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Abstract

We suggest an explicit solution to the graduation problem that aims at designing an optimal curve fit which simultaneously behaves as a smoothing B-spline in each time interval (t_{j-1}, t_j) and as the Hodrick-Prescott filter in the knots t_j, $j = 1, ..., n$. We also outline its numerical performance and compare it to the existing estimates of the standard Hodrick-Prescott filter.

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1. Introduction

Graduation or data smoothing remains a powerful tool to construct life tables in actuarial science and identify trends or business growth, structural breaks and anomalies in econometric data series. This method seems also promising for other types of data series such as share prices and average temperature data. The Hodrick-Prescott filter [12] is perhaps the most widespread graduation method used by almost all central banks to identify business growth in macroeconomic data series. The filter has been proposed originally by Leser [16] for the construction of life tables, building on the graduation method developed by Whittaker [33] and Henderson [11]. Seen as a spline smoothing filter, Spoerl [26] was the first to formulate and solve it as a discrete cubic spline smoothing problem.

The filter as described in Spoerl [26], Leser [16] and later on by Hodrick and Prescott [12] defines a trend $\bar{y} = (\bar{y}_1, ..., \bar{y}_n)$ of a time series $x = (x_1, ..., x_n)$, evaluated at times $t_1 < t_2 < \cdots < t_n$, as the minimizer of

$$\sum_{t=1}^{n} (x_t - \bar{y}_t)^2 + \alpha \sum_{t=3}^{n} (\bar{y}_t - 2\bar{y}_{t-1} + \bar{y}_{t-2})^2, \tag{1.1}$$

for an appropriately chosen positive parameter α, called the *smoothing parameter*. To determine an appropriate value of the smoothing parameter α, Hodrick and Prescott suggested the time series of observations x satisfying the following linear mixed model:

$$\begin{cases} x = \bar{y} + u, \\ P\bar{y} = v, \end{cases} \tag{1.2}$$

where $u \sim N(0, \sigma_u^2 I_n)$ and $v \sim N(0, \sigma_v^2 I_{n-2})$ and P, which stands for the second order backward shift operator, is the following $(n-2) \times n$ -matrix:

$$P := \begin{pmatrix} 1 & -2 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & -2 & 1 \end{pmatrix}. \tag{1.3}$$

With this notation, the minimizer of (1.1) reads

$$\bar{y}(\alpha, x) = (I_n + \alpha P^T P)^{-1} x. \tag{1.4}$$

Using this model, the appropriate smoothing parameter turns out to be the noise-to-signal ratio $\alpha = \sigma_u^2/\sigma_v^2$ (see Schlicht [25]). This parameter is optimal in the sense that

$$\sigma_u^2/\sigma_v^2 = \arg\min_{\alpha}\{\| \mathbb{E}[\bar{y}\,|\,x] - \bar{y}(\alpha,\, x)\|^2\}. \tag{1.5}$$

Also, the optimal trend (HP-trend) that minimizes (1.1) is given by the formula

$$\mathbb{E}[\bar{y}\,|\,x] = \bar{y}\left(\frac{\sigma_u^2}{\sigma_v^2},\, x\right) = \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x. \tag{1.6}$$

The obtained trend, although optimal in the sense of (1.1), is only evaluated at each time t_j. This makes it of limited use for any analysis of the behavior of, e.g., the business growth or the average temperature in any of the time intervals $(t_{j-1},\, t_j)$ unless an appropriate smooth curve fit, such as polynomial splines, between the t_j's is used.

The purpose of this paper is to suggest a smooth curve fit which is *consistent* with the HP-filter, in the sense that the chosen smooth curve $y(t)$ is such that

$$y(t_j) = \bar{y}_j, \quad j = 1, ..., n. \tag{1.7}$$

Namely, we would like to construct a trend $y(t)$ which minimizes the following cost functional:

$$J(y) = \| x - \bar{y} \|^2 + \lambda \int_{t_1}^{t_n} (y^{(2)}(t))^2 dt + \bar{y}^T P^T W P \bar{y}, \tag{1.8}$$

for some smoothing parameters (λ, W), with $\lambda > 0$ and W positive definite matrix. Moreover, the smoothing parameters should satisfy a similar condition to (1.5). Namely,

$$(\lambda^*, W^*) = \underset{\{\lambda>0,\, W \text{ positive definite}\}}{\arg\min} \| \mathbb{E}[\bar{y}\,|\,x] - \bar{y}(\lambda, W, x)\|^2. \tag{1.9}$$

This graduation problem seems new as it aims at designing a curve fit which simultaneously behaves as a smoothing spline in each time interval $(t_{j-1},\, t_j)$ and as the HP-filter in the knots $t_j,\ j = 1, ..., n$.

