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Welfare Theorems with Limited Commitment: Superlinear Pricing*

V. Filipe Martins-da-Rocha[†] Antonio Schwanke[‡]

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Abstract

In dynamic economic models characterized by limited commitment, participation constraints typically bind, resulting in equilibrium interest rates that may fall below growth rates. This undermines the validity of the Second Welfare Theorem under linear pricing conditions. Our research demonstrates that the First and Second Welfare Theorems can be upheld when introducing superlinear price functionals. The study unveils a novel application of nonlinear pricing in financially constrained settings and reveals an intriguing link with the ε -Nash equilibrium concept in the context of repeated games.

Keywords: Limited commitment; constrained inefficiency; First and Second Welfare Theorem, super-linear pricing.

JEL codes: D11; D15; D41; D52; D61.

1 Introduction

Financial frictions play a pivotal role in shaping the dynamics of economic models by introducing real-world complexities that smooth, frictionless models cannot capture. These

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frictions, which can manifest as transaction costs, borrowing constraints, or information asymmetries, serve as key mechanisms that influence how agents interact in financial markets. In a frictionless world, resources would be allocated optimally, and opportunities for risk-sharing would be fully exploited. However, financial frictions create barriers to such optimal allocation. For instance, they can limit the ability of households and firms to access credit, thereby affecting investment and consumption decisions. This has far-reaching implications for economic growth, income distribution, and financial stability.

Financial frictions can be modeled in various ways to capture the complexities and nuances of real-world economic interactions. The existing literature explores transaction costs, information asymmetries, credit constraints, agency problems, search and matching frictions, and liquidity constraints. Our paper is interested in modeling financial frictions arising from a lack of commitment. In many economic settings, agents cannot fully commit to future actions, and the default risk can severely limit their ability to engage in long-term contracts, affecting everything from intertemporal consumption choices to investment decisions.

A fitting context for modeling the repercussions of default is within stochastic and dynamic economic frameworks that necessitate open-ended time horizons. The work of [Kehoe and Levine \(1993\)](#) exemplifies this approach, which integrated participation constraints into the seminal static Arrow-Debreu-MacKenzie model. This model contemplates an exchange economy with fully developed contingent claims markets. In this setting, the penalty for default is perpetual autarky. Given the agents' incapacity to commit to financial agreements, contingent claims are only tradable if they are self-enforcing when compared to autarky.¹

In the absence of participation constraints, [Debreu \(1954, 1959\)](#) (see also [Aliprantis, Border and Burkinshaw \(1997\)](#)) demonstrated that the conventional First and Second Welfare Theorems are extendable to economies with infinitely many commodities. However, the Second Welfare Theorem breaks down when the consumption set is limited to plans adhering to participation constraints. This occurs because the equilibrium interest rates tied to a constrained efficient allocation may fall below growth rates.²

¹[Alvarez and Jermann \(2000\)](#) offered a sequential formulation of this static model, replacing participation constraints with self-enforcing and not-too-tight debt limits.

²There is a growing literature in Macroeconomics exploring fiscal policy in environments with persis-

Our present research shows that the findings of [Debreu \(1954\)](#) can be generalized to scenarios that lack commitment, provided that superlinear price functionals are introduced. The concept of nonlinear pricing recurs across various branches of economic literature. In infinite-dimensional economic models, nonlinear pricing is employed to address the potential lack of appropriate Riesz structure in the positive cone of commodity spaces, particularly in portfolio trading models with incomplete markets.³ Nonlinear pricing also emerges naturally in contexts such as progressive income tax tariffs, price discrimination, and Knightian uncertainty.⁴

By adopting more generalized price functionals, we can establish a version of the Second Welfare Theorem that applies to constrained efficient allocations with implied low interest rates. Moreover, the chosen superlinear pricing functional can be approximated as “close” to linear, as its *lineality space* encompasses economically relevant vectors. We also offer a brief discussion linking our findings to the ε -Nash equilibrium concept in repeated games.

We examine an infinite-horizon stochastic economy featuring a single composite good available for consumption in each period. Agents have the ability to trade contingent plans but lack the capacity to commit to their financial promises. In the event of default, the penalty is perpetual autarky: the agent is restricted to consuming their endowment and must bear the associated income risk. A consumption plan is deemed acceptable only if it meets the criterion of incentive compatibility; that is, the continuation utility under any given contingency must exceed the default value. This framework essentially extends the standard Arrow–Debreu economy, as analyzed by [Debreu \(1954\)](#) (see also [Bewley \(1972\)](#)), by imposing the additional constraint that the consumption set is limited to non-negative, incentive-compatible plans. [Kehoe and Levine \(1993\)](#) adapted the conventional notions of

tently low risk-free interest rates. We refer, among many others, to [Bassetto and Cui \(2018\)](#), [Barro \(2020\)](#), [Brunnermeier, Merkel and Sannikov \(2020\)](#), [Mehrotra and Sergeyev \(2020\)](#), [Reis \(2020\)](#), [Ball and Mankiw \(2021\)](#), [Angeletos, Collard and Dellas \(2022\)](#), and [Aguiar, Amador and Arellano \(2023\)](#). For asset pricing implications, see [Azinovic, Cole and Kübler \(2023\)](#).

³See, for instance, [Aliprantis, Tourky and Yannelis \(2001\)](#), [Aliprantis, Florenzano and Tourky \(2004\)](#) and [Aliprantis, Florenzano and Tourky \(2005\)](#).

⁴See [Berliant and Dunz \(1990\)](#), [Guesnerie and Seade \(1982\)](#) and [Beissner and Riedel \(2019\)](#).

Pareto optimality and competitive equilibrium to this commitment-free environment. A sequential version of this model was later introduced by [Alvarez and Jermann \(2000, 2001\)](#).

In a standard competitive equilibrium with a linear price functional, the present value of aggregate resources is inherently finite. This implies that interest rates, as determined by equilibrium prices, exceed the economy's growth rates, represented by aggregate endowment dynamics. This constraint is overly restrictive and undermines the validity of the Second Welfare Theorem. In support of this, [Bloise and Reichlin \(2011\)](#) constructed a deterministic economy in which a constrained efficient allocation exhibits implied interest rates lower than the economy's growth rates. Similarly, [Martins-da-Rocha and Vailakis \(2015a\)](#) presented another example within a Markovian framework.

To address this problem, we propose employing a perturbation argument. Let c denote a constrained efficient allocation of our initial economy. By artificially introducing a small quantity, $\varepsilon > 0$, of a physical asset that can serve as collateral, the allocation may no longer be constrained efficient in the ε -perturbed economy. This is because agents can leverage this additional asset to transfer consumption across time and states, thereby increasing risk-sharing opportunities. Consequently, a new consumption allocation c^ε emerges that Pareto dominates the initial allocation c . In the ε -perturbed economy, a portion of the debt becomes collateralized, rendering the Second Welfare Theorem applicable as equilibrium interest rates must be strictly larger than endowment growth rates. As a result, a linear functional φ^ε exists that implements the allocation c^ε as a competitive equilibrium (with transfers). As ε approaches zero, the consumption allocation c^ε converges to the original constrained efficient allocation c . The challenging aspect lies in deciphering the asymptotic behavior of the linear functionals φ^ε .

Consider a sequence (ε_n) of perturbation coefficients converging to zero. For any admissible direction of trade x , the sequence of market values $(\varphi^{\varepsilon_n}(x^i - e^i))$, possesses a subsequential limit on the real line. By invoking an extension of the Banach-Mazur Theorem (See Theorem 16.47 in [Aliprantis and Border \(2006\)](#) for a detailed explanation), we demonstrate that the sequence of linear functionals, $(\varphi^{\varepsilon_n})$, converges pointwise to a superlinear functional φ that implements the original consumption allocation c as a competitive equilibrium with transfers.

When the interest rates implied by the allocation c fall below growth rates, the superlinear functional φ cannot be linear. Furthermore, the pricing $\varphi(x)$ for an arbitrary net trade x may not necessarily match its present value under the implied interest rates, as the latter could be undefined. We demonstrate that $\varphi(x)$ can be formulated as the solution of a difference equation, which the present value would also satisfy if it were well-defined.

The remainder of the paper is organized as follows: Section 2 outlines the environment and defines various concepts employed throughout the paper. Section 3 introduces and examines the properties of superlinear prices, confirming the validity of the First Welfare Theorem within this framework. Section 4 presents the main result, while Section 5 briefly compares with existing literature. An illustrative example is discussed in Section 6, and Section 7 offers concluding remarks. Appendices A and B delve into Banach-Mazur limits and omitted technical details, respectively, which are crucial for substantiating the main result.

2 The Model

In the lines of [Kehoe and Levine \(1993\)](#) and [Alvarez and Jermann \(2000\)](#) (see also [Bloise and Reichlin \(2011\)](#) and [Martins-da-Rocha and Vailakis \(2015b\)](#)), we consider an infinite horizon endowment economy with a lack of commitment and self-enforcing participation constraints. Both time and uncertainty are discrete. There is a single non-storable consumption good and infinitely lived agents that share risks but cannot commit to financial contracts.

2.1 Uncertainty

We use an event tree Σ to describe time, uncertainty, and the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \dots\}$ there is a finite set $S^t \subset \Sigma$ of date- t events s^t . Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t . Event $s^{t+\tau}$ is said to follow event s^t , also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in$

$S^{t+\tau} : s^{t+\tau} \succ s^t$ denotes the collection of all date- $(t+\tau)$ events following s^t . Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : s^\tau \succeq s^t\}$. For any subset $A \subseteq \Sigma$, we denote by $\mathbf{1}_A$ the process $x = (x(s^t))_{s^t \in \Sigma}$ defined by $x(s^t) := 1$ if $s^t \in A$ and $x(s^t) := 0$ elsewhere. When $A = \{s^t\}$ is a singleton, we abuse notations and write $\mathbf{1}_{s^t}$ for $\mathbf{1}_{\{s^t\}}$. For any time period $T \geq 1$ and process $x = (x(s^t))_{s^t \in \Sigma}$, we denote by $x^{[T]}$ the T -tail of the process x defined by $x^{[T]}(s^t) = x(s^t)$ if $t \geq T$ and $x^{[T]}(s^t) = 0$ if $t < T$. We let $x_{[T]}$ denote the T -head defined by $x_{[T]}(s^t) = x(s^t)$ if $t < T$ and $x_{[T]}(s^t) = 0$ if $t \geq T$.⁵

2.2 Primitives and Commodity Space

There is a continuum of identical agents of each type i in a finite set I . Each agent i is endowed with a process $e^i = (e^i(s^t))_{s^t \in \Sigma}$ describing the positive amount $e^i(s^t) > 0$ of the consumption good available at event s^t . Let $e := \sum_{i \in I} e^i$ denote the process of aggregate resources in the economy. The commodity space is $\ell^\infty(e)$, the linear space of processes $x \in \mathbb{R}^\Sigma$ satisfying $|x| \leq \lambda e$ for some $\lambda \geq 0$.⁶ The cone of non-negative processes in $\ell^\infty(e)$ is denoted by $\ell_+^\infty(e)$.⁷ A **consumption process** is a vector $c = (c(s^t))_{s^t \in \Sigma} \in \ell_+^\infty(e)$. A collection $(c^i)_{i \in I}$ of consumption processes $c^i \in \mathbb{R}_+^\Sigma$ is called an **allocation**. It is said to be **resource feasible** if $\sum_{i \in I} c^i = e$.

Agent i ranks consumption processes according to the von Neumann-Morgenstern utility function

$$U^i(c) := \sum_{t \geq 0} (\beta^i)^t \sum_{s^t \in S^t} \pi^i(s^t) u^i(c(s^t)),$$

⁵Observe that $x = x_{[T]} + x^{[T]}$.

⁶This restriction of attention to consumption processes in $\ell^\infty(e)$ is made for expositional and mathematical ease.

⁷[Kehoe and Levine \(1993\)](#) and [Bloise and Reichlin \(2011\)](#) assume that endowments are uniformly bounded from above and choose $\ell^\infty := \ell^\infty(\mathbf{1}_\Sigma)$ as the commodity space.

where $\beta^i \in (0, 1)$ is the subjective discount factor, $\pi^i(s^t) > 0$ is agent i 's unconditional prior beliefs that the event s^t represents the history of shocks up to date t , and $u^i : \mathbb{R}_+ \rightarrow [-\infty, \infty)$ is a Bernoulli function assumed to be strictly increasing, concave, continuous on $[0, \infty)$, differentiable on $(0, \infty)$, bounded from above and satisfying Inada's condition at the origin.⁸

2.3 Individual Rationality and Constrained Efficiency

Given a date- t event s^t , we denote by $U^i(c|s^t)$ the lifetime continuation utility conditional to event s^t , defined by⁹

$$U^i(c|s^t) := u^i(c(s^t)) + \sum_{\tau \geq 1} (\beta^i)^\tau \sum_{s^{t+\tau} \succ s^t} \pi^i(s^{t+\tau}|s^t) u^i(c(s^{t+\tau})),$$

where $\pi^i(s^{t+\tau}|s^t) := \pi^i(s^{t+\tau})/\pi^i(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t . We assume that $U^i(e^i|s^0) > -\infty$ for every agent i .¹⁰ Since the Bernoulli function is bounded from above, we get that $U^i(e^i|s^t) > -\infty$ for all event s^t .

