, let us make the following observation for $t \geq 1$. An active vertex remains in $A_{1,t}$ if it had degree 1 at step $t-1$ and was neither selected to a hyperedge, nor deactivated. Recall that a vertex from $A_{1,t-1}$ is chosen at step t in a single trial to the new hyperedge with probability $1/D_{t-1}$ thus the chance that it won't be selected to the hyperedge of cardinality y equals $(1 - 1/D_{t-1})^y$. Also, in each step, with probability p_v , a single new active vertex of degree 1 is added to the hypergraph. Let \mathcal{F}_t denote a σ -algebra associated with the probability space at step t . For $t \geq 1$

$$\mathbb{E}[A_{1,t} | \mathcal{F}_{t-1}] = p_v A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right)^{Y_{t-1}}$$

$$+ p_e A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right)^{Y_t} + p_d A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right) + p_v. \quad (2)$$

Taking the expectation on both sides of (2) we derive upper and lower bounds on $\mathbb{E}[A_{1,t}]$. By Bernoulli's inequality $((1+x)^n \geq 1 + nx$ for $n \in \mathbb{N}$ and $x \geq -1$), Lemma 4 (thus by Asm. (3)) and the independence of Y_t from $A_{1,t-1}$ and D_{t-1}

$$\mathbb{E}[A_{1,t}] \geq p_v \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t - 1}{D_{t-1}}\right) \right]$$

$$+ p_e \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{Y_t}{D_{t-1}}\right) \right]$$

$$+ p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}}\right) \right] + p_v$$

$$= \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu - 1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]}\right) + p_v - o(1). \quad (3)$$

On the other hand, since $(1-x)^n \leq 1/(1+nx)$ for $x \in [0, 1]$ and $n \in \mathbb{N}$, and $A_{1,t-1} \leq t$, by Lemma 3 (thus by Asm. (1), (2) necessary for it) and Lemma 4 (thus by Asm. (3)) we have

$$\begin{aligned} \mathbb{E}[A_{1,t}] &\leq p_v \mathbb{E} \left[\frac{A_{1,t-1}}{1 + (Y_t - 1)/D_{t-1}} \right] + p_e \mathbb{E} \left[\frac{A_{1,t-1}}{1 + Y_t/D_{t-1}} \right] \\ &+ p_d \mathbb{E} \left[A_{1,t-1} \left(1 - \frac{1}{D_{t-1}} \right) \right] + p_v \\ &= \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu-1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v + o(1). \end{aligned} \quad (4)$$

From (3) and (4) we get

$$\mathbb{E}[A_{1,t}] = \mathbb{E}[A_{1,t-1}] \left(1 - \frac{p_v(\mu-1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]} \right) + p_v + o(1).$$

Now, we apply Lemma 1 to the above equation choosing

$$a_t = \mathbb{E}[A_{1,t}], \quad b_t = \frac{p_v(\mu-1) + p_e\mu + p_d}{\mathbb{E}[D_{t-1}]/t}, \quad c_t = p_v + o(1).$$

We have $\lim_{t \rightarrow \infty} c_t = p_v$ and, by Fact 5 (thus by Asm. (1), (4) implying it), $\lim_{t \rightarrow \infty} b_t = \frac{p_v(\mu-1) + p_e\mu + p_d}{\mu(p_v + p_e) - p_d\theta} =: 1/\beta$ thus

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{1,t}]}{t} = \frac{p_v}{1 + 1/\beta} =: \bar{A}_1.$$

Now, we assume that the limit $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k-1,t}]}{t}$ exists and equals \bar{A}_{k-1} and we will show by induction on k that the analogous limit for $\mathbb{E}[A_{k,t}]$ exists. Let us again formulate a master equation, this time for $k > 1$. We have $A_{k,0} = 0$ and for $t \geq 1$ an active vertex appears in $A_{k,t}$ if it was active at step $t-1$, had degree $k-l$ and was chosen exactly l times to a hyperedge, or it had degree k and was not selected for deactivation. Let $B(l, n, p) = \binom{n}{l} p^l (1-p)^{n-l}$. We have

$$\begin{aligned} \mathbb{E}[A_{k,t} | \mathcal{F}_{t-1}] &= p_v \sum_{l=0}^{\min\{Y_t-1, k-1\}} A_{k-l, t-1} B \left(l, Y_t - 1, \frac{k-l}{D_{t-1}} \right) \\ &+ p_e \sum_{l=0}^{\min\{Y_t, k-1\}} A_{k-l, t-1} B \left(l, Y_t, \frac{k-l}{D_{t-1}} \right) + p_d A_{k, t-1} \left(1 - \frac{k}{D_{t-1}} \right). \end{aligned}$$

