

Bijection analysis of finite rotations on \mathbb{Z}^2 : A hierarchical approach

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Motivation

In Euclidean spaces (\mathbb{R}^d , $d \geq 2$), rotations are bijective. This is no longer guaranteed when rotations are considered in \mathbb{Z}^d . We focus on the finite rotations that act on finite subsets of \mathbb{Z}^2 , especially Euclidean balls.

- We shed light on the hierarchical structure of these rotations.
- We propose two algorithmic schemes for the construction of combinatorial models (trees) of the bijective finite rotations.

Rotations on \mathbb{Z}^2 and hinge angles

Let $\theta \in \mathbb{U} = [0, 2\pi)$. Let $p = {}^t(p_x, p_y) \in \mathbb{R}^2$. A (continuous) rotation is defined by

$$\mathcal{R}_\theta(p) = \mathcal{R}_\theta \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} p_x \cos \theta - p_y \sin \theta \\ p_x \sin \theta + p_y \cos \theta \end{pmatrix}$$

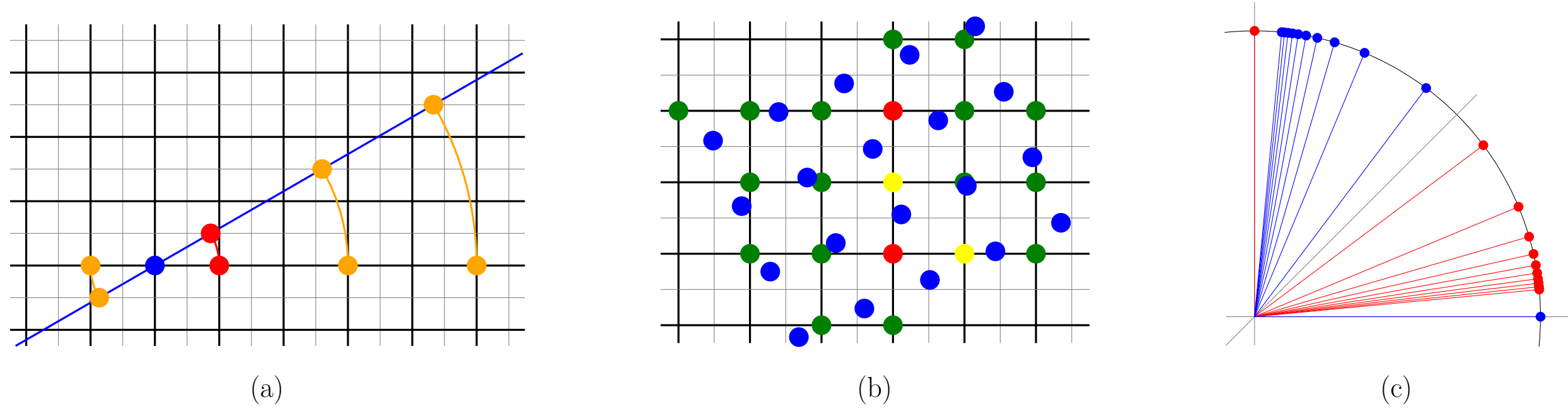
Let $p = {}^t(p_x, p_y) \in \mathbb{Z}^2$. A (discrete) rotation is defined by

$$R_\theta(p) = (D \circ \mathcal{R}_\theta)(p) = \begin{pmatrix} \lfloor p_x \cos \theta - p_y \sin \theta \rfloor \\ \lfloor p_x \sin \theta + p_y \cos \theta \rfloor \end{pmatrix}$$

Let $\theta \in \mathbb{U}$. Let \mathcal{R}_θ be the (continuous) rotation of angle θ . Let $p \in \mathbb{Z}^2$. Let us suppose that there exists $k \in \mathbb{Z}$ such that

$$p_x \cos \theta - p_y \sin \theta = k + 1/2$$

holds. Then we say that θ is a hinge angle. We note $\mathbb{H} \subset \mathbb{U}$ the set of all the hinge angles. We note $\hat{\mathbb{U}} = \mathbb{U} \setminus \mathbb{H}$ the set of all the non-hinge angles.



