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Closure Results for Arbitrarily Partitionable Graphs

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Abstract

A well-known result of Bondy and Chvátal establishes that a graph of order n is Hamiltonian if and only if its n -closure (obtained through repeatedly adding an edge joining any two non-adjacent vertices with degree sum at least n) also is. In this work, we investigate such closure results for arbitrarily partitionable graphs, a weakening of Hamiltonian graphs being those graphs that can be partitioned into arbitrarily many connected graphs of arbitrary orders. Among other results, we establish closure results for arbitrary partitions into connected graphs of order at most 3, for arbitrary partitions into connected graphs of order exactly any λ , and for the property of being arbitrarily partitionable in full.

Keywords: connected partition; arbitrarily partitionable graph; closure; traceability.

1. Introduction

Let $n \geq 1$ be a positive integer. An n -graph G is a graph of order n , while an n -partition $\pi = (\lambda_1, \dots, \lambda_p)$ is a partition of n (i.e., $\lambda_1 + \dots + \lambda_p = n$). A partition (S_1, \dots, S_p) of the vertex set $V(G)$ of G is called a *realisation* of π in G if each part S_i is *connected* and has cardinality λ_i , that is, if, for every $i \in \{1, \dots, p\}$, the graph $G[S_i]$ is a connected graph of order λ_i . We now say G is *arbitrarily partitionable* (AP) if every n -partition is realisable in G , or, in other words, if G can be partitioned into arbitrarily many connected graphs with arbitrary orders. All these notions have been introduced and considered independently by Barth, Baudon, and Puech in [1], and by Horňák and Woźniak in [13], in connection, in particular, with a practical network sharing problem. Since then, quite some interesting aspects of AP graphs have been introduced and investigated in literature, see e.g. the latest references [3, 4, 5, 6, 8, 10, 15] on the topic for further information.

One of the main sources of motivation behind the investigations in the current work, is the fact that APness can be perceived both as a strengthening of *perfect matchings* (sets of $\lfloor n/2 \rfloor$ pairwise disjoint edges¹) and a weakening of *Hamiltonian paths* (paths going through all n vertices exactly once). Indeed, note that an AP n -graph, depending on the parity of n , admits realisations of $(1, 2, \dots, 2)$ or $(2, \dots, 2)$, which form perfect matchings; and that every path is obviously AP, from which we get that any traceable graph, i.e., any graph admitting a Hamiltonian path, is AP. From this, interesting questions on AP graphs arise when wondering how classical results on perfect matchings and Hamiltonian paths adapt to AP graphs. This line of research was initiated by Marczyk, who, in [16], proved:

Theorem 1.1 (Marczyk [16]). *Let G be a connected n -graph in which the degree sum of any two non-adjacent vertices is at least $n - 2$. Then G is AP if and only if G admits a perfect matching.*

¹For convenience, unless stated otherwise, the order of any graph is denoted by n throughout.

30 Clearly, Theorem 1.1 stands as a weakening, to APness, of the well-known sufficient
 31 condition by Ore for graphs to admit Hamiltonian paths or cycles (stating that any con-
 32 nected n -graph in which the degree sum parameter is at least n or $n-1$ admits Hamiltonian
 33 cycles or paths, respectively [18]). A remarkable fact also, is that the statement of Theo-
 34 rem 1.1 involves the two notions (perfect matchings and Hamiltonian paths) between which
 35 APness is comprised; although these two notions are rather distant in general, this shows
 36 there are stronger connections between the two in the context of AP graphs.

37 Marczyk’s Theorem 1.1 opened the way to an interesting line of research on AP graphs,
 38 being to investigate how well-known sufficient conditions for Hamiltonicity or traceability
 39 weaken to APness. In the very line of Theorem 1.1, better results involving the degree sum
 40 of pairs of non-adjacent vertices were established in [12, 17], and such results for triples of
 41 pairwise non-adjacent vertices were considered in [3]. In [14], sufficient conditions in terms
 42 of number of edges have also been established, while toughness properties of AP graphs
 43 have been studied in [5]. Last, in [6], the authors considered several other classical concepts
 44 and results for Hamiltonicity and traceability, such as forbidden induced patterns and the
 45 square operation, and proved that some adapt to APness while some others do not.

46 In the current work, we pursue this line of research by considering yet another classical
 47 aspect borrowed from the study of Hamiltonian and traceable graphs, being that of **graph**
 48 **closures**. Recall that, for a graph G and some $k \geq 1$, the k -closure $\text{clos}(G, k)$ of G is
 49 obtained upon repeatedly adding an edge uv between two non-adjacent vertices u and v
 50 satisfying $d(u) + d(v) \geq k$. Equivalently, note that the k -closure $\text{clos}(G, k)$ of G is obtained
 51 through a k -closing sequence G_0, \dots, G_m , being a sequence of n -graphs where $G_0 = G$ and
 52 $G_m = \text{clos}(G, k)$, and, for every $i \in \{1, \dots, m\}$, the graph G_i is obtained from G_{i-1} by
 53 adding a single edge uv such that $d_{G_{i-1}}(u) + d_{G_{i-1}}(v) \geq k$.

54 Graph closures emerged in a seminal work [7] of Bondy and Chvátal, in which they pro-
 55 vided the celebrated Bondy-Chvátal sufficient condition for Hamiltonicity and traceability:

56 **Theorem 1.2** (Bondy, Chvátal [7]). *An n -graph is Hamiltonian if and only if its n -closure*
 57 *is Hamiltonian. Likewise, an n -graph is traceable if and only if its $(n-1)$ -closure is traceable*

58 Among other remarkable aspects of interest, Bondy and Chvátal’s Theorem 1.2 is im-
 59 portant in that it implies several other classical sufficient conditions for Hamiltonicity and
 60 traceability, such as Dirac’s conditions [11] and Ore’s conditions [18]. It is also worth men-
 61 tioning that the concept of graph closure does not restrict to Hamiltonian cycles and paths
 62 only, and can also be employed to express sufficient conditions for graphs to admit other
 63 types of objects. For more details on this point, and more generally on anything related
 64 to graph closures, we refer the interested reader to the survey [9].

