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Samy Skander Bahoura

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HARNACK INEQUALITIES FOR SOLUTIONS OF PRESCRIBED SCALAR CURVATURE TYPE EQUATIONS.

SAMY SKANDER BAHOURA

ABSTRACT. We give Harnack inequalities for solutions of equations of type prescribed scalar curvature in dimensions $n \geq 4$.

1. INTRODUCTION AND MAIN RESULTS

We consider on a Riemannian manifold (M, g) of dimension $n \geq 4$, the equation:

$$\Delta u + hu = Vu^{(n+2)/(n-2)}, u > 0, \quad (E)$$

with, h a smooth function and $0 < a \leq V(x) \leq b < +\infty$, $\|\nabla V\|_\infty \leq A$.

Equation of this type were considered by many authors, see [1-24]. This equation arise in physics and astronomy. Here we look to a priori estimates of type sup, inf which are characteristic of this equation.

Let (u_k) a sequence of solutions of (E) .

We fix a compact set K of M . We want to prove that: for each compact, for all terms of the sequence (u_k) : $\sup_K u_k$ and $\inf_M u_k$ are linked.

1) If there is a "big" compact K_0 for which there is a subsequence $\sup_{K_0} u_{i_j}$ is bounded, then we have a compactness result for a "big" K_0 and for $K \subset K_0$ the sequence (u_{i_j}) is bounded.

2) If for all compact K , $\sup_K u_k \rightarrow +\infty$. We do a blow-up. We consider $\sup_K u_k = u_k(y_k)$. Consider $R_k \rightarrow 0$, $R_k^{(n-2)/2} = [u_k(y_k)]^{-\epsilon}$ with $0 < \epsilon < 1$. Then:

$$R_k^{(n-2)/2} \sup_{B_{R_k}(y_k)} u_k \geq c_k = [u_k(y_k)]^{1-\epsilon} \rightarrow +\infty.$$

We use the blow-up technique to have, $\exists t_k, \bar{t}_k, u_k(t_k) \geq u_k(\bar{t}_k) \geq u_k(y_k) \rightarrow +\infty$. We do a blow-up, then we consider:

$$n = 4, \quad v_k(y) = r_k u_k(t_k + r_k y) = r_k u(\exp_{t_k}(r_k y)), \quad r_k = [u_k(y_k)]^{-\epsilon},$$

with, $\epsilon > 0$,
and,

$$n \geq 5, \quad v_k(y) = r_k u_k(t_k + (r_k)^{2/(n-2)} y) = r_k u_k(\exp_{t_k}(r_k^{2/(n-2)} y)), \quad r_k = [u_k(t_k)]^{-\epsilon},$$

with, $\epsilon > \frac{(n-4)}{n-2}$.

Note that, here we have considered all terms of the sequence (u_k) .

Let's consider the blow-up functions (v_k) defined previously with the exponential maps for $n \geq 4$, $\exp_{t_k}(y)$, like in the previous papers for the dimensions, 4, 5, 6. Because we consider the compact sets $K, 2K$, and $t_k \in 2K$, the injectivity radius is uniformly bounded below by a positive number. Thus, we can consider all the terms of the sequence (u_k) without extraction. (After supposing the assertion $\inf v_k \geq m > 0$ infinitely many times, we can use extraction, for the points (t_k)).

We fix $m > 0$, we prove the result by assuming $\inf v_k \geq m > 0$, like for the dimensions 4 and 6. After we take $m = r_k \inf u_k > 0$. Suppose by contradiction, that there is infinitely many (v_k) with $\inf v_k \geq m > 0$, the proof imply that $[u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty$ which is impossible. Thus, there is a finite number of terms such that $\inf v_k \geq m > 0$, $k_1, \dots, k_{i(m)}$.