(a) $\pi/6$ is a hinge angle. (b) The image $Y \subset \mathbb{R}^2$ (blue dots) of a finite subset $X \subset \mathbb{Z}^2$ transformed by a (bijective) rotation \mathcal{R}_θ . The image P (green and yellow dots) of X by the (non-bijective) discrete rotation R_θ , i.e. $P = R_\theta(X)$. (c) The first bijective angles depicted on the unit square.

Finite rotations on \mathbb{Z}^2 (and their hierarchical structure)

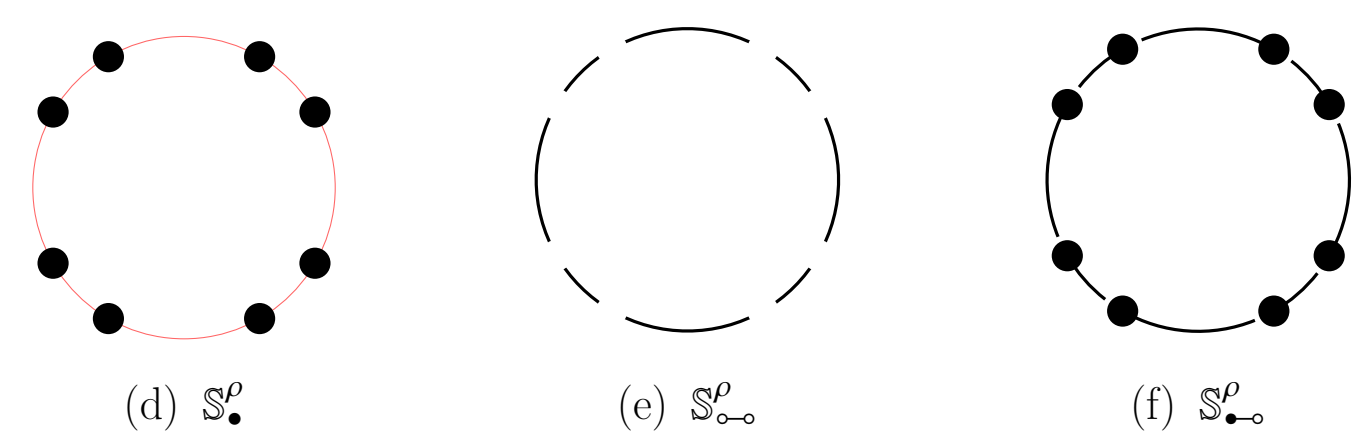
Let $\theta \in \hat{\mathbb{U}}$ be a non-hinge angle. Let R_θ be a discrete rotation. Let $\rho \in \mathbb{R}_+$. Let B^ρ be the Euclidean ball of radius ρ in \mathbb{Z}^2 , i.e.

$$B^\rho = \{q \in \mathbb{Z}^2 \mid \|q\|_2 \leq \rho\}$$

We note $R_\theta^\rho : B^\rho \rightarrow \mathbb{Z}^2$ the rotation defined as $R_\theta^\rho = (R_\theta)|_{B^\rho}$, i.e. as the restriction of the discrete rotation R_θ to the Euclidean ball B^ρ . Such rotation R_θ^ρ is called a ρ -rotation.

We note $\mathbb{H}^\rho \subset \mathbb{H}$ the (finite) subset of hinge angles which relate to the ρ -rotations, i.e. the hinge angles induced at a radius lower than ρ . We set

$$\mathbb{H}^\rho = \{h_j^\rho\}_{j=0}^{\sigma^\rho-1} \quad \mathbb{S}_\bullet^\rho = \left\{ \left\{ h_j^\rho \right\}_{j=0}^{\sigma^\rho-1} \right\} \quad \mathbb{S}_{\bullet-\infty}^\rho = \left\{ \left\{ h_j^\rho, h_{(j+1)[\sigma^\rho]}^\rho \right\}_{j=0}^{\sigma^\rho-1} \right\} \quad \mathbb{S}_{\bullet-\infty}^* = \left\{ \left\{ h_j^\rho, h_{(j+1)[\sigma^\rho]}^\rho \right\}_{j=0}^{\sigma^\rho-1} \right\}$$