When $\lambda = 0$, the problem reduces to the HP-filter, with $W = \alpha I_{n-2}$, and when $W = 0$, the problem reduces to the standard penalized polynomial spline smoothing, that we shortly recall below for the sake of clarity.

Following Schoenberg [27], Reinsch [22] and Wecker and Ansley [29] (see also the nice note by Proietti [20]), given the set of points $t_1 < \cdots < t_n$, a polynomial spline $f(t)$ of degree $p = 2\ell - 1$ with n knots t_1, ..., t_n is a piece-wise polynomial of degree $p = 2\ell - 1$ in each of the intervals $[t_i, t_{i+1})$ so that the first $2\ell - 2$ derivatives of the splines are continuous, whereas the $(p-1)$-st derivative has jumps at the knots. Moreover, a smoothing polynomial spline is (uniquely) the smoothest function that achieves a given degree of fidelity to the data set x. More precisely, the smoothing spline of degree $2\ell - 1$ minimizes

$$\sum_{i=1}^{n} (x_i - f(t_i))^2 + \lambda \int_{t_1}^{t_n} (f^{(\ell)}(t))^2 dt \tag{1.10}$$

among all functions whose first $\ell - 1$ derivatives are continuous and whose ℓth derivative is square integrable, where $\lambda > 0$ controls the balance between the fidelity of the fitted spline to the data set and its roughness.

Poirier [21] (see also Proietti [20]) suggested that the knot t_i is the timing of a structural break and the jump discontinuity at this knot reflects the occurrence of structural changes. Based on this interpretation, splines may 'provide a framework in which structural change is viewed as occurring in a smooth fashion'.

In this respect, a widely used smoothing polynomial spline model can be expressed using the truncated power basis of degree p as follows (see e.g., Green and Silverman [7], Ruppert et al. [23, 24] and Welham et al. [32]):

$$f(t) = \beta_0 + \beta_1(t - t_1) + \cdots + \beta_p(t - t_1)^p + \sum_{i=1}^{n} \eta_i (t - t_i)_+^p, \tag{1.11}$$

where

$$(t - t_i)_+^p = \begin{cases} (t - t_i)^p, & t - t_i \geq 0, \\ 0, & t - t_i < 0. \end{cases}$$

When f is random, the spline coefficient η_i, which regulates the size of the break is interpreted as a random shock that drives the evolution of $f(t)$ and is subject to penalization (see e.g., Ruppert et al. [24]), whereas $(t - t_i)_+^p$ describes its impulse response function, that measures the impact on the future values of *f*.

Denoting $\eta = (\eta_1, ..., \eta_n)$, $\beta = (\beta_0, ..., \beta_p)$, $X(t) = (1, (t - t_1), ..., (t - t_1)^p)$ and $Z(t) = ((t - t_1)_+^p, ..., (t - t_n)_+^p)$, f reads

$$f(t) = X^T(t)\beta + Z^T(t)\eta.$$

The penalty in the minimizer (1.10) can be written as

$$\int_{t_1}^{t_n} (f^{(p-1)}(t))^2 dt = \eta^T G \eta, \tag{1.12}$$

where the matrix *G* is the Gramian associated with *Z*, defined by

$$G = \int_{t_1}^{t_n} \frac{d^{p-1}Z(t)}{dt^{p-1}} \frac{d^{p-1}Z^T(t)}{dt^{p-1}} dt.$$

Hence, the smoothing polynomial spline minimizes

$$\sum_{i=1}^{n} (x_i - f(t_i))^2 + \lambda \eta^T G \eta, \tag{1.13}$$

where, as noted above, the penalization is put on the shocks η. This is also the case for other families of splines such as the cubic splines with periodic constraints used in Kauermann et al. [13], or the P-splines, based on reduced knot B-spline bases, suggested by Eilers and Marx [6] (see also Egerstedt and Martin [5]), Kano et al. [14] and the references therein), whose close connection to the above mentioned family of penalized splines, using a truncated power function basis, is established in Welham et al. [32].