Taking the expectation on both sides we get

$$\mathbb{E}[A_{k,t}] = \mathbb{E}[\psi] + p_v \mathbb{E}[\varphi(Y_t - 1)] + p_e \mathbb{E}[\varphi(Y_t)], \quad \text{where}$$

$$\begin{aligned} \psi &= p_v \sum_{l=0}^1 A_{k-l, t-1} B \left(l, Y_t - 1, \frac{k-l}{D_{t-1}} \right) \\ &+ p_e \sum_{l=0}^1 A_{k-l, t-1} B \left(l, Y_t, \frac{k-l}{D_{t-1}} \right) + p_d A_{k, t-1} \left(1 - \frac{k}{D_{t-1}} \right) \end{aligned}$$

$$\text{and } \varphi(n) = \sum_{l=2}^{\min\{n, k-1\}} A_{k-l, t-1} B \left(l, n, \frac{k-l}{D_{t-1}} \right).$$

We will show that only the term $\mathbb{E}[\psi]$ is significant and $\mathbb{E}[\varphi(Y_t - 1)]$ and $\mathbb{E}[\varphi(Y_t)]$ converge to 0 as $t \rightarrow \infty$. We have

$$\begin{aligned} \varphi(Y_t) &\leq \sum_{l=2}^{k-1} A_{k-l, t-1} \binom{Y_t}{l} \left(\frac{k-l}{D_{t-1}} \right)^l \left(1 - \frac{k-l}{D_{t-1}} \right)^{Y_t-l} \\ &= O(t) \frac{Y_t^2}{D_{t-1}^2}. \end{aligned}$$

Hence by Lemma 3 (thus by Asm. (1), (2)) we get $\mathbb{E}[\varphi(Y_t)] = o(1)$ and, similarly, $\mathbb{E}[\varphi(Y_t - 1)] = o(1)$. The bounds for $\mathbb{E}[\psi]$ derived analogously to the ones for $\mathbb{E}[A_{1,t}]$ give

$$\begin{aligned} \mathbb{E}[A_{k,t}] &= \mathbb{E}[A_{k, t-1}] \left(1 - \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]} \right) \\ &+ \mathbb{E}[A_{k-1, t-1}] \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]} + o(1). \end{aligned} \quad (5)$$

Recall that by the induction assumption $\lim_{t \rightarrow \infty} \mathbb{E}[A_{k-1, t}]/t = \bar{A}_{k-1}$. Now, we apply again Lemma 1 to the above equation choosing

$$\begin{aligned} a_t &= \mathbb{E}[A_{k,t}], \quad b_t = \frac{k(p_v(\mu-1) + p_e\mu + p_d)}{\mathbb{E}[D_{t-1}]/t}, \\ c_t &= \frac{\mathbb{E}[A_{k-1, t-1}]}{t} \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mathbb{E}[D_{t-1}]/t} + o(1). \end{aligned}$$

By Fact 5 (thus by Asm. (1), (4)) we have $\lim_{t \rightarrow \infty} b_t = k/\beta$ and $\lim_{t \rightarrow \infty} c_t = \bar{A}_{k-1} \frac{(k-1)(p_v(\mu-1) + p_e\mu)}{\mu(p_v + p_e) - p_d\theta}$ thus

$$\lim_{t \rightarrow \infty} \frac{a_t}{t} = \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t} = \bar{A}_k = \bar{A}_{k-1} \frac{(k-1)\gamma}{k + \beta}, \quad (6)$$

where $\gamma = \frac{p_v(\mu-1) + p_e\mu}{p_v(\mu-1) + p_e\mu + p_d}$. Thus we got

$$\begin{aligned} \bar{A}_1 &= p_v \beta \frac{1}{(1 + \beta)}, \quad \bar{A}_2 = p_v \beta \frac{\gamma}{(1 + \beta)(2 + \beta)}, \dots, \\ \bar{A}_k &= p_v \beta \frac{\gamma^{k-1} (k-1)!}{(1 + \beta)(2 + \beta) \dots (k + \beta)}. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} \frac{\Gamma(k)k^\alpha}{\Gamma(k+\alpha)} = 1$ for constant $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}]}{t} &= \bar{A}_k = \frac{p_v \beta}{\gamma} \frac{\gamma^k \Gamma(1 + \beta) \Gamma(k)}{\Gamma(k + \beta + 1)} \\ &\sim p_v \cdot c \cdot \gamma^k k^{-(\beta+1)} \quad \text{with } c = \frac{\beta \cdot \Gamma(1 + \beta)}{\gamma}. \end{aligned} \quad (7)$$

