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# Bijectivity analysis of finite rotations on $\mathbb{Z}^2$ : A hierarchical approach<sup>★</sup>

Nicolas Passat<sup>1</sup>[0000-0002-0320-4581], Phuc Ngo<sup>2</sup>[0000-0002-7423-5932], and  
Yukiko Kenmochi<sup>3</sup>[0000-0001-9648-326X]

<sup>1</sup> Université de Reims Champagne-Ardenne, CReSTIC, France

<sup>2</sup> Université de Lorraine, CNRS, LORIA, 54000, Nancy, France

<sup>3</sup> Normandie Univ., UNICAEN, ENSICAEN, CNRS, GREYC, Caen, France

**Abstract.** In this article, we investigate the rotations on  $\mathbb{Z}^2$  (a.k.a discrete rotations). In particular, we focus on the finite rotations that act on finite subsets of  $\mathbb{Z}^2$ , especially Euclidean balls. We shed light on the hierarchical structure of these rotations, induced by their discontinuity (characterized by hinge angles) and the size of the considered ball. We propose efficient algorithmic schemes leading to the construction of combinatorial models (trees) of the bijective finite rotations. These algorithms and structures open the way to a better understanding of the notion of bijectivity with respect to finite vs. infinite discrete rotations.

**Keywords:** Finite rotations · Cartesian grid · Bijectivity · Hinge angles · Hierarchical models

## 1 Introduction

Rotation is a fundamental operation in geometry. In the Euclidean spaces  $\mathbb{R}^d$  ( $d \geq 2$ ), rotations preserve distances and angles; they are also bijective. By contrast, their discrete analogues in  $\mathbb{Z}^d$ , the so-called discrete rotations, exhibit challenges. Not only they do no longer preserve distances and angles, but they are also non-bijective in most cases.

Early in the study of discrete rotations, efforts were geared towards understanding under which hypotheses discrete rotations could be bijective. In [11], a sufficient condition was proposed for bijectivity of discrete rotations on  $\mathbb{Z}^2$ . In [13, 17], a characterization of bijective discrete rotations on  $\mathbb{Z}^2$  was further proposed (whereas it was proved in [10] that there is no bijective discrete rotation on  $\mathbb{Z}^2$  under the specific case where the floor digitization paradigm is considered, see Section 2.3). Extensions of these results were investigated in the hexagonal grid [16] and in  $\mathbb{Z}^3$  [6, 15]. In [3] a family of rotations on  $\mathbb{Z}^2$  based on quasi-shears was proposed, also fulfilling bijectivity properties. This paradigm was extended to hexagonal grids in [5]. In [4], bijective rotations were handled via the composition of bijective reflections. A similar paradigm was investigated via the framework of geometric algebra on  $\mathbb{Z}^2$  [7] and  $\mathbb{Z}^3$  [6].

In this article, we focus on the discrete rotations applied on finite sets (Euclidean balls) of  $\mathbb{Z}^2$ , and we study their bijectivity. Indeed, in practical cases, rotations act on

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images with a finite support, and in this context, the notion of bijectivity may be considered with regards to this support instead of the whole space  $\mathbb{Z}^2$ . We propose an algorithmic framework that allows to build a combinatorial, hierarchical structure which models the space of all the (bijective) discrete rotations on finite Euclidean balls of  $\mathbb{Z}^2$ .

This article is organized as follows. Section 2 provides basic notions on continuous and discrete rotations required to make this article self-contained. In Section 3, we describe the hierarchical structure of the finite discrete rotations, and we provide an efficient algorithm for building it. In Section 4, we describe the hierarchical structures of two families of bijective finite discrete rotations and we provide efficient algorithms for building them. Section 5 illustrates results obtained by applying these algorithms. Section 6 concludes this article.

## 2 Basics on rotations

A point of  $\mathbb{R}^2$  is noted in bold, e.g.  $\mathbf{p} \in \mathbb{R}^2$ . Its coordinates are noted in italic and subscripted by  $x$  and  $y$ , respectively, e.g.  $\mathbf{p} = {}^t(p_x, p_y)$  (or  $(p_x, p_y)$  by abuse of notation). If it is defined by its coordinates, it is noted as the associated couple, e.g.  $(a, b) \in \mathbb{R}^2$ .

### 2.1 Continuous rotations

We note  $\mathbb{U} = [0, 2\pi)$  and we consider  $\mathbb{U}$  as cyclic, i.e. we identify 0 and  $2\pi$ . Let  $\theta \in \mathbb{U}$ . The rotation (with center  $\mathbf{0} = (0, 0)$ ) of angle  $\theta$  is the function  $\mathcal{R}_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined, for all  $\mathbf{p} \in \mathbb{R}^2$ , by

$$\mathcal{R}_\theta(\mathbf{p}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} p_x \cos \theta - p_y \sin \theta \\ p_x \sin \theta + p_y \cos \theta \end{pmatrix} \quad (1)$$

Such rotation is called a *continuous rotation*. We note  $\mathfrak{R}_{\mathbb{R}^2}$  the set of all the continuous rotations.

### 2.2 Hinge angles

Let  $\theta \in \mathbb{U}$  and  $\mathcal{R}_\theta$  the induced continuous rotation. Let  $\mathbf{p} \in \mathbb{Z}^2$ . Let us suppose that there exists  $k \in \mathbb{Z}$  such that one of the following two equations is satisfied

$$p_x \cos \theta - p_y \sin \theta = k + 1/2 \quad (2)$$

$$p_x \sin \theta + p_y \cos \theta = k + 1/2 \quad (3)$$

Then we say that  $\theta$  is a *hinge angle* [14] (induced by  $\mathbf{p}$ ). We note  $\mathbb{H} \subset \mathbb{U}$  the set of all the hinge angles ( $\mathbb{H}$  is dense in  $\mathbb{U}$  [12]). We note  $\mathring{\mathbb{U}} = \mathbb{U} \setminus \mathbb{H}$  the set of non-hinge angles.

A hinge angle  $\theta \in \mathbb{H}$  is determined by a triplet  $(p_x, p_y, k) \in \mathbb{Z}^3$ . However any triplet in  $\mathbb{Z}^3$  does not necessarily define a hinge angle. In particular,  $(p_x, p_y, k) \in \mathbb{Z}^3$  defines a hinge angle iff  $|k + 1/2| \leq \|(p_x, p_y)\|_2$ . We note  $\mathbb{T} \subset \mathbb{Z}^3$  the set of all the triplets that define hinge angles.

We define the function  $\eta : \mathbb{T} \rightarrow \mathbb{H}$  such that for all  $t = (p_x, p_y, k) \in \mathbb{T}$ ,  $\eta(t)$  is the hinge angle induced by  $t$ . The function  $\eta$  is non-injective: many triplets of  $\mathbb{T}$  define the same hinge angle of  $\mathbb{H}$ . The following proposition clarifies this many-to-one relation.

**Proposition 1 ([12])** *Let  $h \in \mathbb{H}$ . There exists a prime generator triplet  $(p_x, p_y, k) \in \mathbb{T}$  such that  $\eta^{-1}(\{h\}) = \{(2n + 1)p_x, (2n + 1)p_y, (2n + 1)k + n\} \mid n \in \mathbb{Z}\}$ .*

### 2.3 Discrete rotations

We now consider rotations (with center  $\mathbf{0}$ ) from  $\mathbb{Z}^2$  to  $\mathbb{Z}^2$ . Let  $\theta \in \mathbb{U}$ . Let  $\mathcal{R}_\theta \in \mathfrak{R}_{\mathbb{R}^2}$  be a continuous rotation (see Eq. (1)). Except in very few cases (i.e. when  $\theta$  is a multiple of  $\pi/2$ ), we have  $\mathcal{R}_\theta(\mathbb{Z}^2) \not\subseteq \mathbb{Z}^2$ . To tackle this issue, i.e. to ensure that the result of a rotation applied on  $\mathbb{Z}^2$  lies in  $\mathbb{Z}^2$ , it is common to compose the result of the continuous rotation with a discretization operator, which is generally set as

$$\left. \begin{array}{l} D : \mathbb{R}^2 \quad \rightarrow \mathbb{Z}^2 \\ (p_x, p_y) \mapsto ([p_x], [p_y]) \end{array} \right\} \quad (4)$$

where  $[\cdot]$  is the rounding function on  $\mathbb{R}$ . (As observed in [10], the floor function, and by symmetry, the ceil one are less interesting from the bijectivity point of view that we adopt in this study.)

