

Online Appendix to

Credit Rationing in Unsecured Debt Markets

This appendix provides detailed proofs and technical arguments omitted from the main text. Additional commentary on self-enforcing debt limits and the weak rollover property can be found in Section [D](#).

A Proof of Proposition [2.1](#)

Fix an event $s^\tau \in \Sigma$ and assume that

$$D^i(s^t) \leq \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \text{for all } s^t \succeq s^\tau$$

with a strict inequality in at least one successor event $s^t \succeq s^\tau$. We shall prove that $V^i(D^i, -D^i(s^\tau)|s^\tau) > V^i(0, 0|s^\tau)$. Denote by (\bar{c}^i, \bar{a}^i) agent i 's optimal choice in the budget set $B^i(0, 0|s^\tau)$. Observe that $V^i(0, 0|s^\tau) = U^i(\bar{c}^i|s^\tau)$. Let $a^i := \bar{a}^i - D^i$ and, for every $s^t \succeq s^\tau$, define $\delta^i(s^t) := -D^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1})$. As $\bar{a}^i \geq 0$, the pair (c^i, a^i) belongs to the budget set $B^i(D^i, -D^i(s^\tau)|s^\tau)$ where the new consumption plan is defined by $c^i := \bar{c}^i + \delta^i$. This implies that $V^i(D^i, -D^i(s^\tau)|s^\tau) \geq U^i(c^i|s^\tau) > U^i(\bar{c}^i|s^\tau)$, where the strict inequality follows from the assumption that $\delta^i(s^t) > 0$ for some $s^t \succeq s^\tau$. As $U^i(\bar{c}^i|s^\tau) = V^i(0, 0|s^\tau)$, we get the desired result.

B Proof of Proposition [3.1](#)

Consider socially feasible pair $(q, (c^i)_{i \in I})$. This means that $(c^i)_{i \in I}$ satisfies the market clearing condition $\sum_{i \in I} c^i = \sum_{i \in I} y^i$, each consumption process c^i is strictly positive and satisfies the participation constraint

$$U^i(c^i|s^t) \geq V_{\text{def}}^i(q|s^t) = V^i(0, 0|s^t)$$

where the price process q is given by

$$q(s^t) = \max_{i \in I} q^i(s^t) \quad \text{where} \quad q^i(s^t) := \beta^i \pi^i(s^t | \sigma(s^t)) \frac{(u^i)'(c^i(s^t))}{(u^i)'(c^i(\sigma(s^t)))}.$$

Fix an arbitrary allocation $(D^i(s^0))_{i \in I}$ satisfying $\sum_{i \in I} D^i(s^0) = 0$. For each $i \in I$, choose a process $(D^i(s^t))_{s^t \succ s^0}$ satisfying the difference equation

$$D^i(s^t) = y^i(s^t) - c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}). \quad (\text{B.1})$$

The existence of D^i can be proved constructively. Assume $D^i(s^t)$ has been defined and pose

$$D^i(s^{t+1}) := \frac{1}{q(s^{t+1})} [D^i(s^t) - (y^i(s^t) - c^i(s^t))].$$

Equivalently, we have

$$p(s^{t+1}) D^i(s^{t+1}) = D^i(s^0) + \sum_{r=0}^t p(\sigma^r(s^t)) (c^i(\sigma^r(s^t)) - y^i(\sigma^r(s^t)))$$

where p is the process of Arrow–Debreu prices defined recursively by $p(s^0) := 1$ and $p(s^t) = q(s^t) p(\sigma(s^t))$ for every $s^t \succ s^0$.

If we let $a^i := -D^i$, then $(q, (c^i, a^i, D^i)_{i \in I})$ is a self-enforcing competitive equilibrium.