For a cubic smoothing spline, corresponding to $p = 3$, the penalization in (1.13), put on the shocks η, does not correspond to the one imposed on the HP-trend in (1.1), and the constructed trend does not agree with the HP filter at the knots, unless the second derivative is replaced by its natural discretized version, which

amounts to fit a cubic spline, using the criterion (1.8) with $\lambda = 0$ and $W = \alpha I_{n-2}$, as suggested in Spoerl [26] and recently proved by Paige and Trindade [19, Theorem 4.1].

In the main result of the paper (Theorem 1), using the control theoretic approach to smoothing splines suggested in Kano et al. [14], Welham et al. [32] or Egerstedt and Martin [5], we make use of B-spline functions to suggest a solution to the constrained minimization problem (1.7)-(1.9). The constructed optimal smooth curve is shown to be *consistent* with the HP-filter, for a given set of discrete data points $x = (x_1, ..., x_n)$, observed at times (knots) $t_1 < t_2 < \cdots < t_n$.

The flexibility of this approach seems suit well our purpose to design an optimal smoothing spline which coincides with the HP-filter. Choosing to use B-splines among other functions is a matter of taste. We simply find them attractive for nonparametric modeling and numerically tractable (see Eilers and Marx [6] for further details).

In Section 2, we review the control theoretic approach to smoothing B-splines which will be applied in Section 3 to construct an optimal smoothing B-spline trend for the HP-filter. Finally, in Section 4, we present some numerical examples. The displayed examples are not aimed at proving the superiority of this graduation technique compared with either the HP-filter or the smoothing splines. A comparison between these smoothing techniques needs a more involved analysis of many data sets than aimed in this paper.

2. Optimal Smoothing B-splines

In this section, we briefly review the control theoretic approach to smoothing splines suggested in Kano et al. [14], Welham et al. [32] or Egerstedt and Martin [5] to construct optimal "smooth" curve for a given set of discrete data points $x = (x_1, ..., x_n)$, observed at times (knots) $t_1 < t_2 < \cdots < t_n$, by using B-spline functions.

Let $B_k(s)$ be a normalized, uniform B-spline of degree k with integer knot points (cf. de Boor [1]) defined by

$$B_k(t) = \begin{cases} N_{k-j,k}(t - j), & j \le t \le j + 1, \quad j = 0, ..., k, \\ 0, & t \le 0, \quad k + 1 \le t. \end{cases} \tag{2.1}$$

The basis elements $N_{k-j,k}(t)$ are obtained recursively by the following scheme:

$$\begin{cases} N_{0,0}(t) = 1, \\ N_{0,k}(t) = \dfrac{1-t}{k} N_{0,k-1}(t), \\ N_{k,k}(t) = \dfrac{1}{k} N_{k-1,k-1}(t), \\ N_{j,k}(t) = \dfrac{k-j+t}{k} N_{j-1,k-1}(t) + \dfrac{1+j-t}{k} N_{j,k-1}(t), \ j = 1, \ldots, k-1, \end{cases} \tag{2.2}$$

together with the normalization

$$\sum_{j=0}^{k} N_{j,k}(t) = 1, \quad 0 \le t \le 1. \tag{2.3}$$

Let $m = n + k - 1$ for $n \ge k - 1$, $k \ge 1$ and consider the curve $f(t)$ generated by weighted and time shifted sum of B-splines, namely

$$f(t) = \sum_{j=1}^{m} \tau_j B_k(\beta(t - t_{j-k})), \quad t_1 \le t \le t_n. \tag{2.4}$$

Here $\beta > 0$ is a constant and t_j 's are equally spaced knot points with

$$t_{i+1} - t_i = \frac{1}{\beta}. \tag{2.5}$$

By a suitable choice of the coefficients, or control points τ_i, $f(t)$ can represent arbitrary polynomial splines of degree k in the interval $[t_1, t_n]$.