Thus we have also, $v_k(0) \leq c(m)$ when $\inf v_k \geq m > 0$. In all cases, we have the following assertion:

$$\inf v_k \geq m > 0 \Rightarrow [u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty,$$

we obtain:

There is a non-increasing positive function $m \rightarrow c(m) > 0$, such that $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} \leq c(m)$. then we apply this with $m = r_k \inf u_k$, we obtain for all terms of the sequence (u_k) :

$$n = 4, [u_k(y_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g),$$

and,

$$n \geq 5, [u_k(t_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(t_k)]^\epsilon}, K, M, g).$$

For $n \geq 5$, we set, $0 < \epsilon_k = \frac{u_k(y_k)}{u_k(t_k)} \leq 1$, $\epsilon_k = \epsilon_k(K)$, we obtain:

$$n \geq 5, [u_k(y_k)]^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g).$$

We have $u_k(y_k) = \sup_K u_k$, we obtain:

Theorem 1.1. *We have:*

1) *There is a compact K_0 of M and a subsequence i_j and a positive constant $C > 0$, such that:*

$$\sup_{K_0} u_{i_j} \leq C, \forall j.$$

Or,

2) *For all compact K , $\sup_K u_k \rightarrow +\infty$ and:*

$$n = 4, (\sup_K u_k)^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{(\sup_K u_k)^\epsilon}, K, M, g),$$

with $\epsilon > 0$,
and,

$$n \geq 5, (\sup_K u_k)^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf u_k}{(\sup_K u_k)^\epsilon}, K, M, g),$$

with, $\epsilon > \frac{(n-4)}{n-2}$

We can see that $\sup_K u_k$ and $\inf_M u_k$ are linked. There is a relation which link these two quantities. For all compact K and all $k \in \mathbb{N}$, $\sup_K u_k$ and $\inf_M u_k$ are linked by the previous relations. There is a positive function $c(\cdot, \cdot, \cdot) > 0$, such that for all k , $\sup_K u_k = c(K, \epsilon_k, \inf_M u_k)$. If we denote $F = \{u_k\}$, $G = \{\epsilon_k = \epsilon_k(K)\}$, $F \times G = \{(u_k, \epsilon_k)\}$, $0 < \epsilon_k = \epsilon_k(K) \leq 1$, we have:

$$\exists c(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon) \in F \times G, \sup_K u = c(K, \epsilon(K), \inf_M u)$$

or,

$$\exists c(\cdot, \cdot, \cdot) > 0, \forall K \subset M, \forall (u, \epsilon) \in F \times G, (\sup_K u)^{1-\epsilon} \leq c(K, \epsilon, \frac{\inf u}{(\sup_K u)^\epsilon}).$$

We write this to highlight the rolling-up phenomenon and the distortion.

2. PROOF OF THE RESULT

For the proof, we use the computations of previous papers with modifications, see [3,5,8,9].

1) *blow-up analysis*:

Let (u_k) a sequence of solutions of (E) . We fix a compact set K of M . We want to prove that: for each compact, for all terms of the sequence (u_k) : $\sup_K u_k$ and $\inf_M u_k$ are linked.

1) If there is a "big" compact K_0 for which there is a subsequence $\sup_{K_0} u_{i_j}$ is bounded, then we have a compactness result for a "big" K_0 and for $K \subset K_0$ the sequence (u_{i_j}) is bounded.

2) If for all compact K , $\sup_K u_k \rightarrow +\infty$. We do a blow-up. We consider $\sup_K u_k = u_k(y_k)$. Consider $R_k \rightarrow 0$, $R_k^{(n-2)/2} = [u_k(y_k)]^{-\epsilon}$ with $0 < \epsilon < 1$. Then:

$$R_k^{(n-2)/2} \sup_{\bar{B}_{R_k}(y_k)} u_k \geq c_k = [u_k(y_k)]^{1-\epsilon} \rightarrow +\infty.$$

We use the blow-up technique to have, $\exists t_k, \bar{t}_k, u_k(t_k) \geq u_k(\bar{t}_k) \geq u_k(y_k) \rightarrow +\infty$.

$$\bar{t}_k, \sup_{\bar{B}_{R_k}(y_k)} u_k = u_k(\bar{t}_k) \geq u_k(y_k) > 0,$$

We consider $s_k(y) = (R_k - d(y, \bar{t}_k))^{(n-2)/2} u_k(y)$, and,

$$t_k, \sup_{\bar{B}_{R_k}(\bar{t}_k)} s_k = s_k(t_k) \geq s_k(\bar{t}_k) = R_k^{(n-2)/2} u_k(\bar{t}_k) \geq R_k^{(n-2)/2} u_k(y_k) > 0,$$

We do a blow-up, then we consider:

$$n = 4, v_k(y) = r_k u_k(t_k + r_k y) = r_k u(\exp_{t_k}(r_k y)), r_k = [u_k(y_k)]^{-\epsilon},$$

with, $\epsilon > 0$,
and,

$$n \geq 5, v_k(y) = r_k u_k(t_k + (r_k)^{2/(n-2)} y) = r_k u_k(\exp_{t_k}(r_k^{2/(n-2)} y)), r_k = [u_k(t_k)]^{-\epsilon},$$

with, $\epsilon > \frac{(n-4)}{n-2}$.