Indeed, the flow budget constraints are satisfied as

$$\begin{aligned} p(s^t) c^i(s^t) + \sum_{s^{t+1} \succ s^t} p(s^{t+1}) a^i(s^{t+1}) &= p(s^t) c^i(s^t) - \sum_{s^{t+1} \succ s^t} p(s^{t+1}) D^i(s^{t+1}) \\ &= p(s^t) y^i(s^t) - p(s^t) D^i(s^t) \\ &= p(s^t) y^i(s^t) + p(s^t) a^i(s^t). \end{aligned}$$

Debt constraints are satisfied as $a^i = -D^i$. This last property also implies that the transversality condition is satisfied. Euler equations are satisfied as $q(s^t) \geq q^i(s^t)$ and debt constraints always bind. Finally, debt limits are self-enforcing as

$$V^i(D^i, -D^i(s^t) | s^t) = V^i(D^i, a^i(s^t) | s^t) = U^i(c^i | s^t) \geq V_{\text{def}}^i(s^t).$$

Debt limits may take negative values and require mandatory savings. There is no inconsistency with our assumption that agents lack commitment. Indeed, when they occur, mandatory savings are self-enforcing: when $D^i(s^t) < 0$, agents prefer to save the amount $-D^i(s^t)$ and get the utility $U^i(D^i, -D^i(s^t) | s^t)$ than default to get $V_{\text{def}}^i(s^t)$.

Remark B.1. If a socially feasible pair $(q, (c^i)_{i \in I})$ can be implemented as a laissez-faire equilibrium $(q, (c^i, a^i, D^i)_{i \in I})$, then each D^i can be chosen to be nonnegative. Indeed, let $\theta^i := a^i + D^i \geq 0$. Because of the market clearing condition, we must have

$$0 = \sum_{i \in I} a^i \geq - \sum_{i \in I} D^i.$$

Let $D := \sum_{i \in I} D^i$. The process D is nonnegative and satisfies exact rollover. Consider any decomposition $D = \sum_{i \in I} \tilde{D}^i$ where each \tilde{D}^i is nonnegative and satisfies the exact rollover property.¹ Pose $\tilde{a}^i := \theta^i - \tilde{D}^i$. As $a^i + D^i = \tilde{a}^i + \tilde{D}^i$, we can verify that $(q, (c^i, \tilde{a}^i, \tilde{D}^i)_{i \in I})$ is a laissez-faire competitive equilibrium.

C Omitted Proofs of Section 5

To simplify the exposition of the theoretical results, we assume in this section that u is bounded. This restriction ensures that the lifetime utility U is continuous (for the product topology) and the demand set is nonempty. The characterization of debt limits can be extended, and our results in this section continue to hold even when u is unbounded. A general treatment of unbounded utility functions requires additional technical assumptions on endowment processes and a suitable modification of the utility function u outside a specific interval so that the equilibrium outcomes remain unaffected. For a detailed discussion, see [Martins-da-Rocha and Santos \(2019\)](#).

C.1 Proof of Theorem 5.1

The proof of Theorem 5.1 exploits two intermediate results. The first and crucial observation, which has no analog in the absence of output contraction, is to show that the present value of foregone endowment imposes a lower bound on not-too-tight debt limits. A direct implication of this property is that the process $PV(\ell^i)$ is finite. This is summarized in the following lemma.

¹We can set $\hat{D}^i := \alpha^i D$ where $\alpha^i \geq 0$ and $\sum_{i \in I} \alpha^i = 1$.

Lemma C.1. *Not-too-tight debt limits are at least as large as the present value of endowment losses, i.e., for each agent i , $D^i(s^t) \geq \text{PV}(\ell^i|s^t)$ at any event s^t .*

A natural approach to prove this result is to show that $D^i(s^t) \geq \ell^i(s^t) + \tilde{D}^i(s^t)$, where $\tilde{D}^i(s^t) := \sum_{s^{t+1} \succ s^t} q(s^{t+1})D^i(s^{t+1})$ is the present value of next period's debt limits, and then use a standard iteration argument. Because, in equilibrium, debt limits are not too tight, this is equivalent to proving that agent i does not have the incentive to default when her net asset position is $\ell^i(s^t) + \tilde{D}^i(s^t)$, i.e.,

$$V^i(D^i, -\ell^i(s^t) - \tilde{D}^i(s^t)|s^t) \geq V_{\ell^i}^i(0, 0|s^t). \quad (\text{C.1})$$