Then, we can define a rotation of angle  $\theta$  from  $\mathbb{Z}^2$  to  $\mathbb{Z}^2$  as a function  $R_\theta : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  such that, for all  $\mathbf{p} \in \mathbb{Z}^2$ , we have

$$R_\theta(\mathbf{p}) = (D \circ \mathcal{R}_\theta)(\mathbf{p}) = \begin{pmatrix} [p_x \cos \theta - p_y \sin \theta] \\ [p_x \sin \theta + p_y \cos \theta] \end{pmatrix} \quad (5)$$

This rotation is called a *discrete rotation*. Such discrete rotation  $R_\theta$  is well-defined iff  $\theta \in \mathring{\mathbb{U}} = \mathbb{U} \setminus \mathbb{H}$  (due to the ambiguous definition of  $[\cdot]$  on  $\mathbb{Z} + 1/2$ ). We note  $\mathfrak{R}_{\mathbb{Z}^2} = \{R_\theta \mid \theta \in \mathring{\mathbb{U}}\}$  the set of all the discrete rotations. We set  $R : \mathring{\mathbb{U}} \rightarrow \mathfrak{R}_{\mathbb{Z}^2}$  the (surjective) function defined for each  $\theta \in \mathring{\mathbb{U}}$  by  $R(\theta) = R_\theta$ .

By contrast with a continuous rotation  $\mathcal{R}_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is bijective, a discrete rotation  $R_\theta : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  may be bijective or not, depending on the value of  $\theta$ . This fact is clarified by the following proposition.

**Proposition 2 ([13])** *Let  $\theta \in \mathring{\mathbb{U}}$ . The discrete rotation  $R_\theta : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is bijective if and only if there exists  $p \in \mathbb{N}$  and  $\varepsilon \in \{-1, 1\}$  such that*

$$\sin \theta \in \left\{ \varepsilon \cdot \frac{2p(p+1)}{2p^2 + 2p + 1}, \varepsilon \cdot \frac{2p+1}{2p^2 + 2p + 1} \right\} \quad (6)$$

The set  $\mathbb{B}$  of the angles characterized by Eq. (6) is composed of specific Pythagorean angles (i.e. angles with rational sine and cosine determined by twin Pythagorean triplets). Pythagorean angles do not intersect  $\mathbb{H}$  [14], and we then have  $\mathbb{B} \cap \mathbb{H} = \emptyset$ .

### 2.4 Discrete finite rotations

Let  $\theta \in \mathring{\mathbb{U}}$ . Let  $R_\theta \in \mathfrak{R}_{\mathbb{Z}^2}$ . Let  $\rho \in \mathbb{R}_+$ . Let  $B^\rho = \{\mathbf{q} \in \mathbb{Z}^2 \mid \|\mathbf{q}\|_2 \leq \rho\}$ . We note  $R_\theta^\rho : B^\rho \rightarrow \mathbb{Z}^2$  the restriction of the discrete rotation  $R_\theta$  to the discrete Euclidean ball  $B^\rho$ . Such rotation  $R_\theta^\rho$  is called a *discrete  $\rho$ -rotation* (or simply a  $\rho$ -rotation). We note  $\mathfrak{R}_{\mathbb{Z}^2}^\rho = \{R_\theta^\rho \mid R_\theta \in \mathfrak{R}_{\mathbb{Z}^2}\}$  the set of all the  $\rho$ -rotations. We set  $R^\rho : \mathring{\mathbb{U}} \rightarrow \mathfrak{R}_{\mathbb{Z}^2}^\rho$  the (surjective) function defined for each  $\theta \in \mathring{\mathbb{U}}$  by  $R^\rho(\theta) = R_\theta^\rho$ .

We define the subset

$$\mathbb{T}^\rho = \{(p_x, p_y, k) \in \mathbb{Z}^3 \mid |k + 1/2| \leq \|\mathbf{p}\|_2 \leq \rho\} \subset \mathbb{T} \quad (7)$$

which gathers the triplets of  $\mathbb{T}$  induced by the points inside  $B^\rho$ . We have  $|\mathbb{T}^\rho| = O(\rho^3)$ . We define the subset  $\mathbb{H}^\rho = \eta(\mathbb{T}^\rho) \subset \mathbb{H}$  of the hinge angles induced the triplets of  $\mathbb{T}^\rho$ . We have  $|\mathbb{H}^\rho| = O(\rho^3)$  [2]. Note that we have  $\rho < 1$  iff  $\mathbb{T}^\rho = \mathbb{H}^\rho = \emptyset$ .

We consider on  $\mathbb{U}$  (seen here as non-cyclic) the restriction  $\leq_{\mathbb{U}}$  (simply noted  $\leq$ ) of the total order  $\leq$  on  $\mathbb{R}$ . By assuming that  $\mathbb{H}^\rho$  is ordered by  $\leq$ , we note  $\mathbb{H}^\rho = \{h_j^\rho\}_{j=0}^{\sigma^\rho-1}$  with  $\sigma^\rho \in \mathbb{N}$ .

We consider on  $\mathbb{T}$  the preorder  $\leq_{\mathbb{T}}$  (simply noted  $\leq$ ) defined, for any  $t_1, t_2 \in \mathbb{T}$  by

$$t_1 \leq_{\mathbb{T}} t_2 \iff \eta(t_1) \leq_{\mathbb{U}} \eta(t_2) \quad (8)$$

By assuming that  $\mathbb{T}^\rho$  is sorted with respect to  $\leq$ , we set  $\mathbb{T}^\rho = \{(p_{x,i}, p_{y,i}, k_i)\}_{i=0}^{s^\rho-1}$  with  $s^\rho \in \mathbb{N}$ . Note that we have  $\sigma^\rho \leq s^\rho$  and  $\sigma^\rho \geq 1$  iff  $\rho \geq 1$ .

If  $\rho < 1$ , we set  $\mathbb{H}_\bullet^\rho = \emptyset$  and  $\mathbb{H}_{\circ-\circ}^\rho = \mathbb{H}_{\bullet-\circ}^\rho = \{\mathbb{U}\}$ . If  $\rho \geq 1$ , we set

$$\mathbb{H}_\bullet^\rho = \left\{ \left\{ h_j^\rho \right\}_{j=0}^{\sigma^\rho-1} \right\} \quad (9)$$

$$\mathbb{H}_{\circ-\circ}^\rho = \left\{ \left( h_j^\rho, h_{j+1[\sigma^\rho]}^\rho \right) \right\}_{j=0}^{\sigma^\rho-1} \quad (10)$$

$$\mathbb{H}_{\bullet-\circ}^\rho = \left\{ \left[ h_j^\rho, h_{j+1[\sigma^\rho]}^\rho \right] \right\}_{j=0}^{\sigma^\rho-1} \quad (11)$$

where  $n[k]$  is  $n$  modulo  $k$ . These three sets can be seen as faces in a complex model of  $\mathbb{U}$ , namely 0-dimensional elements ( $\mathbb{H}_\bullet^\rho$ ), 1-dimensional open elements ( $\mathbb{H}_{\circ-\circ}^\rho$ ) and 1-dimensional semi-open elements ( $\mathbb{H}_{\bullet-\circ}^\rho$ ).

**Property 3** *There is a trivial bijection between  $\mathbb{H}^\rho$  and  $\mathbb{H}_\bullet^\rho$ , and another between  $\mathbb{H}_{\circ-\circ}^\rho$  and  $\mathbb{H}_{\bullet-\circ}^\rho$ . When  $\rho \geq 1$ , there are trivial bijections between  $\mathbb{H}^\rho$ ,  $\mathbb{H}_\bullet^\rho$ ,  $\mathbb{H}_{\circ-\circ}^\rho$  and  $\mathbb{H}_{\bullet-\circ}^\rho$ .*

We set  $\mathring{\mathbb{U}}^\rho = \mathbb{U} \setminus \mathbb{H}^\rho$ . The set  $\mathbb{H}_{\circ-\circ}^\rho$  is a partition of  $\mathring{\mathbb{U}}^\rho$ . The set  $\mathbb{H}_{\bullet-\circ}^\rho$  is a partition of  $\mathbb{U}$ . The union of sets  $\mathbb{H}_\bullet^\rho \cup \mathbb{H}_{\circ-\circ}^\rho$  is also a partition of  $\mathbb{U}$  (that refines the partition  $\mathbb{H}_{\bullet-\circ}^\rho$ ).

Let  $H \in \mathbb{H}_{\circ-\circ}^\rho$ . Let  $\theta_1, \theta_2 \in \mathring{\mathbb{U}}$  be two distinct angles such that  $\theta_1 \in H$ . Let  $R_{\theta_1}, R_{\theta_2} \in \mathfrak{R}_{\mathbb{Z}^2}^\rho$  be the two (distinct) discrete rotations associated to  $\theta_1, \theta_2$ , respectively. Let  $R_{\theta_1}^\rho, R_{\theta_2}^\rho \in \mathfrak{R}_{\mathbb{Z}^2}^\rho$  be the two  $\rho$ -rotations associated to  $\theta_1, \theta_2$ , respectively. We have  $\theta_2 \in H$  if and only if  $R_{\theta_1}^\rho = R_{\theta_2}^\rho$ . This justifies the following property.