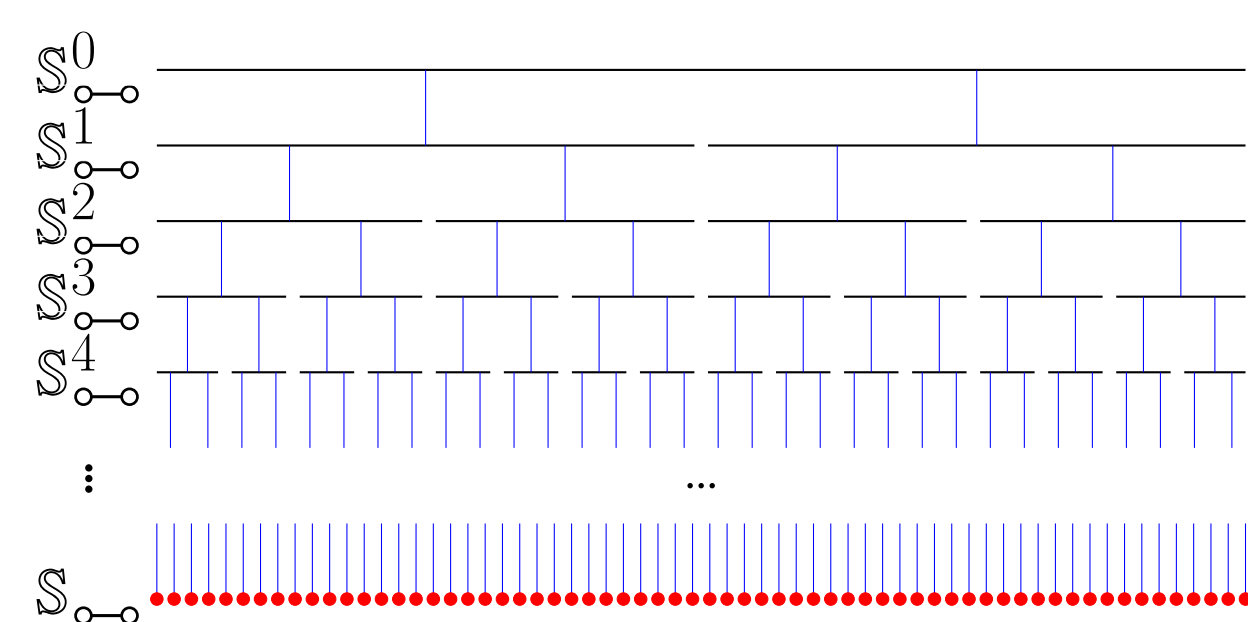
Subdivisions of \mathbb{U} by the three sets \mathbb{S}_\bullet^ρ (d), $\mathbb{S}_{\bullet-\infty}^\rho$ (e) and $\mathbb{S}_{\bullet-\infty}^*$ (f). Note that $\mathbb{S}_\bullet^\rho \cup \mathbb{S}_{\bullet-\infty}^\rho$ is a partition of \mathbb{U} . Note that $\mathbb{S}_{\bullet-\infty}^*$ is also a partition of \mathbb{U} .

Each segment $(h_j^\rho, h_{(j+1)[\sigma^\rho]}^\rho)$ (resp. $[h_j^\rho, h_{(j+1)[\sigma^\rho]}^\rho)$) models a specific ρ -rotation R_θ^ρ .

Let

$$\mathbb{S}_{\bullet-\infty} = \{\theta \mid \theta \in \hat{\mathbb{U}}\} \quad \mathbb{S}_{\bullet-\infty}^* = \mathbb{S}_{\bullet-\infty} \cup \bigcup_{\rho \in \mathbb{N}^{\frac{1}{2}}} \mathbb{S}_{\bullet-\infty}^\rho$$

$\mathbb{S}_{\bullet-\infty}^*$ is in bijection with $\hat{\mathbb{U}}$ and is a partition of $\hat{\mathbb{U}}$. The Hasse diagram $(\mathbb{S}_{\bullet-\infty}^*, \leq)$ of the ordered set $(\mathbb{S}_{\bullet-\infty}^*, \leq)$ is a partition tree.

