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# HARNACK INEQUALITIES FOR EQUATIONS OF TYPE PRESCRIBED SCALAR CURVATURE.

SAMY SKANDER BAHOURA

ABSTRACT. We give Harnack inequalities for solutions of equations of type prescribed scalar curvature in dimensions  $n \geq 4$ .

## 1. INTRODUCTION AND MAIN RESULTS

We consider on a Riemannian manifold  $(M, g)$  of dimension  $n \geq 4$ , the equation:

$$\Delta u + hu = Vu^{(n+2)/(n-2)}, u > 0, \quad (E)$$

with,  $h$  a smooth function and  $0 < a \leq V(x) \leq b < +\infty$ ,  $\|\nabla V\|_\infty \leq A$ .

Equation of this type were considered by many authors, see [1-25]. This equation arise in physics and astronomy. Here we look to a priori estimates of type sup, inf which are characteristic of this equation.

Let  $(u_k)$  a sequence of regular solutions of  $(E)$ .

We fix a compact set  $K$  of  $M$ . We want to prove that: for each compact, for all terms of the sequence  $(u_k)$ :  $\sup_K u_k$  and  $\inf_M u_k$  are linked. Here we prove a weaker inequality for blow-up solutions of the previous equation. Equations of previous type are called, Yamabe equation, prescribed scalar curvature equation, of type prescribed scalar curvature and Schrodinger equation. Here we prove that for blow-up solutions, precisely for each sequence  $(u_k)$ , there is a positive function  $c > 0$ , such that for all compact set, there is a sequence of positive numbers,  $0 < \epsilon_k(K) \leq 1$ , which link  $\sup_K u_k$  and  $\inf_M u_k$ .

Here we have two possibilites up to a subsequence we have a compactness result or an inequality between  $\sup_K u_k$  and  $\inf_M u_k$ . Note that for Li-Zhang result in dimension 3 and 4, they consider the problem around a point, thus the compactness result. Also, for the Harnack inequality, also, see the introduction of [2].

We obtain:

**Theorem 1.1.** *We have:*

1) *There is a compact  $K_0$  of  $M$  and a subsequence  $i_j$  and a positive constant  $C > 0$ , such that:*

$$\sup_{K_0} u_{i_j} \leq C, \quad \forall j.$$

Or,

2) *For all compact  $K$ ,  $\sup_K u_k \rightarrow +\infty$  and:*

$$n = 4, \quad (\sup_K u_k)^{1-\epsilon} \leq c(a, b, A, \frac{\inf_M u_k}{(\sup_K u_k)^\epsilon}, K, M, g),$$

with  $\epsilon > 0$ ,

and,

$$n \geq 5, \quad (\sup_K u_k)^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, n, \epsilon_k \frac{\inf_M u_k}{(\sup_K u_k)^\epsilon}, K, M, g),$$

with,  $\epsilon > \frac{(n-4)}{n-2}$

We can see that  $\sup_K u_k$  and  $\inf_M u_k$  are linked. There is a relation which link these two quantities. For all compact  $K$  and all  $k \in \mathbb{N}$ ,  $\sup_K u_k$  and  $\inf_M u_k$  are linked by the previous relations. There is a positive function  $c(\cdot, \cdot, \cdot) > 0$ , such that for all  $k$ ,  $\sup_K u_k = c(n, K, \epsilon_k, \inf_M u_k)$ . If we denote  $F = \{u_k\}$ ,  $G = \{\epsilon_k = \epsilon_k(K)\}$ ,  $F \times G = \{(u_k, \epsilon_k)\}$ ,  $0 < \epsilon_k = \epsilon_k(K) \leq 1$ , we have:

$$\exists c(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon') \in F \times G, \sup_K u = c(n, K, \epsilon'(K), \inf_M u)$$

or,

$$\exists c(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon') \in F \times G, (\sup_K u)^{1-\epsilon} \leq c(n, K, \epsilon'(K), \frac{\inf_M u}{(\sup_K u)^\epsilon}).$$

We write this to highlight the rolling-up phenomenon and the distortion.

**Remarks:**

a) In the previous theorem, the point 1) assert that, up to a subsequence we have a compactness result.