**Property 4** *There exists a bijection between  $\mathbb{H}_{\circ-\circ}^\rho$  and  $\mathfrak{R}_{\mathbb{Z}^2}^\rho$ .*

It follows that we can symbolically model  $\mathfrak{R}_{\mathbb{Z}^2}^\rho$  by  $\mathbb{H}_{\circ-\circ}^\rho$  (or any other set in bijection with  $\mathbb{H}_{\circ-\circ}^\rho$ , see Property 3).

**Remark 5** *Although the function  $R^\rho$  is initially defined on  $\mathring{\mathbb{U}}$  (i.e. only for non-hinge angles), it is possible to extend it, without loss of correctness, so that it is defined on  $\mathbb{U}$  (i.e. for all angles). Let  $H \in \mathbb{H}_{\circ-\circ}^\rho$ . Let  $h \in H \cap \mathbb{H}$ . Since  $h$  is a hinge angle, the discrete rotation  $R_h$  is undefined, and so is the  $\rho$ -rotation  $R_h^\rho$ . Nonetheless, the hinge angle  $h \in \mathbb{H}$  does not belong to  $\mathbb{H}^\rho$ . We can extend  $R_h^\rho$  by continuity, by setting  $R_h^\rho = R_\theta^\rho$  with  $\theta \in H \cap \mathring{\mathbb{U}}$ . Now, let  $h = h_j^\rho \in \mathbb{H}^\rho$  ( $0 \leq j \leq \sigma^\rho - 1$ ). The  $\rho$ -rotation  $R_h^\rho$  is undefined. However,  $R_\theta^\rho$  is defined (and constant) for all  $\theta \in (h_j^\rho, h_{j+1[\sigma^\rho]}^\rho)$ . We can then extend  $R_h^\rho$  by continuity, by setting  $R_h^\rho = R_\theta^\rho$ . Doing so, we can extend the function  $R^\rho : \mathring{\mathbb{U}} \rightarrow \mathfrak{R}_{\mathbb{Z}^2}^\rho$  as  $R^\rho : \mathbb{U} \rightarrow \mathfrak{R}_{\mathbb{Z}^2}^\rho$ .*

### 3 Hierarchical structure of finite discrete rotations

In this section, we describe a combinatorial (tree) structure for modeling the  $\rho$ -rotations (Section 3.1) and we provide an algorithmic scheme for building it (Section 3.2).

### 3.1 Description

For any  $\rho \in \mathbb{R}_+$ , we set  $\mathbb{I}^\rho = \mathbb{H}_\bullet^\rho \cup \mathbb{H}_{\circ-\circ}^\rho$ . We set  $\mathbb{I} = \{\mathbb{I}^\rho\}_{\rho \in \mathbb{R}_+}$ . Let  $\rho_1, \rho_2 \in \mathbb{R}_+$  and  $r \in \mathbb{N}$ . If  $\sqrt{r} \leq \rho_1, \rho_2 < \sqrt{r+1}$  then we have  $\mathbb{I}^{\rho_1} = \mathbb{I}^{\rho_2}$ . It follows that  $\mathbb{I} = \{\mathbb{I}^{\sqrt{r}}\}_{r \in \mathbb{N}} = \{\mathbb{I}^\rho\}_{\rho \in \sqrt{\mathbb{N}}}$ .

Let  $\rho_1, \rho_2 \in \sqrt{\mathbb{N}}$ . If  $\rho_1 \leq \rho_2$ , then the partition  $\mathbb{I}^{\rho_2}$  refines the partition  $\mathbb{I}^{\rho_1}$ . We note  $\sqsubseteq_{\mathbb{I}}$  (or simply  $\sqsubseteq$ ) this refinement relation on  $\mathbb{I}$ . In particular, for all  $\rho_1, \rho_2 \in \sqrt{\mathbb{N}}$  we have  $\rho_1 \leq \rho_2 \Rightarrow \mathbb{I}^{\rho_1} \sqsubseteq \mathbb{I}^{\rho_2}$ . It is plain that  $(\mathbb{I}, \sqsubseteq)$  is a totally ordered set which admits an infimum and a supremum. The infimum (actually, the minimum) is the trivial partition  $\mathbb{I}^0 = \{\mathbb{U}\}$ . The (unique) discrete 0-rotation associated to  $\mathbb{I}^0$  is the trivial rotation  $R_0^0 : \{\mathbf{0}\} \rightarrow \{\mathbf{0}\}$  that maps  $\mathbf{0}$  onto itself. The supremum is the partition of  $\mathbb{U}$  noted  $\mathbb{I}^\infty = \{\{\theta\} \mid \theta \in \mathring{\mathbb{U}}\}$ . For each  $\theta \in \mathring{\mathbb{U}}$ , the associated discrete rotation is  $R_\theta$ . (The other values  $\theta \in \mathbb{H}$ , i.e. the hinge angles, are not associated to discrete rotations, by definition.)

For any  $\rho \in \sqrt{\mathbb{N}}$ , each set  $\mathbb{I}^\rho$  models both the  $\rho$ -rotations (via  $\mathbb{H}_{\circ-\circ}^\rho$ ) and the hinge angles between these  $\rho$ -rotations (via  $\mathbb{H}_\bullet^\rho$ ). Except for  $\rho = 0$ , there is a trivial bijection between  $\mathbb{H}_\bullet^\rho$ ,  $\mathbb{H}_{\circ-\circ}^\rho$  and  $\mathbb{H}_{\bullet-\circ}^\rho$  (Property 3). There is then a trivial two-to-one function from  $\mathbb{I}^\rho$  to  $\mathbb{H}_{\bullet-\circ}^\rho$  for  $\rho > 0$  (plus a one-to-one function from  $\mathbb{I}^\rho$  to  $\mathbb{H}_{\bullet-\circ}^\rho$ , for  $\rho = 0$ ). One can then equivalently model the structure of the  $\rho$ -rotations either by  $\mathbb{I}^\rho$  or  $\mathbb{H}_{\bullet-\circ}^\rho$ . From now on, we consider  $\mathbb{H}_{\bullet-\circ}^\rho$  (noted  $\mathbb{S}^\rho$ ) instead of  $\mathbb{I}^\rho$ , without loss of generality.

We set  $\mathbb{K} = \bigcup_{\rho \in \sqrt{\mathbb{N}}} \mathbb{S}^\rho$ . Each element  $K \in \mathbb{K}$  is a semi-open interval of  $\mathbb{U}$  that defines a specific discrete  $\rho$ -rotation. By definition, there exists  $r \in \mathbb{N}$  such that  $K \in \mathbb{S}^{\sqrt{r}}$ . This value  $r$  may be non-unique. More precisely, there exists  $\alpha(K), \omega(K) \in \mathbb{N}$ , with  $\alpha(K) \leq \omega(K)$ , such that for all  $r \in \llbracket \alpha(K), \omega(K) \rrbracket$ , we have  $K \in \mathbb{S}^{\sqrt{r}}$ .

We endow  $\mathbb{K}$  with the inclusion order relation  $\sqsubseteq$ , and we note  $\mathfrak{T} = (\mathbb{K}, \triangleleft)$  the Hasse diagram of the ordered set  $(\mathbb{K}, \sqsubseteq)$ . The maximum of  $(\mathbb{K}, \sqsubseteq)$  is  $\mathbb{U}$ .

**Property 6** *Let  $K \in \mathbb{K} \setminus \{\mathbb{U}\}$ . There is a unique  $K' \in \mathbb{K}$  such that  $K \triangleleft K'$ .*

It follows from this property that  $\mathfrak{T} = (\mathbb{K}, \triangleleft)$  has a tree structure with  $\mathbb{U}$  as root.

The set  $\mathbb{K}$  is discrete but infinite, and so is the tree  $\mathfrak{T}$ . Its whole construction is then impossible. However, one may build a finite part of it from its root  $\mathbb{U}$  until a finite, but arbitrary large depth  $\mu$ .

Let  $\mu = \sqrt{m} \in \sqrt{\mathbb{N}}$ . We set  $\mathbb{K}^\mu = \bigcup_{r=0}^m \mathbb{S}^{\sqrt{r}} \subset \mathbb{K}$ . We endow  $\mathbb{K}^\mu$  with the inclusion order relation  $\sqsubseteq$ , and we note  $\mathfrak{T}^\mu = (\mathbb{K}^\mu, \triangleleft)$  the Hasse diagram of  $(\mathbb{K}^\mu, \sqsubseteq)$ . The order set  $(\mathbb{K}^\mu, \sqsubseteq)$  still has  $\mathbb{U}$  as maximum. By contrast with  $(\mathbb{K}, \sqsubseteq)$  it also has minimal elements gathered in  $\mathbb{S}^\mu$ . It is plain that  $\mathfrak{T}^\mu$  is still a tree. By contrast with  $\mathfrak{T}$ , the tree  $\mathfrak{T}^\mu$  is finite, with  $\mathbb{U}$  as root and the elements of  $\mathbb{S}^\mu$  as set of leaves. More precisely, we have the following property.