By definition, the value function $V_{\ell^i}^i$ satisfies:

$$V_{\ell^i}^i(0, 0|s^t) \geq u(y^i(s^t) - \ell^i(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V_{\ell^i}^i(0, 0|s^{t+1}). \quad (\text{C.2})$$

If we had an equality in (C.2), then inequality (C.1) would be straightforward. Indeed, consuming $y^i(s^t) - \ell^i(s^t)$ and borrowing up to each debt limit $D^i(s^{t+1})$ at event s^t leads to the right-hand side continuation utility in (C.2) and satisfies the solvency constraint at event s^t in the budget set defining the left-hand side of (C.1). Unfortunately, in our environment where agents can save upon default condition (C.2) may not hold as an equality.² Overcoming this problem is the technical challenge in the proof of Lemma C.1. The formal argument is presented below.

The second observation is that the process $\text{PV}(\ell^i)$ of present values of endowment losses, when it is finite, is itself not too tight. The following lemma provides the formal statement. The proof follows from a simple translation invariance of the flow budget constraints.

Lemma C.2. *If $\text{PV}(\ell^i|s^0)$ is finite, then the process $\text{PV}(\ell^i)$ is not too tight, i.e.,*

$$V^i(\text{PV}(\ell^i), -\text{PV}(\ell^i|s^t)|s^t) = V_{\ell^i}^i(0, 0|s^t), \quad \forall s^t \succeq s^0.$$

Equipped with Lemma C.1 and Lemma C.2, we can now provide a simple proof of Theorem 5.1.

²In the simpler environment where, upon default, saving is not possible (as it is the case in [Alvarez and Jermann 2000](#)) condition (C.2) always hold as an equality.

Proof of Theorem 5.1. Fix a process D^i of not-too-tight debt limits. Lemma C.1 implies that $\text{PV}(\ell^i|s^0)$ is finite. From Lemma C.2, we also deduce that the process $\underline{D}^i := \text{PV}(\ell^i)$ is not too tight. Martins-da-Rocha and Santos (2019) show that the difference between any two processes of not-too-tight debt limits must be an exact rollover process. Therefore, a process M^i exists satisfying the exact rollover property such that $D^i = \underline{D}^i + M^i$. By Lemma C.1, $D^i \geq \underline{D}^i$, in which case the process M^i must be nonnegative. \square

C.1.1 Proof of Lemma C.1

Since we are exclusively concerned with the single-agent problem, we simplify notation by dropping the superscript i . Let D be a process of not-too-tight limits. We first show that there exists a nonnegative process \underline{D} satisfying

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0. \quad (\text{C.3})$$

Indeed, let Φ be the mapping $B \in \mathbb{R}^\Sigma \mapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leq B \leq \bar{D}$ where

$$\bar{D}(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \succeq s^0.$$

The mapping Φ is continuous (for the product topology), and we have $\Phi[0, \bar{D}] \subseteq [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that Φ admits a fixed point \underline{D} in $[0, \bar{D}]$.

Claim C.1. *The process \underline{D} is tighter than the process D , i.e., $\underline{D} \leq D$.*