Denote

$$\tau = (\tau_1, \ldots, \tau_m)^T \in \mathbb{R}^m, \tag{2.6}$$

$$b(t) = (B_k(\beta(t - t_{1-k})), \ldots, B_k(\beta(t - t_{n-1})))^T \in \mathbb{R}^m \tag{2.7}$$

and

$$B = (b(t_1), \ldots, b(t_n)) \in \mathbb{R}^{n \times m}, \tag{2.8}$$

and, finally, let $\bar{f}$ denote the column vector

$$\bar{f} = (f(t_1), ..., f(t_n))^T. \tag{2.9}$$

In view of (2.4), we have

$$f(t) = \tau^T b(t) \tag{2.10}$$

and

$$\bar{f} = B\tau. \tag{2.11}$$

A smoothing spline is required to have a continuous first derivative and a square integrable second derivative which, in terms of the control points τ, can be expressed as

$$\int_{t_1}^{t_n} (f^{(2)}(t))^2 dt = \tau^T G\tau, \tag{2.12}$$

where $G \in \mathbb{R}^{m\times m}$ is the Gramian associated to b defined by

$$G = \int_{t_1}^{t_n} \frac{d^2 b(t)}{dt^2} \frac{d^2 b^T(t)}{dt^2} dt. \tag{2.13}$$

The criterion (1.10) of finding optimal smoothing splines, translates into finding optimal control points τ which minimize the following cost functional:

$$J(\tau) = \| B\tau - x \|^2 + \lambda\tau^T G\tau, \tag{2.14}$$

where $\lambda > 0$ is a smoothing parameter. Minimizing $J(\tau)$ with respect to τ is by now standard, and the optimal solution is obtained by

$$\tau^* = (\lambda G + B^T B)^{-1} B^T x, \tag{2.15}$$

provided that the matrix $\lambda G + B^T B$ is positive definite. In general, G and $B^T B$ are only positive semi-definite. But, for specific classes of B-splines such as the cubic splines $(k = 3)$ that are most frequently used in many applications, the matrix $\lambda G + B^T B$ is positive definite. Indeed, for the cubic splines, Kano et al. [14] (see Lemma 2 and Theorem 3), obtained the normalized B-uniform spline function $B_3(t)$ and the Gramian G explicitly and analyzed many of their properties.

In particular,

$$B_3(t) = \begin{cases} N_{3,3}(t) = \frac{1}{6}t^3, \quad 0 \le t < 1, \\ N_{2,3}(t-1) = \frac{1}{6}(1 + 3(t-1) + 3(t-1)^2 - 3(t-1)^3), \quad 1 \le t < 2, \\ N_{1,3}(t-2) = \frac{1}{6}(4 - 6(t-2)^2 + 3(t-2)^3), \quad 2 \le t < 3, \\ N_{0,3}(t-3) = \frac{1}{6}(4-t)^3, \quad 3 \le t < 4, \\ 0, \quad \text{otherwise.} \end{cases} \tag{2.16}$$

Moreover, the matrix $B \in \mathbb{R}^{n \times n+2}$ is explicitly obtained as

$$B := \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 \end{pmatrix}. \tag{2.17}$$

The rank of B being n, it follows that BB^T is invertible.

We have the following.

Lemma 2.1 (Kano et al. [14])**.** *For the cubic B-splines, let* Ω *be a positive definite matrix. Then it holds that* $\lambda G + B^T \Omega B$ *is definite positive.*

In the rest of the paper, we will only consider cubic splines, but the results extend easily to general splines provided some further assumptions on the matrix B and the Gramian G.

3. A Smoothing B-spline Trend for the HP-filter

In this section, we apply this control theoretic approach to optimal smoothing B-splines to construct a smoothing *cubic* spline curve $y(t)$ which coincides with optimal trend given by the Hodrick-Prescott filter at the knots $t_1 < t_2 < \cdots < t_n$. This amounts to finding an optimal curve which satisfies the criterion (2.14) and, moreover, is smooth at the knots in the sense that the backward second difference of the $\bar{y}_i := y(t_i)$'s is white noise. Namely, we would like to construct a trend $y(t)$

that satisfies (recall the notation above)

$$\begin{cases} x = \bar{y} + u, \\ P\bar{y} = v, \\ y(t) = b(t)^T \tau, \end{cases} \tag{3.1}$$

where $u \sim N(0, \sigma_u^2 I_n)$ and $v \sim N(0, \sigma_v^2 I_{n-2})$, which minimizes the following cost functional:

$$J(y) = \| x - \bar{y} \|^2 + \lambda \int_{t_1}^{t_n} (y^{(2)}(t))^2 dt + \bar{y}^T P^T W P \bar{y}, \tag{3.2}$$

for some smoothing parameters (λ, W), with $\lambda > 0$ and W positive definite matrix. The smoothing parameters should satisfy a similar condition to (1.5). Namely, using the notation $\bar{y} = B\tau$,

$$(\lambda^*, W^*) = \underset{\{\lambda>0,\, W \text{ positive definite}\}}{\arg\min} \| \mathbb{E}[\bar{y} \,|\, x] - B\hat{\tau}(\lambda, W, x) \|^2. \tag{3.3}$$

In terms of the control points τ, since $y(t) = b(t)^T \tau$, the cost functional (3.2) can be written as a quadratic function of τ as follows:

$$\begin{aligned} J(\tau) &= \| x - \bar{y} \|^2 + \lambda \int_{t_1}^{t_n} (y^{(2)}(t))^2 dt + \bar{y}^T P^T W P \bar{y} \\ &= (x - \bar{y})^T (x - \bar{y}) + \lambda \tau^T G \tau + \tau^T B^T P^T W P B \tau \\ &= x^T x - 2\tau^T B^T x + \tau^T B^T B \tau + \lambda \tau^T G \tau + \tau^T B^T P^T W P B \tau \\ &= x^T x - 2\tau^T B^T x + \tau^T (\lambda G + B^T (I_n + P^T W P) B) \tau \\ &= x^T x - 2\tau^T B^T x + \tau^T L \tau, \end{aligned} \tag{3.4}$$

where

$$L = \lambda G + B^T (I_n + P^T W P) B \tag{3.5}$$

and $G \in \mathbb{R}^{m \times m}$ is the Gramian (2.13) associated to $b(t)$. We notice that the matrix $\Omega = I_n + P^T W P$ is positive definite, since it is the sum of a positive definite and a

positive semi-definite matrix. Hence, by Lemma 2.1, L in (3.5) is positive definite. Thus, the optimal solution of (3.4) is

$$\hat{\tau} = L^{-1}B^T x,$$

which, in terms of (λ, W) and x, reads

$$\hat{\tau}(\lambda, W, x) = (\lambda G + B^T(I_n + P^T W P)B)^{-1}B^T x.$$

Therefore, by (3.1), our cubic B-spline optimal trend is

$$\hat{y}(t) = b(t)^T \hat{\tau}, \quad t \in [t_1, t_n]. \tag{3.6}$$

In view of (1.6), (λ^*, W^*) is optimal for the problem (3.3) if and only if

$$B\hat{\tau}(\lambda^*, W^*, x) = \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x,$$

or, equivalently, if and only if

$$B(\lambda^* G + B^T(I_n + P^T W^* P)B)^{-1}B^T x = \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x. \tag{3.7}$$

We need to show that there is always a pair (λ^*, W^*) such that (3.7) holds and to find the optimal $\hat{\tau}(\lambda^*, W^*, x)$.

Theorem 1. *Given a time series x and a parameter* $\sigma_u^2/\sigma_v^2 > 0$, *there is a set of smoothing parameters* (λ^*, W^*), $\lambda^* > 0$ *and* W^* *positive definite, such that*

$$B\hat{\tau}(\lambda^*, W^*, x) = \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x, \tag{3.8}$$

where $\hat{\tau}(\lambda^*, W^*, x)$ *is explicitly given by*

$$\hat{\tau}(\lambda^*, W^*, x) = B^T(BB^T)^{-1}\left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x. \tag{3.9}$$

Moreover, the set of smoothing parameters (λ^*, W^*) *must satisfy the balance*

equation

$$W^* = \frac{\sigma_u^2}{\sigma_v^2} I_{n-2} - \lambda^* (PP^T)^{-1} P (BB^T)^{-1} BGB^T (BB^T)^{-1} P^T (PP^T)^{-1},$$

which always has a feasible solution.