The point 2) assert that we have a relation between the local supremum and the global infimum. Also, see the introduction of [2].

b) For the point 1) we have one parameter, the local supremum is controled by its self. For the point 2) we have two parameters, the local supremum and the global infimum, there is a relation which link those two quantities. In the paper "Estimations du type  $\sup \times \inf$  sur une variété compacte", we have 3 paramaters, the local supremum, the local infimum and the global supremum. Here, we have, 1 parameter or 2 paramaters each time. At most 2 parameters and at least 1 parameter. Also, see the introduction of [2].

c) In the transformation  $u \rightarrow v = \lambda u(\lambda^{2/(n-2)}x)$ , for the rescaling or blow-up, we have approximatively, the constant  $c(m/\lambda) \equiv c(m)/\lambda$ . Because,  $v \geq m \Rightarrow v(0) \leq c(m)$ , but this it is equivalent to,  $u \geq m/\lambda \Rightarrow u(0) \leq c(m)/\lambda$ , which imply that  $c(m/\lambda) \equiv c(m)/\lambda$ . Thus, the rescaling by  $\lambda$  imply a relation of the type (approximatively):  $c(m/\lambda) \equiv c(m)/\lambda$ .

d) In general as in the paper of Li-Zhang, for the dimensions 3 and 4, of the Yamabe equation, we look to the estimate around a point. The compactness result is important, but also, we look to the solutions which blow-up as mentionned by the example  $x \rightarrow [\epsilon/(\epsilon^2 + |x|^2)]^{(n-2)/2}$ ,  $\epsilon \rightarrow 0$ , thus the point 2). Also, see the introduction of [2].

e) To summarize the points 1) and 2) in one inequality we have:  $\{1\} \text{ or } 2\} \Rightarrow \{1\} + 2\}$ : there is a subsequence  $(u_{i_j})$  such that:

$$n = 4, (\sup_K u_{i_j})^{1-\epsilon} \leq C + c(a, b, A, \frac{\inf_M u_{i_j}}{(\sup_K u_{i_j})^\epsilon}, K, M, g),$$

with  $\epsilon > 0$ ,  
and,

$$n \geq 5, (\sup_K u_{i_j})^{1-\epsilon} \leq C + \epsilon_{i_j}^{1-\epsilon} c(a, b, A, n, \epsilon_{i_j}^\epsilon \frac{\inf_M u_{i_j}}{(\sup_K u_{i_j})^\epsilon}, K, M, g),$$

with,  $\epsilon > \frac{(n-4)}{n-2}$ .

In the Theorem 1.1: we have:  $\{1\} \text{ or } 2\}$  is stronger than  $\{1\} + 2\}$ . We have:  $\{1\} \text{ or } 2\} \geq \{1\} + 2\}$ . We have:  $\{1\} \text{ or } 2\} \Rightarrow \{1\} + 2\}$ .

For all sequence  $(u_k)$  there is a subsequence  $(u_{i_j})$  who satisfies the Harnack inequality (summarizing:  $\{1\} \text{ or } 2\}$  in  $\{1\} + 2\}$ ), we have  $\{1\} \text{ or } 2\}$  in one inequality:

If we denote  $\tilde{F} = \{u_{i_j}\}$ ,  $\tilde{G} = \{\epsilon_{i_j} = \epsilon_{i_j}(K)\}$ ,  $\tilde{F} \times \tilde{G} = \{(u_{i_j}, \epsilon_{i_j})\}$ ,  $0 < \epsilon_{i_j} = \epsilon_{i_j}(K) \leq 1$ , we have:

$$\exists \tilde{c}(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon') \in \tilde{F} \times \tilde{G}, (\sup_K u)^{1-\epsilon} \leq \tilde{c}(n, K, \epsilon'(K), \frac{\inf_M u}{(\sup_K u)^\epsilon}).$$

## 2. PROOF OF THE RESULT

For the proof, we use the computations of previous papers with modifications, see [4,6,9,10].

1) *blow-up analysis:*

Let  $(u_k)$  a sequence of solutions of  $(E)$ . We fix a compact set  $K$  of  $M$ . We want to prove that: for each compact, for all terms of the sequence  $(u_k)$ :  $\sup_K u_k$  and  $\inf_M u_k$  are linked.

1) If there is a "big" compact  $K_0$  for which there is a subsequence  $\sup_{K_0} u_{i_j}$  is bounded, then we have a compactness result for a "big"  $K_0$  and for  $K \subset K_0$  the sequence  $(u_{i_j})$  is bounded.

2) If for all compact  $K$ ,  $\sup_K u_k \rightarrow +\infty$ . We do a blow-up. We consider  $\sup_K u_k = u_k(y_k)$ . Consider  $R_k \rightarrow 0$ ,  $R_k^{(n-2)/2} = [u_k(y_k)]^{-\epsilon}$  with  $0 < \epsilon < 1$ . Then:

$$R_k^{(n-2)/2} \sup_{\bar{B}_{R_k}(y_k)} u_k \geq c_k = [u_k(y_k)]^{1-\epsilon} \rightarrow +\infty.$$

We use the blow-up technique to have,  $\exists t_k, \bar{t}_k, u_k(t_k) \geq u_k(\bar{t}_k) \geq u_k(y_k) \rightarrow +\infty$ .