65 Our main intent in the current work is to initiate the study of graph closures in the
 66 very context of AP graphs. In particular, as explained earlier when introducing Marczyk’s
 67 Theorem 1.1, given any sufficient condition for Hamiltonicity and traceability, it is natural
 68 to wonder whether it weakens to APness. Bondy and Chvátal’s Theorem 1.2 being one of
 69 the most influential results on graph closures, it is thus legitimate to wonder how it adapts
 70 (or not) to APness. As a first step, we consider $(n-2)$ -closures in Section 2, showing
 71 that, for APness, we cannot just weaken Theorem 1.2 to $(n-2)$ -closures. Our arguments
 72 lead us to considering arbitrary partitions into connected graphs of order at most 3 in
 73 Section 3, for which we prove the tight result that it is necessary and sufficient to consider
 74 $(n-1)$ -closures. Then, in Section 4, we consider weaker closures, and establish more general
 75 results for partitions into connected graphs of order any fixed λ , and for the AP property
 76 in full. We finish off in Section 5 with discussions for further work on the topic.

77 **2. On the APness of $(n - 2)$ -closures**

78 Given how sufficient conditions on degree sums of pairs of non-adjacent vertices essen-
 79 tially weaken from n for Hamiltonicity to $n - 1$ for traceability (Ore's Theorem) and to
 80 $n - 2$ for APness (Marczyk's Theorem 1.1), a first, legitimate question is whether Bondy
 81 and Chvátal's Theorem 1.2 weakens the same way to APness, or, in other words, whether
 82 n -graphs are AP if and only if their $(n - 2)$ -closure is. Likewise, just like how Bondy and
 83 Chvátal's Theorem 1.2 implies Ore's Theorem, one can also wonder whether there is some
 84 result on APness and $(n - 2)$ -closures that would imply Marczyk's Theorem 1.1.

85 Previous studies on AP graphs have highlighted that, when realising an n -partition
 86 π in an n -graph, an important parameter to take into account is the *spectrum* $\text{sp}(\pi)$
 87 of π , which is the set of all values appearing in π (thus, $\text{sp}(\pi)$ is essentially π with all
 88 duplicates removed). Perhaps one of the most significant results on partition spectra is one
 89 by Ravoux stating that, in graphs with large diameter, the AP property relies solely on the
 90 realisability of partitions with small spectrum [20]. More generally speaking, as described
 91 in [5], something we learn from previous works on AP graphs is that, 1) the smaller the
 92 spectrum of a partition π is, and 2) the smaller the values of π are, the less chances there
 93 are that π is realisable in a given graph. In other words, to establish that some graphs with
 94 particular properties are not necessarily AP, very generally speaking one should consider
 95 the realisability of partitions with small values and low variety of values.

96 From these thoughts, regarding the hopes we have exposed for $(n - 2)$ -closures earlier,
 97 we get to the matter of establishing whether, in general, n -partitions with small spectrum
 98 are realisable if and only if they are in the $(n - 2)$ -closure. As a starting point, it thus
 99 makes sense to wonder about perfect matchings², or, in other words, about realisations of
 100 partitions with spectrum $\{2\}$. Below, we prove that, indeed, already for this particular
 101 type of partitions we cannot weaken Bondy and Chvátal's Theorem 1.2 to value $n - 2$ as
 102 is. For transparency, let us mention that upcoming Theorem 2.3 is actually a particular
 103 case of a previous result of Plummer and Saito [19]; still, we provide a straight, thorough
 104 proof, as our arguments stand as a good introduction to the ones to be used later on.

105 To begin with, let us remind Tutte's condition for the existence of perfect matchings.

106 **Theorem 2.1** (Tutte [21]). *A graph G with even order has a perfect matching if and only*
 107 *if for every $S \subseteq V(G)$ the graph $G - S$ has at most $|S|$ connected components with odd order.*

108 We now proceed with our first result, Theorem 2.3. Before that, we just need to
 109 introduce an auxiliary result that will show Theorem 2.3 is indeed the best we can hope
 110 for. Recall that for any two positive integers $p, q \geq 1$, we denote by $K_{p,q}$ the complete
 111 bipartite graph with partition classes of cardinality p and q .

112 **Lemma 2.2.** *For any $p \geq 1$, $\text{clos}(K_{p,p+2}, n - 2)$ has perfect matchings while $K_{p,p+2}$ has not.*

113 *Proof.* Set $G = K_{p,p+2}$ for some $p \geq 1$, and denote by U the partition class of G with
 114 cardinality p , and by V that with cardinality $p + 2$. Note that all vertices of U have degree
 115 $|V| = p + 2$, while all vertices of V have degree $|U| = p$. Set $n = |V(G)| = 2p + 2$.

116 First off, note that $G - U$ consists of $p + 2$ connected components, all of which have odd
 117 order, 1. Meanwhile, $|U| = p$. Thus, by Tutte's Theorem, G admits no perfect matchings.

118 Now consider $G' = \text{clos}(G, n - 2)$. Note that, in G , any two non-adjacent vertices u and
 119 v either both belong to U , or both belong to V . Also, if $u, v \in U$, then $d_G(u) + d_G(v) =$
 120 $2(p + 2) = 2p + 4 = n + 2$; while, if $u, v \in V$, then $d_G(u) + d_G(v) = 2p = n - 2$. Thus, for any two

²From here on, perfect matchings are only considered for graphs with even order.

121 non-adjacent vertices u and v of G , we have $d_G(u) + d_G(v) \geq n - 2$, which implies that u
 122 and v are adjacent in G' . Thus G' is complete, and hence it admits perfect matchings. \square

123 **Theorem 2.3.** *Every graph G with even order $n \geq 2$ admits perfect matchings if and only*
 124 *if $\text{clos}(G, n - 1)$ does. Besides, value $n - 1$ in the previous statement is best possible.*

125 *Proof.* Set $G' = \text{clos}(G, n - 1)$. Clearly, if G admits perfect matchings, then so does G' .
 126 Thus, it remains to prove that if G' admits perfect matchings, then so does G . Consider
 127 an $(n - 1)$ -closing sequence G_0, \dots, G_m where $G_0 = G$ and $G_m = G'$. It suffices to prove
 128 that, for any $i \in \{1, \dots, m\}$, from any perfect matching of G_i we can deduce one of G_{i-1} .
 129 Set thus $H = G_{i-1}$ and $H' = G_i$, and let M be a perfect matching of H' .

130 If M is a perfect matching of H , then we are done. Otherwise, it means M contains
 131 uv , the unique edge of H' not in H . Then, by definition, $d_H(u) + d_H(v) \geq n - 1$. Now, for
 132 every neighbour $w \neq v$ of u in H , note that if $uw' \in M$ for some $w' \notin \{u, v, w\}$, and we have
 133 $vw' \in E(H)$, then $M \setminus \{uw, ww'\} \cup \{uw, vw'\}$ is a perfect matching of H . If there is no such
 134 configuration in H , then it means that, for every edge $uw \in E(H)$ such that $uw' \in M$, we
 135 cannot have $vw' \in E(H)$. From this, we deduce that $d_H(v) \leq n - 2 - d_H(u)$, and thus that
 136 $d_H(u) + d_H(v) \leq n - 2$, contradicting that uv is an edge of H' . Thus we can always derive
 137 a perfect matching of H from M , and we are done.