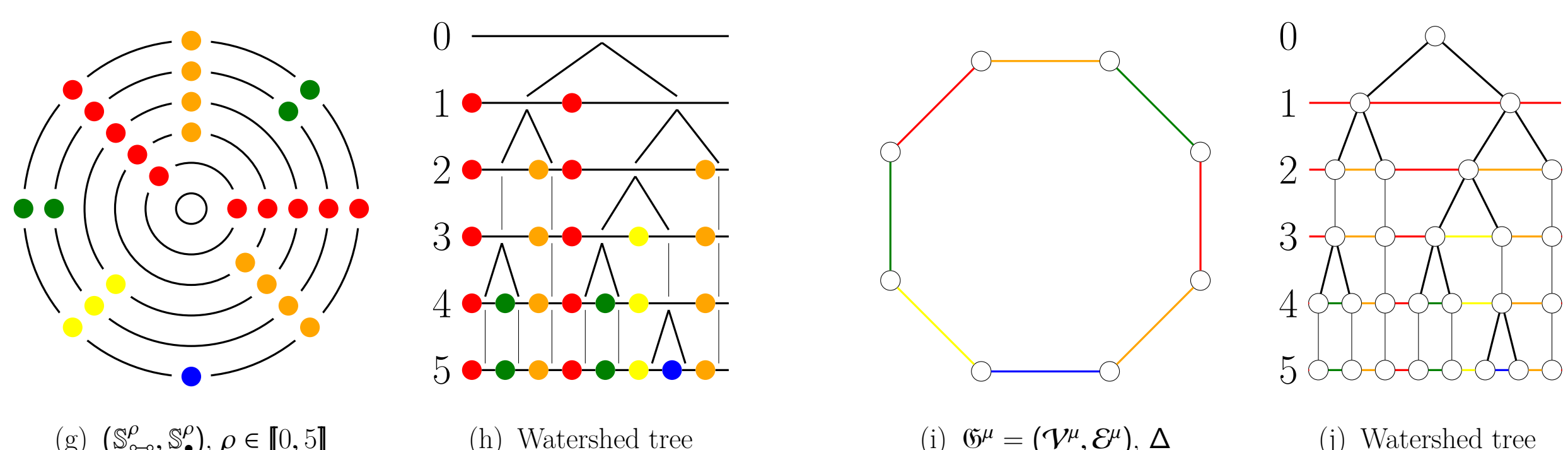


$\mathbb{S}_{\bullet-\infty}^*$ decomposed by increasing values of $\rho \in \mathbb{N}^{\frac{1}{2}}$. A black segment depicts an interval of $\mathbb{S}_{\bullet-\infty}^\rho$. The last line corresponds to $\mathbb{S}_{\bullet-\infty}$.

The combinatorial space of finite rotations is a watershed tree

Let $\mu \in \mathbb{N}^{\frac{1}{2}}$. We now restrict ourselves to the finite subspace of the ρ -rotations for all the values $\rho \in [0, \mu]$. In other words, we consider a finite upper part of $(\mathbb{S}_{\bullet-\infty}^*, \leq)$.

Each element of $\mathbb{S}_{\bullet-\infty}^\mu$ models a μ -rotation. Each element of \mathbb{S}_\bullet^μ models a hinge angle between two μ -rotations. Then $(\mathbb{S}_{\bullet-\infty}^\mu, \mathbb{S}_\bullet^\mu)$ can be seen as a cycle graph $\mathbb{G}^\mu = (\mathcal{V}^\mu, \mathcal{E}^\mu)$. Each hinge angle $h \in \mathbb{H}$ is associated to a value $\Delta(h)$ that corresponds to the lowest radius ρ where h appears. The valued graph (\mathbb{G}^μ, Δ) then defines a saliency map.



