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► **To cite this version:**

Samy Skander Bahoura. Harnack inequalities for solutions of prescribed scalar curvature type equations. 2024. hal-04369041v1

**HAL Id: hal-04369041**

**<https://hal.science/hal-04369041v1>**

Preprint submitted on 2 Jan 2024 (v1), last revised 15 Jan 2024 (v2)

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# HARNACK INEQUALITIES FOR SOLUTIONS OF PRESCRIBED SCALAR CURVATURE TYPE EQUATIONS.

SAMY SKANDER BAHOURA

ABSTRACT. We give Harnack inequalities for solutions of equations of type prescribed scalar curvature in dimensions  $n \geq 4$ .

## 1. INTRODUCTION AND MAIN RESULTS

We consider on a Riemannian manifold  $(M, g)$  of dimension  $n \geq 4$ , the equation:

$$\Delta u + hu = Vu^{(n+2)/(n-2)}, u > 0, \quad (E)$$

with,  $h$  a smooth function and  $0 < a \leq V(x) \leq b < +\infty$ ,  $\|\nabla V\|_\infty \leq A$ .

Equation of this type were considered by many authors, see [1-24]. This equation arise in physics and astronomy. Here we look to a priori estimates of type sup, inf which are characteristic of this equation.

Let  $(u_k)$  a sequence of solutions of  $(E)$ .

We fix a compact set  $K$  of  $M$ . We want to prove that: for each compact, for all terms of the sequence  $(u_k)$ :  $\sup_K u_k$  and  $\inf_M u_k$  are linked.

1) If there is a "big" compact  $K_0$  for which there is a subsequence  $\sup_{K_0} u_{i_j}$  is bounded, then we have a compactness result for a "big"  $K_0$  and for  $K \subset K_0$  the sequence  $(u_{i_j})$  is bounded.

2) If for all compact  $K$ ,  $\sup_K u_k \rightarrow +\infty$ . We do a blow-up. We consider  $\sup_K u_k = u_k(y_k)$ . Consider  $R_k \rightarrow 0$ ,  $R_k^{(n-2)/2} = [u_k(y_k)]^{-\epsilon}$  with  $0 < \epsilon < 1$ . Then:

$$R_k^{(n-2)/2} \sup_{B_{R_k}(y_k)} u_k \geq c_k = [u_k(y_k)]^{1-\epsilon} \rightarrow +\infty.$$

We use the blow-up technique to have,  $\exists t_k, \bar{t}_k, u_k(t_k) \geq u_k(\bar{t}_k) \geq u_k(y_k) \rightarrow +\infty$ . We do a blow-up, then we consider:

$$n = 4, \quad v_k(y) = r_k u_k(t_k + r_k y) = r_k u(\exp_{t_k}(r_k y)), \quad r_k = [u_k(y_k)]^{-\epsilon},$$

with,  $\epsilon > 0$ ,  
and,

$$n \geq 5, \quad v_k(y) = r_k u_k(t_k + (r_k)^{2/(n-2)} y) = r_k u_k(\exp_{t_k}(r_k^{2/(n-2)} y)), \quad r_k = [u_k(t_k)]^{-\epsilon},$$

with,  $\epsilon > \frac{(n-4)}{n-2}$ .

Note that, here we have considered all terms of the sequence  $(u_k)$ .

Let's consider the blow-up functions  $(v_k)$  defined previously with the exponential maps for  $n \geq 4$ ,  $\exp_{t_k}(y)$ , like in the previous papers for the dimensions, 4, 5, 6. Because we consider the compact sets  $K, 2K$ , and  $t_k \in 2K$ , the injectivity radius is uniformly bounded below by a positive number. Thus, we can consider all the terms of the sequence  $(u_k)$  without extraction. (After supposing the assertion  $\inf v_k \geq m > 0$  infinitely many times, we can use extraction, for the points  $(t_k)$ ).

We fix  $m > 0$ , we prove the result by assuming  $\inf v_k \geq m > 0$ , like for the dimensions 4 and 6. After we take  $m = r_k \inf u_k > 0$ . Suppose by contradiction, that there is infinitely many  $(v_k)$  with  $\inf v_k \geq m > 0$ , the proof imply that  $[u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty$  which is impossible. Thus, there is a finite number of terms such that  $\inf v_k \geq m > 0$ ,  $k_1, \dots, k_{i(m)}$ .