We analyze an environment where agents cannot commit to financial contracts. The punishment for default is autarky. Therefore, we should restrict attention to consumption processes $c^i \in \ell_+^\infty(e)$ that satisfy the following participation constraints

$$U(c^i|s^t) \geq U(e^i|s^t), \quad \text{for all } s^t \in \Sigma. \quad (2.1)$$

A consumption process $c^i \in \ell_+^\infty(e)$ satisfying (2.1) is said to be **(interim) individually rational**, and we denote by C^i the set of individually rational consumption processes. The set of individually rational allocations is $C := \prod_{i \in I} C^i$.

An allocation $(\tilde{c}^i)_{i \in I}$ is said to **Pareto dominate** another allocation $(c^i)_{i \in I}$ if $U(\tilde{c}^i) \geq U(c^i)$ for every agent i , with a strict inequality for at least one agent. We recall the concept of constrained Pareto efficiency introduced by [Kehoe and Levine \(1993\)](#).

⁸The function u is said to satisfy the Inada's condition at the origin if $\lim_{\varepsilon \rightarrow 0} [u^i(\varepsilon) - u^i(0)]/\varepsilon = \infty$. This property is automatically satisfied if $u^i(0) = -\infty$.

⁹Observe that $U^i(c|s^0) = U^i(c)$.

¹⁰This assumption is automatically satisfied if either $u^i(0) > -\infty$ or the allocation $(e^i)_{i \in I}$ is uniformly bounded away from zero, in the sense that there exists $\varepsilon > 0$ such that $e^i(s^t) \geq \varepsilon$ for each agent i and event s^t .

Definition 2.1. A resource feasible and individually rational allocation $(c^i)_{i \in I} \in C$ is **constrained Pareto optimal** (or constrained efficient) when there is no other resource feasible and individually rational allocation $(\tilde{c}^i)_{i \in I} \in C$ which Pareto dominates $(c^i)_{i \in I}$.

If an allocation $(c^i)_{i \in I}$ is constrained Pareto optimal, then it must be strictly positive.¹¹ In that case, we can define agent i 's marginal rate of substitution at event s^t by posing

$$\text{MRS}^i(c^i|s^t) := \beta^i \pi^i(s^t|\sigma(s^t)) \frac{(u^i)'(c^i(s^t))}{(u^i)'(c^i(\sigma(s^t)))}.$$

Given a strictly positive allocation $(c^i)_{i \in I}$, we let $p^* = (p^*(s^t))_{s^t \in \Sigma}$ be the process defined recursively by $p^*(s^0) := 1$ and

$$\frac{p^*(s^t)}{p^*(\sigma(s^t))} := \max_{i \in I} \text{MRS}^i(c^i|s^t), \quad \text{for all } s^t \succ s^0.$$

Following [Alvarez and Jermann \(2000\)](#), p^* is called the process of **implied Arrow–Debreu (AD) prices**, and we say that **implied interest rates are higher than growth rates** when the AD market value $p^* \cdot e$ is finite.¹²

2.4 Competitive Equilibrium and Linear Pricing

We analyze constrained competitive equilibria as defined in [Kehoe and Levine \(1993\)](#) (see also and [Alvarez and Jermann \(2000\)](#)). Given a functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ and a real number b^i representing an initial financial transfer, we denote by $B_{\text{AD}}^i(\varphi; b^i)$ the (unconstrained) Arrow–Debreu budget set of all nonnegative consumption processes $c^i \in \mathbb{R}_+^\Sigma$ satisfying the budget restriction $\varphi(c^i - e^i) \leq b^i$. Let $B_{\text{KL}}^i(\varphi; b^i) := B_{\text{AD}}^i(\varphi; b^i) \cap C^i$ be the Kehoe–Levine budget set where agent i is restricted to choose individually rational consumption processes. The standard definition of an Arrow–Debreu competitive equilibrium can be adapted to our environment with limited commitment.

¹¹See Proposition 2.1 in [Martins-da-Rocha and Vailakis \(2015c\)](#) for a detailed proof of this claim.

¹²The choice of the terminology is inspired by the following simplified case. Assume there exist $r > 0$ and $g > 0$ such that $p^*(s^t) = \pi(s^t)(1+r)^{-t}$ and $\sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t)e(s^{t+1}) = (1+g)e(s^t)$. In that case, $p^* \cdot e$ is finite if, and only if, $r > g$.

Definition 2.2. A **Kehoe–Levine (KL) competitive equilibrium** $(\varphi, (c^i)_{i \in I})$ is a family composed of a price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ and a resource feasible and individually rational allocation $(c^i)_{i \in I}$ for which there exists an allocation of initial transfers $(b^i)_{i \in I}$ satisfying $\sum_{i \in I} b^i = 0$ such that, for every agent i ,

(a) $c^i \in B_{\text{KL}}^i(\varphi, b^i)$, i.e., $\varphi(c^i - e^i) \leq b^i$;

(b) if \tilde{c}^i is an individually rational process with $U^i(\tilde{c}^i) > U^i(c^i)$, then $\varphi(\tilde{c}^i - e^i) > b^i$.

In this section, we analyze the properties of competitive equilibria with a linear functional. Assume $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium where the price functional φ is linear. The function φ is necessarily strictly positive in the sense that $\varphi(h) > 0$ for any nonzero and nonnegative process $h \in \mathbb{R}_+^\Sigma \setminus \{0\}$.¹³ We endow the space $\ell^\infty(e)$ with the norm $\|h\|_e$ defined as the lowest $\lambda \geq 0$ satisfying $|h| \leq \lambda e$.¹⁴ Since e belongs to $\|\cdot\|_e$ -interior of $\ell_+^\infty(e)$, the linear functional φ is $\|\cdot\|_e$ -continuous. The $\|\cdot\|_e$ -topological dual of $\ell^\infty(e)$ is denoted by $\text{ba}(e)$, and the subset of non-negative linear functionals in $\text{ba}(e)$ is denoted by $\text{ba}_+(e)$.¹⁵ For any linear functional $\varphi \in \text{ba}_+(e)$, there exists a non-negative charge ν^φ of bounded variation on the σ -algebra 2^Σ (or, equivalently, ν^φ is a finitely additive positive measure), such that $\varphi(h) = \int h_e d\nu^\varphi$ where h_e is the process in ℓ^∞ defined by $h_e(s^t) := h(s^t)/e(s^t)$. In particular, any $\varphi \in \text{ba}_+(e)$ can be decomposed as $\varphi(h) = p^\varphi \cdot h + \varphi^0(h)$ for some non-negative process p^φ satisfying $p^\varphi \cdot e < \infty$ where

$$p^\varphi \cdot h := \sum_{s^t \in \Sigma} p^\varphi(s^t) h(s^t)$$

and some non-negative purely finitely additive linear functional φ^0 .¹⁶

¹³This follows from the assumption that $\beta^i > 0$, $\pi^i(s^t) > 0$ for every $s^t \in \Sigma$ and $u^i : [0, \infty)$ is strictly increasing.

¹⁴Equivalently, we have $\|h\|_e := \sup_{s^t \in \Sigma} |h(s^t)/e(s^t)|$.

¹⁵A linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ is said to be non-negative whenever $\varphi(h) \geq 0$ for every $h \in \ell_+^\infty(e)$.

¹⁶The purely finitely additive linear functional φ^0 can be characterized as follows: it is a linear and $\|\cdot\|_e$ -continuous functional on $\ell^\infty(e)$ such that $\varphi^0(h) = \varphi^0(h^{[T]})$ where we recall that $h^{[T]}$ is the T -tail of h defined by $h^{[T]}(s^t) = h(s^t)$ if $t \geq T$ and 0 elsewhere. Observe moreover that $p^\varphi(s^t) = \varphi(\mathbf{1}_{\{s^t\}})$ for any event s^t .

Proposition 2.1. *A feasible and individually rational allocation can be implemented as a KL competitive equilibrium with linear pricing only if implied interest rates are higher than growth rates.*

Proof. If $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium, then φ is strictly positive and $\|\cdot\|_e$ -continuous. This implies that $\varphi(h) = p^\varphi \cdot h + \varphi^0(h)$ where $p^\varphi \in \mathbb{R}_{++}^\Sigma$ is a strictly positive process satisfying $p^\varphi \cdot e < \infty$ and φ^0 is some non-negative purely finitely additive linear function also called the **bubble component** of φ . We can prove that p^φ dominates the process p^* of implied AD prices, that is $p^\varphi(s^t) \geq p^*(s^t)$ for every $s^t \in \Sigma$.¹⁷ This implies that $p^* \cdot e < \infty$. \square

The above result illustrates that the Second Welfare Theorem is invalid in our environment without commitment when pricing is linear. Indeed, we can find two examples in the literature where a feasible and individually rational allocation $(c^i)_{i \in I}$ could be constrained Pareto optimal and display interest rates lower than growth rates, in the sense that $p^* \cdot e = \infty$.¹⁸

2.5 Quasi-Equilibrium and Uniform Gains To Trade

The standard argument to prove the Second Welfare Theorem with linear pricing involves applying the Convex Separation Theorem. Formally, let $(c^i)_{i \in I}$ be a constrained Pareto optimal allocation. For each agent i , denote by $P^i(c^i)$ the set of all individually rational consumption processes $\tilde{c}^i \in C^i$ satisfying $U^i(\tilde{c}^i) > U^i(c^i)$. We have $\{e\} \cap \sum_{i \in I} P^i(c^i) = \emptyset$. The set $\sum_{i \in I} P^i(c^i)$ is convex and strict monotonicity of preferences implies that $e + \sum_{i \in I} c^i$ belongs to the $\|\cdot\|_e$ interior of $\sum_{i \in I} P^i(c^i)$. Applying the Convex Separation Theorem, we deduce the existence of a nonzero continuous linear functional $\varphi \in \text{ba}(e)$ that supports each set $P^i(c^i)$ at x^i . This implies that $(\varphi, (c^i)_{i \in I})$ is **KL competitive quasi-equilibrium** in the sense that there exists an allocation $(b^i)_{i \in I}$ of initial transfers satisfying $\sum_{i \in I} b^i = 0$ such that, for every agent i ,

¹⁷We refer to Claim 3.1 in [Martins-da-Rocha and Vailakis \(2015b\)](#) for detailed proof.

¹⁸See [Bloise and Reichlin \(2011\)](#) for a deterministic example with time-varying endowments and [Martins-da-Rocha and Vailakis \(2015b\)](#) for an example in a stationary Markovian setting.

(a) $c^i \in B_{\text{KL}}^i(\varphi, b^i)$, i.e., $\varphi(c^i - e^i) \leq b^i$;

(b') if \tilde{c}^i is an individually rational process with $U^i(\tilde{c}^i) > U^i(c^i)$, then $\varphi(\tilde{c}^i - e^i) \geq b^i$.

In the definition of a KL equilibrium, the condition $\varphi(\tilde{c}^i - e^i) \geq b^i$ in (b') is replaced by $\varphi(\tilde{c}^i - e^i) > b^i$. Therefore, if $(\varphi, (c^i)_{i \in I})$ is a KL equilibrium, then it is a KL quasi-equilibrium.

In our environment with limited commitment, the difficult step consists in showing that the quasi-equilibrium $(\varphi, (c^i)_{i \in I})$ is an equilibrium. This is possible if the nonzero price functional φ is nontrivial, as defined below.¹⁹

Lemma 2.1. *Consider a linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ that is **nontrivial** in the sense that there exists an individually rational allocation $(d^i)_{i \in I}$ such that $\sum_{i \in I} \varphi(d^i - e^i) < 0$. Every KL competitive quasi-equilibrium $(\varphi, (c^i)_{i \in I})$ is actually a KL competitive equilibrium.*

Bloise and Reichlin (2011) identified a necessary and sufficient condition on primitives, called “uniform gains to trade” such that every nonzero linear functional that forms a quasi-equilibrium is necessarily nontrivial when autarky is not constrained Pareto efficient. Formally, we say that there are **uniform gains to trade** when there exists an individually rational allocation $(d^i)_{i \in I}$ and $\gamma > 0$ such that $\sum_{i \in I} (d^i - e^i) \leq -\gamma e$. It is straightforward to verify that if $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ is a linear functional such that $(\varphi, (c^i)_{i \in I})$ is a quasi-equilibrium, then $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ must be nontrivial when there are uniform gains to trade.²⁰ Reciprocally, if every quasi-equilibrium is an equilibrium when autarky is not constrained Pareto efficient, then there are uniform gains to trade. This follows from Lemma 4 in Bloise and Reichlin (2011) who proves that if the assumption of uniform gains to trade is not valid, then any feasible and individually rational allocation $(c^i)_{i \in I}$ (even those that are not constrained Pareto efficient) can be decentralized as a quasi-equilibrium $(\varphi, (c^i)_{i \in I})$ for some nonzero linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$.²¹

It follows from the above discussion that the Second Welfare Theorem is valid when there are uniform gains to trade: for every constrained Pareto efficient allocation $(c^i)_{i \in I}$, there

¹⁹The proof of Lemma 2.1 follows standard arguments and is presented in Appendix B.