**Property 7** *The tree  $\mathfrak{T}^\mu$  is a partition tree (a.k.a. a dendrogram).*

This property means that each total cut of this tree (i.e. each maximal set of non-overlapping elements) is a partition of  $\mathbb{U}$ . A popular example of partition tree is the binary partition tree [18] usually considered in mathematical morphology. Here, the tree  $\mathfrak{T}^\mu$  is not a binary tree but a general partition tree. Indeed, for each non-leaf element  $K \in \mathbb{K}$ , there are  $k \geq 2$  elements  $K' \in \mathbb{K}$  such that  $K' \triangleleft K$  (by contrast with  $k = 2$  for the binary partition tree).

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**Algorithm 1:** Construction of the structure of the  $\rho$ -rotations ( $0 \leq \rho \leq \mu$ ).

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**Input :**  $\mu = \sqrt{m} \in \sqrt{\mathbb{N}}$   
**Output:**  $\mathfrak{T}^\mu$

- 1 **begin**
- 2     Build  $B^\mu$  (of size  $O(\mu^2)$ )
- 3     Build  $\mathbb{T}^\mu$  (of size  $O(\mu^3)$ ) from  $B^\mu$
- 4     Sort  $\mathbb{T}^\mu$
- 5     Build  $\mathbb{H}^\mu$  (sorted, of size  $O(\mu^3)$ ) from  $\mathbb{T}^\mu$
- 6     Build  $\mathcal{G}$  (of size  $O(\mu^3)$ ) from  $\mathbb{H}^\mu$
- 7     Build  $\Delta$  (of size  $O(\mu^3)$ ) from  $\mathbb{H}^\mu$
- 8     Build a binary partition tree  $\widehat{\mathfrak{T}}^\mu$  (of size  $O(\mu^3)$ ) from  $(\mathcal{G}, \Delta)$
- 9     Build  $\mathfrak{T}^\mu$  (of size  $O(\mu^3)$ ) from  $\widehat{\mathfrak{T}}^\mu$

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The number of vertices of a partition tree is lower than  $2\lambda - 1$  where  $\lambda$  is the number of leaves (this bound is reached if the partition tree is binary). The size of a general partition tree is then  $O(\lambda)$ . In the case of the tree  $\mathfrak{T}^\mu$ , the set of leaves is  $\mathbb{S}^\mu$ , of size  $O(\mu^3)$ . This is also the size of the tree  $\mathfrak{T}^\mu$ , that represents all the  $\rho$ -rotations for  $0 \leq \rho \leq \mu$ .

### 3.2 Construction

Let  $\mu = \sqrt{m} \in \sqrt{\mathbb{N}}$ , with  $\mu \neq 0$ . Let us focus on the partition  $\mathbb{S}^\mu$  that gathers the leaves of the tree  $\mathfrak{T}^\mu$ . Any element of  $\mathbb{S}^\mu = \mathbb{H}_{\bullet}^\mu$  is a semi-open interval  $S_j^\mu = [h_j^\mu, h_{j+1[\sigma^\mu]}^\mu)$  ( $0 \leq j \leq \sigma^\mu - 1$ ). This semi-open interval is adjacent to exactly two other elements of  $\mathbb{S}^\mu$ , namely  $S_{j+1[\sigma^\mu]}^\mu$  and  $S_{j-1[\sigma^\mu]}^\mu$  (this adjacency is defined by the fact that the closures of the sets overlap). In other words, the topological structure of  $\mathbb{S}^\mu$  is defined by a graph  $\mathcal{G}^\mu = (\mathbb{V}^\mu, \mathbb{E}^\mu)$  which is isomorphic to  $(\mathbb{H}_{\circ}^\mu, \mathbb{H}_{\bullet}^\mu)$ . In particular, it is a  $\sigma^\mu$ -cycle graph.

The edges of  $\mathcal{G}^\mu$  can be endowed with a valuation  $\Delta : \mathbb{H}^\mu \rightarrow \mathbb{N}$  that associates each hinge angle  $h \in \mathbb{H}^\mu$ , to the value  $\Delta(h)$  at which it appears in the combinatorial structure of the discrete rotations, i.e.

$$\Delta(h) = \min \left\{ r \in \mathbb{N} \mid \{h\} \in \mathbb{H}_{\bullet}^{\sqrt{r}} \right\} \quad (12)$$

Note that for any  $K = [h, h'] \in \mathbb{K}^\mu$ , we have  $\alpha(K) = \max\{\Delta(h), \Delta(h')\}$ . We also have the following property, that derives from Proposition 1.

**Property 8** *Let  $h \in \mathbb{H}^\mu$ . Let  $(p_x, p_y, k) \in \mathbb{T}$  be the prime generator of  $h$ , associated to the point  $\mathbf{p} = (p_x, p_y) \in \mathbb{Z}^2$ . We have  $\Delta(h) = \|\mathbf{p}\|_2^2$ .*

The edge-valued graph  $(\mathcal{G}^\mu, \Delta)$  can be seen as a saliency map [9], where the saliency measure is defined by  $\Delta$  on the ordered set  $(\mathbb{N}, \geq)$  (i.e. higher saliencies correspond to lower values). We then have the following property.

**Property 9** *The tree  $\mathfrak{T}^\mu$  is the watershed tree [8] of  $(\mathcal{G}^\mu, \Delta)$ .*

It follows that the tree  $\mathfrak{T}^\mu$  can be built from  $(\mathcal{G}^\mu, \Delta)$  as a watershed tree. This can be done by using the standard binary partition tree construction proposed in [18] (by defining

the priority of vertex merging from the saliency measure) followed by a straightforward collapsing procedure dedicated to turn the obtained binary tree into a general tree.

Based on these considerations, a procedure for building  $\mathfrak{T}^\mu$ , namely the combinatorial structure of all the  $\rho$ -rotations ( $0 \leq \rho \leq \mu$ ) is given in Algorithm 1.

The construction of the Euclidean ball  $B^\mu$  (line 2) has a time cost  $O(\mu^2)$ . The construction of the set  $\mathbb{T}^\mu$  of hinge angle triplets from  $B^\mu$  (line 3) has a time cost  $O(\mu^3)$ . Two hinge angles of  $\mathbb{H}$  can be compared in  $(\mathbb{R}, \leq)$  with a time cost  $O(1)$  [19], based on their generating triplets in  $\mathbb{T}$ . It follows that sorting  $\mathbb{T}^\mu$  (line 4) with respect to  $\leq$  (see Equation (8)) has a time cost  $O(\mu^3 \log \mu)$ . The construction of  $\mathbb{H}^\mu$  (line 5) is done by choosing in  $\mathbb{T}^\mu$  the triplets which are prime generators of hinge angles. Since  $\mathbb{T}^\mu$  is sorted with respect to  $\leq$ , this procedure has a time cost  $O(\mu^3)$ . Due to the bijective links between  $\mathbb{H}^\mu$ ,  $\mathbb{H}_\bullet^\mu$ ,  $\mathbb{H}_{\circ-\circ}^\mu$  and  $\mathbb{H}_{\bullet-\circ}^\mu$ , the construction of  $\mathcal{G}$  from  $\mathbb{H}^\mu$  (line 6) has a time cost  $O(\mu^3)$ . Based on Property 8, the construction of  $\mathcal{A}$  (line 7) from  $\mathbb{H}^\mu$  (modeled by the prime generators of  $\mathbb{T}^\mu$ ) has a time cost  $O(\mu^3)$ . The construction of a binary partition tree  $\widetilde{\mathfrak{T}}^\mu$  (line 8) from the edge-valued graph  $(\mathcal{G}, \mathcal{A})$  presents a time cost  $O(\mu^3 \log \mu)$  [18]. The final conversion of this binary partition tree  $\widetilde{\mathfrak{T}}^\mu$  into the general partition tree  $\mathfrak{T}^\mu$  (line 9) is done by collapsing the redundant vertices within  $\widetilde{\mathfrak{T}}^\mu$ . (These vertices are characterized by the fact that both their creation and their merging are carried out for a same saliency value of  $\mathcal{A}$ .) This conversion has a time cost  $O(\mu^3)$ . This justifies the following property.

**Property 10** *The construction of  $\mathfrak{T}^\mu$  (of size  $O(\mu^3)$ ) has a time cost  $O(\mu^3 \log \mu)$ .*

Based on Algorithm 1, the hierarchical structure of all the  $\rho$ -rotations (with  $0 \leq \rho \leq \mu \in \sqrt{\mathbb{N}}$ ) can then be built with a quasi-linear time cost with respect to its size.