Proof of Claim C.1. Fix a node s^t . Since $V_\ell(0, 0|s^t) = V(D, -D(s^t)|s^t)$ and $V(D, \cdot|s^t)$ is strictly increasing, it is sufficient to show that $V(D, -\underline{D}(s^t)|s^t) \geq V_\ell(0, 0|s^t)$. Denote by (c, \tilde{a}) the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some

arbitrary event s^t .³ We let \widehat{D} be the process defined by $\widehat{D}(s^t) := \min\{D(s^t), \underline{D}(s^t)\}$ for all s^t . Observe that

$$\begin{aligned} y(s^t) - \underline{D}(s^t) &= y(s^t) - \ell(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \widehat{D}(s^{t+1}) \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) [\tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})] \\ &= c(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) \end{aligned}$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})$. Since $\widehat{D} \leq D$, we have $a(s^{t+1}) \geq -D(s^{t+1})$. At any successor event $s^{t+1} \succ s^t$, we have

$$\begin{aligned} y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\ &\geq y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\ &\geq y(s^{t+1}) - \ell(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) \widehat{D}(s^{t+2}) \\ &\geq c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) [\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\ &\geq c(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) a(s^{t+2}) \end{aligned}$$

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.⁴ Observe that $a(s^{t+2}) \geq -D(s^{t+2})$ as $\widehat{D} \leq D$.

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that (c, a) belongs to the budget set $B(D, -\underline{D}(s^t)|s^t)$. It follows that

$$V(D, -\underline{D}(s^t)|s^t) \geq U(c|s^t) = V_\ell(0, 0|s^t)$$

implying the desired result: $\underline{D}(s^t) \leq D(s^t)$. □

It follows from Claim C.1 that \underline{D} satisfies

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \underline{D}(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \quad (\text{C.4})$$

³That is, the process \tilde{a} supports consumption c such that $U(c|s^t) := V_\ell(0, 0|s^t)$.

⁴To get the second weak inequality, we use equation (C.3).

Applying equation (C.4) recursively, we get

$$\begin{aligned}
p(s^t)\underline{D}(s^t) &= p(s^t)\ell(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} p(s^{t+1})\ell(s^{t+1}) + \dots \\
&\quad \dots + \sum_{s^T \in S^T(s^t)} p(s^T)\ell(s^T) + \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})\underline{D}(s^{T+1})
\end{aligned}$$

for any $T > t$. Since \underline{D} is nonnegative, it follows that

$$p(s^t)\underline{D}(s^t) \geq \sum_{\tau=t}^T \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)\ell(s^\tau).$$

Passing to the limit when T goes to infinity, we get that $\text{PV}(\ell|s^t)$ is finite for any event s^t (in particular for s^0). Recalling that $D \geq \underline{D}$, we also get that $D(s^t) \geq \text{PV}(\ell|s^t)$.

C.1.2 Proof of Lemma C.2

Denote by (c, \tilde{a}) the optimal consumption and bond holdings in the budget set $B_\ell(0, 0|s^t)$ for some arbitrary event s^t . We pose $\underline{D} := \text{PV}(\ell)$ and observe that

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}).$$

It is easy to show that (c, a) is optimal in the budget set $B(\underline{D}, -\underline{D}(s^0)|s^t)$ where $a := \tilde{a} - \underline{D}$. We then deduce that $V^i(\underline{D}, -\underline{D}(s^t)|s^t) = V_\ell(0, 0|s^t)$, so proving the claim.

C.2 Proof of Proposition 5.1

Let $(q, (c^i, a^i, D^i)_{i \in I})$ be an equilibrium with limited pledgeability. Since pledgeable income is nonnegligible, we must have

$$\sum_{i \in I} \text{PV}(y^i|s^0) \leq \frac{1}{\varepsilon} \sum_{i \in I} \text{PV}(\ell^i|s^0).$$

By the decomposition property property (5.3), we have that $\text{PV}(\ell^i|s^0) < \infty$ for each agent i , so we deduce that the aggregate wealth of the economy $\sum_{i \in I} \text{PV}(y^i|s^0)$ must be finite. Since consumption markets clear, we obtain that the present value of optimal consumption is finite

for all agents. In addition, due to the Inada’s condition, the optimal consumption is strictly positive.⁵ Lemma A.1 in [Martins-da-Rocha and Vailakis \(2017\)](#) then implies that the market transversality condition holds true:⁶

$$\lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0. \quad (\text{C.5})$$

The decomposition property property (5.3) implies that, for each i , there exists a nonnegative discounted martingale process M^i such that $D^i = \text{PV}(\ell^i) + M^i$. Condition C.5 can then be rewritten as follows:

$$\lim_{t \rightarrow \infty} \sum_{s^t \in S^t} p(s^t)a^i(s^t) = -p(s^0)M^i(s^0).$$

Since bond markets clear, we deduce that $\sum_{i \in I} M^i(s^0) = 0$, proving the desired result: $M^i = 0$ for each i .