²⁰Indeed, if $(\varphi, (c^i)_{i \in I})$ is a quasi-equilibrium, then φ is increasing. Since $\varphi \neq 0$, we must have $\varphi(e) > 0$.

²¹In that case, the separating linear functional φ is a pure bubble, i.e., $p^\varphi = 0$.

exists a nonzero linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ such that $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium. Moreover, the linear functional φ cannot have a bubble component and must coincide with the vector p^* of implied AD prices.²²

This paper investigates the validity of the Second Welfare Theorem when gains to trade are not uniform or when there are no gains to trade. We shall prove that we can establish the Second Welfare if we allow for more general price functionals. Naturally, we also show that our general price functionals are linear when there are uniform gains to trade.

3 Superlinear Pricing

The definition of a KL competitive equilibrium is built upon the existence of a price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ that associates a value $\varphi(z)$ to any possible net-trade $z \in \ell^\infty(e)$.²³ When there is full commitment, we can restrict attention to linear price functionals and obtain the First and Second Welfare Theorems. In our environment with limited commitment, the Second Welfare Theorem fails if we impose linearity of the price functional. To address this issue, we propose to relax the assumption of linearity. In contrast to [Beissner and Riedel \(2019\)](#) who introduced an equilibrium concept with sublinear price functionals, we follow the opposite route by considering superlinear price functionals. We also allow for infinite values. Formally, $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is a **superlinear price functional** when, for any $x, y \in \ell^\infty(e)$,

(i) $\varphi(0) = 0$;

(ii) $\varphi(\lambda x) = \lambda\varphi(x)$ for any $\lambda > 0$;²⁴

(iii) if $\{\varphi(x), \varphi(y)\} \neq \{-\infty, \infty\}$, then $\varphi(x) + \varphi(y) \leq \varphi(x + y)$.²⁵

²²See Lemma 3.2 in [Martins-da-Rocha and Vailakis \(2015b\)](#).

²³The concept of price functional is at the core of the theory of value proposed by [Debreu \(1959\)](#).

²⁴We take the convention that $\lambda\infty = \infty$ and $\lambda(-\infty) = -\infty$ for any $\lambda > 0$.

²⁵We take the convention that $\alpha + \infty = \infty$ for any $\alpha \in \mathbb{R} \cup \{\infty\}$; and $\alpha + (-\infty) = -\infty$ for any $\alpha \in \mathbb{R} \cup \{-\infty\}$.

The first argument in favor of superlinear price functionals is that it does not invalidate the First Welfare Theorem.

Theorem 3.1 (First Welfare Theorem). *If $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium with a superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, then the consumption allocation $(c^i)_{i \in I}$ is constrained Pareto efficient.*

Proof. Assume, by way of contradiction, that there exists an alternative resource feasible and individually rational allocation $(d^i)_{i \in I}$ that Pareto dominates $(c^i)_{i \in I}$. There must exist a third resource feasible and individually rational allocation $(f^i)_{i \in I}$ such that $U^i(f^i) > U^i(c^i)$ for every $i \in I$.²⁶ Since $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium, we must have $\varphi(f^i - e^i) > b^i$ for each i . In particular, $\varphi(f^i - e^i) \neq -\infty$ for each i . Superlinearity of φ implies that

$$\varphi(0) = \varphi\left(\sum_{i \in I} f^i - e^i\right) \geq \sum_{i \in I} \varphi(f^i - e^i) > \sum_{i \in I} b^i$$

and we get a contradiction since $\sum_{i \in I} b^i = 0$. □

The main contribution of this paper is to show that the Second Welfare Theorem is also valid if the price functional can be chosen to be superlinear. The smaller the set of possible decentralizing price functionals, the stronger the result. In particular, we would like price functionals to be as close as possible to linear functionals. The lineality space associated with a superlinear functional is a relevant measure of the extent of nonlinearity. Given a superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$, the **lineality space**, denoted by $\text{Lin}(\varphi)$, is the set of all vectors $v \in \ell^\infty(e)$ satisfying $\varphi(x + \lambda v) = \varphi(x) + \lambda \varphi(v)$ for any $\lambda \in \mathbb{R}$ and any vector $x \in \ell^\infty(e)$. It is straightforward to verify that $\text{Lin}(\varphi)$ is a vector subspace of $\ell^\infty(e)$ and φ is linear on $\text{Lin}(\varphi)$. Moreover, a vector v belongs to

²⁶There exists an agent $j \in I$ such $U^j(d^j) > U^j(c^j)$. If $d^j(s^0) > 0$, then we can reduce $d^j(s^0)$ by some small enough $\varepsilon > 0$ such that agent j 's participation constraint at $t = 0$ remains valid. The amount $\varepsilon > 0$ can be distributed to the other agents and strictly increase their utility. If $d^j(s^0) = 0$, then for at least one successor event $s^1 \succ s^0$, we must have $U^j(d^j|s^1) > U^j(c^j|s^1)$. If $d^j(s^1) > 0$, we can reproduce the previous argument reducing agent j 's consumption at s^1 . If $d^j(s^1) = 0$, then there must be at least one successor event $s^2 \succ s^1$ such that $U^j(d^j|s^2) > U^j(c^j|s^2)$. It follows from Lemma 2.2 in the supplementary material of [Martins-da-Rocha and Vailakis \(2015b\)](#) that this procedure necessarily ends in a finite time.

the lineality space if, and only if, $\varphi(v) \in \mathbb{R}$ and $\varphi(-v) = -\varphi(v)$.²⁷ We do not want to allow for unnecessary distortions on price functionals and introduce the following concept: a superlinear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is **Malinvaud linear** when the lineality space $\text{Lin}(\varphi)$ contains the vector space ℓ^f of processes $x = (x(s^t))_{s^t \in \Sigma}$ with finite support, that is, $\{s^t \in \Sigma : x(s^t) \neq 0\}$ is finite.²⁸ If φ is Malinvaud linear, then we let $p^\varphi \in \mathbb{R}^\Sigma$ be the process defined by $p^\varphi(s^t) := \varphi(\mathbf{1}_{s^t})$. The following proposition states important properties satisfied at equilibrium by Malinvaud linear price functionals.

Proposition 3.1. *Consider a KL competitive equilibrium $(\varphi, (c^i)_{i \in I})$ with a superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$. If φ is Malinvaud linear, then*

- (i) every equilibrium net-trade $c^i - e^i$ belongs to the lineality space $\text{Lin}(\varphi)$;
- (ii) if $\tilde{c}^i \in C^i$ satisfies $U^i(\tilde{c}^i) > U^i(c^i)$, then $\varphi(\tilde{c}^i - c^i) > 0$;
- (iii) φ is **strictly increasing** in the sense that $\varphi(y) > \varphi(x)$ for any $y > x$;
- (iv) p^φ is strictly positive and satisfies

$$\frac{p^\varphi(s^t)}{p^\varphi(\sigma(s^t))} \geq \frac{p^*(s^t)}{p^*(\sigma(s^t))} := \max_{i \in I} \text{MRS}^i(c^i | s^t), \quad \text{for all } s^t \succ s^0.$$

Proof. We start by proving (i). Agent i 's equilibrium net trade $c^i - e^i$ is denoted by z^i . Since U^i is strictly increasing, we have $\varphi(z^i + \varepsilon \mathbf{1}_{s^0}) > b^i$ for every $\varepsilon > 0$. Malinvaud linearity implies that $\varphi(z^i) + \varepsilon \varphi(\mathbf{1}_{s^0}) > b^i$. Passing to the limit when ε vanishes, we get that $\varphi(z^i) \geq b^i$. By definition of equilibrium, we also have the converse inequality. This proves that $\varphi(z^i) = b^i$ and $\sum_{i \in I} \varphi(z^i) = 0$. Since $\sum_{i \in I} z^i = 0$, this is sufficient to deduce that $\varphi(z^i) = -\varphi(-z^i)$.²⁹ This last property guarantees $z^i \in \text{Lin}(\varphi)$.

²⁷If $\varphi(v) \in \mathbb{R}$, superlinearity implies that $\varphi(v) + \varphi(-v) \leq \varphi(0) = 0$.

²⁸Inspired by Malinvaud (1953), Bloise, Reichlin and Tirelli (2013) introduced the concept of Malinvaud efficiency in models with limited commitment. It is a form of short-term efficiency where an allocation cannot be improved over any arbitrary finite horizon.

²⁹Indeed, by superlinearity, we have $\varphi(z^i) = -\sum_{j \neq i} \varphi(z^j) \geq -\varphi(-z^i)$. Superlinearity also implies that $\varphi(z^i) + \varphi(-z^i) \leq 0$.

To show (ii), consider an arbitrary individually rational consumption process $\tilde{c}^i \in C^i$ such that $U^i(\tilde{c}^i) > U^i(c^i)$. By definition of equilibrium, we necessarily have $\varphi(\tilde{c}^i) > b^i$. We also have $b^i = \varphi(c^i - e^i)$. Since $z^i = c^i - e^i$ belongs to the lineality space of φ , we deduce that

$$\varphi(\tilde{c}^i - c^i) = \varphi(\tilde{c}^i - e^i - z^i) = \varphi(\tilde{c}^i - e^i) - \varphi(z^i) > b^i - \varphi(z^i) = 0.$$

To prove (iii), we pick $y > x$ and pose $v := y - x$. Strict monotonicity of preferences imply that $\tilde{c}^i := c^i + v$ is individually rational and satisfies $U^i(\tilde{c}^i) > U^i(c^i)$. From (ii), we deduce that $\varphi(v) > 0$. Moreover, superlinearity implies that $\varphi(y) = \varphi(x + v) \geq \varphi(x) + \varphi(v) > \varphi(x)$.

The arguments to show that (iv) follows from (ii) are routine (see [Bloise and Reichlin \(2011\)](#) or [Martins-da-Rocha and Vailakis \(2015b\)](#)) since φ is Malinvaud linear. \square

4 Second Welfare Theorem

This paper's main contribution is showing that every constrained Pareto optimal allocation can be decentralized into a KL competitive equilibrium by a superlinear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$. However, the set of superlinear functionals is large, and some of them are so distant from linear that they do not display any economic content. To illustrate this issue, we introduce the following important notations.

We denote by H^i the set of all directions $h^i \in \mathbb{R}^\Sigma$ such that $e^i + \varepsilon h^i \in C^i$ for some $\varepsilon > 0$ small enough. The set H^i is a convex cone containing \mathbb{R}_+^Σ called the set of **admissible directions**. If $c^i \in C^i$ is an individually rational consumption process, net trade $c^i - e^i$ must belong to H^i . In other words, our assumption of lack of commitment restricts agent i to choosing contracts in H^i . We let H be the set of **socially admissible directions** defined by

$$H := \sum_{i \in I} H^i.$$

The set H is a convex cone containing each H^i .

For the definition of a KL competitive equilibrium $(\varphi, (c^i)_{i \in I})$, only the restriction of φ to the cone H is economically relevant since each agent i can only choose a contract x^i satisfying $e^i + x^i \in C^i$. Given this observation, it can be trivial to find a superlinear functional

that decentralizes a constrained efficient allocation as a KL competitive equilibrium. Indeed, consider an economy where the autarchic allocation $(e^i)_{i \in I}$ is constrained efficient.³⁰ Consider $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{\infty, -\infty\}$ defined by

$$\varphi(x) = \begin{cases} \infty & \text{if } x \in H \setminus \{0\}, \\ 0 & \text{if } x = 0, \\ -\infty & \text{if } x \notin H. \end{cases}$$

If we chose initial wealth equal to zero (i.e., $b^i = 0$), the pair $(\varphi, (e^i)_{i \in I})$ is a KL competitive equilibrium, but the superlinear function φ is very distant from a linear functional since its linearity is minimal: $\text{Lin}(\varphi) = \{0\}$. Moreover, the functional φ has no economic content as it does not depend on the agents' marginal rates of substitution at equilibrium.