## 4 Hierarchical structure of bijective discrete rotations

Based on the tree  $\mathfrak{T}^\mu$ , we now investigate the structure of the discrete rotations with regard to the bijectivity property. In particular, we adopt two points of view. First, we focus on the  $\rho$ -rotations which are the restrictions of bijective discrete rotations (Section 4.1). Second, we focus on the injective  $\rho$ -rotations, which are then bijective from their support to their finite image set (Section 4.2).

### 4.1 Structure of the restrictions of bijective discrete rotations

In a first time, we consider the  $\rho$ -rotations which are restrictions of bijective discrete rotations. We note  $\mathfrak{B} \subset \mathfrak{R}_{\mathbb{Z}^2}$  the set of the bijective discrete rotations, characterized by Proposition 2.

Let  $\mu = \sqrt{m} \in \sqrt{\mathbb{N}}$ . Our purpose is to define, for all  $0 \leq \rho \leq \mu$ , the subset  $\mathfrak{B}^\rho = \{R_\theta^\rho \in \mathfrak{R}_{\mathbb{Z}^2}^\rho \mid R_\theta \in \mathfrak{B}\}$ , i.e. the subset of the  $\rho$ -rotations which are restrictions of bijective discrete rotations. (We bijectively associate the set  $\mathfrak{B}^\rho \subseteq \mathfrak{R}_{\mathbb{Z}^2}^\rho$  to the subset  $\mathfrak{S}^\rho \subseteq \mathbb{S}^\rho$ .)

In a first time, we investigate the specific subset  $\mathfrak{B}^\mu \subset \mathfrak{R}_{\mathbb{Z}^2}^\mu$ . This subset  $\mathfrak{B}^\mu$  can be defined by determining which rotations in  $\mathfrak{R}_{\mathbb{Z}^2}^\mu$  (corresponding to leaves of the tree  $\mathfrak{T}^\mu$ , i.e. to intervals of  $\mathbb{H}_{\bullet-\circ}^\mu$ ) are bijective.



From the definition of  $\mathfrak{B}^\mu$ , a rotation corresponding to the interval  $[h_j^\mu, h_{j+1[\sigma^\mu]}^\mu)$  is bijective iff there exists an angle  $\theta \in \mathbb{B}$  (see Section 2.3) such that  $h_j^\mu < \theta < h_{j+1[\sigma^\mu]}^\mu$ . It is then sufficient to sort the set  $\mathbb{H}^\mu \cup \mathbb{B}$  to determine the bijective  $\mu$ -rotations, i.e. to build  $\mathfrak{S}^\mu$ . (We recall that  $\mathbb{H}^\mu \cap \mathbb{B} = \emptyset$ .) Since  $\mathbb{B}$  is infinite, such sorting is not tractable. Nonetheless, we can consider only a finite subset of  $\mathbb{B}$ , as shown by the following discussion.

From Proposition 2, we know that the sine of angles that define bijective discrete rotations are characterized by specific rational values, namely twin Pythagorean triplets. Due to symmetry considerations we restrict, without loss of generality, our discussion to the subset of angles  $(0, \pi/4]$ , where the angles  $\theta$  leading to bijective discrete rotations are characterized by

$$\sin \theta \in \left\{ s_p = \frac{2p+1}{2p^2+2p+1} \right\}_{p \in \mathbb{N}^*} \quad (13)$$

Let  $\theta \in (0, \pi/4]$  such that  $\sin \theta = s_p$  for a given  $p \in \mathbb{N}^*$ . Let us suppose that  $p^2 > 4m$ . Then, we have  $\sin \theta < 1/p < 1/2\mu$ . In such conditions, for any point  $\mathbf{p} \in \mathbb{B}^\mu$ , we have  $\|\mathbf{p} - \mathcal{R}_\theta(\mathbf{p})\|_2 < 1/2$ , and thus  $R_\theta^\mu(\mathbf{p}) = \mathbf{p}$ . In other words, we have  $R_\theta^\mu = R_0^\mu$ . It follows that for all  $p \in \mathbb{N}^*$  such that  $p > 2\mu$ , the angles  $\theta_p$  generating bijective discrete rotations belong to exactly one interval of  $\mathbb{S}^\mu$ , namely  $[h_{\sigma^\mu-1}^\mu, h_0^\mu)$  (which trivially corresponds to the bijective discrete rotation of angle 0).

As a consequence, we can restrict the bijectivity analysis to a finite subset noted  $\mathbb{B}^\mu$  generated by the values of  $p \leq 2\sqrt{m} = 2\mu$ . This set  $\mathbb{B}^\mu$  has a size  $O(\mu)$ , and it is thus sufficient to sort the (finite) set  $\mathbb{H}^\mu \cup \mathbb{B}^\mu$  to define  $\mathfrak{B}^\mu$ . We recall that from Algorithm 1,  $\mathbb{H}^\mu$  is already sorted. Sorting (by dichotomy)  $\mathbb{H}^\mu \cup \mathbb{B}^\mu$  then has a time cost  $O(\mu \log \mu)$ .

Let  $r \in \llbracket 0, m-1 \rrbracket$ . Let  $S \in \mathbb{S}^{\sqrt{r}}$ . From the definition of  $\mathfrak{B}^{\sqrt{r}}$ , we have

$$S \in \mathfrak{S}^{\sqrt{r}} \iff \exists S' \in \mathfrak{S}^{\sqrt{r+1}}, S' \triangleleft S \quad (14)$$

where  $\triangleleft$  is the Hasse relation of the tree  $\mathfrak{T}^\mu$ . Once the bottom structure  $\mathfrak{S}^\mu$  (that defines the bijective leaves of the tree  $\mathfrak{T}^\mu$ ) is obtained, we can define the successive sets  $\mathfrak{S}^{\sqrt{r}}$  for  $r$  from  $m-1$  to 0 by a propagation from the leaves of the tree  $\mathfrak{T}^\mu$  to all the other vertices in a bottom-up fashion, with a time cost  $O(\mu)$ . We then derive the whole set  $\mathfrak{S}_\mu = \bigcup_{r=0}^m \mathfrak{S}^{\sqrt{r}}$  (and equivalently  $\mathfrak{B}_\mu = \bigcup_{r=0}^m \mathfrak{B}^{\sqrt{r}}$ ), the structure of which is given by the restriction of the relation  $\triangleleft$  to  $\mathfrak{S}_\mu$ . In particular,  $(\mathfrak{S}_\mu, \triangleleft_{\mathfrak{S}_\mu})$  is a partial tree of  $\mathfrak{T}^\mu$ . This analysis justifies the following property.

**Property 11** *The tree  $(\mathfrak{S}_\mu, \triangleleft_{\mathfrak{S}_\mu})$  that models the structure of all the  $\rho$ -rotations ( $0 \leq \rho \leq \mu$ ), which are restrictions of bijective discrete rotations, has a size  $O(\mu)$  and can be built with a time cost  $O(\mu \log \mu)$ .*

## 4.2 Structure of the injective / bijective $\rho$ -rotations

In a second time, we consider the  $\rho$ -rotations which are injective, and thus bijective from their support set to their image set. For any  $\rho \in \sqrt{\mathbb{N}}$ , we note  $\mathfrak{I}^\rho \subseteq \mathfrak{R}_{\mathbb{Z}^2}^\rho$  the set of these injective  $\rho$ -rotations.

Let  $\rho \in \sqrt{\mathbb{N}}$ . Let  $\theta \in \mathring{\mathbb{U}}$ . Let  $R_\theta^\rho \in \mathfrak{R}_{\mathbb{Z}^2}^\rho$  be the  $\rho$ -rotation induced by  $\theta$ . We have

$$R_\theta^\rho \in \mathfrak{B}^\rho \implies R_\theta^\rho \in \mathfrak{I}^\rho \quad (15)$$

However, the reciprocal is not always true. We have  $\mathfrak{I}^\rho \supseteq \mathfrak{B}^\rho$  but we may have  $\mathfrak{I}^\rho \not\subseteq \mathfrak{B}^\rho$ .

Let  $\mu = \sqrt{m} \in \sqrt{\mathbb{N}}$ . We set  $\mathfrak{Z}_\mu = \bigcup_{r=0}^m \mathfrak{Z}^{\sqrt{r}}$ . By contrast with  $\mathfrak{B}_\mu$  which was built in a bottom-up fashion from  $\mathfrak{B}^\mu$  to  $\mathfrak{B}^0$  (i.e. from the leaves of  $\mathfrak{T}^\mu$  to its root), here  $\mathfrak{Z}_\mu$  will be built in a top-down fashion from  $\mathfrak{Z}^0$  to  $\mathfrak{Z}^\mu$  (i.e. from the root of  $\mathfrak{T}^\mu$  to its leaves).