(g) $(\mathbb{S}_{\bullet-\infty}^\mu, \mathbb{S}_\bullet^\mu)$, $\rho \in [0, \mu]$ (h) Watershed tree (i) $\mathbb{G}^\mu = (\mathcal{V}^\mu, \mathcal{E}^\mu)$, Δ (j) Watershed tree

The combinatorial space of all the ρ -rotations for $0 \leq \rho \leq \mu$ is the watershed tree [1] of the saliency graph (\mathbb{G}^μ, Δ) .

Bijection analysis of finite rotations on \mathbb{Z}^2

A discrete rotation R_θ is bijective [2] iff $\theta \in \mathbb{B}$ with

$$\mathbb{B} = \left\{ \theta \in \mathbb{U} \mid \sin \theta \in \left\{ \pm \frac{2p(p+1)}{2p^2+2p+1}, \pm \frac{2p-1}{2p^2+2p+1} \mid p \in \mathbb{N} \right\} \right\}$$

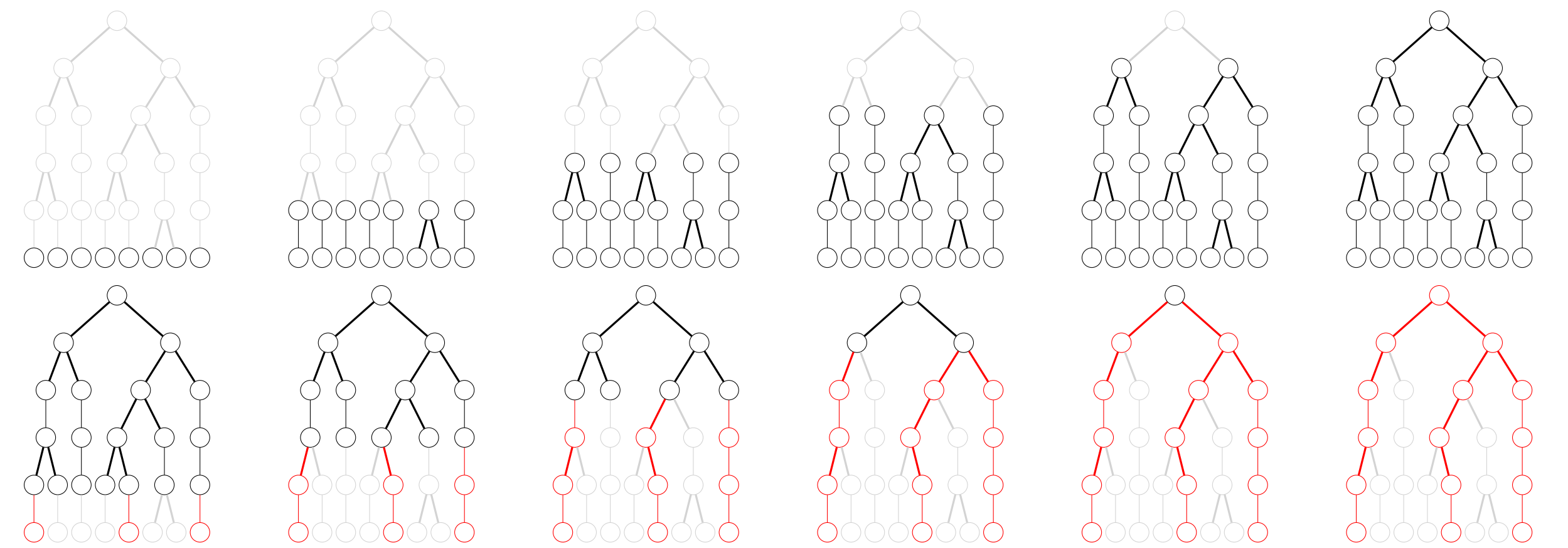
We can define a bijective ρ -rotations R_θ^ρ by two ways:

- (B₁) R_θ^ρ is the restriction to B^ρ of a bijective discrete rotation R_θ ;
- (B₂) R_θ^ρ is an injective mapping from B^ρ to $R_\theta(B^\rho)$.