Thus we have also,  $v_k(0) \leq c(m)$  when  $\inf v_k \geq m > 0$ . In all cases, we have the following assertion:

$$\inf v_k \geq m > 0 \Rightarrow [u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty,$$

we obtain:

There is a non-increasing positive function  $m \rightarrow c(m) > 0$ , such that  $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} \leq c(m)$ . then we apply this with  $m = r_k \inf u_k$ , we obtain for all terms of the sequence  $(u_k)$ :

$$n = 4, [u_k(y_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g),$$

and,

$$n \geq 5, [u_k(t_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(t_k)]^\epsilon}, K, M, g).$$

For  $n \geq 5$ , we set,  $0 < \epsilon_k = \frac{u_k(y_k)}{u_k(t_k)} \leq 1$ ,  $\epsilon_k = \epsilon_k(K)$ , we obtain:

$$n \geq 5, [u_k(y_k)]^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g).$$

We have  $u_k(y_k) = \sup_K u_k$ , we obtain:

**Theorem 1.1.** *We have:*

1) *There is a compact  $K_0$  of  $M$  and a subsequence  $i_j$  and a positive constant  $C > 0$ , such that:*

$$\sup_{K_0} u_{i_j} \leq C, \forall j.$$

Or,

2) *For all compact  $K$ ,  $\sup_K u_k \rightarrow +\infty$  and:*

$$n = 4, (\sup_K u_k)^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{(\sup_K u_k)^\epsilon}, K, M, g),$$

with  $\epsilon > 0$ ,  
and,

$$n \geq 5, (\sup_K u_k)^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf u_k}{(\sup_K u_k)^\epsilon}, K, M, g),$$

with,  $\epsilon > \frac{(n-4)}{n-2}$

We can see that  $\sup_K u_k$  and  $\inf_M u_k$  are linked. There is a relation which link these two quantities. For all compact  $K$  and all  $k \in \mathbb{N}$ ,  $\sup_K u_k$  and  $\inf_M u_k$  are linked by the previous relations. There is a positive function  $c(\cdot, \cdot, \cdot) > 0$ , such that for all  $k$ ,  $\sup_K u_k = c(K, \epsilon_k, \inf_M u_k)$ . If we denote  $F = \{u_k\}$ ,  $G = \{\epsilon_k = \epsilon_k(K)\}$ ,  $F \times G = \{(u_k, \epsilon_k)\}$ ,  $0 < \epsilon_k = \epsilon_k(K) \leq 1$ , we have:

$$\exists c(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon) \in F \times G, \sup_K u = c(K, \epsilon(K), \inf_M u)$$

or,

$$\exists c(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon) \in F \times G, (\sup_K u)^{1-\epsilon} \leq c(K, \epsilon, \frac{\inf u}{(\sup_K u)^\epsilon}).$$

We write this to highlight the rolling-up phenomenon and the distortion.

## 2. PROOF OF THE RESULT

For the proof, we use the computations of previous papers with modifications, see [3,5,8,9].

1) *blow-up analysis*:

Let  $(u_k)$  a sequence of solutions of  $(E)$ . We fix a compact set  $K$  of  $M$ . We want to prove that: for each compact, for all terms of the sequence  $(u_k)$ :  $\sup_K u_k$  and  $\inf_M u_k$  are linked.

1) If there is a "big" compact  $K_0$  for which there is a subsequence  $\sup_{K_0} u_{i_j}$  is bounded, then we have a compactness result for a "big"  $K_0$  and for  $K \subset K_0$  the sequence  $(u_{i_j})$  is bounded.