Note that, here we have considered all terms of the sequence (u_k) .

Let's consider the blow-up functions (v_k) defined previously with the exponential maps for $n \geq 4$, $\exp_{t_k}(y)$, like in the previous papers for the dimensions, 4, 5, 6. Because we consider the compact sets $K, 2K$, and $t_k \in 2K$, the injectivity radius is uniformly bounded below by a positive number. Thus, we can consider all the terms of the sequence (u_k) without extraction. (After supposing the assertion $\inf v_k \geq m > 0$ infinitely many times, we can use extraction, for the points (t_k)).

We consider, $\delta_0 = \delta_0(K) = \inf\{\delta_P/4, P \in K\}$, with δ_P continuous in P and smaller than the injectivity radius in P for each P . We have a finite cover of K by small balls of radius $\delta_0/2$, we have a finite set of points $z_j \in K$: $K \subset \cup_{\{j=0, \dots, l\}} B(z_j, \delta_0/2) \subset \cup_{\{j=0, \dots, l\}} \bar{B}(z_j, 3\delta_0) = K_{\delta_0}$ is compact.

We take $R_k^{(n-2)/2} = \inf\{u_k(y_k)^{-\epsilon}, (\delta_0/2)^{(n-2)/2}\}$. The small balls are all compact, thus, \bar{t}_k exist and t_k exist. We take $r_k = \inf\{\delta_P/4, P \in K_{\delta_0}, u_k(t_k)^{-\epsilon}\}$. Thus \bar{t}_k, t_k and $\exp_{t_k}(\cdot)$ and v_k are defined for all $k \geq 0$.

We fix $m > 0$, we prove the result by assuming $\inf v_k \geq m > 0$, like for the dimensions 4 and 6. After we take $m = r_k \inf u_k > 0$. Suppose by contradiction, that there is infinitely many (v_k) with $\inf v_k \geq m > 0$, the proof imply that $[u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty$ which is impossible. Thus, there is a finite number of terms such that $\inf v_k \geq m > 0$, $k_1, \dots, k_{i(m)}$. Thus we have also, $v_k(0) \leq c(m)$ when $\inf v_k \geq m > 0$. In all cases, we have the following assertion:

$$\inf v_k \geq m > 0 \Rightarrow [u_k(\cdot)]^{1-\epsilon} = v_k(0) \leq c(m) < +\infty,$$

we obtain:

There is a non-increasing positive function $m \rightarrow c(m) > 0$, such that $\inf v_k \geq m > 0 \Rightarrow (u_k(\cdot))^{1-\epsilon} \leq c(m)$. then we apply this with $m = r_k \inf u_k$, we obtain for all terms of the sequence (u_k) :

$$n = 4, [u_k(y_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g),$$

and,

$$n \geq 5, [u_k(t_k)]^{1-\epsilon} \leq c(a, b, A, \frac{\inf u_k}{[u_k(t_k)]^\epsilon}, K, M, g).$$

For $n \geq 5$, we set, $0 < \epsilon_k = \frac{u_k(y_k)}{u_k(t_k)} \leq 1$, $\epsilon_k = \epsilon_k(K)$, we obtain:

$$n \geq 5, [u_k(y_k)]^{1-\epsilon} \leq \epsilon_k^{1-\epsilon} c(a, b, A, \epsilon_k^\epsilon \frac{\inf u_k}{[u_k(y_k)]^\epsilon}, K, M, g).$$

We have $u_k(y_k) = \sup_K u_k$.