Building the (bijective) finite rotations on \mathbb{Z}^2 (bottom-up)

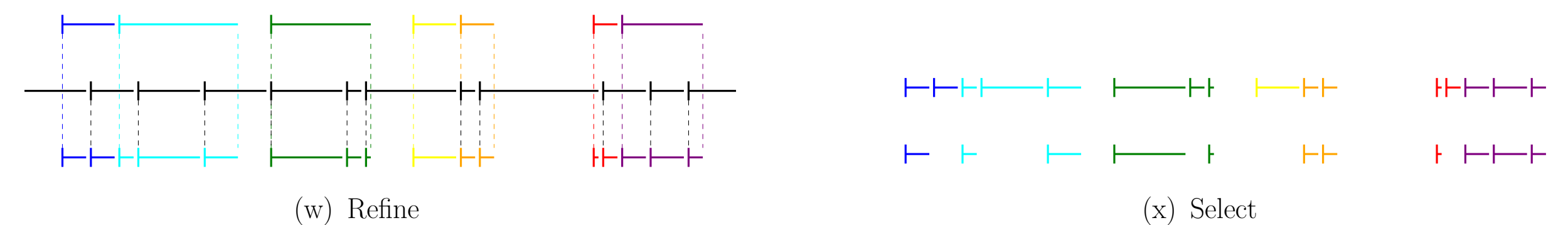
A first way of building the combinatorial space of the (bijective) discrete rotations is to:

- Build the watershed tree of $(\mathbb{S}_{\bullet-\infty}^\rho, \mathbb{S}_\bullet^\rho)$, $\rho \in [0, \mu]$ with a time cost $\mathcal{O}(\mu^3 \log \mu)$
- Determine the bijective rotations in the tree from the leaves up to the root:
 - With definition (B₁), this can be done with a time cost $\mathcal{O}(\mu^3)$.
 - With definition (B₂), this can be done with a reasonable time cost.



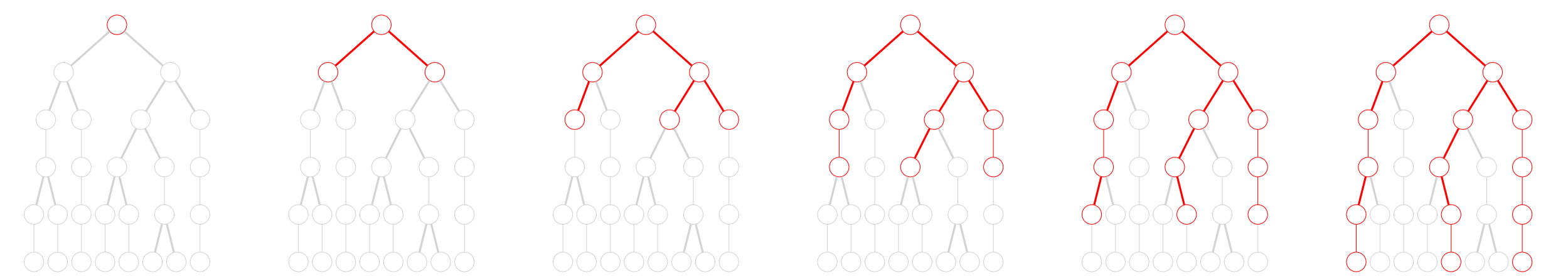
Building the (bijective) finite rotations on \mathbb{Z}^2 (top-down)

A second way of building the combinatorial space of the (bijective) discrete rotations is to build only the bijective ρ -rotations, from $\rho = 0$ down to $\rho = \mu$ by following an iterative “refine and select” paradigm.