D Self-Enforcing Debt Limits Need Not Satisfy Weak Rollover

In the laissez-faire equilibrium, d^{lf} serves as the debt limit for borrowing against the high and low income regimes. In our policy experiment the low-income agent at period $t = 2$ issues debt $d_3(\varepsilon)$ against next period’s high-income realization. However, we have not addressed how much debt, denoted $\tilde{d}_3(\varepsilon)$, can be issued at $t = 2$ by the high-income agent against next period’s low-income realization. This omission is intentional, as its level is irrelevant in the equilibrium under study—agents save against their low income. Consequently, without loss of generality, we can set $\tilde{d}_3(\varepsilon)$ equal to its not-too-tight level, given the future debt limits—all of which are d^{lf} for $t > 3$. The characterization result of [Hellwig and Lorenzoni \(2009\)](#) then applies, so that $\tilde{d}_3(\varepsilon) = q_4(\varepsilon)d^{\text{lf}}$.

⁵See the supplemental material of [Martins-da-Rocha and Santos \(2019\)](#) for detailed proof.

⁶The market transversality condition differs from the individual transversality condition. Indeed, due to the lack of commitment, agent i ’s debt limits may bind, in which case we do not necessarily have that $p(s^t) = \beta^t \pi(s^t)u'(c^i(s^t))/u'(c^i(s^0))$.

The low-income agent at period $t = 1$ borrows up to the not-too-tight limit $d_2(\varepsilon)$ against next period's high-income realization. Since this agent will have high income at $t = 2$, from her perspective, borrowing capacity is constrained by the sequence of not-too-tight debt limits $(d_2(\varepsilon), \tilde{d}_3(\varepsilon), d^{\text{lf}}, \dots, d^{\text{lf}}, \dots)$. Again, [Hellwig and Lorenzoni \(2009\)](#) applies, so that

$$d_2(\varepsilon) = q_3(\varepsilon)\tilde{d}_3(\varepsilon) = q_3(\varepsilon)q_4(\varepsilon)d^{\text{lf}}.$$

Figure [D.1\(a\)](#) plots the functions $\varepsilon \mapsto d_2(\varepsilon)$ and $\varepsilon \mapsto q_3(\varepsilon)\tilde{d}_3(\varepsilon)$ to demonstrate their equivalence. We also plot in Figure [D.1\(b\)](#) the function $\varepsilon \mapsto d_3(\varepsilon)$ and $\varepsilon \mapsto q_4(\varepsilon)d^{\text{lf}}$ to illustrate that $d_3(\varepsilon)$ is too tight.