II) *Auxiliary function and moving-plane method:*

We use the computations of previous papers with modifications, see [3,5,8,9]. We consider:

$$w_k(t, \theta) = e^{(n-2)t/2} v_k(e^t \theta) - \frac{m}{2} e^{(n-2)t/2},$$

We have $\lambda_k = \frac{-2}{n-2} \log v_k(0)$.

1) The biggest term is the term of V (for $n \geq 6$): $w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}})$, $t_0 \geq t \geq \xi_k$.

Because we must compare:

$$(w_k^{\xi_k})^{(n+2)/(n-2)} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \text{ and } (w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)},$$

and, we have used the mean value theorem for $f(t) = t^{(n+2)/(n-2)}$, and $w_k^{\xi_k} \leq w_k$ to have:

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)} \leq c(w_k^{\xi_k})^{4/(n-2)} (w_k^{\xi_k} - w_k),$$

and,

$$(w_k^{\xi_k})^{4/(n-2)} (w_k^{\xi_k} - w_k) \leq c(w_k^{\xi_k})^{4/(n-2)} (e^{(n-2)t^{\xi_k}/2} - e^{(n-2)t/2}).$$

Now, we write:

$$e^t = e^{(n-2)t/2} e^{(4-n)t/2} \leq e^{(4-n)\xi_k/2} e^{(n-2)t/2},$$

we integrate between t and t^{ξ_k} , we obtain:

$$(e^t - e^{t^{\xi_k}}) \leq c e^{(4-n)\xi_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

But,

$$w_k^{\xi_k} \leq c e^{(n-2)(\xi_k - \lambda_k)/2},$$

Thus the biggest term is:

$$w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{2/(n-2)} e^{(\xi_k - (n-2)\lambda_k)/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

but $\xi_k \leq \lambda_k + 2$, we obtain:

$$w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{2/(n-2)} e^{-(n-4)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

Thus,

$$n \geq 5, w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq \frac{c}{[u_k(t_k)]^{\epsilon - (n-4)/(n-2)}} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

$$n = 4, w_k^{\xi_k} r_k^{2/(n-2)} (e^t - e^{t^{\xi_k}}) \leq \frac{c}{[u_k(y_k)]^\epsilon} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

These terms are controlled by the term: $-\frac{m}{2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2})$.

2) Also, we have for $n \geq 6$:

Because we must compare:

$$w_k^{\xi_k} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \text{ and } (w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)},$$

We must look to the term:

$$(w_k^{\xi_k})^{1-(4/(n-2))} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) = (w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}), t_0 \geq t \geq \xi_k,$$

We write:

$$e^{2t} = e^{(n-2)t/2} e^{(6-n)t/2} \leq e^{(6-n)\xi_k/2} e^{(n-2)t/2},$$

But,

$$w_k^{\xi_k} \leq c e^{(n-2)(\xi_k - \lambda_k)/2},$$

Thus,

$$(w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \leq c r_k^{4/(n-2)} e^{-(n-6)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

we obtain:

$$(w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^t - e^{t^{\xi_k}}) \leq c r_k^{4/(n-2)} e^{-(n-6)\lambda_k/2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

Thus,

$$n \geq 6, (w_k^{\xi_k})^{(n-6)/(n-2)} r_k^{4/(n-2)} (e^{2t} - e^{2t^{\xi_k}}) \leq \frac{c}{[u_k(t_k)]^{\epsilon - (n-6)/(n-2)}} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2}),$$

But, $\epsilon > \frac{n-4}{n-2}$, these terms are controlled by the term: $-\frac{m}{2} (e^{(n-2)t/2} - e^{(n-2)t^{\xi_k}/2})$.

3) For $n = 5$: we have the terms: we use the binomial formula: we write:

$$(w_k^{\xi_k})^{7/3} - w_k^{7/3} = ((w_k^{\xi_k})^{1/3})^7 - (w_k^{1/3})^7,$$

$$x^7 - y^7 \equiv (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6), x = (w_k^{\xi_k})^{1/3}, y = w_k^{1/3},$$

but,

$$w_k^{\xi_k} - w_k = (x^3 - y^3) \equiv (x - y)(x^2 + xy + y^2),$$

Thus,

$$(w_k^{\xi_k})^{7/3} - w_k^{7/3} = (x^3 - y^3) \times \frac{(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)}{(x^2 + xy + y^2)},$$