To avoid considering arbitrary price functionals as in the example above, we restrict attention to superlinear functionals that are almost-linear on the economically relevant subset of socially admissible directions.

Definition 4.1. A superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is said **almost-linear** when the restriction of φ to the convex cone H of socially admissible directions satisfies the following properties

- (a) $\varphi(0) = 0$ and $\varphi(H) \subseteq \mathbb{R} \cup \{\infty\}$;
- (b) $\varphi(\lambda x) = \lambda \varphi(x)$ for any $\lambda \neq 0$ and any $x \in H$;³¹
- (c) $\varphi(x) + \varphi(y) = \varphi(x + y)$ for any $x, y \in H$.³²

Let H^φ be the convex cone of socially admissible directions $x \in H$ satisfying $\varphi(x) \in \mathbb{R}$. We use the terminology “almost-linear” because the linearity space $\text{Lin}(\varphi)$ contains $H^\varphi - H^\varphi$, the linear subspace generated by H^φ .³³ It is because we may have $\varphi(x) = \infty$ for some

³⁰As the example in [Martins-da-Rocha and Vailakis \(2015a\)](#).

³¹We take the convention that $\lambda \infty = \infty$ for any $\lambda > 0$ and $\lambda \infty = -\infty$ for any $\lambda < 0$.

³²We take the convention that $\alpha + \infty = \infty$ for any $\alpha \in \mathbb{R} \cup \{\infty\}$.

³³Indeed, if $x \in H$ and $\varphi(x) \in \mathbb{R}$, then condition (b) implies $\varphi(-x) = -\varphi(x)$.

admissible contracts $x \in H$ that we have to consider superlinear functionals when extending the price functional to the whole commodity space.

If $(\tilde{c}^i)_{i \in I}$ is an individually rational consumption allocation satisfying the weak market clearing condition $\sum_{i \in I} \tilde{c}^i \leq \sum_{i \in I} e^i$, then each $\tilde{c}^i - e^i$ is called a **feasible net trade**. When φ is almost-linear and nondecreasing, every feasible net trade belongs to H^φ .³⁴ Indeed, each process $\tilde{c}^i - e^i$ belongs to H which implies that $\varphi(\tilde{c}^i - e^i)$ belongs to $\mathbb{R} \cup \{\infty\}$. As φ is additive on H , we must have $\sum_{i \in I} \varphi(\tilde{c}^i - e^i) \leq 0$. Therefore, we cannot have $\varphi(\tilde{c}^i - e^i) = \infty$.

A nonnegative $g \in \ell_+^\infty(e)$ is said to be a **gain-to-trade vector** when there exists an individually rational allocation $(f^i)_{i \in I}$ such that $g + \sum_{i \in I} f^i = e$. When φ is almost-linear, the linearity space $\text{Lin}(\varphi)$ contains the gain-to-trade vectors. Denote by $G \subseteq \ell^\infty(e)$ the set of all vectors $x \in \ell^\infty(e)$ such that $\alpha|x|$ is a gain-to-trade vector for some $\alpha > 0$ small enough. The set G is a vector subspace of $\ell^\infty(e)$ called the **gains-to-trade ideal**.³⁵

The lineality space $\text{Lin}(\varphi)$ can be interpreted as a “measure” of the nonlinearity of a superlinear functional φ . The larger the lineality space, the closer to a linear functional. We prove below that the separating superlinear functional can be chosen such that its linearity space contains all finite horizon vectors $x \in \ell^f$ (and therefore is Malinvaud linear),³⁶ all feasible net trades, and the gains-to-trade ideal G .

Theorem 4.1 (Second Welfare Theorem). *For every constrained Pareto efficient allocation $(c^i)_{i \in I}$, there exists a superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ such that $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium. Moreover, the price functional φ can be chosen to be almost-linear and satisfy the following properties:*

(i) $\varphi(x) = p^* \cdot x$ when x is a finite horizon vector in ℓ^f ;

(ii) the linearity space $\text{Lin}(\varphi)$ contains the gains-to-trade ideal G ;

³⁴A super linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is said to be **nondecreasing** when $\varphi(y) \geq \varphi(x)$ for any $y \geq x$.

³⁵The set G is the union of all ideals $\ell^\infty(g)$ generated by a gain-to-trade vector g .

³⁶Recall that ℓ^f is the vector space of all vectors $x \in \ell^\infty(e)$ such that $\{s^t \in \Sigma : x(s^t) \neq 0\}$ is finite. Equivalently, a vector $x \in \ell^\infty(e)$ belongs to ℓ^f if, and only if, for some T large enough, the tailed process $x^{[T]}$ is null.

(iii) for every individually rational $\tilde{c}^i \in C^i$ and any event s^t , we have

$$\frac{U^i(\tilde{c}^i|s^t) - U^i(c^i|s^t)}{(u^i)'(c^i(s^t))} \leq \frac{1}{p(s^t)} \varphi((\tilde{c}^i - c^i)\mathbf{1}_{\Sigma(s^t)})$$

in particular $\lambda^i [U^i(\tilde{c}^i) - U^i(c^i)] \leq \varphi(\tilde{c}^i - c^i)$, where $\lambda^i := 1/(u^i)'(c^i(s^0))$.

Proof of Theorem 4.1. Let $(c^i)_{i \in I}$ be a constrained Pareto efficient allocation. Following [Martins-da-Rocha and Vailakis \(2015b\)](#), we denote by $\mathcal{E}(\varepsilon)$ the ε -perturbed economy in which we introduce an ad-hoc seizable physical asset in positive net supply that delivers the fraction $\varepsilon e(s^t)$ of the aggregate endowment at every event s^t . The outside option in the ε -perturbed economy has the same value as in the original economy $\mathcal{E}(0)$ without seizable physical assets. Therefore, a consumption process c^i for agent i is individually rational in $\mathcal{E}(\varepsilon)$ if, and only if, it is individually rational in $\mathcal{E}(0)$. An allocation $(c^i)_{i \in I}$ of individually rational consumption processes is said to be ε -feasible when it is feasible for the perturbed economy $\mathcal{E}(\varepsilon)$, i.e., $\sum_{i \in I} c^i = (1 + \varepsilon)e$. It is said ε -constrained Pareto optimal if it is ε -feasible and there is no other ε -feasible and individually rational consumption allocation that Pareto dominates $(c^i)_{i \in I}$. Applying Theorem 4.1 in [Martins-da-Rocha and Vailakis \(2015b\)](#), we deduce the existence of a nonnegative sequence (ε_n) decreasing to 0 and a sequence (c_n) of ε_n -constrained Pareto optimal allocations $c_n := (c_n^i)_{i \in I}$ that converges (for the product topology) to the allocation $c := (c^i)_{i \in I}$.³⁷ The ε -perturbed economy has been constructed to exhibit uniform gains to trade (see Proposition 4.1 in [Martins-da-Rocha and Vailakis \(2015b\)](#)). Applying Proposition 3.1 in [Martins-da-Rocha and Vailakis \(2015b\)](#), we deduce that the implied AD prices p_n^* associated to c_n exhibit high interest rates.³⁸ By construction, we have $\text{MRS}^i(c_n^i|s^t) \leq p_n^*(s^t)/p_n^*(\sigma(s^t))$. Since c_n is constrained Pareto optimal in $\mathcal{E}(\varepsilon_n)$, it follows from Remark 3.3 in [Martins-da-Rocha and Vailakis \(2015b\)](#) that $\text{MRS}^i(c_n^i|s^t) = p_n^*(s^t)/p_n^*(\sigma(s^t))$ when $U^i(c_n^i|s^t) > U^i(e^i|s^t)$. All the conditions of Proposition B.1 in the appendix are met. Therefore, for every individually rational consumption process $\tilde{c}^i \in C^i$

$$\lambda_n^i [U^i(\tilde{c}^i) - U^i(c_n^i)] \leq p_n^* \cdot (\tilde{c}^i - c_n^i), \quad \text{where} \quad \lambda_n^i := \frac{1}{(u^i)'(c_n^i(s^0))}. \quad (4.1)$$

³⁷It follows from Proposition 2.1 in the supplementary material of [Martins-da-Rocha and Vailakis \(2015b\)](#) that c_n^i and c^i are strictly positive for every i .

³⁸Recall that $p_n^* = (p_n^*(s^t))_{s^t \in \Sigma}$ is the strictly positive process defined recursively by $p_n^*(s^0) = 1$ and for every $s^t \succ s^0$, $p_n^*(s^t)/p_n^*(\sigma(s^t)) = \max_{i \in I} \text{MRS}^i(c_n^i|s^t)$.

Summing over i and using the market clearing condition $\sum_{i \in I} c_n^i = (1 + \varepsilon_n)e = \varepsilon_n e + \sum_{i \in I} c^i$, we get

$$\varepsilon_n(p_n^* \cdot e) + \sum_{i \in I} \lambda_n^i [U^i(\tilde{c}^i) - U^i(c_n^i)] \leq \sum_{i \in I} p_n^* \cdot (\tilde{c}^i - c^i). \quad (4.2)$$

Pose $\lambda^i := 1/(u^i)'(c^i(s^0))$. Since (c_n^i) converges to c^i for the product topology, we have $\lim \lambda_n^i = \lambda^i$ and $\lim U^i(c_n^i) = U^i(c^i)$.³⁹ We also have that, for every event s^t , the sequence $(p_n^*(s^t))$ converges to $p^*(s^t)$, where we recall that p^* is the vector of implied AD prices associated to $c = (c^i)_{i \in I}$. Choosing $\tilde{c}^i = c^i$ in (4.2), we deduce that $\lim \varepsilon_n(p_n^* \cdot e) = 0$.⁴⁰ If we let

$$\chi_n := \varepsilon_n(p_n^* \cdot e) + \sum_{i \in I} \lambda_n^i [U^i(c^i) - U^i(c_n^i)]$$

then, for every individually rational consumption process $\tilde{c}^i \in C^i$

$$\chi_n + \sum_{i \in I} \lambda_n^i [U^i(\tilde{c}^i) - U^i(c^i)] \leq \sum_{i \in I} p_n^* \cdot (\tilde{c}^i - c^i)$$

with $\lim \chi_n = 0$. This implies that for every n , every agent i and every individually rational consumption process $\tilde{c}^i \in C^i$

$$\chi_n + \lambda_n^i [U^i(\tilde{c}^i) - U^i(c^i)] \leq p_n^* \cdot (\tilde{c}^i - c^i). \quad (4.3)$$

At this point, the natural idea is to pass to the limit in the above inequality. The LHS converges to $\lambda^i [U^i(\tilde{c}^i) - U^i(c^i)]$. The issue is the RHS. For every event s^t , the sequence $(p_n^*(s^t))$ converges to $p^*(s^t)$. This does not necessarily imply that $\lim p_n^* \cdot (\tilde{c}^i - c^i) = p^* \cdot (\tilde{c}^i - c^i)$. However, it follows from Proposition A.1 that there exists a superlinear functional $\psi : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ that is increasing, linear when restricted to the set ℓ^∞ of bounded sequences, additive when restricted to bounded from below sequences, and such that $\psi((x_n)) = \lim x_n$ for any converging sequence (x_n) . For any process $h \in \mathbb{R}^\Sigma$, we let

$$\varphi(h) = \psi((p_n^* \cdot h)).$$

³⁹Lemma 2.1 in the supplementary material of [Martins-da-Rocha and Vailakis \(2015b\)](#) shows that $U^i(\cdot)$ is continuous for the product topology on the set C^i of individually rational consumption processes.

⁴⁰This result is valid even if implied interest rates at $(c^i)_{i \in I}$ are lower than growth rates. In that case, we have $p^* \cdot e = \infty$ and, consequently, $\lim p_n^* \cdot e = \infty$. Our arguments show that (ε_n) converges faster to 0 than $(1/(p_n^* \cdot e))$.

Applying ψ to both sides of (4.3), we get that $\lambda^i[U^i(\tilde{c}^i) - U^i(c^i)] \leq \varphi(\tilde{c}^i - c^i)$ for any individually rational consumption process \tilde{c}^i . This proves the second part of (iii). To prove the first part, we fix an arbitrary event s^t and apply Proposition B.1 conditional to event s^t . Then, for every individually rational consumption process $\tilde{c}^i \in C^i$, we have

$$\frac{U^i(\tilde{c}^i|s^t) - U^i(c_n^i|s^t)}{(u^i)'(c_n^i(s^t))} \leq \frac{1}{p_n^*(s^t)} p_n^* \cdot (\tilde{c}^i - c_n^i) \mathbf{1}_{\Sigma(s^t)}. \quad (4.4)$$

Reproducing the above arguments developed for the initial event s^t , we get the first part of (iii).