Let  $\beta^\mu : \mathbb{K}^\mu \rightarrow \{\perp, \top\}$  be the Boolean function that characterizes the (non-)injectivity of any  $K \in \mathbb{K}^\mu$ . Building  $\mathfrak{Z}_\mu$  is then equivalent to building  $\beta^\mu$ . In the sequel, we then build  $\beta^\rho$  for  $\rho$  from 0 to  $\mu$ . The definition of  $\beta^0$  is trivial, since  $\mathbb{K}^0 = \{\mathbb{U}\}$  contains a unique element that models the rotation  $R_0^0$ . Since  $R_0^0 \in \mathfrak{B}^0 \subseteq \mathfrak{Z}^0$ , we have  $\beta^0(\mathbb{U}) = \top$ .

Now, let  $\rho = \sqrt{r} \in \sqrt{\mathbb{N}}$  with  $r \geq 1$ , and let us suppose that  $\beta^{\sqrt{r-1}}$  is already known. The current iteration of the algorithm consists of building  $\beta^{\sqrt{r}}$  from  $\beta^{\sqrt{r-1}}$ . Given a Euclidean ball  $B^\rho$  with radius  $\rho$ , a  $\rho$ -rotation is defined as a function  $R_\theta^\rho : B^\rho \rightarrow \mathbb{Z}^2$ . In particular, its support  $B^{\sqrt{r}}$  is larger than the support  $B^{\sqrt{r-1}}$  of the discrete  $\sqrt{r-1}$ -rotations. We note  $C^\rho = B^{\sqrt{r}} \setminus B^{\sqrt{r-1}} = \{\mathbf{p} \in \mathbb{Z}^2 \mid \|\mathbf{p}\|_2 = \rho\}$ .

Let  $\mathbf{q} \in C^\rho$ . This point  $\mathbf{q}$  generates triplets  $(q_x, q_y, k^q) \in \mathbb{T}^\rho$  that induce hinge angles. We note  $\mathbb{H}^q = \{h_i^q\}_{i=1}^{\sigma^q}$  ( $\sigma^q \geq 0$ ) the set of all these (sorted) hinge angles associated with  $\mathbf{q}$ . The set  $\mathbb{H}^q$  allows to define a partition  $\mathbb{H}_{\bullet}^q = \{H_j^q = [h_j^q, h_{j+1[\sigma^q]}^q)\}_{j=0}^{\sigma^q-1}$  the same way as in Equation (11). The set  $\mathbb{H}_{\bullet}^q$  can be endowed with a valuation function  $\nu^q : \mathbb{H}_{\bullet}^q \rightarrow \mathbb{Z}^2$  defined, for all  $j \in \llbracket 0, \sigma^q - 1 \rrbracket$ , by  $\nu^q(H_j^q) = R_\theta(\mathbf{q})$  for  $\theta \in H_j^q$ .

Let  $\mathbf{b} \in \mathbb{Z}^2$  be a point such that  $\|\mathbf{q} - \mathbf{b}\|_2 = 1$  and  $\mathbf{b} \in B^{\sqrt{r-1}}$ . There exists one or two such points. Hereafter, the procedure is described by assuming that  $\mathbf{b}$  is unique; it is simply repeated if there exist two points  $\mathbf{b}$ . The point  $\mathbf{b}$  is (one of) the only (two) point(s) for which we may have  $\mathbf{q}, \mathbf{b} \in B^{\sqrt{r}}$  and  $R_\theta(\mathbf{q}) = R_\theta(\mathbf{b})$ , leading to a non-injective configuration with respect to  $\mathbf{q}$ . We define  $\mathbb{T}^b, \mathbb{H}^b, \mathbb{H}_{\bullet}^b$  and  $\nu^b$  the same way as for  $\mathbf{p}$ . From the union set  $\mathbb{H}^b \cup \mathbb{H}^q$ , we can build the partition  $\overline{\mathbb{H}}_{\bullet}^q$  that refines  $\mathbb{H}_{\bullet}^q$  and  $\mathbb{H}_{\bullet}^b$  (as their supremum for  $\sqsubseteq$ ). We extend the valuation functions  $\nu^q$  and  $\nu^b$  from their initial support to  $\overline{\mathbb{H}}_{\bullet}^q$  so that for all  $j \in \llbracket 0, \overline{\sigma}^q - 1 \rrbracket$ , we have  $\nu^q(\overline{H}_j^q) = \nu^q(H_j^q)$  with  $\overline{H}_j^q \subseteq H_j^q \in \mathbb{H}_{\bullet}^q$  and  $\nu^b(\overline{H}_j^q) = \nu^b(H_{j'}^b)$  with  $\overline{H}_j^q \subseteq H_{j'}^b \in \mathbb{H}_{\bullet}^b$ . We can endow the new partition  $\overline{\mathbb{H}}_{\bullet}^q$  with the local injectivity characterization function  $\iota^q : \overline{\mathbb{H}}_{\bullet}^q \rightarrow \{\perp, \top\}$ , derived from  $\nu^q$  and  $\nu^b$  and defined, for all  $j \in \llbracket 0, \overline{\sigma}^q - 1 \rrbracket$ , by  $\iota^q(\overline{H}_j^q) = (\nu^q(\overline{H}_j^q) \neq (\nu^b(\overline{H}_j^q))$ .

For all the points  $\mathbf{q} \in C^\rho$ , we then have access to a partition  $\overline{\mathbb{H}}_{\bullet}^q$  of  $\mathbb{U}$  and a local injectivity characterization function  $\iota^q : \overline{\mathbb{H}}_{\bullet}^q \rightarrow \{\perp, \top\}$ . Following the same approach as described above, one can define the partition  $\mathbb{S}^{\sqrt{r}} = \mathbb{H}_{\bullet}^{\sqrt{r}}$  as the supremum of the partition  $\mathbb{S}^{\sqrt{r-1}} = \mathbb{H}_{\bullet}^{\sqrt{r-1}}$  and all the partitions  $\overline{\mathbb{H}}_{\bullet}^q$  induced by all the points  $\mathbf{q} \in C^\rho$ . The function  $\beta^{\sqrt{r-1}}$  and all the functions  $\iota^q$  can be extended from their initial support to  $\mathbb{H}_{\bullet}^{\sqrt{r}}$ . The function  $\beta^{\sqrt{r}} : \mathbb{H}_{\bullet}^{\sqrt{r}} \rightarrow \{\perp, \top\}$  is finally defined, for all  $j \in \llbracket 0, \sigma^{\sqrt{r}} - 1 \rrbracket$  by

$$\beta^{\sqrt{r}}(H_j^{\sqrt{r}}) = \beta^{\sqrt{r-1}}(H_j^{\sqrt{r-1}}) \wedge \bigwedge_{\mathbf{q} \in C^{\sqrt{r}}} \iota^q(H_j^{\sqrt{r-1}}) \quad (16)$$

For each  $r \in \llbracket 0, m \rrbracket$ , the set  $\mathfrak{Z}^\rho$  is given by  $(\beta^{\sqrt{r}})^{-1}(\{\top\})$ .

The overall procedure is summarized in Algorithm 2. For the sake of efficiency, it is relevant to design partial partitions  $\mathbb{H}_{\bullet}^{\sqrt{r}}$  that only contain the intervals for which  $\beta^{\sqrt{r}}$  has the  $\top$  value, instead of total partitions of  $\mathbb{U}$  (storing the intervals for which  $\beta^{\sqrt{r}}$  has the  $\perp$  value is useless, since a function is non-injective whenever one of its restricted functions is non-injective). We assume that we handle partial partitions in the

**Algorithm 2:** Construction of the injective / bijective  $\rho$ -rotations.

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Input :  $\mu = \sqrt{m} \in \sqrt{\mathbb{N}}$ 
Output:  $\mathfrak{S}_\mu$  (given by  $\beta^\rho$  for  $0 \leq \rho \leq \mu$ )
1 begin
2   Build  $\mathbb{H}^0, \mathbb{H}_{\bullet\circ}^0$  and  $\beta^0$ 
3   for  $r \in \llbracket 1, m \rrbracket$  do
4     Build  $C^{\sqrt{r}}$ 
5     foreach  $\mathbf{q} \in C^{\sqrt{r}}$  do
6       Build  $\mathbb{H}^{\mathbf{q}}, \mathbb{H}_{\bullet\circ}^{\mathbf{q}}$  and  $\nu^{\mathbf{q}}$ 
7       foreach  $\mathbf{b} \in \mathbb{Z}^2$  s.t.  $\|\mathbf{b}\|_2 < \sqrt{r}$  and  $\|\mathbf{p} - \mathbf{b}\|_2 = 1$  do
8         Build  $\mathbb{H}^{\mathbf{b}}, \mathbb{H}_{\bullet\circ}^{\mathbf{b}}$  and  $\nu^{\mathbf{b}}$ 
9         Build / update  $\overline{\mathbb{H}}^{\mathbf{q}}, \overline{\mathbb{H}}_{\bullet\circ}^{\mathbf{q}}$  and  $\iota^{\mathbf{q}}$ 
10        Build / update  $\mathbb{H}^{\sqrt{r}}, \mathbb{H}_{\bullet\circ}^{\sqrt{r}}$  and  $\beta^{\sqrt{r}}$ 