2) If for all compact  $K$ ,  $\sup_K u_k \rightarrow +\infty$ . We do a blow-up. We consider  $\sup_K u_k = u_k(y_k)$ . Consider  $R_k \rightarrow 0$ ,  $R_k^{(n-2)/2} = [u_k(y_k)]^{-\epsilon}$  with  $0 < \epsilon < 1$ . Then:

$$R_k^{(n-2)/2} \sup_{\bar{B}_{R_k}(y_k)} u_k \geq c_k = [u_k(y_k)]^{1-\epsilon} \rightarrow +\infty.$$

We use the blow-up technique to have,  $\exists t_k, \bar{t}_k, u_k(t_k) \geq u_k(\bar{t}_k) \geq u_k(y_k) \rightarrow +\infty$ .

$$\bar{t}_k, \sup_{\bar{B}_{R_k}(y_k)} u_k = u_k(\bar{t}_k) \geq u_k(y_k) > 0,$$

We consider  $s_k(y) = (R_k - d(y, \bar{t}_k))^{(n-2)/2} u_k(y)$ , and,

$$t_k, \sup_{\bar{B}_{R_k}(\bar{t}_k)} s_k = s_k(t_k) \geq s_k(\bar{t}_k) = R_k^{(n-2)/2} u_k(\bar{t}_k) \geq R_k^{(n-2)/2} u_k(y_k) > 0,$$

We do a blow-up, then we consider:

$$n = 4, v_k(y) = r_k u_k(t_k + r_k y) = r_k u(\exp_{t_k}(r_k y)), r_k = [u_k(y_k)]^{-\epsilon},$$

with,  $\epsilon > 0$ ,  
and,

$$n \geq 5, v_k(y) = r_k u_k(t_k + (r_k)^{2/(n-2)} y) = r_k u_k(\exp_{t_k}(r_k^{2/(n-2)} y)), r_k = [u_k(t_k)]^{-\epsilon},$$

with,  $\epsilon > \frac{(n-4)}{n-2}$ .

Note that, here we have considered all terms of the sequence  $(u_k)$ .

Let's consider the blow-up functions  $(v_k)$  defined previously with the exponential maps for  $n \geq 4$ ,  $\exp_{t_k}(y)$ , like in the previous papers for the dimensions, 4, 5, 6. Because we consider the compact sets  $K, 2K$ , and  $t_k \in 2K$ , the injectivity radius is uniformly bounded below by a positive number. Thus, we can consider all the terms of the sequence  $(u_k)$  without extraction. (After supposing the assertion  $\inf v_k \geq m > 0$  infinitely many times, we can use extraction, for the points  $(t_k)$ ).

We consider,  $\delta_0 = \delta_0(K) = \inf\{\delta_P/4, P \in K\}$ , with  $\delta_P$  continuous in  $P$  and smaller than the injectivity radius in  $P$  for each  $P$ . We have a finite cover of  $K$  by small balls of radius  $\delta_0/2$ , we have a finite set of points  $z_j \in K$ :  $K \subset \cup_{\{j=0, \dots, l\}} B(z_j, \delta_0/2) \subset \cup_{\{j=0, \dots, l\}} \bar{B}(z_j, 3\delta_0) = K_{\delta_0}$  is compact.

We take  $R_k^{(n-2)/2} = \inf\{u_k(y_k)^{-\epsilon}, (\delta_0/2)^{(n-2)/2}\}$ . The small balls are all compact, thus,  $\bar{t}_k$  exist and  $t_k$  exist. We take  $r_k = \inf\{\delta_P/4, P \in K_{\delta_0}, u_k(t_k)^{-\epsilon}\}$ . Thus  $\bar{t}_k, t_k$  and  $\exp_{t_k}(\cdot)$  and  $v_k$  are defined for all  $k \geq 0$ .

We fix  $m > 0$ , we prove the result by assuming  $\inf v_k \geq m > 0$ , like for the dimensions 4 and 6. After we take  $m = r_k \inf u_k > 0$ . Suppose by contradiction, that there is infinitely many  $(v_k)$  with  $\inf v_k \geq m > 0$ , the proof imply that  $[u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty$  which is impossible. Thus, there is a finite number of terms such that  $\inf v_k \geq m > 0$ ,  $k_1, \dots, k_{i(m)}$ . Thus we have also,  $v_k(0) \leq c(m)$  when  $\inf v_k \geq m > 0$ . In all cases, we have the following assertion:

$$\inf v_k \geq m > 0 \Rightarrow [u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty,$$

we obtain:

There is a non-increasing positive function  $m \rightarrow c(m) > 0$ , such that  $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} \leq c(m)$ . then we apply this with  $m = r_k \inf u_k$ , we obtain for all terms of the sequence  $(u_k)$ :

$$n = 4, [u_k(y_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g),$$

and,

$$n \geq 5, [u_k(t_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(t_k)]^\epsilon}, K, M, g).$$

For  $n \geq 5$ , we set,  $0 < \epsilon_k = \frac{u_k(y_k)}{u_k(t_k)} \leq 1$ ,  $\epsilon_k = \epsilon_k(K)$ , we obtain:

$$n \geq 5, [u_k(y_k)]^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g).$$

We have  $u_k(y_k) = \sup_K u_k$ .

II) *Auxiliary function and moving-plane method:*

We use the computations of previous papers with modifications, see [3,5,8,9]. We consider:

$$w_k(t, \theta) = e^{(n-2)t/2} v_k(e^t \theta) - \frac{m}{2} e^{(n-2)t/2},$$

We have  $\lambda_k = \frac{-2}{n-2} \log v_k(0)$ .

1) The biggest term is the term of  $V$  (for  $n \geq 6$ ):  $w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}})$ ,  $t_0 \geq t \geq \xi_k$ .

Because we must compare:

$$(w_k^{\xi_k})^{(n+2)/(n-2)} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \text{ and } (w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)},$$

and, we have used the mean value theorem for  $f(t) = t^{(n+2)/(n-2)}$ , and  $w_k^{\xi_k} \leq w_k$  to have:

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)} \leq c(w_k^{\xi_k})^{4/(n-2)} (w_k^{\xi_k} - w_k),$$

and,

$$(w_k^{\xi_k})^{4/(n-2)} (w_k^{\xi_k} - w_k) \leq c(w_k^{\xi_k})^{4/(n-2)} (e^{(n-2)t^{\xi_k}/2} - e^{(n-2)t/2}).$$

Now, we write:

$$e^t = e^{(n-2)t/2} e^{(4-n)t/2} \leq e^{(4-n)\xi_k/2} e^{(n-2)t/2},$$

we integrate between  $t$  and  $t^{\xi_k}$ , we obtain:

$$(e^t - e^{t^{\xi_k}}) \leq c e^{(4-n)\xi_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

But,

$$w_k^{\xi_k} \leq c e^{(n-2)(\xi_k - \lambda_k)/2},$$

Thus the biggest term is:

$$w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{2/(n-2)} e^{(\xi_k - (n-2)\lambda_k)/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

but  $\xi_k \leq \lambda_k + 2$ , we obtain:

$$w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{2/(n-2)} e^{-(n-4)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

Thus,

$$n \geq 5, w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq \frac{c}{[u_k(t_k)]^{\epsilon - (n-4)/(n-2)}} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

$$n = 4, w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq \frac{c}{[u_k(y_k)]^\epsilon} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

These terms are controlled by the term:  $-\frac{m}{2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2})$ .

2) Also, we have for  $n \geq 6$ :

Because we must compare:

$$w_k^{\xi_k} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \text{ and } (w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)},$$

We must look to the term:

$$(w_k^{\xi_k})^{1-(4/(n-2))} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) = (w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}), t_0 \geq t \geq \xi_k,$$

We write:

$$e^{2t} = e^{(n-2)t/2} e^{(6-n)t/2} \leq e^{(6-n)\xi_k/2} e^{(n-2)t/2},$$

But,

$$w_k^{\xi_k} \leq c e^{(n-2)(\xi_k - \lambda_k)/2},$$

Thus,

$$(w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \leq c r_k^{4/(n-2)} e^{-(n-6)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

we obtain:

$$(w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{4/(n-2)} e^{-(n-6)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

Thus,

$$n \geq 6, (w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \leq \frac{c}{[u_k(t_k)]^{\epsilon - (n-6)/(n-2)}} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

But,  $\epsilon > \frac{n-4}{n-2}$ , these terms are controlled by the term:  $-\frac{m}{2}(e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2})$ .