Now, let us evaluate $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{k,t}]}{t}$. We have $I_{k,0} = 0$ for all $k \geq 1$. For $t \geq 1$ the expected number of inactive vertices of degree $k \geq 1$ at step t , given \mathcal{F}_{t-1} , can be expressed as

$$\mathbb{E}[I_{k,t} | \mathcal{F}_{t-1}] = I_{k, t-1} + p_d A_{k, t-1} \frac{k}{D_{t-1}},$$

since inactive vertices of degree k remain in $I_{k,t}$ forever and a vertex of degree k becomes inactive if it was selected in step $t-1$ for deactivation. Taking the expectation on both sides, by Lemma 4 (thus by Asm. (3)), we obtain

$$\mathbb{E}[I_{k,t}] = \mathbb{E}[I_{k, t-1}] + p_d \mathbb{E}[A_{k, t-1}] \frac{k}{\mathbb{E}[D_{t-1}]} + o(1).$$

Then, by Fact 5 (thus by Asm. (1) and (4)),

$$\begin{aligned} \lim_{t \rightarrow \infty} (\mathbb{E}[I_{k,t}] - \mathbb{E}[I_{k, t-1}]) &= \lim_{t \rightarrow \infty} p_d k \frac{\mathbb{E}[A_{k, t-1}] \cdot t}{t \cdot \mathbb{E}[D_{t-1}]} + o(1) \\ &= \bar{A}_k k \delta, \quad \text{where } \delta = \frac{p_d}{(p_v + p_e)\mu - p_d\theta}. \end{aligned}$$

And, by Stolz-Cesàro theorem (Thm. 1), we obtain

$$\bar{I}_k := \lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{k,t}]}{t} = \lim_{t \rightarrow \infty} (\mathbb{E}[I_{k,t}] - \mathbb{E}[I_{k, t-1}]) = \bar{A}_k k \delta. \quad (8)$$

Finally, by (7) and (8)

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathbb{E}[N_{k,t}]}{t} &= \lim_{t \rightarrow \infty} \frac{\mathbb{E}[A_{k,t}] + \mathbb{E}[I_{k,t}]}{t} = \bar{A}_k + \bar{I}_k \\ &= \bar{A}_k(1 + k\delta) \sim p_v \cdot c \cdot k^{-\beta} \gamma^k \left(\frac{1}{k} + \delta \right). \end{aligned}$$

□

IV. ESTIMATING THE LIMITING VALUE θ

This section is devoted to estimating θ which appears as one of the parameters in the degree distribution of our hypergraph model H (consult Thm. 2). Recall that Θ_t stands for the degree of a vertex chosen for deactivation at time t and it appears in the fourth assumption needed to prove Thm. 2

4. $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \theta \in \mathbb{R}_{>0}$.

Let us start with showing that $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ is of order $\Theta(t)$.

Lemma 6. *Let $\mathbb{E}[Y_t] = \mu$ for all $t > 0$. Then $p_d \leq \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] \leq 1 + \frac{p_v(\mu-1) + p_e\mu}{p_d}$.*

Proof. By equation (1) we get $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \frac{1}{p_d} (1 + (p_v + p_e)\mu t - \mathbb{E}[D_t])$. Note that $\mathbb{E}[D_t] \geq \mathbb{E}[A_t] = 1 + (p_v - p_d)t$ (we assume $p_v > p_d$) thus on one hand $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] \leq t \left(1 + \frac{p_v(\mu-1) + p_e\mu}{p_d} \right)$ and on the other $\sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] \geq \mathbb{E}[I_t] = p_d t$. □

Unfortunately, we were not able to prove that the limit $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ exists. We support this assumption by simulations in Sec. V. Whereas in this section we show, assuming that the limit exists, how to estimate it.

Throughout this section $F(a, b; c; z)$ denotes the Gaussian hypergeometric function, i.e., for $a, b, c, z \in \mathbb{C}$, $|z| < 1$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$.