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above complexity analysis. Under this hypothesis, the space cost of the partial partitions associated to  $\mathbb{H}^{\sqrt{r}}, \mathbb{H}_{\bullet\circ}^{\sqrt{r}}$  is assumed to be  $\mathcal{O}(\sqrt{r})$ . (At this stage, this assumption relies on our experimental analysis of the evolution of the population of injective rotations; this remains to be theoretically confirmed.) The construction of  $\mathbb{H}^0, \mathbb{K}^0$  and  $\beta^0$  (line 2) has a time cost  $\mathcal{O}(1)$ . The external loop (line 3) iterates  $\mathcal{O}(m)$  times. The construction of  $C^{\sqrt{r}}$  (of size  $\mathcal{O}(1)$ ) (line 4) has a time cost  $\mathcal{O}(1)$ . The medial loop (line 5) iterates  $\mathcal{O}(1)$  times. At iteration  $r$ , the construction of  $\mathbb{H}^{\mathbf{q}}, \mathbb{H}_{\bullet\circ}^{\mathbf{q}}$  and  $\nu^{\mathbf{q}}$ , of size  $\mathcal{O}(\sqrt{r})$ , (line 6) has a time cost  $\mathcal{O}(\sqrt{r})$ . The internal loop (line 7) iterates  $\mathcal{O}(1)$  times. At iteration  $r$  of the medial loop, the construction of  $\mathbb{H}^{\mathbf{b}}, \mathbb{H}_{\bullet\circ}^{\mathbf{b}}$  and  $\nu^{\mathbf{b}}$  (of size  $\mathcal{O}(\sqrt{r})$ ) (line 8) while the updating of  $\overline{\mathbb{H}}^{\mathbf{q}}, \overline{\mathbb{H}}_{\bullet\circ}^{\mathbf{q}}$  and  $\iota^{\mathbf{q}}$  (line 9), that consists of merging pairs of partitions of size  $\mathcal{O}(\sqrt{r})$  also has a time cost  $\mathcal{O}(\sqrt{r})$ . At iteration  $r$  of the medial loop, the update of  $\mathbb{H}^{\sqrt{r}}, \mathbb{H}_{\bullet\circ}^{\sqrt{r}}$  and  $\beta^{\sqrt{r}}$  that consists of merging pairs partitions of size  $\mathcal{O}(\sqrt{r-1})$  and  $\mathcal{O}(\sqrt{r})$ , has a time cost  $\mathcal{O}(\sqrt{r})$ . This analysis motivates the following conjecture.

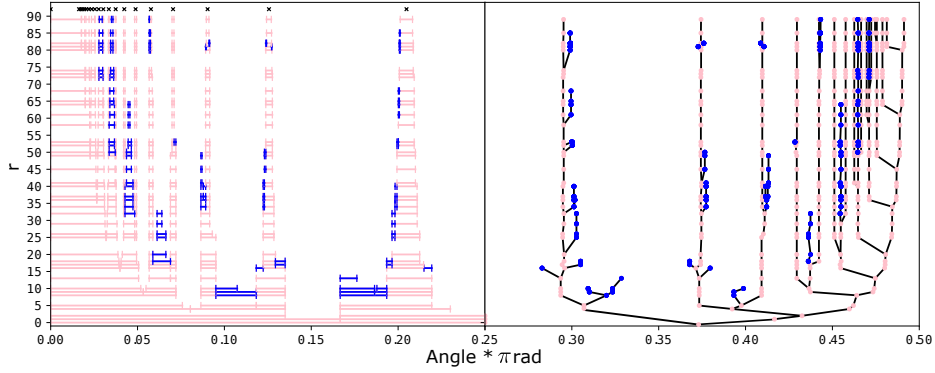
**Conjecture 12** *The tree  $(\mathfrak{S}_\mu, \triangleleft_{\mathfrak{S}_\mu})$  that models the structure of all the injective  $\rho$ -rotations ( $0 \leq \rho \leq \mu$ ) has a size  $\mathcal{O}(\mu)$  and can be built with a time cost  $\mathcal{O}(\mu^{\frac{3}{2}})$ .*

## 5 Experimental results

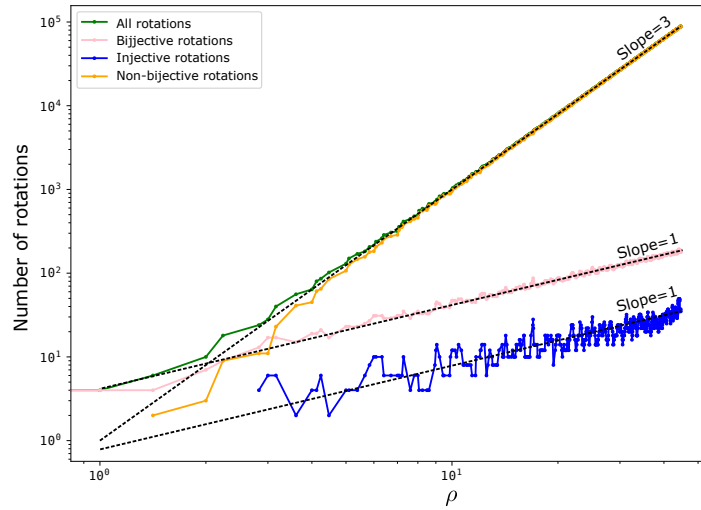
We now illustrate some results obtained by application of the proposed algorithms.

In Figure 1, we show the first stages of the hierarchical structure of the bijective and injective  $\rho$ -rotations. The non-bijective  $\rho$ -rotations were also computed, but they are not visualized for the sake of readability.

Figure 2 provides the growth of the size of the various families of  $\rho$ -rotations with respect to the radius  $\rho$  of the Euclidean balls. Theoretically, the numbers of the  $\rho$ -rotations and the non-bijective  $\rho$ -rotations grow in  $\mathcal{O}(\rho^3)$ , while the number of the bijective  $\rho$ -rotations (i.e. the restrictions of bijective discrete rotations) grows in  $\mathcal{O}(\rho)$ . These



**Fig. 1.** Hierarchical structure of the bijective  $\rho$ -rotations. Here, the values of  $\rho = \sqrt{r}$  are given for  $0 \leq r \leq 90$  (y axis). The visualized angles of  $\mathbb{U}$  are restricted to  $[0, \pi/2]$  (x axis), due to symmetry considerations. For the sake of visualisation, only the bijective (Section 4.1) and injective (Section 4.2)  $\rho$ -rotations are visualized. On the left side (angles in  $[0, \pi/4]$ ), the intervals of  $\mathbb{S}^\rho$  corresponding to the  $\rho$ -rotations are depicted. On the (symmetric) right side (angles in  $[\pi/4, \pi/2]$ ), the hierarchical structure between these intervals /  $\rho$ -rotations is depicted. The pink elements correspond to  $\rho$ -rotations which are the restrictions of discrete rotations (in particular, the bijective angles associated to these rotations are depicted by black  $\times$  in the upper part of the right side). The blue elements correspond to  $\rho$ -rotations which are injective on  $B^\rho$  and thus bijective from their finite support to their image.



**Fig. 2.** Evolution of the size of the different families of  $\rho$ -rotations (all, bijective, injective, non-bijective) (y-axis) with respect to  $\rho$  (x-axis) in log-log scale. The lines of slope 3 and 1 are depicted to emphasize the  $O(\rho^3)$  and  $O(\rho)$  behaviours.

behaviours are experimentally confirmed. The number of the injective  $\rho$ -rotations is experimentally assessed as growing in  $O(\rho)$ . This result, although intuitive, remains to be proved. This is why the results stated at the end of Section 4 are given as a conjecture.

## 6 Conclusion

In this article, we investigated the notion of bijectivity for the discrete rotations on finite subsets of  $\mathbb{Z}^2$ . We emphasized their hierarchical structure, which allowed us to design efficient solutions for building them, by taking advantage of algorithmics on trees developed in particular in mathematical morphology.

In further works, we will aim to improve these algorithmic developments. We will also more deeply study the notion of injective rotations, and their behaviour for large values of  $\rho$ . In particular, we will aim to formally prove how the population of these rotations evolves when considering Euclidean balls of increasing radii. To complete this framework, we will also investigate alternative definitions of bijectivity. Finally, we will build upon this framework to deal with the problem of approximation of non-bijective rotations [7]. We may also further investigate other properties of finite discrete rotations, that were also pioneered e.g. in [1] from the point of view of dynamic systems.