3) For  $n = 5$ : we have the terms: we use the binomial formula: we write:

$$(w_k^{\xi_k})^{7/3} - w_k^{7/3} = ((w_k^{\xi_k})^{1/3})^7 - (w_k^{1/3})^7,$$

$$x^7 - y^7 \equiv (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6), x = (w_k^{\xi_k})^{1/3}, y = w_k^{1/3},$$

but,

$$w_k^{\xi_k} - w_k = (x^3 - y^3) \equiv (x - y)(x^2 + xy + y^2),$$

Thus,

$$(w_k^{\xi_k})^{7/3} - w_k^{7/3} = (x^3 - y^3) \times \frac{(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)}{(x^2 + xy + y^2)},$$

Here, we have  $x \leq y$ , thus:

$$(x^2 + xy + y^2) \leq cy^2, (x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6) \geq c'x^2y^4, c, c' > 0$$

Thus, because  $y \geq \frac{m}{2}e^{3t/2}$  we obtain:

$$(w_k^{\xi_k})^{7/3} - w_k^{7/3} \leq c(w_k^{\xi_k} - w_k)(w_k^{\xi_k})^{2/3}w_k^{2/3} \leq -ce^t(w_k^{\xi_k})^{2/3}(e^{3t/2} - e^{3t^{\xi_k}/2}), c > 0$$

For the case:  $A = (w_k^{\xi_k})^{1/3}r_k^{4/3}(e^{2t} - e^{2t^{\xi_k}})$ ,  $t_0 \geq t \geq \xi_k$

We have:

$$|A| \leq e^t(w_k^{\xi_k})^{1/3}r_k^{4/3}(e^t - e^{t^{\xi_k}})$$

The dominant term is:

$$B = (w_k^{\xi_k})^{1/3}r_k^{4/3}(e^t - e^{t^{\xi_k}})$$

We have:

$$e^t = e^{-t/2} e^{3t/2} \leq c e^{-\xi_k/2} e^{3t/2}, w_k^{\xi_k} \leq c e^{3(\xi_k - \lambda_k)/2}$$

Thus,

$$|B| \leq c r_k^{4/3} e^{-\lambda_k/2} (e^{3t/2} - e^{3t\xi_k/2})$$

$$\lambda_k = -(2/3)(1 - \epsilon) \log u_k(t_k), r_k = u_k(t_k)^{-\epsilon},$$

$$e^{-\lambda_k/2} = u_k(t_k)^{(1/3)(1-\epsilon)}, r_k^{4/3} = u_k(t_k)^{-4\epsilon/3}, r_k^{4/3} e^{-\lambda_k/2} = u_k(t_k)^{(1/3)(1-5\epsilon)}$$

The condition is  $1 - 5\epsilon < 0$ ,  $\epsilon > \frac{1}{5}$ , but  $\epsilon > \frac{n-4}{n-2} = \frac{5-4}{5-2} = 1/3 > 1/5$ .

4) We have the same thing for the dimension 4.

5) When we use the auxiliary function  $\frac{m}{2} e^{(n-2)t/2}$ , there is a term:

$$r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}),$$

To correct this term, we consider a part of the term :

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)}$$

We use the binomial formula as for the previous case of dimension 5. We have:

$$x = (w_k^{\xi_k})^{1/(n-2)}, y = (w_k)^{1/(n-2)},$$

$$x^{n+2} - y^{n+2} = (x - y)(y^{n+1} + \dots), x^{n-2} - y^{n-2} = (x - y)(y^{n-3} + \dots + x^{n-3})$$

Thus,

$$x^{n+2} - y^{n+2} = (x^{n-2} - y^{n-2}) \frac{(y^{n+1} + \dots)}{(y^{n-3} + \dots)}$$

Because  $x \leq y$  and,

$$(y^{n+1} + \dots) \geq y^{n+1}, (x^{n-3} + \dots + y^{n-3}) \leq c y^{n-3}, c > 0,$$

We obtain:

$$x^{n+2} - y^{n+2} \leq c(x^{n-2} - y^{n-2})y^4, c > 0,$$

Thus,

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)} \leq c(w_k^{\xi_k} - w_k)w_k^{4/(n-2)}, c > 0$$

Because,  $w_k \geq \frac{m}{2} e^{(n-2)t/2}$ , we obtain:

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)} \leq -c e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}), c > 0,$$

Thus the term:  $r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2})$  is controlled by the term,  $-c e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}), c > 0$ .