To prove (i), we fix an arbitrary finite horizon process $x \in \ell^f$. Observe that $\lim p_n^* \cdot x = p^* \cdot x$. This implies that $\varphi(x) = p^* \cdot x$. The same argument applies to $-x$ and we get $\varphi(-x) = -p^* \cdot x = -\varphi(x)$. This proves that $x \in \text{Lin}(\varphi)$.

We shall now prove that φ is additive on the set H .

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To show that the lineality space $\text{Lin}(\varphi)$ contains all feasible net trades, we fix an arbitrary allocation $(\tilde{c}^i)_{i \in I}$ of individually rational consumption processes satisfying $\sum_{i \in I} \tilde{c}^i \leq e$. For each i , we let $x_n^i := p_n^* \cdot (\tilde{c}^i - c^i)$. Since the RHS of (4.3) converges, the sequence (x_n^i) is bounded from below. Since $\sum_{i \in I} x_n^i \leq 0$, we deduce that each sequence (x_n^i) is actually bounded and belongs to ℓ^∞ . This implies $\varphi(\tilde{c}^i - c^i) = \Lambda((x_n^i))$ where Λ is the Banach-Mazur limit mapping used to define ψ in Proposition A.1. Given that Λ is linear, we have that $\varphi(\tilde{c}^i - c^i) = -\varphi(c^i - \tilde{c}^i)$. This is sufficient to deduce that $\tilde{c}^i - c^i$ belongs to the lineality space $\text{Lin}(\varphi)$. Choosing $\tilde{c}^i = e^i$, we deduce that $e^i - c^i$ also belongs to $\text{Lin}(\varphi)$. We can then deduce that $\tilde{c}^i - e^i = \tilde{c}^i - c^i + c^i - e^i$ also belongs to $\text{Lin}(\varphi)$.

Fix now a vector g in the gains-to-trade ideal G . There exists $\alpha > 0$ such that $g := \alpha|x|$ is a gain-to-trade vector. This implies that there exists a individually rational allocation $(f^i)_{i \in I}$ such that $g + \sum_{i \in I} f^i = e$. Let's start by proving that $p^* \cdot g$ is finite. Choosing $\tilde{c}^i = f^i$ in (4.4), we have

$$p_n^* \cdot g \leq -\chi_n + \sum_{i \in I} \lambda_n^i [U^i(c^i) - U^i(f^i)].$$

For any arbitrary T , we have

$$p_n^* \cdot g_{[T]} \leq p_n^* \cdot g \leq -\chi_n + \sum_{i \in I} \lambda_n^i [U^i(c^i) - U^i(f^i)].$$

Passing to the limit when n tends to infinite, we have

$$p^* \cdot g_{[T]} \leq \sum_{i \in I} \lambda^i [U^i(c^i) - U^i(f^i)].$$

This is sufficient to deduce that $p^* \cdot g < \infty$. We propose now to prove that $\varphi(-x) = -\varphi(x)$. This is sufficient to deduce that $G \subseteq \text{Lin}(\varphi)$. Recall that $\alpha|x| + \sum_{i \in I} f^i = e$. For every $n \in \mathbb{N}$, we have

$$p_n^* \cdot \alpha|x| = \sum_{i \in I} p_n^*(e^i - f^i).$$

As each f^i is individually rational, we already know that the sequence $(p_n^* \cdot (f^i - e^i))$ is bounded from below. As $\sum_{i \in I} f^i - e^i \leq 0$, we deduce that each sequence $(p_n^* \cdot (f^i - e^i))$ is actually bounded. We have thus proved that the sequence $(p_n^* \cdot |x|)$ is bounded. As $|p_n^* \cdot x| \leq p_n^* \cdot |x|$, we also have that the sequence $(p_n^* \cdot x)$ is bounded. This implies $\varphi(x) = \Lambda((p_n^* \cdot x))$ where Λ is the Banach-Mazur limit mapping used to define ψ in Proposition A.1. Given that Λ is linear, we have

$$\varphi(-x) = \Lambda(-(p_n^* \cdot x)) = -\Lambda((p_n^* \cdot x)) = -\varphi(x).$$

This is sufficient to get the desired result.

We still have to prove that $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium. Fix an agent i and pose $b^i := \varphi(c^i - e^i)$. Since $c^i - e^i$ is a feasible net trade, we have $c^i - e^i \in \text{Lin}(\varphi)$ and $b^i \in \mathbb{R}$. Consider an arbitrary individually rational consumption process $\tilde{c}^i \in C^i$ satisfying $U^i(\tilde{c}^i) > U^i(c^i)$. We shall prove that $\varphi(\tilde{c}^i - e^i) > b^i$. By (ii), we have $\varphi(\tilde{c}^i - c^i) > 0$. Since $e^i - c^i$ belongs to the lineality space of φ , we have

$$\varphi(\tilde{c}^i - e^i) = \varphi((\tilde{c}^i - c^i) + (c^i - e^i)) = \varphi(\tilde{c}^i - c^i) + \varphi(c^i - e^i) = \varphi(\tilde{c}^i - c^i) + b^i$$

and we get the desired result. □

The Second Welfare Theorem under uniform gains to trade in [Bloise and Reichlin \(2011\)](#) and [Martins-da-Rocha and Vailakis \(2015b\)](#) is a direct corollary of Theorem 4.1.⁴¹

⁴¹Proposition 1.2 in the supplementary material of [Martins-da-Rocha and Vailakis \(2015b\)](#) extends Lemma 2 in [Bloise and Reichlin \(2011\)](#) by allowing for a larger set of primitives. See the discussion in ([Martins-da-Rocha and Vailakis 2015b](#), Appendix A.4).

Corollary 4.1. *Assume there are uniform gains to trade. For every constrained Pareto efficient allocation $(c^i)_{i \in I}$, there exists linear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ such that $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium. Moreover, the price functional φ can be chosen to satisfy $\varphi(x) = p^* \cdot x$ for every $x \in \ell^\infty(e)$.*

Proof. It follows from Theorem 4.1 that there exists a superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ such that $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium. The superlinear functional φ can be chosen such that $\text{Lin}(\varphi)$ contains the gains-to-trade ideal G and satisfies $\varphi(x) = p^* \cdot x$ for every $x \in \ell^f$. The assumption of uniform gains to trade implies that $G = \ell^\infty(e)$. This means that φ must be linear. Following the discussion in Section 2.4, we must have $\varphi(x) = p^* \cdot x + \varphi^0(x)$, where φ^0 is a nonnegative finitely additive linear functional on $\ell^\infty(e)$. It follows from Lemma 3.2 in [Martins-da-Rocha and Vailakis \(2015a\)](#) that $\varphi^0 = 0$: when there are uniform gains to trade, any linear price functional supporting a constrained efficient allocation cannot involve a bubble component. \square

5 Relations to the Literature

5.1 ε -Optimality and Asymptotically-Linear Equilibrium

In the literature of extensive-form games, [Radner \(1980\)](#) introduced the ε -Nash equilibrium solution that requires every player to be within ε of his optimal payoff. This concept of ε -optimality has been popular to analyze repeated-games.⁴² In the general equilibrium literature, it has been used by [Kubler and Polemarchakis \(2004\)](#) to prove the existence of stationary Markov equilibria in economies of overlapping generations. Adapting the notion of ε -optimality to our setting, we introduce the following concept.

Definition 5.1. A resource feasible and individually rational allocation $(c^i)_{i \in I}$ is said to be an ε -linear equilibrium allocation (with $\varepsilon \geq 0$) when there exists a linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ and an allocation of initial transfers $(b^i)_{i \in I}$ satisfying $\sum_{i \in I} b^i = 0$ such that, for every agent i ,

⁴²We refer, among many others, to [Radner \(1981\)](#), [Watson \(1994\)](#), [Lehrer and Sorin \(1998\)](#), [Nachbar \(1997, 2005\)](#), [Mailath, Postlewaite and Samuelson \(2005\)](#), [Noguchi \(2015\)](#), and [Norman \(2022\)](#).

(a) $c^i \in B_{\text{KL}}^i(\varphi, b^i)$, i.e., $\varphi(c^i - e^i) \leq b^i$;

(b'') if \tilde{c}^i is an individually rational process with $U^i(\tilde{c}^i) > U^i(c^i) + \varepsilon$, then $\varphi(\tilde{c}^i - e^i) > b^i$.

When $\varepsilon = 0$, we use the term **linear equilibrium**. Observe that $(c^i)_{i \in I}$ is a linear equilibrium allocation when it can be implemented through a KL competitive equilibrium $(\varphi, (c^i)_{i \in I})$ with a linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$. The only difference between an ε -linear equilibrium and a linear equilibrium is the optimality condition (b'') that is weaker than (b) in Definition 2.2. In particular, a linear equilibrium is also an ε -linear equilibrium for any $\varepsilon > 0$. The converse is not necessarily true. To analyze the asymptotic behavior of ε -linear equilibria when ε vanishes, we introduce the following concept.

Definition 5.2. A resource feasible and individually rational allocation $(c^i)_{i \in I}$ is said to be an **asymptotically linear equilibrium** when it is an ε -linear equilibrium for any $\varepsilon > 0$ small enough.

It turns out that the First and Second Welfare Theorems can be extended to asymptotically linear equilibria.

Theorem 5.1 (First Welfare Theorem). *Every asymptotically linear equilibrium is constrained Pareto efficient.*

Proof. The proof is a straightforward adaptation of the standard argument. Consider a resource feasible and individually rational allocation $(c^i)_{i \in I}$ that is an asymptotically linear equilibrium. Let $(\tilde{c}^i)_{i \in I}$ be another resource feasible and individually rational allocation. Assume, by way of contradiction, that $(\tilde{c}^i)_{i \in I}$ Pareto dominates the allocation $(c^i)_{i \in I}$. As in the proof of Theorem 3.1, we can show that there must exist a third resource feasible and individually rational allocation $(f^i)_{i \in I}$ such that $U^i(f^i) > U^i(c^i)$ for every i . Choose $\varepsilon > 0$ small enough such that $\varepsilon < U^i(f^i) - U^i(c^i)$. Since $(c^i)_{i \in I}$ is an asymptotically linear equilibrium, there exists a linear functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R}$ such that $\varphi(f^i) > \varphi(c^i)$ for every i . This contradicts feasibility of $(c^i)_{i \in I}$ and $(f^i)_{i \in I}$. \square

Theorem 5.2 (Second Welfare Theorem). *Every constrained Pareto optimal allocation is an asymptotically linear equilibrium.*

Proof. Consider a resource feasible and individually rational allocation $(c^i)_{i \in I}$ that is constrained Pareto optimal. Following the arguments of the proof of Theorem 4.1, we can show the existence of: a nonnegative sequence (ε_n) decreasing to 0, a sequence (c_n) of strictly positive and individually rational allocations $c_n := (c_n^i)_{i \in I}$ that converges (for the product topology) to the allocation $c := (c^i)_{i \in I}$, a sequence (p_n) of strictly positive linear vectors $p_n \in \text{ba}_+(e)$ satisfying $p_n(s^0) = 1$, and a sequence (χ_n) converging to 0, such that

$$\chi_n + \lambda_n^i [U^i(\tilde{c}^i) - U^i(c^i)] \leq p_n \cdot (\tilde{c}^i - c^i) \quad (5.1)$$

for every individually rational consumption process $\tilde{c}^i \in C^i$, where $\lambda_n^i = 1/(u^i)'(c_n^i(s^0))$. To prove that $(c^i)_{i \in I}$ is an asymptotically linear equilibrium, we fix an arbitrary $\varepsilon > 0$. Choose n large enough such that, for every $i \in I$, we have $\chi_n^i + \lambda_n^i \varepsilon > 0$.⁴³ To prove that $(c^i)_{i \in I}$ is an ε -linear equilibrium, we fix an individually rational consumption process $(\tilde{c}^i)_{i \in I}$ satisfying $U^i(\tilde{c}^i) > U^i(c^i) + \varepsilon$. Equation (5.1) implies that $p_n \cdot (\tilde{c}^i - c^i) > 0$. We have thus proved conditions (a) and (b'') in Definition 5.1 with φ defined by $\varphi(h) := p_n \cdot h$ and $b^i := p_n \cdot (e^i - c^i)$. \square

5.2 Sup-Convolution

To simplify the technical arguments presented in this section, we assume that $u(0) > -\infty$. Consider a KL competitive equilibrium $(\varphi, (c^i)_{i \in I})$ satisfying the properties of Theorem 4.1. For every agent i and every individually rational consumption plan $\tilde{c}^i \in C^i$, we have

$$\lambda^i [U^i(\tilde{c}^i) - U^i(c^i)] \leq \varphi(\tilde{c}^i - c^i) \quad (5.2)$$

where $\lambda^i := 1/(u^i)'(c^i(s^0))$. Denote by $H_{c^i}^i$ the set of all directions $h^i \in \mathbb{R}^\Sigma$ such that $c^i + \varepsilon h^i \in C^i$ for some $\varepsilon > 0$ small enough. The set $H_{c^i}^i$ is a convex cone containing \mathbb{R}_+^Σ called the set of **admissible directions** from c^i . Since φ is positively homogeneous, (5.2) implies that for every $\varepsilon > 0$ small enough,

$$\lambda^i \frac{1}{\varepsilon} [U^i(c^i + \varepsilon h^i) - U^i(c^i)] \leq \varphi(h^i).$$

⁴³Such an integer n exists because $\lim[\chi_n + \lambda_n^i(\varepsilon/2)] = \lambda^i(\varepsilon/2)$ where $\lambda^i = 1/(u^i)'(c^i(s^0)) > 0$.