## Appendix

### Proof of Property 3

- The bijection from  $\mathbb{H}^\rho$  to  $\mathbb{H}_\bullet^\rho$  is given by  $h \mapsto \{h\}$ . The inverse bijection is given by  $\{h\} \mapsto h$ . In the case where  $\rho < 1$ , both bijections are equal to  $\emptyset$ .
- The bijection from  $\mathbb{H}_{\infty}^\rho$  to  $\mathbb{H}_{\infty}^\rho$  is given by  $(h_1, h_2) \mapsto [h_1, h_2]$ . The inverse bijection is given by  $[h_1, h_2] \mapsto (h_1, h_2)$ . In the case where  $\rho < 1$ , there is no hinge angle  $h$  and then  $\mathbb{H}_{\infty}^\rho$  and  $\mathbb{H}_{\infty}^\rho$  can be modeled as singleton sets that contain a unique element  $\mathbb{U}$  (from which we remove an arbitrary point for  $\mathbb{H}_{\infty}^\rho$ ), and the bijection still holds.
- Let us suppose that  $\rho \geq 1$ . In that case, there is at least one hinge angle, and we can define a bijection e.g. from  $\mathbb{H}_\bullet^\rho$  to  $\mathbb{H}_{\infty}^\rho$ . This bijection is given by  $\{h_j^\rho\} \mapsto (h_j^\rho, h_{j+1[\sigma^\rho]}^\rho)$ . The inverse bijection is given by  $(h_j^\rho, h_{j+1[\sigma^\rho]}^\rho) \mapsto \{h_j^\rho\}$ . The other bijections follow by transitivity. ■

### Proof of Property 4

Let  $H \in \mathbb{H}_{\infty}^\rho$ . Let  $\theta_1, \theta_2 \in H$ . Let  $R_{\theta_1}, R_{\theta_2} \in \mathfrak{R}_{\mathbb{Z}^2}$  be the two discrete rotations associated to  $\theta_1, \theta_2$ , respectively. Let  $R_{\theta_1}^\rho, R_{\theta_2}^\rho \in \mathfrak{R}_{\mathbb{Z}^2}^\rho$  be the two  $\rho$ -rotations associated to  $\theta_1, \theta_2$ , respectively. There is no hinge angle of  $\mathbb{H}^\rho$  in the interval defined by  $\theta_1, \theta_2$ . It follows that  $R_{\theta_1}^\rho$  and  $R_{\theta_2}^\rho$  are equal (and also equal to all the  $R_\theta^\rho$  for  $\theta$  in this interval). This allows us to define a function from  $\mathbb{H}_{\infty}^\rho$  to  $\mathfrak{R}_{\mathbb{Z}^2}^\rho$ . The surjectivity of this function derives from the surjectivity of  $R^\rho$ . Now, let  $H_1, H_2 \in \mathbb{H}_{\infty}^\rho$  be two distinct intervals of  $\mathbb{H}_{\infty}^\rho$ , and let  $\theta_1 \in H_1$  and  $\theta_2 \in H_2$ . There exists (at least) one hinge angle of  $\mathbb{H}^\rho$  that separates  $H_1$  and  $H_2$ . It follows that there exists (at least) one point of  $B^\rho$  for which  $R_{\theta_1}^\rho$  and  $R_{\theta_2}^\rho$  differ. It follows that the function defined above from  $\mathbb{H}_{\infty}^\rho$  to  $\mathfrak{R}_{\mathbb{Z}^2}^\rho$  is injective, and thus bijective. ■

### Proof of Property 6

Let  $K \in \mathbb{K} \setminus \{\mathbb{U}\}$ . Then we have  $K \in \mathbb{H}_{\bullet, \circ}^{\alpha(K)}$  with  $\alpha(K) = \sqrt{r}$  for a given  $r \in \mathbb{N}$  distinct from 0 (since  $K \neq \mathbb{U}$ ). The set  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r}}$  is a partition of  $\mathbb{U}$  and it refines any partition  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r'}}$  for  $r' \in \mathbb{N}$ ,  $r' \leq r$ . In particular,  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r-1}}$  is the finest partition refined by  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r}}$ . Since  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r-1}}$  is refined by  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r}}$ , then there exists  $K' \in \mathbb{H}_{\bullet, \circ}^{\sqrt{r-1}}$  such that  $K \subseteq K'$ . But since  $\sqrt{r} = \alpha(K)$ , we have  $K \neq K'$  and then  $K \subset K'$ . Moreover, since  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r-1}}$  is the finest partition refined by  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r}}$ , there is no  $K'' \in \mathbb{K}$  such that  $K \subset K'' \subset K'$ , and it follows that  $K \triangleleft K'$ . The fact that  $K'$  is unique derives from the fact that each  $\mathbb{H}_{\bullet, \circ}^{\sqrt{r}}$  ( $r \in \mathbb{N}$ ) is a partition of  $\mathbb{U}$ , which implies that there exists, for each  $r' < r$  a unique  $K'''$  such that  $K \subset K'''$ . ■

### Proof of Property 7

The proof is by induction on  $\mu \in \sqrt{\mathbb{N}}$ . If  $\mu = 0$ , then  $\mathfrak{T}^\mu$  is composed of only one element, namely  $\mathbb{U}$ , and the only total cut of the tree is  $\{\mathbb{U}\}$  which is a partition of  $\mathbb{U}$ . Now, let us suppose that  $\mu > 0$  and that we already have that  $\mathfrak{T}^\mu$  is a partition tree. We set  $\mu = \sqrt{m}$ , with  $m \in \mathbb{N}$ . Let us consider the “next” tree, namely  $\mathfrak{T}^{\mu'}$  with  $\mu' = \sqrt{m+1}$ . This tree  $\mathfrak{T}^{\mu'}$  is obtained by adding to the nodes of  $\mathfrak{T}^\mu$  the elements of  $\mathbb{S}^{\sqrt{m+1}} = \mathbb{H}_{\bullet, \circ}^{\sqrt{m+1}}$ , and by enriching the relation  $\triangleleft$  by linking each element  $K$  of  $\mathbb{H}_{\bullet, \circ}^{\sqrt{m+1}}$  that was not yet in  $\mathbb{H}_{\bullet, \circ}^{\sqrt{m}}$  with a unique element  $K'$  of  $\mathbb{H}_{\bullet, \circ}^{\sqrt{m}}$  (see Property 6). Let us consider a total cut of this new tree. If this cut does not contain any element from  $\mathbb{H}_{\bullet, \circ}^{\sqrt{m+1}}$ , then it is a partition of  $\mathbb{U}$  by the induction hypothesis. Now, let us consider that this total cut contains an element  $K$  of  $\mathbb{H}_{\bullet, \circ}^{\sqrt{m+1}}$ . Since this is a cut, it does not contain the unique element  $K'$  such that  $K \triangleleft K'$ , but since it is a total cut, it must then contain all the elements  $K''$  that also satisfy  $K'' \triangleleft K'$ . All these elements form a partition of  $K'$ . By substituting  $K'$  to these elements in the cut, and by doing the same for all the other elements  $K$  of  $\mathbb{H}_{\bullet, \circ}^{\sqrt{m+1}}$ , we build a new cut of the tree  $\mathfrak{T}^\mu$ , which is a partition of  $\mathbb{U}$ , by induction hypothesis. By construction, this partition is refined by the initial cut and the initial cut of  $\mathfrak{T}^{\mu'}$  is then also a partition of  $\mathbb{U}$ . It follows that  $(\mathbb{K}^\mu, \subseteq)$  is a partition tree. ■

### Proof of Property 8

Let  $\{h\} \in \mathbb{H}_{\bullet, \circ}^\mu$ . Let  $t = (p_x, p_y, k) \in \mathbb{T}$  be the prime generator of  $h$ , associated to the point  $\mathbf{p} = (p_x, p_y) \in \mathbb{Z}^2$ . From Proposition 1, for any triplet  $t' = (q_x, q_y, k') \in \mathbb{T}$  such that  $\eta(t) = \eta(t')$ , there exists  $n \in \mathbb{Z}$  such that  $q_x = (2n+1)p_x$  and  $q_y = (2n+1)p_y$ . It follows that  $\|\mathbf{p}\|_2 \leq \|\mathbf{q}\|_2$ . Since  $\eta(t) = h$ , we have  $\{h\} \in \mathbb{H}_{\bullet, \circ}^{\|\mathbf{p}\|_2}$ . But since  $\mathbf{p}$  is the point of lowest norm that satisfies this property, we have  $\|\mathbf{p}\|_2^2 = \min \left\{ r \in \mathbb{N} \mid \{h\} \in \mathbb{H}_{\bullet, \circ}^{\sqrt{r}} \right\}$  and the result follows. ■

### Proof of Property 9

From Property 7,  $\mathfrak{T}^\mu$  is a partition tree. With the above definitions and analogies, each thresholding of the graph  $\mathcal{G}^\mu$  with respect to  $\Delta$  at value  $\rho$  builds a partition of nodes of  $\mathcal{G}^\mu$  which models the partition  $\mathbb{S}^\rho$ . As a consequence,  $\mathfrak{T}^\mu$  is the watershed tree of  $(\mathcal{G}^\mu, \Delta)$ . ■

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