We obtain the same proof in the previous papers, the dimensions 4, 6, see [5,8,9].

## REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag 1998
- [2] S.S.Bahoura. Différentes estimations du  $\sup u \times \inf u$  pour l'équation de la courbure scalaire prescrite en dimension  $n \geq 3$ . J. Math. Pures Appl. (9), 82 (1) (2003), pp. 43-66
- [3] S.S.Bahoura. Majorations du type  $\sup u \times \inf u \leq c$  pour l'équation de la courbure scalaire sur un ouvert de  $\mathbb{R}^n$ ,  $n \geq 3$ . J. Math. Pures. Appl.(9) 83 2004 no, 9, 1109-1150.
- [4] S.S.Bahoura. Estimations du type  $\sup \times \inf$  sur une variété compacte. Bull.Sci.math. 130 (7), 2006, pp 624-636.
- [5] S.S.Bahoura. Estimations uniformes pour l'équation de Yamabe en dimensions 5 et 6. Journal.Func.Anal. 242 (2), 2007, pp 550-562.
- [6] S.S.Bahoura. Lower bounds for  $\sup + \inf$  and  $\sup * \inf$  and an extension of Chen-Lin result in dimension 3. Acta. Math. Scientia. 28 (4), 2008, pp 749-758.
- [7] S.S.Bahoura. Harnack inequalities for Yamabe type equations. Bull.Sci.Math. 133 (8), 2009, pp 875-892.
- [8] S.S.Bahoura. A uniform estimate for scalar curvature equation on manifolds of dimension 4. J.Math.Anal.Appl. Volume 388, 1, (2012) pp 386-392.
- [9] S.S. Bahoura. An estimate on Riemannian manifolds of dimension 4. Analysis in Theory and Applications. No 32, 3, (2016) pp 272-282.
- [10] H. Brezis, YY. Li, I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [11] H.Brezis and F.Merle, Uniform estimates and blow-up behavior for solutions of  $-\Delta u = Ve^u$  in two dimensions, Commun Partial Differential Equations 16 (1991), 1223-1253.
- [12] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42 (3) (1989) 271-297.
- [13] C.C.Chen.C.S.Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in  $\mathbb{R}^2$ . Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [14] C-C.Chen, C-S. Lin. Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. L(1997) 0971-1017.
- [15] B. Gidas, W-M. Ni, L. Nirenberg. Symmetry and Related Properties via the Maximum Principle. Commun. Math. Phys. 68, 209-243 (1979).
- [16] D. Gilbarg, N.S. Trudinger. Elliptic Partial Differential Equations of Second order, Berlin Springer-Verlag, Second edition, Grundlehern Math. Wiss.,224, 1983.
- [17] E. Hebey, Analyse non lineaire sur les Variétés, Editions Diderot.
- [18] E. Hebey, M. Vaugon. The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds. Duke Math. J. 79 (1995), no. 1, 235–279.
- [19] J.M. Lee, T.H. Parker. The Yamabe problem. Bull.Amer.Math.Soc (N.S) 17 (1987), no.1, 37 -91.
- [20] YY. Li. Harnack Type Inequality: the Method of Moving Planes. Commun. Math. Phys. 200,421-444 (1999).
- [21] YY. Li. Prescribing scalar curvature on  $\mathbb{S}_n$  and related Problems. C.R. Acad. Sci. Paris 317 (1993) 159-164. Part I: J. Differ. Equations 120 (1995) 319-410. Part II: Existence and compactness. Comm. Pure Appl.Math.49 (1996) 541-597.
- [22] YY. Li, L. Zhang. A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2004), no. 2, 133–151.
- [23] F.C. Marques. A Priori Estimates for the Yamabe Problem in the non-locally conformally flat case. J. Diff. Geom. 71 (2005) 315-346.
- [24] I. Shafrir. A sup+inf inequality for the equation  $-\Delta u = Ve^u$ . C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

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