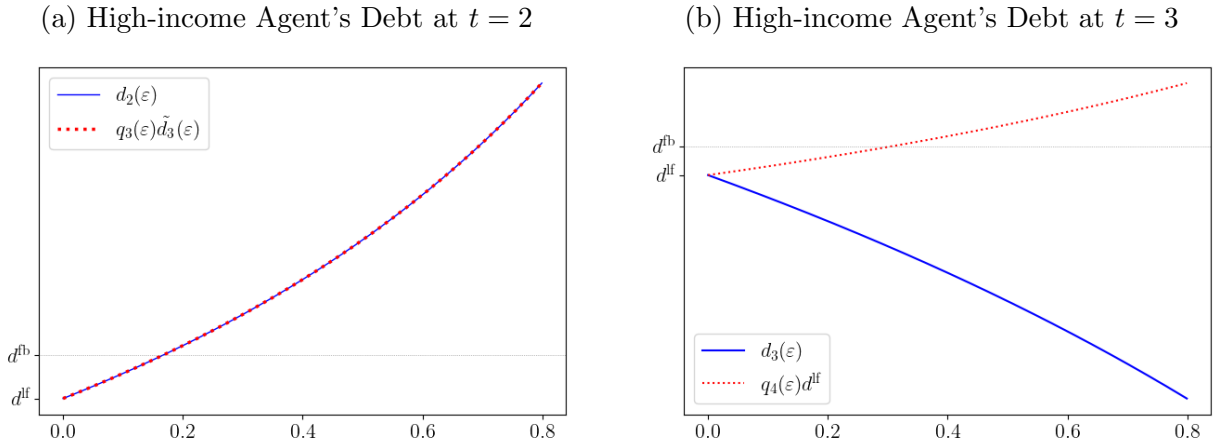


Figure D.1: Debt Limits and Weak Rollover

Let $\tilde{d}_2(\varepsilon)$ denote the debt limit restricting how much debt the high-income agent at period $t = 1$ can issue against next period's low-income realization. Following the same reasoning as before, we can set $\tilde{d}_2(\varepsilon)$ to its not-too-tight level. Notice that we cannot appeal to [Hellwig and Lorenzoni \(2009\)](#) to determine its level as we did with $\tilde{d}_3(\varepsilon)$ since, at period $t = 2$, this agent is constrained by $d_3(\varepsilon)$ that is too-tight. To compute $\tilde{d}_2(\varepsilon)$, we have to solve for the low-income agent's continuation utility when she starts at $t = 2$ with asset holdings equal to $-\tilde{d}_2(\varepsilon)$. Our educated guess is: at $t = 2$ she borrows up to her debt limit $d_3(\varepsilon)$ contingent to next period's high-income realization. Her continuation utility is then

$$u(y_L - \tilde{d}_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon)) + \beta u(y_H - x_3(\varepsilon)) + \beta^2 u(y_L + x^{\text{lf}}) + \beta^3 U_H^{\text{lf}}.$$

This educated guess is correct if, and only if, the first order condition at $t = 2$ is valid, i.e.,

$$q_3(\varepsilon) \geq \beta \frac{u'(y_H - x_3(\varepsilon))}{u'(y_L - \tilde{d}_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon))}.$$

The above inequality is satisfied for $\varepsilon > 0$ small enough as it is satisfied with a strict inequality when $\varepsilon = 0$. To compute the default value $V_{2,L}(\varepsilon)$ of low-income agent at $t = 2$, we make the educated guess that $V_{2,L}(\varepsilon) = u(y_L) + V_{3,H}(\varepsilon)$, where $V_{3,H}(\varepsilon)$ is the default value of the high-income agent at $t = 3$.⁷ This educated guess is correct if, and only if, the following FOC is satisfied

$$q_3(\varepsilon) \geq \beta \frac{u'(y_H - q_4(\varepsilon)\theta_4(\varepsilon))}{u'(y_L)}.$$

Again, the above inequality is satisfied for $\varepsilon > 0$ small enough as it is satisfied with a strict inequality when $\varepsilon = 0$.

As d^{lf} is not-too-tight, we have $U_H^{\text{lf}} = V_H^{\text{lf}}$. This implies that the not-too-tight debt level $\tilde{d}_2(\varepsilon)$ is determined by the equation

$$\begin{aligned} u(y_L - \tilde{d}_2(\varepsilon) + q_3(\varepsilon)d_3(\varepsilon)) + \beta u(y_H - x_3(\varepsilon)) + \beta^2 u(y_L + x^{\text{lf}}) = \\ u(y_L) + \beta u(y_H - q_4(\varepsilon)\theta_4(\varepsilon)) + \beta^2 u(y_L + \theta_4(\varepsilon)), \quad (\text{D.1}) \end{aligned}$$

where we recall that $\theta_4(\varepsilon) \geq 0$ is the optimal saving decision out of equilibrium of the high-income agent at date $t = 4$. It is characterized by the following FOC

$$q_4(\varepsilon) = \beta \frac{u'(y_L + \theta_4(\varepsilon))}{u'(y_H - q_4(\varepsilon)\theta_4(\varepsilon))} \quad \text{and} \quad q^{\text{lf}} \geq \beta \frac{u'(y_H - x^{\text{lf}})}{u'(y_L + \theta_4(\varepsilon))}.$$