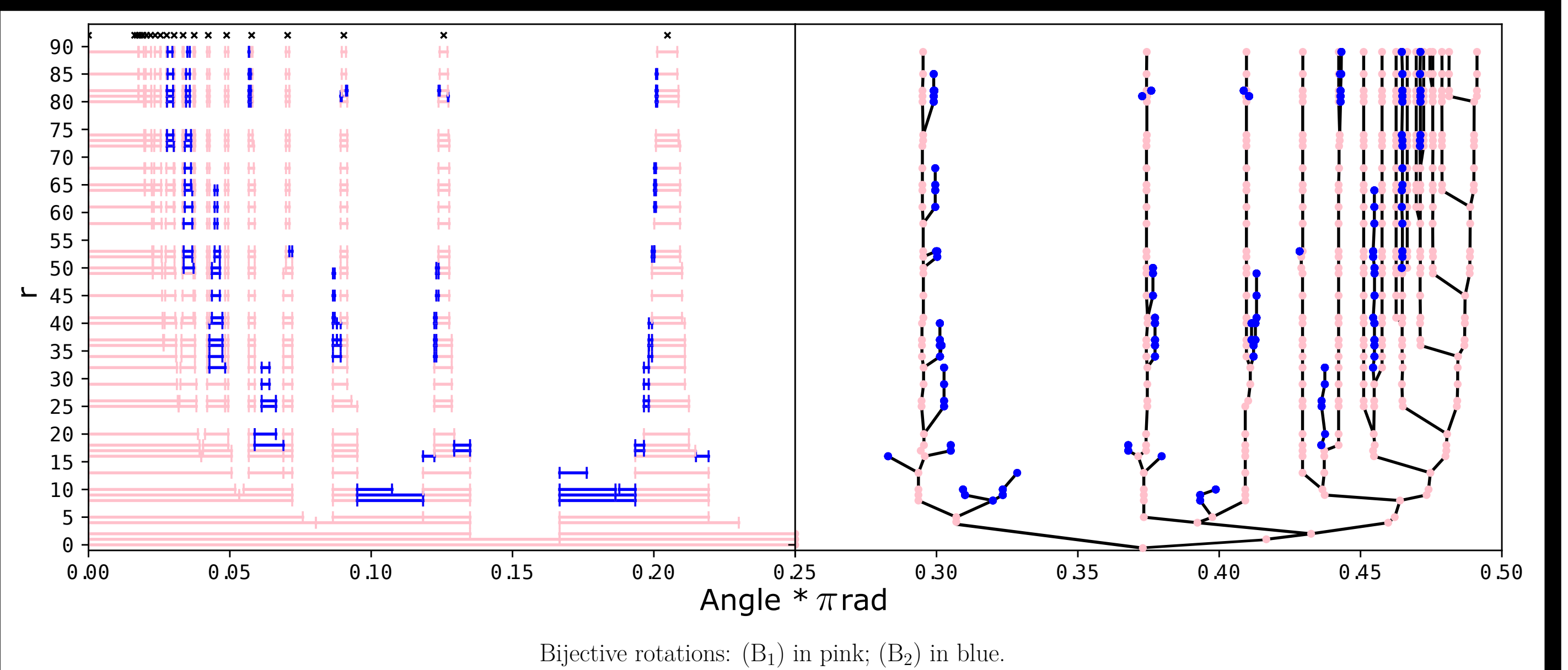


The time cost for building the combinatorial space depends on the family of considered ρ -rotations:

- $\mathcal{O}(\mu^3 \log \mu)$ for all the ρ -rotations;
- $\mathcal{O}(\mu^3)$ for all the bijective ρ -rotations with definition (B₁);
- $\mathcal{O}(\mu^{k+2} \log \mu)$ for all the bijective ρ -rotations with definition (B₂) with $1 \leq k \leq 3$ (not yet known!);