Lemma 7. *Assume that $\mathbb{P}[D_t \neq \mathbb{E}[D_t] + o(t)] = o(1/t)$. Then*

$$\mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t}}{D_t} \right] = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t}]}{\mathbb{E}[D_t]} + o(1).$$

The proof can be found in [19].

Theorem 3. *Assume that the conditions (1–4) from Sec. III hold. Then θ is a fixed point of the function $R(x) := \frac{F(2, 2; \rho(x); \gamma)}{F(1, 2; \rho(x); \gamma)}$, where $\gamma = \frac{p_v(\mu-1) + p_e\mu}{p_v(\mu-1) + p_e\mu + p_d}$ and $\rho(x) = 2 + \frac{\mu(p_v + p_e) - p_d x}{p_v(\mu-1) + p_e\mu + p_d}$.*

Proof. Recall that $\theta = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ and Θ_t is the degree of a vertex chosen for deactivation at time t . Let \mathcal{F}_t denote a σ -algebra associated with the probability space at step t . We have $\mathbb{E}[\Theta_t | \mathcal{F}_{t-1}] = \sum_{k \geq 1} k \frac{k A_{k,t-1}}{D_{t-1}}$, hence

taking expectation on both sides, applying Lemma 7 (thus by Asm. (3)) and noting that $D_t = \sum_{k \geq 1} k A_{k,t}$ we get

$$\begin{aligned} \mathbb{E}[\Theta_t] &= \mathbb{E} \left[\frac{\sum_{k \geq 1} k^2 A_{k,t-1}}{D_{t-1}} \right] = \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t-1}]}{\mathbb{E}[D_{t-1}]} + o(1) \\ &= \frac{\mathbb{E}[\sum_{k \geq 1} k^2 A_{k,t-1}]}{\mathbb{E}[\sum_{k \geq 1} k A_{k,t-1}]} + o(1). \end{aligned}$$

Now, by eq. (7) (thus by Asm. (1-4) needed to prove Thm. 2)

$$\lim_{t \rightarrow \infty} \mathbb{E}[\Theta_t] = \frac{\sum_{k \geq 1} k^2 \bar{A}_k}{\sum_{k \geq 1} k \bar{A}_k} = \frac{F(2, 2; \rho(\theta); \gamma)}{F(1, 2; \rho(\theta); \gamma)}.$$

Finally, setting $a_t = \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$ and $b_t = t$ in Stolz-Cesàro Theorem (Thm. 1) we obtain

$$\theta = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau] = \frac{F(2, 2; \rho(\theta); \gamma)}{F(1, 2; \rho(\theta); \gamma)}.$$

□

From now on we consider the behavior of $R(x)$ only in the interval $[0, \hat{\theta}]$, where $\hat{\theta} = \frac{(p_v + p_e)\mu}{p_d}$ since we know that the limiting value θ we are looking for belongs there. Indeed, by Lemma 6 we know that it is at least p_d and at most $\frac{(p_v + p_e)\mu - p_v + p_d}{p_d}$ and we work by $p_v > p_d$ to ensure that, on average, we add more vertices to the network than we deactivate. Recall that the function $F(a, b; c; z)$ is defined for $|z| < 1$ and $c \notin \mathbb{Z}_{\leq 0}$. Therefore, since $0 < \gamma < 1$ and $\rho(x)$ is positive on $[0, \hat{\theta}]$, both $F(1, 2; \rho(x); \gamma)$ and $F(2, 2; \rho(x); \gamma)$ are always defined, continuous and positive on $[0, \hat{\theta}]$. This implies that $R(x)$ is continuous on $[0, \hat{\theta}]$. Below we will justify that $R(x)$ has just one fixed point in the interval $[0, \hat{\theta}]$ and that a fixed-point iteration method will converge here. We start with recalling Banach Fixed Point Theorem.

Theorem 4 (Banach Fixed Point Theorem). *Let (S, d) be a non-empty complete metric space with a contraction mapping $R : S \rightarrow S$. Then R admits a unique fixed point s^* in S ($R(s^*) = s^*$). Furthermore, s^* can be found as follows: start with an arbitrary element $s_0 \in S$ and define a sequence $\{s_n\}_{n \geq 1}$ by $s_n = R(s_{n-1})$ for $n \geq 1$. Then $\lim_{n \rightarrow \infty} s_n = s^*$.*

Thus we aim at showing that $R(x)$ is a contraction mapping on $[0, \hat{\theta}]$. From now on let $F_1(x) = F(1, 2; \rho(x); \gamma)$ and $F_2(x) = F(2, 2; \rho(x); \gamma)$ for γ and $\rho(x)$ as in Thm. 3.