The concavity of U^i implies that the LHS is an increasing function of ε . Passing to the limit, we obtain that

$$\lambda^i dU_{c^i}^i(h^i) = \lambda^i \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [U^i(c^i + \varepsilon h^i) - U^i(c^i)] \leq \varphi(h^i) \quad (5.3)$$

where $dU_{c^i}^i(h^i) \in \mathbb{R} \cup \{\infty\}$ is the directional derivative of U^i at c^i in the direction h^i . Given the additive form we imposed on U^i , we can characterize the directional derivative as follows.

Lemma 5.1. *For every i and every admissible direction $h^i \in H_{c^i}^i$, we have*

$$\lambda^i dU_{c^i}^i(h^i) = p^i \cdot h^i = \lim_{\tau \rightarrow \infty} \sum_{t=1}^{\tau} \sum_{s^t \in S^t} p^i(s^t) h^i(s^t)$$

where $p^i = (p^i(s^t))_{s^t \succeq s^0}$ is the process of individual prices defined by

$$p^i(s^t) = (\beta^i)^t \pi^i(s^t) \frac{(u^i)'(c^i(s^t))}{(u^i)'(c^i(s^0))}.$$

Proof. Given our assumptions on primitives, we have

$$U^i(c^i + \varepsilon h^i) - U^i(c^i) = \lim_{\tau \rightarrow \infty} \sum_{t=0}^{\tau} (\beta^i)^t \sum_{s^t \in S^t} \pi^i(s^t) [u^i(c^i(s^t) + \varepsilon h^i(s^t)) - u^i(c^i(s^t))].$$

The concavity of u^i implies that $u^i(c^i(s^t) + \varepsilon h^i(s^t)) - u^i(c^i(s^t)) \leq \varepsilon (u^i)'(c^i(s^t)) h^i(s^t)$. Since $U^i(0) > -\infty$, we deduce that

$$\lim_{\tau \rightarrow \infty} \sum_{t=0}^{\tau} (\beta^i)^t \sum_{s^t \in S^t} \pi^i(s^t) (u^i)'(c^i(s^t)) c^i(s^t) \leq U^i(c^i) - U^i(0).$$

Since $h^i = [\tilde{c}^i - c^i]/\varepsilon$ with $\tilde{c}^i := c^i + \varepsilon h^i \geq 0$, we deduce that

$$\lim_{\tau \rightarrow \infty} \sum_{t=0}^{\tau} (\beta^i)^t \sum_{s^t \in S^t} \pi^i(s^t) (u^i)'(c^i(s^t)) h^i(s^t) \quad \text{exists in } \mathbb{R} \cup \{\infty\}$$

and we get the desired result. □

Given an individually rational allocation $\mathbf{c} = (c^i)_{i \in I}$, we let

$$\mathcal{H}_{\mathbf{c}} := \prod_{i \in I} H_{c^i}^i$$

be the set of allocations of admissible directions. We denote by $H_{\mathbf{c}}$ the set of **socially admissible directions** from the allocation $\mathbf{c} = (c^i)_{i \in I}$ defined by

$$H_{\mathbf{c}} := \sum_{i \in I} H_{c^i}^i.$$

The set $H_{\mathbf{c}}$ is a convex cone containing each H_{c^i} . If $\tilde{\mathbf{c}} = (\tilde{c}^i)_{i \in I}$ is an allocation of individually rational allocations, then each net trade $\tilde{c}^i - c^i$ belongs to $H_{\mathbf{c}}$.

For any arbitrary socially admissible direction $h \in H_{\mathbf{c}}$, there exists an allocation $(h^i)_{i \in I} \in \mathcal{H}_{\mathbf{c}}$ such that $h = \sum_{i \in I} h^i$. Equation (5.3) implies that

$$\sum_{i \in I} p^i \cdot h^i \leq \varphi(h).$$

Since $\varphi(h)$ is independent of the decomposition $h = \sum_{i \in I} h^i$, we must have

$$\varphi_{\mathbf{c}}^*(h) := \sup \left\{ \sum_{i \in I} p^i \cdot h^i : h = \sum_{i \in I} h^i \quad \text{and} \quad (h^i)_{i \in I} \in \mathcal{H}_{\mathbf{c}} \right\} \leq \varphi(h)$$

for any socially admissible direction $h \in H_{\mathbf{c}}$.

For any $\lambda > 0$, we have $\lambda \sum_{i \in I} h^i = \sum_{i \in I} \lambda h^i$ and $p^i \cdot (\lambda h^i) = \lambda p^i \cdot h^i$. This implies that the functional $\varphi_{\mathbf{c}}^* : H_{\mathbf{c}} \rightarrow \mathbb{R} \cup \{\infty\}$ is positively homogeneous in the sense that $\varphi_{\mathbf{c}}^*(\lambda h) = \lambda \varphi_{\mathbf{c}}^*(h)$ for any socially admissible direction $h \in H_{\mathbf{c}}$ and any $\lambda > 0$. Moreover, as $\mathcal{H}_{\mathbf{c}} + \mathcal{H}_{\mathbf{c}} \subseteq \mathcal{H}_{\mathbf{c}}$, we deduce that $\varphi_{\mathbf{c}}^*$ is superadditive in the sense that $\varphi_{\mathbf{c}}^*(h) + \varphi_{\mathbf{c}}^*(h') \leq \varphi_{\mathbf{c}}^*(h + h')$ for any h and h' in $H_{\mathbf{c}}$. To conclude that $\varphi_{\mathbf{c}}^*$ is superlinear,⁴⁴ we should verify that $\varphi_{\mathbf{c}}^*(0) = 0$.⁴⁵ It turns out that this property characterizes constrained efficiency.

Proposition 5.1. *A feasible, individually rational and strictly positive allocation $\mathbf{c} = (c^i)_{i \in I}$ is constrained Pareto optimal if, and only if, $\varphi_{\mathbf{c}}^*(0) = 0$.*

Proof. The “only if” part follows from the fact that $\varphi_{\mathbf{c}}^*(h) \leq \varphi(h)$ where φ is an equilibrium price functional derived from Theorem 4.1. To prove the converse, assume that $\mathbf{c} = (c^i)_{i \in I}$ is a feasible and individually rational allocation that satisfies $\varphi_{\mathbf{c}}^*(0) = 0$. Assume, by way of contradiction, that there exists a feasible and individually rational allocation $(\tilde{c}^i)_{i \in I}$ such that $U^i(\tilde{c}^i) \geq U^i(c^i)$ for every $i \in I$, with a strict inequality for at least one agent $k \in I$. Observe that $h^i := \tilde{c}^i - c^i$ is an admissible direction in H_{c^i} . The concavity of U^i implies that

$$U^i(c^i + h^i) - U^i(c^i) \leq dU_{c^i}^i(h^i) = (u^i)'(c^i(s^0))p^i \cdot h^i$$

⁴⁴We can extend $\varphi_{\mathbf{c}}^*$ to the whole domain \mathbb{R}^{Σ} by posing $\varphi_{\mathbf{c}}^*(h) := -\infty$ if $h \notin H_{\mathbf{c}}$.

⁴⁵Observe that $\varphi_{\mathbf{c}}^*(0) \in \{0\} \cup \{\infty\}$.

and we deduce that

$$0 < \sum_{i \in I} \frac{1}{(u^i)'(c^i(s^0))} [U^i(c^i + h^i) - U^i(c^i)] \leq \sum_{i \in I} p^i \cdot h^i \leq \varphi_c^* \left(\sum_{i \in I} h^i \right).$$

The feasibility of $(\tilde{c}^i)_{i \in I}$ implies that $\sum_{i \in I} h^i = 0$. The contradiction stems from the assumption that $\varphi_c^*(0) = 0$. \square

The definition of φ_c^* takes the form of a sup-convolution of personalized prices. Indeed, if we pose

$$\varphi^i(x) := \begin{cases} p^i \cdot x & \text{if } x \in H_{c^i}^i, \\ -\infty & \text{otherwise,} \end{cases}$$

then, taking the convention that $(-\infty) + \infty = -\infty$, we can extend φ^* to the whole space \mathbb{R}^Σ by posing

$$\varphi_c^*(x) = \sup \left\{ \sum_{i \in I} \varphi^i(x^i) : x = \sum_{i \in I} x^i \right\}.$$

The construction of a price functional as the sup-convolution of personalized prices already appeared in the literature as a solution concept to tackle the lack of lattice structure of the commodity space. [Aliprantis, Tourky and Yannelis \(2001\)](#) first introduced these techniques that were subsequently developed by [Aliprantis, Florenzano and Tourky \(2004, 2005\)](#). The lack of lattice structure typically occurs in models where the positive cone of an infinite dimensional commodity space has no interior points (as in the models of commodity differentiation). Still, it may also occur for some finite-dimensional commodity spaces as illustrated by [Aliprantis, Monteiro and Tourky \(2004\)](#).

In our model, the commodity space is \mathbb{R}^Σ endowed with the standard lattice structure defined by the positive cone \mathbb{R}_+^Σ , which has an interior point (for the sup-metric). The need for superlinear price functionals stems from the financial frictions of the limited commitment assumption. Indeed, the standard arguments of the Walrasian general equilibrium model cannot be applied in our setting because the consumption set C^i is a strict subset of the positive cone \mathbb{R}_+^Σ .

6 Example

In this section, we restrict attention to stationary Markovian economies. Uncertainty is assumed to be represented by a simple Markov process on a finite state space Z . An event s^t is then a $t+1$ -vector (s_0, s_1, \dots, s_t) where each shock $s_\tau \in Z$ and $s_0 \in Z$ is fixed. In addition, the conditional probability $\pi(s^{t+1}|s^t)$ is assumed to depend only on s_t and s_{t+1} . We abuse notation and denote this conditional probability by $\pi(s_{t+1}|s_t)$.⁴⁶ A process $x = (x(s^t))_{s^t \in \Sigma}$ is said to be **stationary Markovian** if $x(s^t)$ is a time invariant function of the current shock s_t . We make a slight abuse of terminology and use the notation $x(s_t)$ for $x(s^{t-1}, s_t)$.

We assume that agent i 's endowment process is stationary Markovian. For any event $s^t = (s^{t-1}, s_t)$, the endowment $e^i(s^{t-1}, s_t)$ is denoted by $e^i(s_t)$. It follows that the reservation utility process is also stationary Markovian. Indeed, for every event $s^t = (s^{t-1}, s_t)$ we have $U(e^i|s^t) = V^i(s_t)$ where $V^i = (V^i(z))_{z \in Z} \in \mathbb{R}^Z$ is the unique solution of the following recursive equations

$$\forall z \in Z, \quad V^i(z) = u(e^i(z)) + \beta \sum_{z' \in Z} \pi(z'|z) V^i(z').$$

When the process of endowments is stationary Markovian and the autarchic allocation $(e^i)_{i \in I}$ is not constrained Pareto efficient, then the Second Welfare Theorem is valid with linear prices.⁴⁷ If the autarchic allocation is constrained Pareto efficient and the implied riskless interest rate at autarky is nonpositive, then decentralization as a competitive equilibrium only occurs with a superlinear price functional. To illustrate this property, we consider the standard stationary Markovian symmetric economy with two agents and two shocks.

There are two agents $I = \{i_1, i_2\}$. In each period, one agent receives the high endowment $1 + \sigma$ and the other receives the low endowment $1 - \sigma$ where $\sigma \in [0, 1]$. Agents switch endowments with probability $1 - \delta$ where $\delta \in (0, 1)$. Formally, uncertainty is captured by the Markov process s_t , with state space $Z = \{z_{i_1}, z_{i_2}\}$ and symmetric transition probabilities $\text{Prob}(s_{t+1} = z_i | s_t = z_i) = \delta$. The endowment $e^i(s^t)$ only depends on the current shock s_t ,

⁴⁶This implies that $\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\dots\pi(s_1|s_0)$.