$$\bar{t}_k, \sup_{\bar{B}_{R_k}(y_k)} u_k = u_k(\bar{t}_k) \geq u_k(y_k) > 0,$$

We consider  $s_k(y) = (R_k - d(y, \bar{t}_k))^{(n-2)/2} u_k(y)$ , and,

$$t_k, \sup_{\bar{B}_{R_k}(\bar{t}_k)} s_k = s_k(t_k) \geq s_k(\bar{t}_k) = R_k^{(n-2)/2} u_k(\bar{t}_k) \geq R_k^{(n-2)/2} u_k(y_k) > 0,$$

We do a blow-up, then we consider:

$$n = 4, v_k(y) = r_k u_k(t_k + r_k y) = r_k u(\exp_{t_k}(r_k y)), r_k = [u_k(y_k)]^{-\epsilon},$$

with,  $\epsilon > 0$ ,  
and,

$$n \geq 5, v_k(y) = r_k u_k(t_k + (r_k)^{2/(n-2)} y) = r_k u_k(\exp_{t_k}(r_k^{2/(n-2)} y)), r_k = [u_k(t_k)]^{-\epsilon},$$

with,  $\epsilon > \frac{(n-4)}{n-2}$ .

Note that, here we have considered all terms of the sequence  $(u_k)$ .

Let's consider the blow-up functions  $(v_k)$  defined previously with the exponential maps for  $n \geq 4$ ,  $\exp_{t_k}(y)$ , like in the previous papers for the dimensions, 4, 5, 6. Because we consider the compact sets  $K, 2K$ , and  $t_k \in 2K$ , the injectivity radius is uniformly bounded below by a positive number. Thus, we can consider all the terms of the sequence  $(u_k)$  without extraction. (After supposing the assertion  $\inf v_k \geq m > 0$  infinitely many times, we can use extraction, for the points  $(t_k)$ ).

We consider,  $\delta_0 = \delta_0(K) = \inf\{\delta_P/4, P \in K\}$ , with  $\delta_P$  continuous in  $P$  and smaller than the injectivity radius in  $P$  for each  $P$ . We have a finite cover of  $K$  by small balls of radius  $\delta_0/2$ , we have a finite set of points  $z_j \in K$ :  $K \subset \cup_{\{j=0, \dots, l\}} B(z_j, \delta_0/2) \subset \cup_{\{j=0, \dots, l\}} \bar{B}(z_j, 3\delta_0) = K_{\delta_0}$  is compact.

We take  $R_k^{(n-2)/2} = \inf\{u_k(y_k)^{-\epsilon}, (\delta_0/2)^{(n-2)/2}\}$ . The small balls are all compact, thus,  $\bar{t}_k$  exist and  $t_k$  exist. We take for  $n = 4$ ,  $r_k = \inf\{\delta_P/4, P \in K_{\delta_0}, u_k(y_k)^{-\epsilon}\}$  and, for  $n \geq 5$ ,  $r_k = \inf\{\delta_P/4, P \in K_{\delta_0}, u_k(t_k)^{-\epsilon}\}$ . Thus  $\bar{t}_k, t_k$  and  $\exp_{t_k}(\cdot)$  and  $v_k$  are defined for all  $k \geq 0$ .

We fix  $m > 0$ , we prove the result by assuming  $\inf v_k \geq m > 0$ , like for the dimensions 4 and 6. After we take  $m = r_k \inf u_k > 0$ . Suppose by contradiction, that there are infinitely many  $(v_k)$  with  $\inf v_k \geq m > 0$ , the proof imply that  $[u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty$  which is impossible. Thus, there is a finite number of terms such that  $\inf v_k \geq m > 0$ ,  $k_1, \dots, k_{j(m)}$ . Thus we have also,  $v_k(0) \leq c(m)$  when  $\inf v_k \geq m > 0$ . In all cases, we have the following assertion:

$$\inf v_k \geq m > 0 \Rightarrow [u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty,$$

we obtain:

There is a non-increasing positive function  $m \rightarrow c(m) > 0$ , such that  $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} \leq c(m)$ . then we apply this with  $m = r_k \inf_M u_k$ , we obtain for all terms of the sequence  $(u_k)$ :

$$n = 4, [u_k(y_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf_M u_k}{[u_k(y_k)]^\epsilon}, K, M, g),$$

and,

$$n \geq 5, [u_k(t_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf_M u_k}{[u_k(t_k)]^\epsilon}, K, M, g).$$

For  $n \geq 5$ , we set,  $0 < \epsilon_k = \frac{u_k(y_k)}{u_k(t_k)} \leq 1$ ,  $\epsilon_k = \epsilon_k(K)$ , we obtain:

$$n \geq 5, [u_k(y_k)]^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf_M u_k}{[u_k(y_k)]^\epsilon}, K, M, g).$$

We have  $u_k(y_k) = \sup_K u_k$ .