138 The last part of the statement follows *e.g.* from Lemma 2.2. \square

139 Due to Theorem 2.3, it is not true that any n -graph is AP if and only if its $(n - 2)$ -
 140 closure is AP, which would have stood as a smooth and natural weakening of Bondy and
 141 Chvátal's Theorem 1.2 to APness. The fact that perfect matchings are an obstruction to
 142 such a result is actually not that surprising either, as, recall, Marczyk's Theorem 1.1, to
 143 weaken Ore's Theorem from $n - 1$ to $n - 2$, already had to exclude such partitions from the
 144 equation. In the present case, however, things are a bit different, as we can actually adapt
 145 Theorem 2.3 to partitions with spectrum $\{3\}$, see upcoming Theorem 2.5.

146 As earlier, we start off by introducing some construction that will show the next result
 147 is best possible. In the present case, the construction actually holds for any size value
 148 $\lambda \geq 2$, although upcoming Theorem 2.5 deals only with the particular case where $\lambda = 3$.
 149 For any three positive integers $p, q, r \geq 1$, we denote by $T(p, q, r)$ the graph obtained from
 150 the disjoint union of three cliques K_p, K_q , and K_r on p, q , and r vertices, respectively, by
 151 adding a universal vertex v . Note that v is a cut-vertex of $T(p, q, r)$, whose removal results
 152 in exactly three connected components, being K_p, K_q , and K_r .

153 **Theorem 2.4.** *For every $\lambda \geq 2$ and $p \geq 1$ with $p \equiv \lambda - 1 \pmod{\lambda}$, $\text{clos}(T(1, p, p), n - 2)$ has*
 154 *realisations of $(\lambda, \dots, \lambda)$ while $T(1, p, p)$ has not.*

155 *Proof.* For any $\lambda \geq 2$ and $p \geq 1$ with $p \equiv \lambda - 1 \pmod{\lambda}$, set $G = T(1, p, p)$ and $n = |V(G)| =$
 156 $2p + 2$. Since G has a cut-vertex v whose removal results in three connected components,
 157 $K, K',$ and K'' , having order 1, p , and p , respectively, and $p \equiv \lambda - 1 \pmod{\lambda}$, it should be
 158 clear that, in any realisation \mathcal{R} of $(\lambda, \dots, \lambda)$ in G , there must be one part containing v and
 159 $\lambda - 1$ vertices of K' , and similarly one part containing v and $\lambda - 1$ vertices of K'' , which is
 160 impossible. Thus \mathcal{R} cannot exist, and G does not admit any realisation of $(\lambda, \dots, \lambda)$.

161 Let us now consider $G' = \text{clos}(G, n - 2)$. Note that, in G , all vertices of K' and K''
 162 have degree $p = \frac{n-2}{2}$, vertex v has degree $n - 1$, and the unique vertex of K has degree 1.
 163 Thus, in G' , any two vertices of K' and K'' are adjacent. This implies G' is traceable, from
 164 which we get that G' is AP, and thus admits realisations of $(\lambda, \dots, \lambda)$, as claimed. \square

165 **Theorem 2.5.** *Every graph G of order $n \equiv 0 \pmod{3}$ at least 3 admits realisations of*
 166 *$(3, \dots, 3)$ if and only if $\text{clos}(G, n-1)$ does. Besides, value $n-1$ in the previous state-*
 167 *ment is best possible.*

168 *Proof.* Set $G' = \text{clos}(G, n-1)$. As in the proof of Theorem 2.3, we focus on an $(n-1)$ -
 169 closing sequence G_0, \dots, G_m where $G_0 = G$ and $G_m = G'$, our goal being to prove that, for
 170 any $i \in \{1, \dots, m\}$, any realisation \mathcal{R} of $(3, \dots, 3)$ in some $H' = G_i$ can be turned into one
 171 of $H = G_{i-1}$. Assume thus \mathcal{R} does not hold as is in H ; thus, there is a part $S = \{u, v, w\}$ of
 172 \mathcal{R} that contains the unique edge vw of H' not in H , thereby making $H[S]$ not connected.
 173 Then we can assume $H[S]$ contains uv , while w is isolated in $H[S]$.

174 Let n_3, n_2 , and n_1 be the number of parts of \mathcal{R} different from S in which, in H , vertex v
 175 has exactly three, two, or one neighbour, respectively. Then $d_H(v) = 1 + 3n_3 + 2n_2 + n_1$. We
 176 now analyse the number of possible edges incident to w going to some part $S' = \{x, y, z\}$
 177 of \mathcal{R} different from S in which v also has neighbours (all these adjacencies being in H).

- 178 • If v is adjacent to all of x, y , and z , then we claim a realisation of $(3, \dots, 3)$ in H can
 179 be deduced if w , in H , is also adjacent to any vertex of S' . Indeed, assume, w.l.o.g.,
 180 that wx is an edge of H . Since $H[S']$ is connected, H' (and thus H) necessarily
 181 contains one of xy or xz . In the former case, replacing S and S' in \mathcal{R} with $\{u, v, z\}$
 182 and $\{w, x, y\}$ results in a realisation of $(3, \dots, 3)$ in H . In the latter case, we can
 183 instead replace parts S and S' with $\{u, v, y\}$ and $\{w, x, z\}$.
- 184 • If v is adjacent only to, say, x and y in S' , then we claim a realisation of $(3, \dots, 3)$
 185 in H can be deduced if w , in H , is also adjacent to any two vertices in S' . First off,
 186 note that, for similar arguments as in the previous case, we would be done if these
 187 at least two neighbours of w in S' include z . So suppose w is only adjacent to x and
 188 y . Since $H'[S']$ is connected, we have that xz or yz lies in H' (and thus in H). In
 189 the former case, replace, in \mathcal{R} , parts S and S' with $\{u, v, y\}$ and $\{w, x, z\}$ to get a
 190 realisation in H . In the latter case, replace S and S' with $\{u, v, x\}$ and $\{w, y, z\}$.
- 191 • If v is adjacent only to, say, x in S' , then, for similar reasons as earlier, we claim a
 192 realisation of $(3, \dots, 3)$ in H can be obtained from \mathcal{R} if w is adjacent, in H , to all
 193 three vertices of S' . This is because, in that case, replacing, in \mathcal{R} , parts S and S'
 194 with $\{u, v, x\}$ and $\{w, y, z\}$ yields a desired realisation.