Finally, the cubic B-spline optimal trend is

$$\hat{y}(t) = b(t)^T B^T (BB^T)^{-1} \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P \right)^{-1} x. \tag{3.10}$$

Proof. Consider the problem

$$B\hat{\tau}(\lambda, W, x) = B(\lambda G + B^T (I_n + P^T W P) B)^{-1} B^T x = \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P \right)^{-1} x. \tag{3.11}$$

If there are (λ^*, W^*) such that (3.11) holds, then it is solved using the fact that

$$(\lambda^* G + B^T (I_n + P^T W^* P) B)^{-1} B^T = B^T (BB^T)^{-1} \left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P \right)^{-1},$$

which, in turn, may be rewritten as

$$W^* = \frac{\sigma_u^2}{\sigma_v^2} I_{n-2} - \lambda^* (PP^T)^{-1} P (BB^T)^{-1} BGB^T (BB^T)^{-1} P^T (PP^T)^{-1}, \tag{3.12}$$

using the fact that BB^T and PP^T are invertible. This is the relation that must hold between W^* and λ^* for the solution to hold. The first part of the proof is dedicated to proving that there exists a set (λ^*, W^*) such that (3.12) has a solution.

The first part of the expression on the right hand side of (3.12) is positive definite, since I_{n-2} is just the identity matrix and the parameter $\sigma_u^2/\sigma_v^2 > 0$. Since $(PP^T)^{-1} = ((PP^T)^{-1})^T$, $(BB^T)^{-1} = ((BB^T)^{-1})^T$ and G is positive semi-definite and symmetric, as seen in Kano et al. [14],

$$\lambda^* (PP^T)^{-1} P (BB^T)^{-1} BGB^T (BB^T)^{-1} P^T (PP^T)^{-1} := \lambda^* A$$

is positive semi-definite.

For $n \geq 3$, set

$$\lambda_{n-2} = \max_{v_i,\, i=1,\, 2,\, ...,\, n-2} v_i^T A v_i,$$

where

$$v_i = (\underbrace{0,\ 0,\ ...,\ 0}_{i-1 \text{ zeros}},\ 1,\ 0,\ ...,\ 0)^T \in \mathbb{R}^{n-2}$$

are the eigenvectors of I_{n-2}.

Let z be an arbitrary vector in $\mathbb{R}^{n-2}$, expressed in the basis vectors of I_{n-2},

$$z = \sum_{i=1}^{n-2} c_i v_i, \text{ for some } c_i \in \mathbb{R}, \quad i = 1, ..., n-2.$$

Then $\frac{\sigma_u^2}{\sigma_v^2} I_{n-2} - \lambda^* A$ is positive definite if

$$z^T \left(\frac{\sigma_u^2}{\sigma_v^2} I_{n-2} - \lambda^* A \right) z > 0, \text{ for all } z \neq 0. \tag{3.13}$$

Defining the semi-norm $\| z \|^2 = z^T A z$ and using the Cauchy-Schwarz inequality, gives

$$z^T A z = \left(\sum_{i=1}^{n-2} c_i v_i \right)^T A \left(\sum_{i=1}^{n-2} c_i v_i \right) = \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} (c_i v_i)^T A (c_j v_j)$$

$$= \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} c_i c_j v_i^T A v_j \leq \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} | c_i c_j | \sqrt{v_i^T A v_i} \sqrt{v_j^T A v_j} \leq \lambda_{n-2} \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} | c_i c_j |$$

$$\leq \lambda_{n-2} \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} c_i^2 = \lambda_{n-2} (n-2) \sum_{i=1}^{n-2} c_i^2, \text{ for any } z \in \mathbb{R}^{n-2},$$

where we have used the fact that $\max_{v_i,\, i=1,\, 2,\, ...,\, n-2} v_j^T A v_j \leq \lambda_{n-2}$. Therefore,

$$z^T\left(\frac{\sigma_u^2}{\sigma_v^2} I_{n-2} - \lambda^* A\right) z \geq \frac{\sigma_u^2}{\sigma_v^2}\sum_{i=1}^{n-2} c_i^2 - \lambda^*\lambda_{n-2}(n-2)\sum_{i=1}^{n-2} c_i^2$$

$$= \left(\frac{\sigma_u^2}{\sigma_v^2} - (n-2)\lambda^*\lambda_{n-2}\right)\sum_{i=1}^{n-2} c_i^2, \text{ for any } z \in \mathbb{R}^{n-2},$$

which is strictly positive for any

$$0 < \lambda^* < \frac{1}{(n-2)\lambda_{n-2}}\frac{\sigma_u^2}{\sigma_v^2}.$$

This proves that there exists a parameter set (λ^*, W^*) such that $\lambda^* > 0$ and W^* positive definite that solves equation (3.11).