Here in the blow-up analysis, we supposed that there are infinitely many  $v_k$  with  $\inf v_k \geq m > 0$ , without loss of generality we assume that the subsequence is the sequence, also for the points  $t_k$  there is a subsequence which converge to a point  $\tilde{t} \in K_{\delta_0}$ , without loss of generality we assume that the subsequence is the sequence. In the blow-up function (for  $k$  large), the blow-up of  $v_k$  is the blow-up of  $u_k$ , then we use the diagonal process to extract a subsequence which converge on compact sets to  $v$  and we use Caffarelli-Gidas-Spruck result to have  $v = (\frac{1}{1+V(\tilde{t})|x|^2})^{(n-2)/2}$ . Without loss of generality we assume that this subsequence is the sequence and  $V(\tilde{t}) = 1$ . Also we have:  $\inf v_k \geq m > 0, t_k \rightarrow \tilde{t} \in K_{\delta_0}$ .

## II) Auxiliary function and moving-plane method:

We use the computations of previous papers with modifications, see [4,6,8,9,10]. We consider:

$$z_k(t, \theta) = e^{(n-2)t/2} u_k(\exp_{t_k}(e^t \theta)),$$

and the blow-up function,

$$w_k(t, \theta) = e^{(n-2)t/2} v_k(e^t \theta) = z_k(t + \frac{2}{n-2} \log r_k, \theta),$$

We have  $\lambda_k = \frac{-2}{n-2} \log v_k(0)$ ,  $N = \frac{2n}{n-2}$ . We have:

Let,

$$b_1 = J(t_k, e^t, \theta) = \sqrt{\det(g_{ij, t_k})(e^t \theta)}, \quad a(t_k, t, \theta) = \log J(t_k, e^t, \theta).$$

**Lemma 2.1.** *The function  $z_k$  is solution of:*

$$(1) \quad -\partial_{tt} z_k - \partial_t a \partial_t z_k + \Delta_\theta z_k + c z_k = V_k z_k^{(n+2)/(n-2)},$$

with,

$$c = c(t_k, t, \theta) = \frac{(n-2)^2}{4} + \partial_t a + h e^{2t}.$$

Proof of the lemma, see [6,8,9,10].

Now we have,  $\partial_t a = \frac{\partial_t b_1}{b_1}$ ,  $b_1(t_k, t, \theta) = J(t_k, e^t, \theta) > 0$ ,

We can write,

$$-\frac{1}{\sqrt{b_1}} \partial_{tt} (\sqrt{b_1} z_k) + \Delta_\theta z_k + [c(t) + b_1^{-1/2} b_2(t, \theta)] z_k = V_k z_k^{(n+2)/(n-2)},$$

where,  $b_2(t, \theta) = \partial_{tt} (\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2$ .

Let,

$$\tilde{z}_k = \sqrt{b_1} z_k,$$

and the blow-up function (rescaled function) and the function with the auxiliary function:

$$\tilde{w}_k = (\sqrt{b_1})(t + \frac{2}{n-2} \log r_k, \theta) \cdot w_k, \quad \bar{w}_k(t, \theta) = \tilde{w}_k(e^t \theta) - \frac{m}{2} e^{(n-2)t/2},$$

we have:

**Lemma 2.2.** *The function  $\tilde{z}_k$  is solution of:*

$$\begin{aligned} & -\partial_{tt} \tilde{z}_k + \Delta_\theta(\tilde{z}_k) + 2\nabla_\theta(\tilde{z}_k) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2) \tilde{z}_k = \\ (2) \quad & = V_k \left( \frac{1}{b_1} \right)^{N-2} \tilde{z}_k^{(n+2)/(n-2)}, \end{aligned}$$

$$\text{where, } c_2 = \left[ \frac{1}{\sqrt{b_1}} \Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right].$$

Proof of the lemma, see [6,8,9,10].

We have,

$$c(t_k, t, \theta) = \frac{(n-2)^2}{4} + \partial_t a + h e^{2t}, \quad (\alpha_1)$$

$$b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2, \quad (\alpha_2)$$

$$c_2 = \left[ \frac{1}{\sqrt{b_1}} \Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right], \quad (\alpha_3)$$