Now, if none of the situations above occurs, then we deduce that

$$d_H(w) \leq n - 3 - 3n_3 - 2n_2 - n_1 = n - 2 - (1 + 3n_3 + 2n_2 + n_1) = n - 2 - d_H(v),$$

195 and hence $d_H(w) \leq n - 2 - d_H(v)$, and $d_H(v) + d_H(w) \leq n - 2$, a contradiction.

196 To conclude the proof, remark that *e.g.* Theorem 2.4 shows the second part of the
 197 statement also holds true. □

198 Although the first part of the statement of Theorem 2.5 holds only for partitions con-
 199 taining value $\lambda = 3$ only, Theorem 2.4 implies that if the first part of Theorem 2.5 also held
 200 for any $\lambda \geq 4$, then value $n-1$ would be best possible. Anyhow, Theorem 2.5 shows that
 201 Bondy and Chvátal's Theorem 1.2 does not weaken to APness by just considering threshold
 202 $n-2$, and this is not due to perfect matchings only, as Theorem 2.3 could indicate.

203 3. Connected partitions into parts of size at most 3

204 In previous Theorems 2.3 and 2.5, we were only concerned with connected partitions
 205 into parts all having the same size, λ , for very small values of λ , namely 2 and 3. To get
 206 a flavour of what it would take, with the same approach, to generalise our arguments to
 207 any partition, in the next result we consider a generalisation of Theorems 2.3 and 2.5 to
 208 partitions with spectrum included in $\{1, 2, 3\}$, or, in other words, to connected partitions
 209 into parts of size at most 3. As will be recalled right after the proof, such a result is not
 210 interesting only for generalisation purposes, but also because such partitions, in previous
 211 works on APness, have been proved to be of prime interest for certain graph classes.

212 **Theorem 3.1.** *For every $n \geq 2$ and every n -partition π with spectrum included in $\{1, 2, 3\}$,
 213 every graph G of order n admits realisations of π if and only if $\text{clos}(G, n-1)$ does. Besides,
 214 value $n-1$ in the previous statement is best possible.*

215 *Proof.* The proof goes similarly as the proofs of Theorems 2.3 and 2.5. Since one of the two
 216 directions is obvious, we focus on proving the less straight direction. Set $G' = \text{clos}(G, n-1)$,
 217 and let π be an n -partition with spectrum included in $\{1, 2, 3\}$. We consider an $(n-1)$ -
 218 closing sequence G_0, \dots, G_m where $G_0 = G$ and $G_m = G'$. Again, it suffices to prove that,
 219 for any $i \in \{1, \dots, m\}$, any realisation of π in G_i yields one of G_{i-1} . So, we consider
 220 $H = G_{i-1}$ and $H' = G_i$, and let \mathcal{R} be a realisation of π in H' . If \mathcal{R} does not hold directly
 221 in H , then it is because of a part S containing the unique edge of H' that does not belong
 222 to H (that is, $H[S]$ is not connected while $H'[S]$ is). Then, $|S| \in \{2, 3\}$.

223 To begin with, we assume S has cardinality 2, and set $S = \{u, v\}$. Then, $uv \in E(H')$ and
 224 $uv \notin E(H)$. As in the proofs of Theorems 2.3 and 2.5, we analyse the possible neighbours
 225 of v in H w.r.t. those of u .

- 226 • If u has, in H , a neighbour in a part S' of cardinality 2 of \mathcal{R} different from S , and v
 227 is adjacent to the other vertex of S' , then, through the exact same arguments as in
 228 the proof of Theorem 2.3, we can deduce, from \mathcal{R} , a realisation of π in H .
- 229 • If u is adjacent, in H , to a vertex w such that $S' = \{w\}$ is a part of \mathcal{R} , then note
 230 that, from \mathcal{R} , replacing S and S' with $\{u, w\}$ and $\{v\}$ results in a realisation in H .
- 231 • Assume last that u has, in H , neighbours in a part $S' = \{w, x, y\}$ of cardinality 3.
 - 232 – If u is adjacent to all of w, x , and y , and v is also adjacent to some vertex of
 233 S' , then a realisation of π in H can be obtained from \mathcal{R} as follows. Assuming,
 234 w.l.o.g., that v is adjacent to w , then just replace parts S and S' with $\{u, x, y\}$
 235 and $\{v, w\}$. This indeed results in a desired realisation.
 - 236 – If u is adjacent, in H , to exactly two vertices in S' , say w and x w.l.o.g., then we
 237 claim a realisation of π in H can be deduced from \mathcal{R} in case v is also adjacent
 238 to two vertices of S' in H . We distinguish two cases.
 - 239 * Assume first that v is adjacent to y . In that case, it suffices to replace parts
 240 S and S' of \mathcal{R} with $\{u, w, x\}$ and $\{v, y\}$.
 - 241 * Assume second that v is adjacent to w and x (and not to y). Since $H'[S']$
 242 (and thus $H[S']$) is connected, then one of w and x is adjacent to y in H'
 243 (and H). Assuming w.l.o.g. that xy is an edge of H , we can here replace
 244 parts S and S' of \mathcal{R} with $\{u, w\}$ and $\{v, x, y\}$ to be done.
 - 245 – If u is adjacent, in H , to only one vertex of S' , say w w.l.o.g., and v is adjacent,
 246 in H , to all of w, x , and y , then we can obtain a desired realisation by replacing
 247 parts S and S' of \mathcal{R} with $\{u, w\}$ and $\{v, x, y\}$.

Now denote by m_1 and m_2 the number of parts of size 2 of \mathcal{R} in which u has (in H) exactly one or two neighbours, respectively, and by n_1 , n_2 , and n_3 the number of parts of size 3 in which u has (in H) exactly one, two, or three neighbours. Then

$$d_H(u) = m_1 + 2m_2 + n_1 + 2n_2 + 3n_3.$$

Also, since none of the cases above apply, we have

$$d_H(v) \leq n - 2 - m_1 - 2m_2 - n_1 - 2n_2 - 3n_3 = n - 2 - d_H(u),$$

248 and thus $d_H(u) + d_H(v) \leq n - 2$, a contradiction.

249 Now suppose $S = \{u, v, w\}$, where $H[S]$ contains the edge uv while w is isolated (because H' contains the edge vw while H does not contain it). For the same reasons as in
250
251 the proof of Theorem 2.5, we can assume that, in \mathcal{R} , there is no $S' \neq S$ of size 3 in which:

- 252 • v has three neighbours and w has one neighbour;
- 253 • v has two neighbours and w has two neighbours;
- 254 • v has one neighbour and w has three neighbours.