The proofs of Lemmas 8, 9, and 10 can be found in [19].

Lemma 8. *The function $R(x) = \frac{F_2(x)}{F_1(x)}$ can be expressed as $R(x) = x - \frac{p_v}{p_d} + \frac{1}{1-\gamma} \frac{\rho(x)-1}{F_1(x)}$, where $\gamma, \rho(x)$ are as in Thm. 3.*

Lemma 9. *The function $R(x)$ strictly increases on $[0, \hat{\theta}]$.*

Lemma 10. *$R(x)$ is a contraction mapping on $[0, \hat{\theta}]$.*

Corollary 1. *Assume that the conditions (1–4) from Sec. III hold (in particular, $\theta = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=1}^t \mathbb{E}[\Theta_\tau]$). Then θ is a unique fixed point of $R(x)$ in $[0, \hat{\theta}]$, such that $\lim_{n \rightarrow \infty} \theta_n = \theta$, where $\theta_{n+1} = R(\theta_n)$ and θ_0 can take any value in $[0, \hat{\theta}]$.*

Proof. The proof follows directly from the fact that θ is a fixed point of $R(x)$ (Thm. 3), the fact that $R(x)$ is a contraction mapping defined on a complete metric space (Lemma 10), and the Banach fixed-point theorem (Thm. 4). \square

Remark 6. The speed of convergence of the fixed-point iteration method may be described by a Lipschitz constant for R , denoted here by q : $d(\theta, \theta_{n+1}) \leq \frac{q}{1-q} d(\theta_{n+1}, \theta_n)$. If we conjecture that $R(x)$ is convex on $[0, \hat{\theta}]$ then we easily get ($R(x)$ is increasing) that the best Lipschitz constant for R is $q = \sup_{x \in [0, \hat{\theta}]} R'(x) = R'(\hat{\theta}) = 1 + \frac{1-\gamma}{\gamma} \ln(1-\gamma)$. However, proving the convexity of $R(x)$ seems very demanding.

In the next section we present the results of applying the fixed-point iteration method to estimate θ for the exemplary random hypergraph following our model.

V. EXPERIMENTAL RESULTS

In order to verify the results obtained and the legitimacy of our assumptions, we ran numerous simulations of the model testing different sets of parameters. In this section we present the results of simulated $\tilde{H} = H(H_0, p_v = 0.3, p_e = 0.5, p_d = 0.2, Y_t)$, where the distribution of Y_t was experimentally obtained from a real collaboration network G . G was built upon data extracted from Scopus [28], 239,414 computer science articles published between 1990 and 2018 by 258,145 authors. Each author was treated as a node and every publication corresponded to a hyperedge between its co-authors.

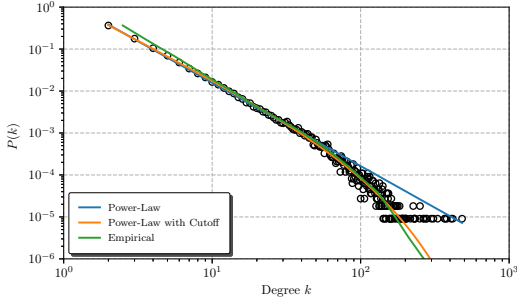


Fig. 1: The degree distribution of the Scopus network G .

We used statistical tools from [13] to fit and compare the theoretical distributions with the real degree distribution of G . One finds the result in Fig. 1 which shows that a power-law with an exponential cutoff is a good fit here (this is just one of many examples of real-life networks that follow this distribution [9]).

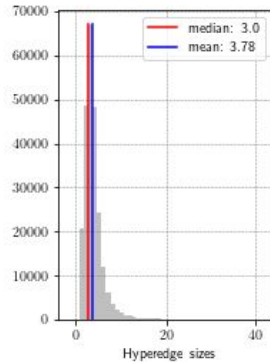


Fig. 2: The distribution of sizes of hyperedges in the Scopus network G .

Fig. 2 shows the distribution of hyperedge sizes in G - the one chosen for Y_t in our experiment.