We have if we denote the previous operator:  $L(t, \theta) = -\partial_{tt}(\cdot) + \Delta_\theta(\cdot) + 2\nabla_\theta(\cdot) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2} b_2 - c_2)(\cdot)$   
and,

$$\tilde{b}_1 = b_1(t + \frac{2}{n-2} \log r_k, \theta),$$

We have:

$$L(t, \theta)(\tilde{z}_k) = V_k \left( \frac{1}{b_1} \right)^{N-2} \tilde{z}_k^{(n+2)/(n-2)},$$

and for the blow-up function (the rescaled function), we replace  $t$  by  $t + \frac{2}{n-2} \log r_k$ :

$$L(t + \frac{2}{n-2} \log r_k, \theta)[\tilde{w}_k(t, \theta)] = V_k(t + \frac{2}{n-2} \log r_k, \theta) \left( \frac{1}{\tilde{b}_1} \right)^{N-2} \tilde{w}_k^{(n+2)/(n-2)}$$

We set,

$$\tilde{L}(t, \theta) = L(t + \frac{2}{n-2} \log r_k, \theta), \quad \tilde{V}_k = V_k(t + \frac{2}{n-2} \log r_k, \theta)$$

Thus,

$$\tilde{L}(t, \theta)[\tilde{w}_k(t, \theta)] = \tilde{V}_k \left( \frac{1}{\tilde{b}_1} \right)^{N-2} \tilde{w}_k^{(n+2)/(n-2)}$$

and,

$$\bar{w}_k(t, \theta) = \tilde{w}_k(e^t \theta) - \frac{m}{2} e^{(n-2)t/2},$$

We have:

**Proposition 2.3.** We have for  $\lambda_k = \frac{-2}{n-2} \log v_k(0)$ :

$$1) \quad \tilde{w}_k(\lambda_k, \theta) - \tilde{w}_k(\lambda_k + 4, \theta) \geq \tilde{k} > 0, \quad \forall \theta \in \mathbb{S}_{n-1}.$$

For all  $\beta > 0$ , there exist  $c_\beta > 0$  such that:

$$2) \quad \frac{1}{c_\beta} e^{(n-2)t/2} \leq \tilde{w}_k(\lambda_k + t, \theta) \leq c_\beta e^{(n-2)t/2}, \quad \forall t \leq \beta, \quad \forall \theta \in \mathbb{S}_{n-1}.$$

We want to apply the Hopf maximum principle.

$$\bar{w}_k(t, \theta) = \tilde{w}_k(e^t \theta) - \frac{m}{2} e^{(n-2)t/2},$$

Like in [9] we have the some properties for  $\bar{w}_k$ , we have:

**Lemma 2.4.** There is  $\nu < 0$  such that for  $\lambda \leq \nu$ :

$$\bar{w}_k^\lambda(t, \theta) - \bar{w}_k(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\lambda, t_0] \times \mathbb{S}_{n-1}.$$

Let  $\xi_k$  be the following real number,

$$\xi_k = \sup\{\lambda \leq \lambda_k + 2, \bar{w}_k^\lambda(t, \theta) - \bar{w}_k(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\lambda, t_0] \times \mathbb{S}_{n-1}\}.$$

We have the same computations as for the previous papers, see [4,6,8,9,10]. We have the increment of functions and operators.

$$\begin{aligned} \tilde{L}(t, \theta)[\tilde{w}_k^{\xi_k}(t, \theta) - \tilde{w}_k(t, \theta)] &= [\tilde{L}(t, \theta) - \tilde{L}(t^{\xi_k}, \theta)][\tilde{w}_k^{\xi_k}(t, \theta)] + \tilde{L}(t^{\xi_k}, \theta)[\tilde{w}_k^{\xi_k}(t, \theta)] - \tilde{L}(t, \theta)[\tilde{w}_k(t, \theta)] = \\ &= [\tilde{L}(t, \theta) - \tilde{L}(t^{\xi_k}, \theta)][\tilde{w}_k^{\xi_k}(t, \theta)] + \tilde{V}_k^{\xi_k} \left( \frac{1}{\tilde{b}_1^{\xi_k}} \right)^{N-2} (\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - \tilde{V}_k \left( \frac{1}{\tilde{b}_1} \right)^{N-2} \tilde{w}_k^{(n+2)/(n-2)} \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{L}(t, \theta)[\tilde{w}_k^{\xi_k}(t, \theta) - \tilde{w}_k(t, \theta)] &= \\ &= [\tilde{L}(t, \theta) - \tilde{L}(t^{\xi_k}, \theta)][\tilde{w}_k^{\xi_k}(t, \theta)] + \tilde{V}_k^{\xi_k} \left( \frac{1}{\tilde{b}_1^{\xi_k}} \right)^{N-2} (\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - \tilde{V}_k \left( \frac{1}{\tilde{b}_1} \right)^{N-2} \tilde{w}_k^{(n+2)/(n-2)}, \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{L}(t, \theta)[\bar{w}_k^{\xi_k}(t, \theta) - \bar{w}_k(t, \theta)] &= \\ &= [\tilde{L}(t, \theta) - \tilde{L}(t^{\xi_k}, \theta)][\tilde{w}_k^{\xi_k}(t, \theta)] + \\ &+ \tilde{V}_k^{\xi_k} \left( \frac{1}{\tilde{b}_1^{\xi_k}} \right)^{N-2} (\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - \tilde{V}_k \left( \frac{1}{\tilde{b}_1} \right)^{N-2} \tilde{w}_k^{(n+2)/(n-2)} + \\ &+ O(1) r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}), \end{aligned}$$