⁴⁷Proposition 4 in [Bloise and Reichlin \(2011\)](#) shows that the condition of uniform gains to trade is satisfied if, and only if, the autarchic allocation $(e^i)_{i \in I}$ is not constrained Pareto efficient.

with $e^i(s^t) = 1 + \sigma$ if $s_t = z_i$ and $e^i(s^t) = 1 - \sigma$ if $s_t \neq z_i$. The initial state at $t = 0$ is z_{i_1} , i.e., agent i_1 starts with high income (and agent i_2 starts with low income).

Define $V_h(\sigma) := U(e^i|(s^{t-1}, z_i))$ and $V_\ell(\sigma) := U(e^i|(s^{t-1}, z_j))$ the autarchic continuation utility in the high-endowment and low-endowment state respectively, where $z_j \neq z_i$. We easily compute

$$(1 - \beta)V_h(\sigma) = \bar{\alpha}u(1 + \sigma) + \underline{\alpha}u(1 - \sigma) \quad \text{and} \quad (1 - \beta)V_\ell(\sigma) = \underline{\alpha}u(1 + \sigma) + \bar{\alpha}u(1 - \sigma)$$

where

$$\bar{\alpha} := \frac{1 - \beta\delta}{(1 - \beta\delta) + (\beta - \beta\delta)} \quad \text{and} \quad \underline{\alpha} := 1 - \bar{\alpha}.$$

Since $\bar{\alpha} > \underline{\alpha}$ and u is strictly concave, the function V_ℓ is strictly decreasing on $[0, 1]$, and there exists σ_M such that V_h is strictly increasing on $[0, \sigma_M]$ and strictly decreasing on $[\sigma_M, 1]$. The threshold σ_M is determined by the equation

$$\frac{u'(1 - \sigma_M)}{u'(1 + \sigma_M)} = \frac{\bar{\alpha}}{\underline{\alpha}} = \frac{1 - \beta\delta}{\beta - \beta\delta}.$$

Observe that implied Arrow–Debreu prices satisfy

$$\frac{p^*(s^{t+1})}{p^*(s^t)} = \beta\delta =: q^{nc}, \quad \text{if } s_{t+1} = s_t$$

and

$$\frac{p^*(s^{t+1})}{p^*(s^t)} = \beta(1 - \delta) \frac{u'(1 - \sigma)}{u'(1 + \sigma)} =: q^c(\sigma), \quad \text{if } s_{t+1} \neq s_t.$$

The risk-less interest rate $q^{nc} + q^c(\sigma)$ is constant and satisfies

$$\forall \sigma < \sigma_M, \quad q^{nc} + q^c(\sigma) < q^{nc} + q^c(\sigma_M) = 1.$$

This implies that if $\sigma < \sigma_M$, the autarchic allocation displays implied interest rates higher than growth and, therefore, is constrained Pareto efficient. The interesting case corresponds to $\sigma = \sigma_M$. [Martins-da-Rocha and Vailakis \(2015b\)](#) proved that the autarchic allocation is constrained efficient when $\sigma = \sigma_M$. Applying [Theorem 4.1](#), we deduce that there exists a superlinear price functional $\varphi : \ell^\infty(e) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ that is almost-linear and such that $(\varphi, (e^i)_{i \in I})$ is a KL competitive equilibrium.

We aim at computing the value $\varphi(z^i)$ for some very specific processes z^i . To simplify the analysis, we focus on stationary Markovian processes z^i characterized by a pair (z_ℓ, z_h) as follows:

$$z^i(s^t) = \begin{cases} z_h & \text{if } e^i(s_t) = e_h, \\ z_\ell & \text{if } e^i(s_t) = e_\ell; \end{cases}$$

According to the arguments of the proof of Theorem 4.1, to construct a decentralizing price functional φ that is almost-linear, it is sufficient to identify a sequence (ε_n) of positive numbers decreasing to 0 and a sequence (c_n) of ε_n -constrained Pareto optimal allocations $c_n = (c^i)_{i \in I}$ that converges to the autarchic allocation e . According to the arguments in (Martins-da-Rocha and Vailakis 2015b, Section 4.3), the allocation c_n can be chosen stationary Markovian. This means that there exists $c_{n,h} > c_{n,\ell} > 0$ such that

$$c_n^i(s^t) = \begin{cases} c_{n,h} & \text{if } e^i(s^t) = e_h, \\ c_{n,\ell} & \text{if } e^i(s^t) = e_\ell. \end{cases}$$

The corresponding implied AD prices $p_n^* = (p_n^*(s^t))_{s^t \succeq s^0}$ satisfy the Markovian property

$$p_n^*(s^t) = q_n^*(s^t) p_n^*(\sigma(s^t)), \quad \text{for all } s^t \succ s^0,$$

where the price $q_n^*(s^t)$ is given by⁴⁸

$$q_n^*(s^t) = \begin{cases} q_n^c := (1 - \delta) \beta \frac{u'(c_{n,h})}{u'(c_{n,\ell})}, & \text{if } s^t = \sigma(s^t), \\ q^{nc} = \delta \beta, & \text{otherwise.} \end{cases}$$

Agent i_1 starts with high income. Therefore, we let $w_{n,h} := p_n^* \cdot z^{i_1}$ and $w_{n,\ell} := p_n^* \cdot z^{i_2}$. As z^i is stationary Markovian, we have

$$w_{n,h} = z_h + q^{nc} w_{n,h} + q_n^c w_{n,\ell} \quad \text{and} \quad w_{n,\ell} = z_\ell + q^{nc} w_{n,\ell} + q_n^c w_{n,h}.$$

We deduce that

$$\begin{cases} (1 - q^{nc} - q_n^c)(1 - q^{nc} + q_n^c) w_{n,h} &= (1 - q^{nc}) z_h + q_n^c z_\ell, \\ (1 - q^{nc} - q_n^c)(1 - q^{nc} + q_n^c) w_{n,\ell} &= (1 - q^{nc}) z_\ell + q_n^c z_h. \end{cases}$$

⁴⁸The price q_n^* can be interpreted as the price at event $\sigma(s^t)$ of the Arrow security paying contingent to event s^t .

The implied AD prices of ε_n -perturbed economy display high-interest rates. This means that $q^{nc} + q_n^c < 1$ and we have

$$w_{n,h} = \frac{(1 - q^{nc})z_h + q_n^c z_\ell}{(1 - q^{nc} - q_n^c)(1 - q^{nc} + q_n^c)} \quad \text{and} \quad w_{n,\ell} = \frac{(1 - q^{nc})z_\ell + q_n^c z_h}{(1 - q^{nc} - q_n^c)(1 - q^{nc} + q_n^c)}.$$

As the sequence (c_n) converges to the autarkic allocation e , we have $\lim q_n^c = q^c$. This implies that

$$(\varphi(z^{i_1}), \varphi(z^{i_2})) = \lim(w_{n,h}, w_{n,\ell}) = \begin{cases} (\infty, \infty) & \text{if } z_h + z_\ell > 0, \\ \left(\frac{z_h}{2(1 - \beta\delta)}, \frac{z_\ell}{2(1 - \beta\delta)} \right) & \text{if } z_h + z_\ell = 0, \\ (-\infty, -\infty) & \text{if } z_h + z_\ell < 0. \end{cases}$$

Consider now another specific case where the process z is deterministic, i.e., $z(s^t) = z_t$ for some sequence $(z_t)_{t \geq 0} \in \mathbb{R}^N$. For each n , we denote by $\kappa_n := q^{nc} + q_n^c$ the price of the riskless zero-coupon bond. Equivalently, κ_n is the discount factor associated with the AD prices $p_n^* = (p_n^*(s^t))_{s^t \geq s^0}$. As the process z is deterministic, we have

$$p_n^* \cdot z = \sum_{t \geq 0} (\kappa_n)^t z_t.$$

Consider the case where $z_t = (-1)^t$. Observe that the “standard” linear market value $p^* \cdot z$ is not well defined since the sequence $(\zeta^\tau)_{\tau \geq 1}$ of partial sums defined by

$$\zeta^\tau := p^* \cdot z_{[\tau+1]} = \sum_{t=0}^{\tau} \sum_{s^t \in S^t} p^*(s^t) z(s^t) = \sum_{t=0}^{\tau} (-1)^t = \begin{cases} 1 & \text{if } \tau \text{ is even,} \\ 0 & \text{if } \tau \text{ is odd,} \end{cases}$$

has no limit. Nonetheless, the superlinear functional φ assigns a real value to the process z . Indeed, for every n , we have

$$p_n^* \cdot z = \sum_{t \geq 0} (\kappa_n)^{2t} - \sum_{t \geq 0} (\kappa_n)^{2t+1} = (1 - \kappa_n) \frac{1}{1 - (\kappa_n)^2} = \frac{1}{1 + \kappa_n}.$$

This implies that $\lim p_n^* \cdot z = 1/2$ and, consequently, $\varphi(z) = 1/2$. Observe in particular that

$$0 = \liminf_{\tau \rightarrow \infty} (p^* \cdot z_{[\tau]}) < \varphi(z) < \limsup_{\tau \rightarrow \infty} (p^* \cdot z_{[\tau]}) = 1.$$

This implies that we cannot obtain the value $\varphi(z)$ from a subsequential limit of the sequence of partial sums $(p^* \cdot z_{[\tau]})_{\tau \geq 1}$. This implies that

$$\varphi(z^{[\tau]}) = \begin{cases} 1/2 & \text{if } \tau \text{ is even,} \\ -1/2 & \text{if } \tau \text{ is odd.} \end{cases}$$

In particular, the market value $\varphi(z^{[T]})$ of the T -tail of the process z does not vanish when T tends to infinite.

7 Conclusion

This paper examines the validity of the Second Welfare Theorem in the context of limited commitment. By allowing for nonlinear pricing functionals, we show the validity of the Second Welfare Theorem, even when equilibrium interest rates fall below growth rates. The separating pricing functional can be chosen to be “close to linear” in the sense that its restriction to socially admissible directions is additive and positively homogeneous.

Our findings build upon and extend the work of [Kehoe and Levine \(1993\)](#), [Alvarez and Jermann \(2000\)](#), [Bloise and Reichlin \(2011\)](#) and [Martins-da-Rocha and Vailakis \(2015a\)](#). While their versions of the Second Welfare Theorem serve as a direct corollary to our Theorem 4.1, they do not address the challenges posed by low-interest rate environments.

Appendix

A Extensions of Banach-Mazur Limits

Denote by $\ell^\infty \subseteq \mathbb{R}^{\mathbb{N}}$ the subspace of bounded real sequences. A **Banach-Mazur limit** is a linear functional $\Lambda : \ell^\infty \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) Λ is positive in the sense that $\Lambda(x) \geq 0$ for every nonnegative sequence $x \in \mathbb{R}_+^{\mathbb{N}}$;
- (ii) Λ is normalized in the sense that $\Lambda(\mathbf{1}_{\mathbb{N}} := (1, 1, \dots)) = 1$;
- (iii) $\Lambda((x_n)) = \Lambda((x_{n+1}))$ for every $(x_n) \in \ell^\infty$.

A Banach-Mazur limit exists (see Theorem 16.47 in [Aliprantis and Border \(2006\)](#)) and extends the “limit operator” since $\liminf x_n \leq \Lambda((x_n)) \leq \limsup x_n$ for any bounded sequence $(x_n) \in \ell^\infty$. We extend a Banach-Mazur limit to the space $\mathbb{R}^{\mathbb{N}}$ as follows.

Proposition A.1. *Fix a Banach-Mazur limit $\Lambda : \ell^\infty \rightarrow \mathbb{R}$ and consider $\psi : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be defined by*

$$\psi(x) := \sup\{\Lambda(y) : y \in \ell^\infty \text{ and } y \leq x\} \quad (\text{A.1})$$

with the convention that $\sup(\emptyset) = -\infty$. The functional ψ is increasing, superlinear, coincides with Λ on ℓ^∞ , and the lineality space of ψ contains ℓ^∞ . Moreover, ψ is additive on the subset of sequences bounded from below.⁴⁹

Proof. For any $x \in \mathbb{R}^{\mathbb{N}}$, we denote by $A(x)$ the set of all vectors $y \in \ell^\infty$ such that $y \leq x$. Observe that $\psi(x) = \sup\{\Lambda(y) : y \in A(x)\}$. The function ψ is increasing as $A(x) \subseteq A(x')$ for any $x, x' \in \mathbb{R}^{\mathbb{N}}$ such that $x' \geq x$.