To illustrate that not-too-tight limits cannot necessarily be rolled over, we plot in Figure D.2(a) the functions $\varepsilon \mapsto \tilde{d}_2(\varepsilon)$ and $\varepsilon \mapsto q_3(\varepsilon)d_3(\varepsilon)$. To reinforce this feature, Figure D.2(b) plots the functions $\varepsilon \mapsto d_1(\varepsilon)$ and $\varepsilon \mapsto q_2(\varepsilon)\tilde{d}_2(\varepsilon)$, showing that they do not coincide. The interesting implication is that the converse of Proposition 2.1 is not necessarily true: self-enforcing debt limits need not satisfy weak rollover.

⁷Recall that $V_{3,H}(\varepsilon) = u(y_H - q_4(\varepsilon)\theta_4(\varepsilon)) + \beta u(y_L + \theta_4(\varepsilon)) + \beta^2 V_H^{\text{lf}}$.

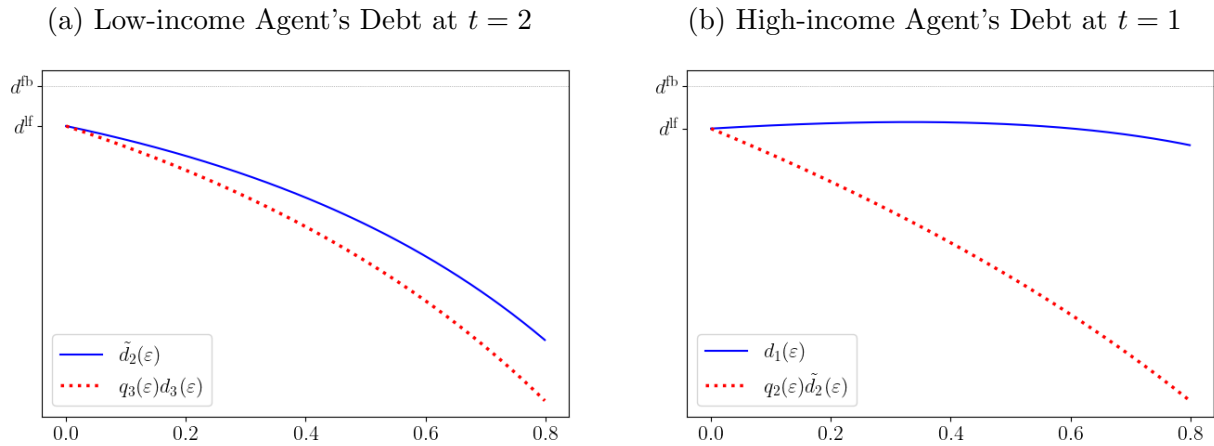


Figure D.2: Debt Limits and Weak Rollover

References

- Alvarez, F. and Jermann, U. J.: 2000, Efficiency, equilibrium, and asset pricing with risk of default, *Econometrica* **68**(4), 775–797.
- Hellwig, C. and Lorenzoni, G.: 2009, Bubbles and self-enforcing debt, *Econometrica* **77**(4), 1137–1164.
- Martins-da-Rocha, V. F. and Santos, M.: 2019, Self-enforcing debt and rational bubbles. Available at SSRN: <https://ssrn.com/abstract=3169229>.
- Martins-da-Rocha, V. F. and Vailakis, Y.: 2017, Borrowing in excess of natural ability to repay, *Review of Economic Dynamics* **23**, 42–59.