We have:

$$[\tilde{L}(t, \theta) - \tilde{L}(t^{\xi_k}, \theta)][\tilde{w}_k^{\xi_k}(t, \theta)] = O(1) \tilde{w}_k^{\xi_k} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}),$$

and,

$$\tilde{V}_k^{\xi_k} \left( \frac{1}{\tilde{b}_1^{\xi_k}} \right)^{N-2} (\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - \tilde{V}_k \left( \frac{1}{\tilde{b}_1} \right)^{N-2} \tilde{w}_k^{(n+2)/(n-2)} =$$

(3)

$$= O(1)r_k^{4/(n-2)}\tilde{w}_k^{\xi_k}(e^{2t}-e^{2t^{\xi_k}})+O(1)\tilde{w}_k^{\xi_k}r_k^{2/(n-2)}(e^t-e^{t^{\xi_k}})+\tilde{V}_k\left(\frac{1}{\tilde{b}_1}\right)^{N-2}[(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)}-\tilde{w}_k^{(n+2)/(n-2)}]$$

Thus,

$$\begin{aligned} & \tilde{L}(t, \theta)[\tilde{w}_k^{\xi_k}(t, \theta) - \bar{w}_k(t, \theta)] = \\ & = \tilde{V}_k\left(\frac{1}{\tilde{b}_1}\right)^{N-2}[(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - \tilde{w}_k^{(n+2)/(n-2)}] + O(1)r_k^{4/(n-2)}\tilde{w}_k^{\xi_k}(e^{2t} - e^{2t^{\xi_k}}) + \\ (4) \quad & + O(1)\tilde{w}_k^{\xi_k}r_k^{2/(n-2)}(e^t - e^{t^{\xi_k}}) + O(1)r_k^{4/(n-2)}e^{2t}(e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}), \end{aligned}$$

We want to prove that by using the Hopf maximum principle, (like in [3,5,7,8]):

$$\min_{\theta \in \mathbb{S}_{n-1}} \bar{w}_k(t_0, \theta) \leq \max_{\theta \in \mathbb{S}_{n-1}} \bar{w}_k(2\xi_k - t_0, \theta),$$

For this, we argue by contradiction and we assume that:

$$\min_{\theta \in \mathbb{S}_{n-1}} \bar{w}_k(t_0, \theta) > \max_{\theta \in \mathbb{S}_{n-1}} \bar{w}_k(2\xi_k - t_0, \theta),$$

Thus, our assumption is:

$$\bar{w}_k(2\xi_k - t_0, \theta) - w_k(t_0, \theta) < 0, \forall \theta \in \mathbb{S}_{n-1}.$$

Now, we want to prove that:

$$[\tilde{w}_k^{\xi_k}(t, \theta) - \bar{w}_k(t, \theta)] \leq 0 \Rightarrow \tilde{L}(t, \theta)[\tilde{w}_k^{\xi_k}(t, \theta) - \bar{w}_k(t, \theta)] \leq 0,$$

For this:

1) The biggest term is the term of  $V$  (for  $n \geq 6$ ):  $\tilde{w}_k^{\xi_k}r_k^{2/(n-2)}(e^t - e^{t^{\xi_k}})$ ,  $t_0 \geq t \geq \xi_k$ .

Because we must compare:

$$(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)}r_k^{2/(n-2)}(e^t - e^{t^{\xi_k}}) \text{ and } (\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - (\tilde{w}_k)^{(n+2)/(n-2)},$$

and, we have used the mean value theorem for  $f(t) = t^{(n+2)/(n-2)}$ , and  $\tilde{w}_k^{\xi_k} \leq \tilde{w}_k$  to have:

$$(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - (\tilde{w}_k)^{(n+2)/(n-2)} \leq c(\tilde{w}_k^{\xi_k})^{4/(n-2)}(\tilde{w}_k^{\xi_k} - \tilde{w}_k),$$

and,

$$(\tilde{w}_k^{\xi_k})^{4/(n-2)}(\tilde{w}_k^{\xi_k} - \tilde{w}_k) \leq c(\tilde{w}_k^{\xi_k})^{4/(n-2)}(e^{(n-2)t^{\xi_k}/2} - e^{(n-2)t/2}).$$