Taking a set (λ^*, W^*) such that (3.12) holds, (3.11) is solved by taking

$$\hat{\tau}(\lambda, W, x) = B^T(BB^T)^{-1}\left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x.$$

Hence, the cubic B-spline that coincides with the Hodrick-Prescott trend at the times $(t_1, t_2, ..., t_n)$, using the smoothing parameter σ_u^2/σ_v^2, may be written as

$$\hat{y}(t) = b(t)^T B^T(BB^T)^{-1}\left(I_n + \frac{\sigma_u^2}{\sigma_v^2} P^T P\right)^{-1} x. \qquad \square$$

4. Numerical Illustration

Given the time series $x = (x_1, ..., x_n)$, we will compare the results of the spline approximation (3.10) of the trend $\bar{y}$ defined in (1.2) using different estimators of the optimal smoothing parameter σ_u^2/σ_v^2. The first estimator we use is the consistent estimator of the noise-to-signal ratio $\alpha_{DDR} = \widehat{\sigma_u^2/\sigma_v^2}$ proposed in Dermoune et al. [4],

$$\widehat{\sigma_u^2/\sigma_v^2} = \max\left\{0, -\frac{1}{4}\left(\frac{3}{2} + \frac{(n-3)\sum_{j=1}^{n-2}\xi_j^2}{(n-2)\sum_{j=1}^{n-3}\xi_j\xi_{j+1}}\right)^{-1}\right\}, \qquad (4.1)$$

where $\xi = Px$. Secondly, we look at the Generalized Cross Validation (GCV)

estimator of σ_u^2/σ_v^2, α_{GCV}, which is the minimizer of the GCV score, as described in Wahba [28], and solved efficiently in Weinert [30, 31]. The third smoothing parameter we use is the one proposed by Hodrick and Prescott [12], $\alpha_{HP} = 1600$, for quarterly time series.

The resulting smoothing parameters are presented in Table 1 and the fitted trends using these parameters are presented in Figures 1-4, where we see that the cubic spline coincides perfectly with the HP-trend at the knot points, as predicted in Theorem 1.

Table 1. Estimated values of the smoothing parameter α as defined in Section 4

	US GNP	GBP/USD	S&P500	NIKKEI225
α_{DDR}	0.321	0.446	0.267	0.539
α_{GCV}	0.240	0.198	3.65e-3	0.263
α_{HP}	1600	1600	1600	1600

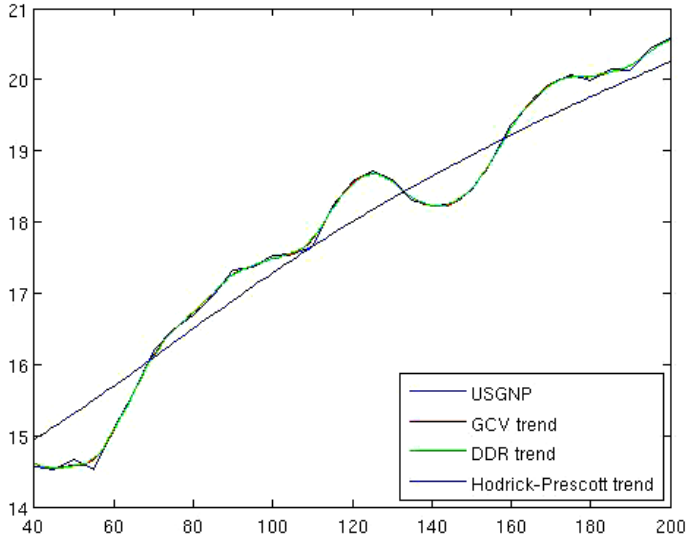

Figure 1. US quarterly GNP from 1947 Q1 to 2010 Q2. Index number 2005 = 100. The cross is the resulting HP-trend while the solid line is the time series and the dotted line is the cubic spline using α_{DDR}.