To prove the superlinearity of ψ , we fix $x, x' \in \mathbb{R}^{\mathbb{N}}$ and $\lambda > 0$. Since $A(\lambda x) = \lambda A(x)$ for any $\lambda > 0$, we deduce that $\psi(\lambda x) = \lambda \psi(x)$. We shall now prove that $\psi(x) + \psi(x') \leq \psi(x + x')$ when $\{\psi(x), \psi(x')\} \neq \{-\infty, \infty\}$. Without any loss of generality, assume that $A(x) \neq \emptyset$ and $A(x') \neq \emptyset$.⁵⁰ This implies that $\psi(x) > -\infty$ and $\psi(x') > -\infty$. Fix $y \in A(x)$ and $y' \in A(x')$. Since $y + y' \in A(x + x')$ and Λ is linear, we deduce that $\Lambda(y) \leq \psi(x + x') - \Lambda(y')$. This implies that $\psi(x) \leq \psi(x + x') - \Lambda(y')$. Since $\psi(x) > -\infty$, we deduce that $\Lambda(y') \leq \psi(x + x') - \psi(x)$. This implies that $\psi(x') \leq \psi(x + x') - \psi(x)$. The last property we shall prove, $\psi(0) = 0$, follows from the fact that Λ is positive.

To prove that ψ and Λ coincide on ℓ^∞ , we fix an arbitrary $x \in \ell^\infty$. For any $y \in A(x)$, we have $\Lambda(y) \leq \Lambda(x)$ since Λ is linear and positive. The desired follows from the fact that $x \in A(x)$.

To show that $\ell^\infty \subseteq \text{Lin}(\psi)$, we fix an arbitrary $x \in \ell^\infty$. As ψ and Λ coincide on ℓ^∞ and Λ is linear, we must have $\psi(-x) = -\psi(x)$. This is sufficient to deduce that x belongs to the lineality space of ψ .

⁴⁹Formally, if x and x' are two sequences in $\mathbb{R}^{\mathbb{N}}$ that are bounded from below, then $\psi(x) + \psi(x') = \psi(x + x')$.

⁵⁰If $A(x) = \emptyset$, then $\psi(x) = -\infty$ and we must have $\psi(x') \neq \infty$. This implies that $\psi(x) + \psi(x') \leq \psi(x + x')$. If $A(x') = \emptyset$, then interchange the roles of x and x' .

To show that ψ is additive on the subset of bounded from below sequences, we let $x = (x_n)$ and $\tilde{x} = (\tilde{x}_n)$ be two sequences in $\mathbb{R}^{\mathbb{N}}$ such that there exist \underline{x} and $\underline{\tilde{x}}$ in \mathbb{R} satisfying $x_n \geq \underline{x}$ and $\tilde{x}_n \geq \underline{\tilde{x}}$ for every $n \in \mathbb{N}$. As $\mathbf{1}_{\mathbb{N}} = (1, 1, \dots)$ is a bounded sequence, it belongs to the lineality space $\text{Lin}(\psi)$. This implies that

$$\psi(x - \underline{x}\mathbf{1}_{\mathbb{N}}) = \psi(x) + \psi(-\underline{x}\mathbf{1}_{\mathbb{N}}) = \psi(x) + \Lambda(-\underline{x}\mathbf{1}_{\mathbb{N}}) = \psi(x) - \underline{x}.$$

Similarly, we have $\psi(\tilde{x} - \underline{\tilde{x}}\mathbf{1}_{\mathbb{N}}) = \psi(\tilde{x}) - \underline{\tilde{x}}$. Therefore, to prove that $\psi(x + \tilde{x}) = \psi(x) + \psi(\tilde{x})$, we can assume without any loss of generality that x and \tilde{x} are nonnegative sequences. As ψ is superlinear, we already know that $\psi(x) + \psi(\tilde{x}) \leq \psi(x + \tilde{x})$. To prove the converse inequality, we let $z \in A(x + \tilde{x})$ be an arbitrary bounded sequence satisfying $z \leq x + \tilde{x}$. Replacing z by the sequence $(z_n^+ := \max\{z_n, 0\})$ if necessary, we can assume that z is nonnegative. For each $n \in \mathbb{N}$, we have $0 \leq z_n \leq x_n + \tilde{x}_n$. We can decompose $z_n = y_n + \tilde{y}_n$ where y_n and \tilde{y}_n are nonnegative numbers satisfying $y_n \leq x_n$ and $\tilde{y}_n \leq \tilde{x}_n$.⁵¹ Observe that $y := (y_n)$ belongs to $A(x)$ and $\tilde{y} := (\tilde{y}_n)$ belongs to $A(\tilde{x})$. We deduce that

$$\Lambda(z) = \Lambda(y) + \Lambda(\tilde{y}) \leq \psi(x) + \psi(\tilde{x}).$$

As the above inequality is valid for any $z \in A(x + \tilde{x})$, we deduce that $\psi(x + \tilde{x}) \leq \psi(x) + \psi(\tilde{x})$. \square

B Proofs

B.1 Proof of Lemma 2.1

If φ is nontrivial, then there must exist at least one agent k such that $\varphi(d^k - e^k) < b^k$. Fix an individually rational consumption process \tilde{c}^k such that $U^k(\tilde{c}^k) > U^k(c^k)$. For every $\varepsilon \in (0, 1)$, we let $c^k(\varepsilon) := (1 - \varepsilon)\tilde{c}^k + \varepsilon d^k$. Observe that $c^k(\varepsilon) \in C^k$. Continuity of U^k implies that there exists $\varepsilon > 0$ small enough such that $U^k(c^k(\varepsilon)) > U^k(c^k)$. From the definition of a quasi-equilibrium, we must have $\varphi(c^k(\varepsilon) - e^k) \geq b^k$. The condition $\varphi(d^k - e^k) < b^k$ implies

⁵¹Indeed, it is sufficient to pose $y_n := z_n \alpha_n$ and $\tilde{y}_n := z_n \tilde{\alpha}_n$ where $\alpha_n := x_n / (x_n + \tilde{x}_n)$ and $\tilde{\alpha}_n := \tilde{x}_n / (x_n + \tilde{x}_n)$.

that $\varphi(\tilde{c}^k - e^k) > b^k$. The function U^k is strictly increasing. Observing that $b^k = \varphi(c^k - e^k)$, we deduce that φ is strictly increasing. We can now follow standard arguments to prove that $(\varphi, (c^i)_{i \in I})$ is a KL competitive equilibrium. Indeed, fix an arbitrary agent i and individually rational consumption process $\tilde{c}^i \in C^i$ such that $U^i(\tilde{c}^i) > U^i(c^i)$. If $\tilde{c}^i(s^0) > 0$, there exists $\varepsilon > 0$ small enough such that $\tilde{c}^i - \varepsilon \mathbf{1}_{s^0}$ is still an individually rational consumption process that satisfies $U^i(\tilde{c}^i - \varepsilon \mathbf{1}_{s^0}) > U^i(c^i)$. By definition of a quasi-equilibrium, we must have $\varphi(\tilde{c}^i - \varepsilon \mathbf{1}_{s^0} - e^i) \geq b^i = \varphi(c^i - e^i)$. Since $\varphi(\mathbf{1}_{s^0}) > 0$, this implies that $\varphi(\tilde{c}^i - e^i) > \varphi(c^i - e^i)$. If $\tilde{c}^i(s^0) = 0$, we can apply Lemma 2.2 in the supplementary material of [Martins-da-Rocha and Vailakis \(2015b\)](#) to get the existence of a successor event $s^\tau \succ s^0$ such that

- (a) $\tilde{c}^i(s^\tau) > 0$ and $U^i(\tilde{c}^i|s^\tau) > U^i(e^i|s^\tau)$;
- (b) for every intermediate event s^t satisfying $s^0 \prec s^t \prec s^\tau$, we have $\tilde{c}^i(s^t) = 0$ and $U^i(\tilde{c}^i|s^t) > U^i(e^i|s^t)$.

Given the above properties, we can find $\varepsilon > 0$ small enough such that $\tilde{c}^i - \varepsilon \mathbf{1}_{s^\tau}$ is still an individually rational consumption process that satisfies $U^i(\tilde{c}^i - \varepsilon \mathbf{1}_{s^\tau}) > U^i(c^i)$. By definition of a quasi-equilibrium, we must have $\varphi(\tilde{c}^i - \varepsilon \mathbf{1}_{s^\tau} - e^i) \geq b^i = \varphi(c^i - e^i)$. Since $\varphi(\mathbf{1}_{s^\tau}) > 0$, this implies that $\varphi(\tilde{c}^i - e^i) > \varphi(c^i - e^i)$.

B.2 Missing Result

Proposition B.1. *Fix a process of strictly positive AD prices $p = (p(s^t))_{s^t \succeq s^0}$ with $p(s^0) = 1$ such that $p \cdot e < \infty$. Fix a strictly positive and individually rational consumption process $c^i \in C^i$ that satisfies the following property for every event $s^t \succ s^0$*

$$\text{MRS}^i(c^i|s^t) \leq \frac{p(s^t)}{p(\sigma(s^t))} \quad \text{with equality if } U(c^i|s^t) > U^i(e^i|s^t).$$

For every individually rational consumption process $\tilde{c}^i \in C^i$ and every event s^t , we have

$$\frac{1}{u'(c^i(s^0))} (\beta^i)^t \pi^i(s^t) [U^i(\tilde{c}^i|s^t) - U^i(c^i|s^t)] \leq p \cdot ([\tilde{c}^i - c^i] \mathbf{1}_{\Sigma(s^t)}).$$

Proof. To simplify the presentation, we let $q^i(s^t) := \text{MRS}^i(c^i|s^t)$ for every event $s^t \succ s^0$. Let p^i be the process defined by $p^i(s^0) := 1$ and $p^i(s^t) := (\beta^i)^t \pi^i(s^t) u'(c^i(s^t)) / u'(c^i(s^0))$

for every $s^t \succ s^0$. Observe that $p^i(s^t) = q^i(s^t)p^i(\sigma(s^t))$. We also define the Arrow price $q(s^t) := p(s^t)/p(\sigma(s^t))$ for every $s^t \succ s^0$. Observe that

$$q^i(s^t) \leq q(s^t) \quad \text{with equality if } U(c^i|s^t) > U^i(e^i|s^t). \quad (\text{B.1})$$

Fix an arbitrary individually rational consumption process $\tilde{c}^i \in C^i$. For every event $s^t \succeq s^0$, we pose

$$b^i(s^t) := \frac{1}{p^i(s^t)} \sum_{s^r \in \Sigma(s^t)} p^i(s^r)(\tilde{c}^i(s^r) - c^i(s^r)).$$

The process b^i is well-defined because $p \cdot e < \infty$.⁵² Concavity of the Bernoulli function u^i implies that

$$\frac{1}{u'(c^i(s^0))} (\beta^i)^t \pi^i(s^t) [U^i(\tilde{c}^i|s^t) - U^i(c^i|s^t)] \leq p^i(s^t) b^i(s^t).$$

Properties described in (B.1) imply that⁵³

$$b^i(s^r) \leq (\tilde{c}^i(s^r) - c^i(s^r)) + \sum_{s^{r+1} \succ s^r} q(s^{r+1}) b^i(s^{r+1}).$$

Multiplying by $p(s^r)$ and summing the above inequalities over all events $s^r \succeq s^t$ that occur prior to some date $\xi > t$, we get

$$p(s^t) b^i(s^t) \leq \sum_{r=t}^{\xi-1} \sum_{s^r \in S^r(s^t)} p(s^r) [\tilde{c}^i(s^r) - c^i(s^r)] + \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi) b^i(s^\xi).$$

Since $p \cdot e < \infty$, we can also verify that⁵⁴

$$\lim_{\xi \rightarrow \infty} \sum_{s^\xi \in S^\xi(s^t)} p(s^\xi) b^i(s^\xi) = 0.$$

We have thus proved that

$$\frac{1}{u'(c^i(s^0))} (\beta^i)^t \pi^i(s^t) [U^i(\tilde{c}^i|s^t) - U^i(c^i|s^t)] \leq \sum_{r \geq t} \sum_{s^r \in S^r(s^t)} p(s^r) [\tilde{c}^i(s^r) - c^i(s^r)].$$

Observe that the RHS of the above inequality is exactly $p \cdot ([\tilde{c}^i - c^i] \mathbf{1}_{\Sigma(s^t)})$. \square

⁵²See Footnote 8 in the supplemental material in [Martins-da-Rocha and Vailakis \(2015b\)](#) for details.

⁵³See the proof of Lemma 2 in [Bloise, Reichlin and Tirelli \(2013\)](#) or the proof of Claim 2.1 in the supplemental material in [Martins-da-Rocha and Vailakis \(2015b\)](#) for details.

⁵⁴See the proof of Claim 2.1 in the supplemental material in [Martins-da-Rocha and Vailakis \(2015b\)](#) for details.

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