Now, we write:

$$e^t = e^{(n-2)t/2}e^{(4-n)t/2} \leq e^{(4-n)\xi_k/2}e^{(n-2)t/2},$$

we integrate between  $t$  and  $t^{\xi_k}$ , we obtain:

$$(e^t - e^{t^{\xi_k}}) \leq ce^{(4-n)\xi_k/2}(e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

But,

$$\tilde{w}_k^{\xi_k} \leq ce^{(n-2)(\xi_k - \lambda_k)/2},$$

Thus the biggest term is:

$$\tilde{w}_k^{\xi_k}r_k^{2/(n-2)}(e^t - e^{t^{\xi_k}}) \leq cr_k^{2/(n-2)}e^{(\xi_k - (n-2)\lambda_k)/2}(e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

but  $\xi_k \leq \lambda_k + 2$ , we obtain:

$$\tilde{w}_k^{\xi_k}r_k^{2/(n-2)}(e^t - e^{t^{\xi_k}}) \leq cr_k^{2/(n-2)}e^{-(n-4)\lambda_k/2}(e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

Thus,



$$n \geq 5, \tilde{w}_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq \frac{c}{[u_k(t_k)]^{\epsilon - (n-4)/(n-2)}} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

$$n = 4, \tilde{w}_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq \frac{c}{[u_k(y_k)]^\epsilon} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

These terms are controlled by the term:  $-\frac{m}{2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2})$ .

2) Also, we have for  $n \geq 6$ :

Because we must compare:

$$\tilde{w}_k^{\xi_k} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \text{ and } (\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - (\tilde{w}_k)^{(n+2)/(n-2)},$$

We must look to the term:

$$(\tilde{w}_k^{\xi_k})^{1-(4/(n-2))} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) = (\tilde{w}_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}), t_0 \geq t \geq \xi_k,$$

We write:

$$e^{2t} = e^{(n-2)t/2} e^{(6-n)t/2} \leq e^{(6-n)\xi_k/2} e^{(n-2)t/2},$$

But,

$$\tilde{w}_k^{\xi_k} \leq c e^{(n-2)(\xi_k - \lambda_k)/2},$$

Thus,

$$(\tilde{w}_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \leq c r_k^{4/(n-2)} e^{-(n-6)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

we obtain:

$$(\tilde{w}_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{4/(n-2)} e^{-(n-6)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

Thus,

$$n \geq 6, (\tilde{w}_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \leq \frac{c}{[u_k(t_k)]^{\epsilon - (n-6)/(n-2)}} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

But,  $\epsilon > \frac{n-4}{n-2}$ , these terms are controlled by the term:  $-\frac{m}{2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2})$ .

3) For  $n = 5$ : we have the terms: we use the binomial formula: we write:

$$(\tilde{w}_k^{\xi_k})^{7/3} - \tilde{w}_k^{7/3} = ((\tilde{w}_k^{\xi_k})^{1/3})^7 - (\tilde{w}_k^{1/3})^7,$$

$$x^7 - y^7 \equiv (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6), x = (\tilde{w}_k^{\xi_k})^{1/3}, y = \tilde{w}_k^{1/3},$$

but,

$$\tilde{w}_k^{\xi_k} - \tilde{w}_k = (x^3 - y^3) \equiv (x - y)(x^2 + xy + y^2),$$

Thus,

$$(\tilde{w}_k^{\xi_k})^{7/3} - \tilde{w}_k^{7/3} = (x^3 - y^3) \times \frac{(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)}{(x^2 + xy + y^2)},$$

Here, we have  $x \leq y$ , thus:

$$(x^2 + xy + y^2) \leq c y^2, (x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6) \geq c' x^2 y^4, c, c' > 0$$

Thus, because  $y \geq \frac{m}{2} e^{3t/2}$  we obtain:

$$(\tilde{w}_k^{\xi_k})^{7/3} - \tilde{w}_k^{7/3} \leq c (\tilde{w}_k^{\xi_k} - \tilde{w}_k) (\tilde{w}_k^{\xi_k})^{2/3} \tilde{w}_k^{2/3} \leq -c e^t (\tilde{w}_k^{\xi_k})^{2/3} (e^{3t/2} - e^{3t^{\xi_k}/2}), c > 0$$

For the case:  $A = (\tilde{w}_k^{\xi_k})^{1/3} r_k^{4/3} (e^{2t} - e^{2t\xi_k})$ ,  $t_0 \geq t \geq \xi_k$   
We have:

$$|A| \leq e^t (\tilde{w}_k^{\xi_k})^{1/3} r_k^{4/3} (e^t - e^{t\xi_k})$$

The dominant term is:

$$B = (\tilde{w}_k^{\xi_k})^{1/3} r_k^{4/3} (e^t - e^{t\xi_k})$$

We have:

$$e^t = e^{-t/2} e^{3t/2} \leq c e^{-\xi_k/2} e^{3t/2}, w_k^{\xi_k} \leq c e^{3(\xi_k - \lambda_k)/2}$$