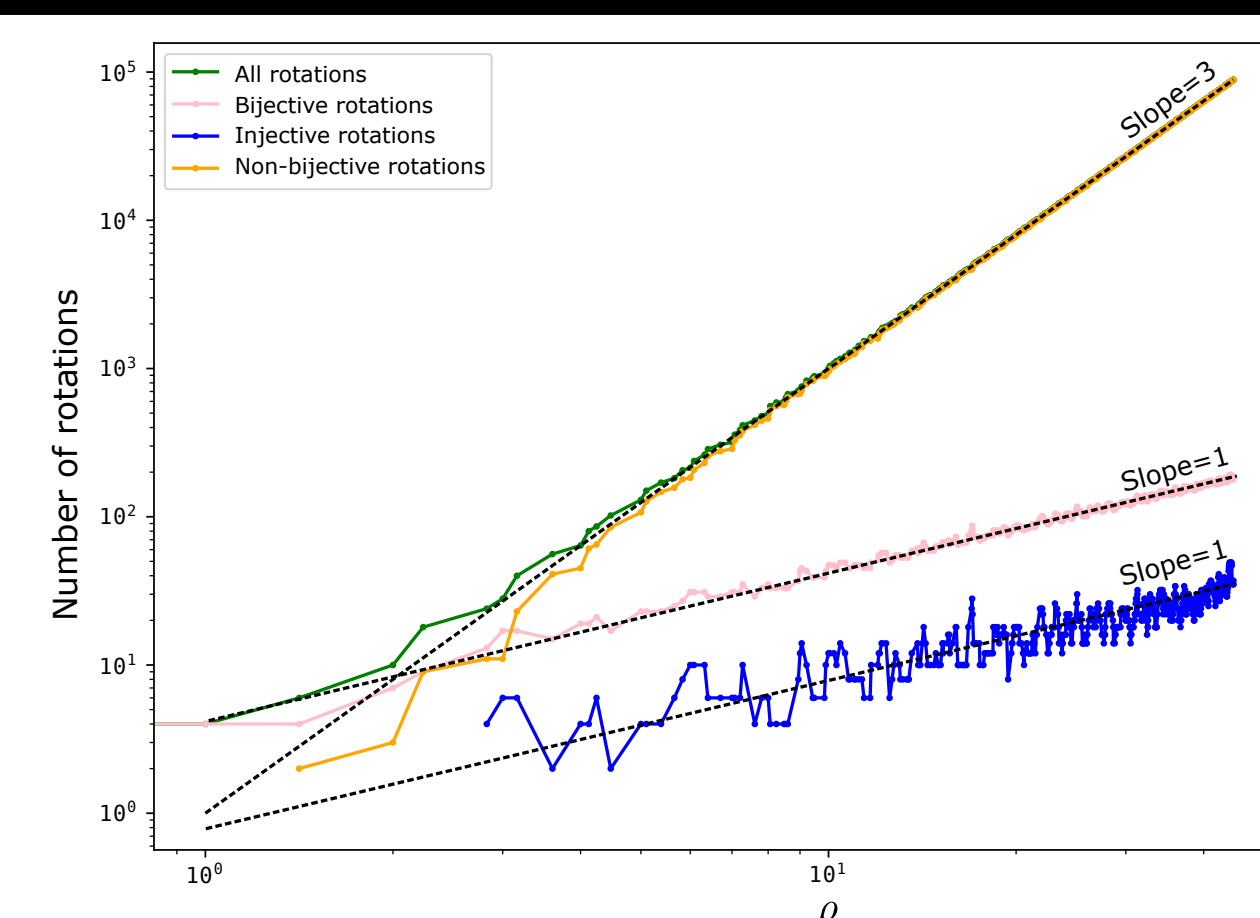


The hierarchical structure of the bijective rotations on \mathbb{Z}^2



Bijection rotations: (B₁) in pink; (B₂) in blue.

Complexity of the (bijective) rotations on \mathbb{Z}^2



Evolution of the size of the different families of ρ -rotations (y-axis) with respect to ρ (x-axis) in log-log scale

References

- [1] L. Najman and M. Schmitt. Geodesic saliency of watershed contours and hierarchical segmentation. IEEE Transactions on Pattern Analysis and Machine Intelligence, 18:1163–1173, 1996.
- [2] B. Nouvel and É. Rémy. Characterization of bijective discretized rotations. In International Workshop on Combinatorial Image Analysis (IWCIA), Proceedings, volume 3322 of Lecture Notes in Computer Science, pages 248–259. Springer, 2004.

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