Here, we have $x \leq y$, thus:

$$(x^2 + xy + y^2) \leq cy^2, (x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6) \geq c'x^2y^4, c, c' > 0$$

Thus, because $y \geq \frac{m}{2} e^{3t/2}$ we obtain:

$$(w_k^{\xi_k})^{7/3} - w_k^{7/3} \leq c(w_k^{\xi_k} - w_k)(w_k^{\xi_k})^{2/3} w_k^{2/3} \leq -ce^t (w_k^{\xi_k})^{2/3} (e^{3t/2} - e^{3t^{\xi_k}/2}), c > 0$$

For the case: $A = (w_k^{\xi_k})^{1/3} r_k^{4/3} (e^{2t} - e^{2t^{\xi_k}})$, $t_0 \geq t \geq \xi_k$

We have:

$$|A| \leq e^t (w_k^{\xi_k})^{1/3} r_k^{4/3} (e^t - e^{t^{\xi_k}})$$

The dominant term is:

$$B = (w_k^{\xi_k})^{1/3} r_k^{4/3} (e^t - e^{t^{\xi_k}})$$

We have:

$$e^t = e^{-t/2} e^{3t/2} \leq c e^{-\xi_k/2} e^{3t/2}, w_k^{\xi_k} \leq c e^{3(\xi_k - \lambda_k)/2}$$

Thus,

$$|B| \leq c r_k^{4/3} e^{-\lambda_k/2} (e^{3t/2} - e^{3t\xi_k/2})$$

$$\lambda_k = -(2/3)(1 - \epsilon) \log u_k(t_k), r_k = u_k(t_k)^{-\epsilon},$$

$$e^{-\lambda_k/2} = u_k(t_k)^{(1/3)(1-\epsilon)}, r_k^{4/3} = u_k(t_k)^{-4\epsilon/3}, r_k^{4/3} e^{-\lambda_k/2} = u_k(t_k)^{(1/3)(1-5\epsilon)}$$

The condition is $1 - 5\epsilon < 0$, $\epsilon > \frac{1}{5}$, but $\epsilon > \frac{n-4}{n-2} = \frac{5-4}{5-2} = 1/3 > 1/5$.

4) We have the same thing for the dimension 4.

5) When we use the auxiliary function $\frac{m}{2} e^{(n-2)t/2}$, there is a term:

$$r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}),$$

To correct this term, we consider a part of the term :

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)}$$

We use the binomial formula as for the previous case of dimension 5. We have:

$$x = (w_k^{\xi_k})^{1/(n-2)}, y = (w_k)^{1/(n-2)},$$

$$x^{n+2} - y^{n+2} = (x - y)(y^{n+1} + \dots), x^{n-2} - y^{n-2} = (x - y)(y^{n-3} + \dots + x^{n-3})$$

Thus,

$$x^{n+2} - y^{n+2} = (x^{n-2} - y^{n-2}) \frac{(y^{n+1} + \dots)}{(y^{n-3} + \dots)}$$

Because $x \leq y$ and,

$$(y^{n+1} + \dots) \geq y^{n+1}, (x^{n-3} + \dots + y^{n-3}) \leq c y^{n-3}, c > 0,$$

We obtain:

$$x^{n+2} - y^{n+2} \leq c(x^{n-2} - y^{n-2})y^4, c > 0,$$

Thus,

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)} \leq c(w_k^{\xi_k} - w_k)w_k^{4/(n-2)}, c > 0$$

Because, $w_k \geq \frac{m}{2} e^{(n-2)t/2}$, we obtain:

$$(w_k^{\xi_k})^{(n+2)/(n-2)} - (w_k)^{(n+2)/(n-2)} \leq -c e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}), c > 0,$$

Thus the term: $r_k^{4/(n-2)} e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2})$ is controlled by the term, $-c e^{2t} (e^{(n-2)t/2} - e^{(n-2)t\xi_k/2}), c > 0$.

We obtain the same proof in the previous papers, the dimensions 4, 6, see [5,8,9].

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DEPARTMENT OF MATHEMATICS, PIERRE ET MARIE CURIE UNIVERSITY, 75005, PARIS, FRANCE.
Email address: samybahoura@gmail.com