255 We now lead a similar analysis regarding the possible neighbours of v in H in parts $S' \neq S$
256 of \mathcal{R} of cardinality 1 or 2.

- 257 • Assume first that v has, in H , a neighbour x in a part $S' = \{x\}$ of size 1. Then a
258 realisation of π in H is obtained upon replacing S and S' with $\{u, v, x\}$ and $\{w\}$.
- 259 • Assume second that v has, in H , neighbours in a part $S' = \{x, y\}$ of cardinality 2.
260 Recall that xy is an edge of both H' and H .
 - 261 – Assume first v is adjacent to both x and y in H . If w is adjacent to any of x and
262 y in H , then a realisation of π in H is obtained from \mathcal{R} upon replacing parts S
263 and S' with either $\{u, v, x\}$ and $\{w, y\}$, or $\{u, v, y\}$ and $\{w, x\}$.
 - 264 – Assume second v is adjacent only to x in G . If w is adjacent to x and y , then a
265 realisation is obtained when replacing parts S and S' with $\{u, v, x\}$ and $\{w, y\}$.

Now set m_1 and m_2 the number of parts of cardinality 2 of \mathcal{R} in which v has (in H) exactly one or two neighbours, respectively, and n_1 , n_2 , and n_3 the number of parts of cardinality 3 in which v has (in H) exactly one, two, or three neighbours. Then

$$d_H(v) = 1 + m_1 + 2m_2 + n_1 + 2n_2 + 3n_3.$$

Meanwhile, by the arguments above, we have

$$d_H(w) \leq n - 3 - m_1 - 2m_2 - n_1 - 2n_2 - 3n_3 = n - 2 - d_H(v),$$

266 from which we deduce $d_H(v) + d_H(w) \leq n - 2$, a final contradiction.

267 The very last part of the statement follows *e.g.* from Lemma 2.2 and Theorem 2.4. \square

268 Theorem 3.1 makes more particular sense in the context of graph classes for which the
269 AP property is known to rely only on the realisability of partitions with spectrum included
270 in $\{1, 2, 3\}$. Such matters relate to questions raised first by Barth and Fournier in [2] on
271 the complexity of determining whether a graph is AP. Note that it is not clear whether
272 this problem lies in NP or co-NP, since the APness of an n -graph relies on the realisability

273 of an exponential number of n -partitions, while the non-APness of an n -graph implies an
 274 exponential number of partitions of the vertex set do not stand as realisations of particular
 275 n -partitions. Still, following the terminology from [4], it is believed that, perhaps, every
 276 n -graph G admits a *polynomial kernel of partitions*, i.e., a set K of a polynomial number
 277 (function of n) of n -partitions such that G is AP if and only if all partitions of K are
 278 realisable in G . If this was true, then this would imply deciding APness lies in NP.

279 Polynomial kernels of partitions have been proved to exist for a few graph classes, such
 280 as subdivided stars [1, 2], graphs with large diameter [20], superclasses of split graphs [6,
 281 8, 15], and others [4, 6]. In particular, the polynomial kernel from [8] for split graphs
 282 contains partitions with spectrum included in $\{1, 2, 3\}$ only, which kernel also holds for
 283 superclasses of split graphs [15], in particular for $\{2K_2, C_4\}$ -free graphs. Thus, a direct,
 284 general consequence of Theorem 3.1 and of these thoughts, is the following:

285 **Corollary 3.2.** *If \mathcal{G} is a class of graphs of order $n \geq 2$ admitting n -partitions with spectrum*
 286 *included in $\{1, 2, 3\}$ as a polynomial kernel, then every $G \in \mathcal{G}$ is AP if and only if $\text{clos}(G, n-$
 287 $1)$ is.*

288 Note that Corollary 3.2 would extend the same way to any n -graph class for which
 289 n -partitions with spectrum included in $\{1, 2, 3\}$ stand as a polynomial kernel. Recall also
 290 that building graph closures can be done in polynomial time (see e.g. [9]).

291 4. More general results for weaker closures

292 In this section, we investigate how the proof arguments from the proofs of previous
 293 Theorems 2.3, 2.5, and 3.1 could be generalised to other partitions with spectrum of size 1
 294 (Subsection 4.1) and even to APness (Subsection 4.2), provided we consider weaker closures.

295 4.1. Connected partitions into parts of size λ

296 Before proceeding with our main result here, let us establish the following useful lemma.

297 **Lemma 4.1.** *If G is an n -graph with two non-adjacent vertex u and v such that $d(u) +$
 298 $d(v) \geq n + c$, then u and v have at least $c + 2$ common neighbours. In particular, $c \leq n - 4$.*

Proof. Since u and v are not adjacent, we have $N(u) \cup N(v) \subseteq V(G) \setminus \{u, v\}$, which set
 has cardinality $n - 2$. Now, if u and v had at most $c + 1$ common neighbours only, then we
 would have

$$d(u) + d(v) \leq 2(c + 1) + (n - 2) - (c + 1) = n + c - 1,$$

299 a contradiction.

300 The last part is because having $d(u) + d(v) \geq 2n - 3$ would imply u and v have at least
 301 $n - 1$ common neighbours, which is impossible in loopless graphs. \square

302 We now essentially adapt the first part of the statement of Theorem 2.5 to partitions
 303 with spectrum $\{\lambda\}$ for any $\lambda \geq 4$. Recall that Theorem 2.4 is precisely about such partitions,
 304 and thus indicates that, even in this context, we must at least consider $(n - 1)$ -closures.

305 **Theorem 4.2.** *For every $\lambda \geq 4$, every graph G of order $n \equiv 0 \pmod{\lambda}$ at least λ admits*
 306 *realisations of $(\lambda, \dots, \lambda)$ if and only if $\text{clos}(G, 2n - (\frac{2n}{\lambda} + \lambda - 1))$ does.*

307 *Proof.* Set $\alpha = 2n - (\frac{2n}{\lambda} + \lambda - 1)$, and $G' = \text{clos}(G, \alpha)$. Consider an α -closing sequence
 308 G_0, \dots, G_m where $G_0 = G$ and $G_m = G'$. It again suffices to prove that any realisation \mathcal{R}
 309 of $(\lambda, \dots, \lambda)$ of $H' = G_i$ for any $i \in \{1, \dots, m\}$ yields one of $H = G_{i-1}$. If \mathcal{R} does not hold

310 as is in H , then it is because a part S of \mathcal{R} is not connected in H (while it is in H'). That
 311 is, $H'[S]$ contains the only edge uv that is present in H' but not in H .