Thus,

$$|B| \leq c r_k^{4/3} e^{-\lambda_k/2} (e^{3t/2} - e^{3t\xi_k/2})$$

$$\lambda_k = -(2/3)(1 - \epsilon) \log u_k(t_k), r_k = u_k(t_k)^{-\epsilon},$$

$$e^{-\lambda_k/2} = u_k(t_k)^{(1/3)(1-\epsilon)}, r_k^{4/3} = u_k(t_k)^{-4\epsilon/3}, r_k^{4/3} e^{-\lambda_k/2} = u_k(t_k)^{(1/3)(1-5\epsilon)}$$

The condition is  $1 - 5\epsilon < 0$ ,  $\epsilon > \frac{1}{5}$ , but  $\epsilon > \frac{n-4}{n-2} = \frac{5-4}{5-2} = 1/3 > 1/5$ .

4) We have the same thing for the dimension 4.

5) When we use the auxiliary function  $\frac{m}{2} e^{(n-2)t/2}$ , there is a term:

$$r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}),$$

To correct this term, we consider a part of the term :

$$(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - (\tilde{w}_k)^{(n+2)/(n-2)}$$

We use the binomial formula as for the previous case of dimension 5. We have:

$$x = (\tilde{w}_k^{\xi_k})^{1/(n-2)}, y = (\tilde{w}_k)^{1/(n-2)},$$

$$x^{n+2} - y^{n+2} = (x - y)(y^{n+1} + \dots), x^{n-2} - y^{n-2} = (x - y)(y^{n-3} + \dots + x^{n-3})$$

Thus,

$$x^{n+2} - y^{n+2} = (x^{n-2} - y^{n-2}) \frac{(y^{n+1} + \dots)}{(y^{n-3} + \dots)}$$

Because  $x \leq y$  and,

$$(y^{n+1} + \dots) \geq y^{n+1}, (x^{n-3} + \dots + y^{n-3}) \leq c y^{n-3}, c > 0,$$

We obtain:

$$x^{n+2} - y^{n+2} \leq c(x^{n-2} - y^{n-2})y^4, c > 0,$$

Thus,

$$(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - (\tilde{w}_k)^{(n+2)/(n-2)} \leq c(\tilde{w}_k^{\xi_k} - \tilde{w}_k) \tilde{w}_k^{4/(n-2)}, c > 0$$

Because,  $\tilde{w}_k \geq \frac{m}{4} e^{(n-2)t/2}$ , we obtain:

$$(\tilde{w}_k^{\xi_k})^{(n+2)/(n-2)} - (\tilde{w}_k)^{(n+2)/(n-2)} \leq -c e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}), c > 0,$$

Thus the term:  $r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2})$  is controlled by the term,  $-c e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}), c > 0$ .

We obtain the same proof in the previous papers, the dimensions 4, 6, see [6,9,10].

If we use the Hopf maximum principle, we obtain (like in [4,6,8,9]):

$$\min_{\theta \in \mathbb{S}_{n-1}} \bar{w}_k(t_0, \theta) \leq \max_{\theta \in \mathbb{S}_{n-1}} \bar{w}_k(2\xi_k - t_0, \theta),$$

thus for  $k$  large:

$$(u_k(\cdot))^{1-\epsilon} = v_k(0) \leq c,$$

It is a contradiction.

Finally, for each  $m > 0$  there is a finite  $v_k$  such that  $\inf v_k \geq m > 0, k_1, \dots, k_m \in \mathbb{N}$ .

Here also, we have the existence of  $c(m) > 0$  such that  $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} = v_k(0) \leq c$ . We prove this by contradiction, suppose that for fixed  $m > 0$ , for all  $c > 0$  there is  $i_c \in \mathbb{N}$  with  $\inf v_{i_c} \geq m > 0$  and  $v_{i_c}(0) \geq c$ , if we take  $c \rightarrow +\infty$ , because the number of indices is bounded and we have a sequence of integers, this sequence converge and in fact is constant because we consider integers. Thus there is an index  $k$  such that  $v_k(0) \geq c \rightarrow +\infty$  and  $\inf v_k \geq m > 0$ , and thus  $v_k$  is singular at 0, but this is impossible because  $v_k$  is regular.

We obtain:

There is a non-increasing positive function  $m \rightarrow c(m) > 0$ , such that  $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} \leq c(m)$ . then we apply this with  $m = r_k \inf_M u_k$ , we obtain the inequality for all terms of the sequence  $(u_k)$ .

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