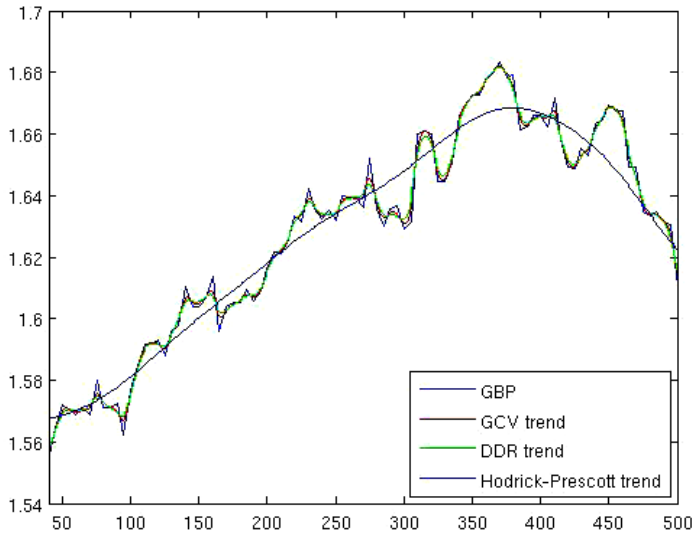

Figure 2. Daily GBP/USD close prices from 03/31/03 to 10/01/10. The cross is the resulting HP-trend while the solid line is the time series and the dotted line is the cubic spline using α_{DDR}.

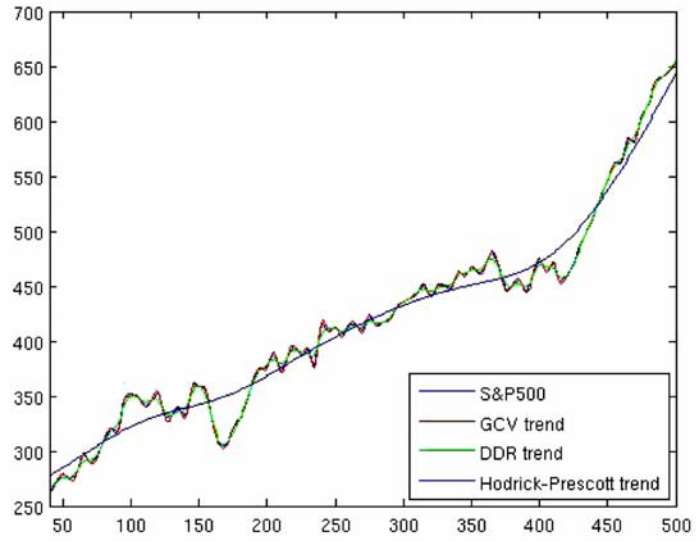

Figure 3. Monthly close prices of S&P500 from Dec. 1987 to Aug. 2010. The cross is the resulting HP-trend while the solid line is the time series and the dotted line is the cubic spline using α_{DDR}.

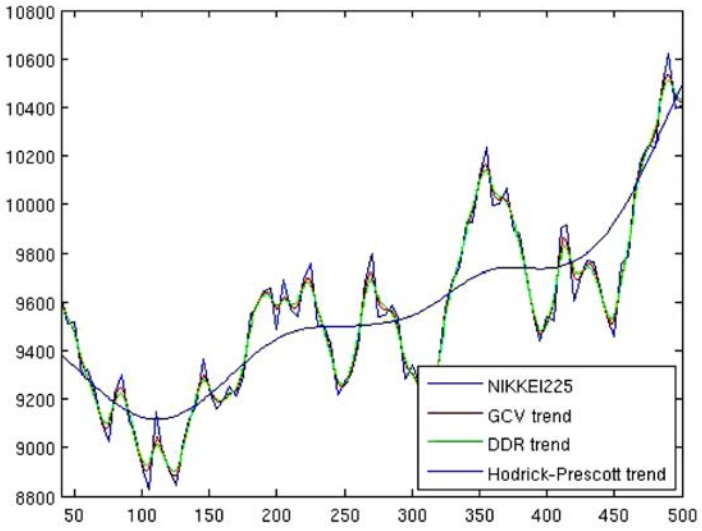

Figure 4. Daily adjusted close prices of NIKKEI225 from 01/04/02 to 10/01/10. The cross is the resulting HP-trend while the solid line is the time series and the dotted line is the cubic spline using α_{DDR}.

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