Since $H[S]$ is not connected, and $H[S] = H'[S] - uv$, vertices u and v cannot have any
 common neighbour in S . We claim there is necessarily a part $S' \neq S$ of \mathcal{R} in which u and v
 have at least $\lambda - 1$ common neighbours. Indeed, note first that the value of λ ensures that
 $n \geq 2\lambda$, as, for $n = \lambda$, we have $\alpha = \lambda - 1$, while H in the present case is not connected and
 thus $d_H(u) + d_H(v) \leq n - 2 = \lambda - 2$, a contradiction. Thus, $|\mathcal{R}| \geq 2$, and if u and v had at
 most $\lambda - 2$ common neighbours in all of the $\frac{n}{\lambda} - 1 \geq 1$ parts of \mathcal{R} different from S , then u
 and v would have at most

$$\left(\frac{n}{\lambda} - 1\right)(\lambda - 2) = n - \left(\frac{2n}{\lambda} + \lambda - 2\right)$$

312 common neighbours, while, by Lemma 4.1, vertices u and v have at least $n - (\frac{2n}{\lambda} + \lambda - 3)$
 313 common neighbours in H , a contradiction.

314 Thus, let $S' \neq S$ be any part of \mathcal{R} in which u and v have at least $\lambda - 1$ common neighbours.
 315 Let also S_u and S_v be the vertex sets of the exactly two connected components of $H[S']$,
 316 where S_u contains u while S_v contains v . In case u and v have exactly $\lambda - 1$ common
 317 neighbours in S' , then let also w be the unique vertex of S' not adjacent to both u and v ;
 318 otherwise, if u and v are both adjacent to all vertices of S' , then let w be any vertex of S' .
 319 In both cases, let w' be any neighbour of w in $H[S']$.

320 Since $\lambda \geq 4$, we must have $\lambda - |S_u| \geq 2$ or $\lambda - |S_v| \geq 2$. Assume w.l.o.g. the former
 321 inequality holds. Now replace S and S' in \mathcal{R} with both $S_u \cup \{w, w'\} \cup X$ and $S_v \cup Y$, where
 322 X is any set of $\lambda - (|S_u| + 2) \geq 0$ vertices of $S' \setminus \{w, w'\}$, and $Y = S' \setminus \{w, w'\} \setminus X$. Since
 323 ww' is an edge of H , and u and v are adjacent to all vertices in $S' \setminus \{w\}$, it can be observed
 324 that this results in a realisation of $(\lambda, \dots, \lambda)$ in H . \square

325 Note that $2n - (\frac{2n}{\lambda} + \lambda - 1)$ is a non-decreasing function of λ , so the most interesting
 326 case, yielding the stronger closure, is for $\lambda = 4$, for which case Theorem 4.2 deals with
 327 $(\frac{3}{2}n - 3)$ -closures. Recall that, regardless of the actual value of λ , in any case we cannot
 328 hope to establish results involving closures stronger than $(n - 1)$ -closures.

329 4.2. APness

330 We here consider the AP property in full, that is, we wonder about generalisations of
 331 Bondy and Chvátal's Theorem 1.2 to AP graphs. By the last part of Lemma 4.1, recall
 332 that we must consider degree sum thresholds at most $2n - 4$. In what follows, we prove that
 333 an n -graph G is indeed AP if and only if its $(2n - 4)$ -closure is. This result being surely far
 334 from optimal, we then improve it down to $(2n - 5)$ -closures and even to $(2n - 6)$ -closures,
 335 for the sake mainly of showcasing why it might be tedious to go even lower.

336 **Theorem 4.3.** *Every graph G of order $n \geq 3$ is AP if and only if $\text{clos}(G, 2n - 4)$ is.*

337 *Proof.* Set $G' = \text{clos}(G, 2n - 4)$. Again, it suffices to prove that if the $(2n - 4)$ -closure
 338 $\text{clos}(G, 2n - 4)$ is AP, then G is also AP. Actually, we can consider a $(2n - 4)$ -closing sequence
 339 G_0, \dots, G_m where $G_0 = G$ and $G_m = G'$, and prove that, for $H' = G_i$ and $H = G_{i-1}$ for some
 340 $i \in \{1, \dots, m\}$, any realisation \mathcal{R} of any n -partition π in H' yields one of H . If \mathcal{R} holds as
 341 is in H , then we are done. Otherwise, some part S of \mathcal{R} contains the only edge uv of H'
 342 not in H , and we have that $H[S]$ is not connected.

343 Let us denote by S_u and S_v the subsets of vertices of S such that S_u contains the
 344 vertices of $H[S]$ belonging to the same connected component as u , and S_v contains those
 345 belonging to the same connected component as v . Then, $S = S_u \cup S_v$, and we have $u \in S_u$

346 and $v \in S_v$. By Lemma 4.1, vertices u and v have at least $n - 2$ common neighbours in H ,
347 and, actually, since u and v are not adjacent in H , we have $N_H(u) = N_H(v) = V(H) \setminus \{u, v\}$.
348 In particular, since $H[S]$ is not connected, it must be that $|S_u| = |S_v| = 1$, as otherwise u
349 and v would have, in H , a common neighbour in either S_u or S_v , making $H[S]$ connected.
350 Thus, $|S| = 2$, and since $n - 2 \geq 1$ we have $|\pi| \geq 2$. Now, consider $S' \neq S$ another part of \mathcal{R} .
351 Then, as mentioned above, in H , vertices u and v are adjacent to all vertices of S' . It then
352 suffices, to obtain a realisation of π in H , to start from \mathcal{R} , and, denoting by w any vertex
353 of S' , to replace parts S and S' with, say, $\{u, w\}$ and $\{v\} \cup S' \setminus \{w\}$. \square

354 **Theorem 4.4.** *Every graph G of order $n \geq 4$ is AP if and only if $\text{clos}(G, 2n - 5)$ is.*

355 *Proof.* The proof goes similarly as that of Theorem 4.3 (from a $(2n - 5)$ -closing sequence).
356 This time, in H vertices u and v have at least $n - 3$ common neighbours. Since $H[S]$ is not
357 connected, vertices u and v have no common neighbours in S , which implies $2 \leq |S| \leq 3$.

358 If $|S| = 3$, then we can assume, w.l.o.g., that $S_u = \{u\}$ and $S_v = \{v, w\}$. Thus vw is
359 an edge of H , and u and v admit all vertices of $V(H) \setminus \{u, v, w\}$ as common neighbours.
360 Then, again, we can just consider any part $S' \neq S$ of \mathcal{R} (such exists since $2n - 5 \geq 3$), and,
361 denoting by x any vertex of S' , a realisation of π in H is obtained when starting from \mathcal{R} ,
362 and replacing parts S and S' with $\{v, w, x\}$ and $\{u\} \cup S' \setminus \{x\}$, respectively.