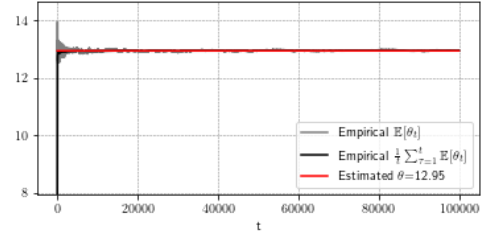


Fig. 3: The empirical average deactivated degree (grey and black curves) as compared to the estimated θ (red line) in \tilde{H} .

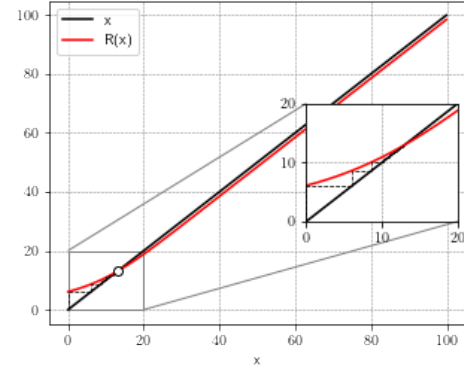


Fig. 4: The visualization of the fixed-point iteration method applied to the model \tilde{H} , starting from $\theta_0 = 0$.

The evolution of the average degree of a vertex selected for deactivation in \tilde{H} compared to the value of θ calculated using the fixed-point iteration method (Cor. 1) is presented in Fig. 3. It shows the convergence of the empirical average degree of a deactivated vertex to the estimated value of θ which supports both, our Asm. (4) as well as the method for evaluating θ (see Fig. 4 for its visualization).

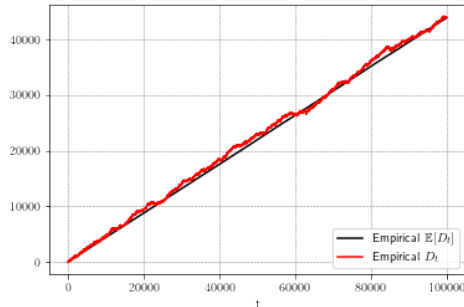


Fig. 5: The empirical $\mathbb{E}[D_t]$ and the trajectory of D_t for \tilde{H} .

Furthermore, we checked empirically the value of $\mathbb{E}[D_t]$ in \tilde{H} (we ran 1000 simulations up to 100,000 steps). The empirical $\mathbb{E}[D_t]$ appeared to be linear with the slope $\hat{\alpha} = 0.438948$ (Fig. 5). We then calculated the slope of the theoretical $\mathbb{E}[D_t]$ using the fixed-point iteration method to compute θ and, then, plugging it into equation from Lemma 5. It yielded

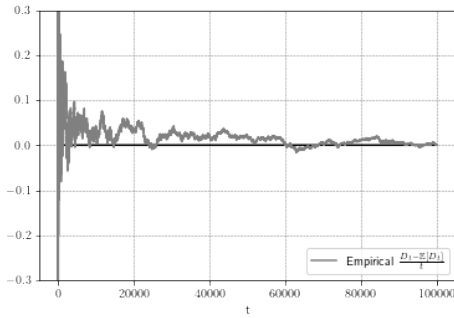


Fig. 6: The concentration of D_t for the hypergraph \tilde{H} .

$\alpha = 0.438184$ which closely corresponds to $\hat{\alpha}$. Finally, the result seen in Fig. 6 supports our Asm. (3) about the concentration of D_t .

For the results of simulations conducted with different sets of parameters (e.g. Y_t following some theoretical distribution, like Poisson), check the extended version of this paper [19].

VI. CONCLUSIONS

To the best of our knowledge, we have presented the first complex network model which allows for multiary relations and deactivation of elements¹. Both those eventualities occur naturally in real-life systems. We thus believe that the model will find a wide range of applications in many research domains. We have also proved that its degree distribution follows a power-law with an exponential cutoff, which, according to the broad study of Broido and Clauset [9], is the distribution most often observed in nature.

Challenging further research would be to check how the choice of a deletion procedure (e.g., deleting vertices at random or inversely to the degree instead of in proportion to the degree) influences the limiting degree distribution. It would be also useful to determine which deletion procedures are the more adequate for the different real life complex networks.

The other interesting direction of study is to make the attachment rule dependent not only on the degrees of vertices but also on their additional own characteristic (called *fitness* in the literature [8]).

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¹In our model the deactivated elements do not establish new connections any more but they do not disappear from the system and still contribute to the total sum of degrees. We find this setting very useful for real-life applications.

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