363 Now assume $|S| = 2$, i.e., S contains u and v only. If $N_H(u) = N_H(v) = V(H) \setminus \{u, v\}$,
364 then a realisation of π in H can be obtained similarly as in the proof of Theorem 4.3.
365 Hence, we can last assume u and v have only $n - 3$ common neighbours in H . Let us thus
366 denote by w the only vertex of $V(H) \setminus \{u, v\}$ not adjacent to both u and v in H . Again,
367 since $2n - 5 \geq 3$, we must have $|\pi| \geq 2$. If \mathcal{R} contains a part $S' \neq S$ not containing w , then
368 we can again reach our conclusion similarly as in the proof of Theorem 4.3. Otherwise,
369 $|\pi| = 2$, and \mathcal{R} thus contains only two parts, S and S' , where $w \in S'$. By arguments above,
370 u and v are adjacent to all vertices of $S' \setminus \{w\}$.

- 371 • If $|S'| = 1$, then $n = 3$, a case not covered by the statement.
- 372 • If $|S'| = 2$, then $S' = \{w, x\}$ where x is a common neighbour of u and v in H . In that
373 case, $n = 4$, and since $d_H(u) + d_H(v) \geq 2n - 5 = 3$, for uv to be an edge of H' it must
374 be that uw is an edge of H , w.l.o.g. Then a realisation of π in H is obtained upon
375 considering parts $\{u, w\}$ and $\{v, x\}$.
- 376 • If $|S'| \geq 3$, then consider any vertex x of $S' \setminus \{w\}$ such that $H[S' \setminus \{x\}]$ is connected.
377 Such an x can be obtained e.g. by considering a leaf different from w of a spanning
378 tree of $H[S']$. Since $|S'| \geq 3$, recall that $H[S' \setminus \{x\}]$ contains a vertex adjacent to
379 both u and v . Then a realisation of π in H is obtained from \mathcal{R} when considering
380 parts $\{u, x\}$ and $\{v\} \cup S' \setminus \{x\}$.

381 This concludes the proof. \square

382 **Theorem 4.5.** *Every n -graph G with $n \geq 5$ is AP if and only if $\text{clos}(G, 2n - 6)$ is.*

383 *Proof.* We follow the lines of the proofs of Theorems 4.3 and 4.4 (but this time considering
384 a $(2n - 6)$ -closing sequence). Here, in H , vertices u and v have at least $n - 4$ common
385 neighbours. Because $H[S]$ is not connected, vertices u and v have no common neighbours
386 in S , from which we deduce $2 \leq |S| \leq 4$. Also, keep in mind throughout that $n \geq 5$.

387 Similarly as in the proof of Theorem 4.4, if $|S| = 4$, then u and v are adjacent to all
388 vertices of $V(H) \setminus S$. If, say, $|S_u| = 1$ and $|S_v| = 3$, then, again, it suffices to consider any
389 part $S' \neq S$ of \mathcal{R} (it can be checked that such exists, as $n \geq 5$ which implies $2n - 6 \geq 4$), and,

390 denoting by w any vertex of S' (being a common neighbour of u and v in H), a realisation
 391 of π in H is obtained when replacing parts S and S' of \mathcal{R} with $S_v \cup \{w\}$ and $S' \cup \{u\} \setminus \{w\}$.
 392 Otherwise, $|S_u| = |S_v| = 2$; we consider a few cases.

- 393 • If there is a part $S' \neq S$ of \mathcal{R} with $|S'| \geq 4$, then a realisation of π in H is obtained
 394 upon replacing S and S' with $S_u \cup \{w, x\}$ (where w and x are any two vertices of S')
 395 and $S_v \cup S' \setminus \{w, x\}$.
- 396 • If there is a part $S' \neq S$ with $|S'| = 3$, then, setting $S' = \{w, x, y\}$, it here suffices to
 397 replace S and S' with, say, $S_u \cup \{w, x\}$ and $S_v \cup \{y\}$.
- 398 • If there is a part $S' \neq S$ with $|S'| = 2$, then, setting $S' = \{w, x\}$, it here suffices to
 399 replace S and S' with, say, S_u and $S_v \cup \{w, x\}$.
- 400 • If there is a part $S' \neq S$ with $|S'| = 1$, then, setting $S' = \{w\}$, it here suffices to replace
 401 S and S' with, say, $S_u \setminus \{u\}$ and $S_v \cup \{w, u\}$.

402 Thus, when $|S| = 4$, a realisation of π in H can be deduced in all cases.

403 If $|S| = 3$, then set $S = \{u, v, w\}$, where $S_u = \{u\}$ and $S_v = \{v, w\}$ (thus vw is an edge
 404 of H). If u and v are, in H , both adjacent to all vertices of $V(H) \setminus S$ (there are such,
 405 since $2n - 6 \geq 4$), then a realisation of π in H can be obtained similarly as in the proof of
 406 Theorem 4.4. Thus, we can assume there is some $x \in V(H) \setminus S$ such that all vertices of
 407 $V(H) \setminus S \setminus \{x\}$ are common neighbours of u and v . Actually, again, if there is some part
 408 $S' \neq S$ of \mathcal{R} not containing x , then a realisation can be obtained. So we get to assuming
 409 that $|\pi| = 2$, and that, besides S , the only part S' of \mathcal{R} contains x .

- 410 • If $|S'| = 1$, then $n = 4$, a case not covered by the statement.
- 411 • If $|S'| \geq 2$, then S' contains a vertex y adjacent to x being a common neighbour of u
 412 and v . Furthermore, u and v are both adjacent to all vertices of $S' \setminus \{x\}$. Then the
 413 two parts $\{u, x, y\}$ and $\{v, w\} \cup S' \setminus \{x, y\}$ form a realisation of π in H .

414 It now remains to consider when $|S| = 2$, that is, when $S = \{u, v\}$. In this case, there are
 415 at most two vertices of $V(H) \setminus \{u, v\}$ that are not common neighbours of u and v in H . We
 416 can actually suppose that there exist exactly two such vertices w and x of $V(H) \setminus \{u, v\}$
 417 that are not common neighbours, as, if there is none or only one such vertex, then we can
 418 proceed similarly as in the proof of Theorem 4.4. Likewise, if there is a part $S' \neq S$ of
 419 \mathcal{R} that contains only common neighbours of u and v in H , then a realisation can also be
 420 deduced. So, besides S , we can last assume either there is only one part S' (that is, $|\pi| = 2$,
 421 and S' contains both w and x), or there are exactly two parts S_w and S_x (that is, $|\pi| = 3$,
 422 and S_w contains w while S_x contains x).

- 423 • Assume first $|\pi| = 2$. Thus, \mathcal{R} contains a unique part $S' \neq S$, that contains both w
 424 and x . If there is a spanning tree of $H[S']$ that contains a leaf y different from w and
 425 x , then recall that that leaf is adjacent to both u and v in H , and the two parts $\{u, y\}$
 426 and $\{v\} \cup S' \setminus \{y\}$ then form a realisation of π in H (unless $S' = \{w, x, y\}$, in which
 427 case necessarily wx is an edge, and we could instead consider the two parts $\{u, v, y\}$
 428 and $\{w, x\}$). So we can now assume that any vertex of $S' \setminus \{w, x\}$ is a cut-vertex of
 429 any spanning tree of $H[S']$, which implies any spanning tree T of $H[S']$ must be a
 430 path with end-vertices w and x . In that case, a realisation of π in H is obtained upon
 431 considering parts $\{w, w'\}$ and $\{u, v\} \cup S' \setminus \{w, w'\}$, where w' is the unique neighbour
 432 of w in T . Note indeed that if this was not the case, then we would have $w' = x$, thus
 433 $|S'| = 2$ and $n = 4$, which peculiar case is not covered by the statement.

434 • Assume now $|\pi| = 3$. Thus, \mathcal{R} contains exactly two parts S_w and S_x different from S ,
435 where S_w and S_x contain w and x , respectively. Also, in H vertices u and v are both
436 adjacent to all vertices of $S_w \setminus \{w\}$ and $S_x \setminus \{x\}$. Here, assume, say, that $H[S_w]$
437 contains a spanning tree with a leaf y different from w . Then, a realisation of π in
438 H is obtained upon replacing, in \mathcal{R} , parts S and S_w with $\{u, y\}$ and $\{v\} \cup S_w \setminus \{y\}$.
439 Now, if none of $H[S_w]$ and $H[S_x]$ admits such a spanning tree, then $|S_w| = |S_x| = 1$
440 and $n = 4$, which peculiar case is not covered by the statement.

441 We are thus done in all cases. □

442 5. Conclusions

443 In this work, inspired by Bondy and Chvátal's influential Theorem 1.2, we have in-
444 vestigated closure results for connected partitions of graphs, in particular in the context
445 of the AP property which has been regarded as a weakening of Hamiltonicity. Looking
446 at Theorem 1.2, a natural first question (in view of Marczyk's Theorem 1.1) was whether
447 $(n - 2)$ -closures are necessary and sufficient for the AP property. In Section 2, we proved
448 that this is not true, because of several particular types of n -partitions. For such parti-
449 tions, we proved, in particular throughout Section 3, that it is sufficient and necessary to
450 consider $(n - 1)$ -closures. In last Section 4, we then considered other n -partitions, includ-
451 ing the whole set of n -partitions (and thus the AP property), and proved that it is here
452 sufficient to consider weaker closures (namely $(2n - c)$ -closures for some c).

453 Let us mention that the fact that $(n - 2)$ -closures do not suffice for APness (Theorems 2.3
454 and 2.5) is not too surprising, given that Marczyk's Theorem 1.1 already had to exclude
455 perfect matchings from the equation. Given all connections between the results involved,
456 the fact that perfect matchings form an obstruction to APness and $(n - 2)$ -closures can
457 thus be regarded as a counterpart. What is more surprising is that perfect matchings, in
458 our case, are not the only obstruction, recall Theorem 2.5.

459 Given the results we came up with, we thus have no obvious objection again the fact
460 that, perhaps, in general $(n - 1)$ -closures are necessary and sufficient.

461 **Question 5.1.** *Is it true that any graph is AP if and only if its $(n - 1)$ -closure is?*

462 If true, a notable consequence of Question 5.1 is that it would imply Marczyk's Theo-
463 rem 1.1, just like how Bondy and Chvátal's Theorem 1.2 implies Ore's Theorem. A way to
464 progress towards Question 5.1 could be to first generalise Theorems 4.3 to 4.5 to $(2n - c)$ -
465 closures for any fixed $c \geq 4$. However, as highlighted by the proofs of Theorems 4.3 to 4.5,
466 proceeding this way is not quite obvious. In particular, in our arguments, it is important
467 that we keep control over the structure of $H[S]$, which is less and less obvious as c in-
468 creases, but easy to do when c is very small (as in the cases we considered). Also, the fact
469 that uv is an edge of H' but not of H is a very local property, which becomes less and
470 less easy to exploit as the diameter of $H[S]$ increases (which might be the case when c
471 is large). Altogether, an annoying point is that $|S|$ being large implies u and v have lots
472 of common neighbours in $V(H) \setminus S$ but the structure of $H[S]$ might get hard to handle,
473 while $|S|$ being small implies the structure of $H[S]$ is easier to deal with but u and v have
474 less common neighbours in $V(H) \setminus S$, which is a crucial point.

475 To summarise, while some of our arguments in the proofs of Theorems 4.3 to 4.5 could
476 obviously be generalised to larger values of c , some others do not; thus, to go further, it
477 would be crucial to come up with other arguments. For instance, it might be useful to
478 exploit that if, for some c , the closure $\text{clos}(G, 2n - c)$ is AP but $H[S]$ has a very faulty
479 structure, then we could deduce other n -partitions showing that $\text{clos}(G, 2n - c)$ is not AP.

480 One has to take into account also that if c is too large w.r.t. n , then we might fall
 481 into pathological cases, as better highlighted by Theorem 4.5. Note indeed that if we
 482 consider as G the disjoint union of two edges, then $n = 4$ and $2n - 6 = 2$, while, for any
 483 two non-adjacent vertices u and v of G we have $d_G(u) + d_G(v) = 2$. Thus in that case
 484 $\text{clos}(G, 2n - 6)$ is complete and thus AP, while G is obviously not AP (consider partitioning
 485 it following $(1, 3)$). So the statement of Theorem 4.5 is actually best possible (w.r.t. n),
 486 and, as considering larger values of c , we must make sure to focus on large enough values of
 487 n only, or add additional constraints (such as focusing at least on connected graphs only).

488 Other directions of interest include other results of the type of Theorem 3.1, to establish
 489 other results of the sort of Corollary 3.2, for other polynomial kernels of partitions. As a
 490 more general perspective, we wonder whether it could be worth studying the APness of
 491 graphs w.r.t. the traceability or the Hamiltonicity of some of their closures, and *vice versa*.

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