# A note on the relaxation process in a class of non-equilibrium two-phase flow models

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# A NOTE ON THE RELAXATION PROCESS IN A CLASS OF NON-EQUILIBRIUM TWO-PHASE FLOW MODELS*

JEAN-MARC HÉRARD †

**Abstract.** We focus here on the relaxation process in a class of two-phase flow models, considering first gas-liquid flows, and then liquid-vapour mixtures. The whole analysis enables to exibit a few conditions on the flow in order to guarantee the time decay of some variables. The former may depend on initial conditions but also on equations of state within each phase. The present analysis aims at providing some better understanding of inner processes, and it is also useful for numerical purposes, as emphasized in appendix B. It is a sequel of paper [25] where the sole pressure relaxation process in some multiphase flow models is investigated.

**Keywords.** Entropy ; Relaxation ; Two-phase flows ; Liquid-gas models ; Liquid-vapour mixture.

**AMS subject classifications.** 76T10 ; 76L05 ; 35Q35 ; 35L60.

## 1. Introduction

The main objective of the present note is to examine the relaxation process in a class of non-equilibrium two-phase flow models. For that purpose, we consider in the sequel a class of two-phase flows that we briefly recall below (see also, among others [1, 3, 7, 11, 12, 14, 15, 17, 18, 20, 21, 28, 30, 31, 34]). Both gas-liquid flows (without any mass transfer), and liquid-vapour flows with a unique component (for instance water) will be examined in the sequel. Actually, we wish to examine whether solutions of the governing set of equations comply with the expected relaxation process for pressure, velocity and temperature gaps $\Delta P, \Delta U, \Delta T$ between phases (and also Gibbs potential gaps when focusing on liquid-vapour flows including mass transfer).

In the case of gas-liquid flows, the statistical fractions of immiscible liquid phase and gas phase are noted $\alpha_l(\mathbf{x},t)$ and $\alpha_g(\mathbf{x},t)$, and they are such that:

$$\alpha_l(\mathbf{x},t)+\alpha_g(\mathbf{x},t)=1$$

If we turn to liquid-vapour flows, we will note $\alpha_l(\mathbf{x},t)$ and $\alpha_v(\mathbf{x},t)$ statistical fractions for the liquid phase and vapour phase respectively. Of course they will comply with the constraint:

$$\alpha_l(\mathbf{x},t)+\alpha_v(\mathbf{x},t)=1$$

In order to ease notations, we will priviledge the liquid phase in both cases.

Within the $k-$phase, $U_k$, $P_k$, $\rho_k$, $m_k=\alpha_k\rho_k$ and $E_k$ will respectively denote the phasic mean velocity, mean pressure, mean density, mass fraction and mean total energy, setting:

$$E_k=\rho_k(\epsilon_k(\rho_k,P_k)+U_k^2/2)$$

The gas-liquid state variable $W^{lg}$ will be:

$$W^{lg}=(\alpha_l,m_l,m_g,m_lU_l,m_gU_g,\alpha_lE_l,\alpha_gE_g)$$

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while the liquid-vapour state variable will correspond to:

$$W^{lv} = (\alpha_l, m_l, m_v, m_l U_l, m_v U_v, \alpha_l E_l, \alpha_v E_v)$$

The governing set of balance equations for mass, momentum and energy, within phase $k$, and for the statistical fraction $\alpha_l$, write:

$$\begin{cases} \partial_t(\alpha_l) + \mathcal{V}_I(W)\nabla\alpha_l = \phi_l(W) \ ; \\ \partial_t(m_k) + \nabla.(m_k \mathbf{U}_k) = \Gamma_k(W) \ ; \\ \partial_t(m_k \mathbf{U}_k) + \nabla.(m_k \mathbf{U}_k \otimes \mathbf{U}_k + \alpha_k P_k \mathbf{Id}) - \Pi_I(W)\nabla\alpha_k = \mathbf{S_{Q_k}}(W) \ ; \\ \partial_t(\alpha_k E_k) + \nabla.(\alpha_k \mathbf{U}_k(E_k + P_k)) + \Pi_I(W)\partial_t(\alpha_k) = S_{E_k}(W) \ . \end{cases} \tag{1.1}$$

for $k \in (l,g)$, or $k \in (l,v)$, while setting $\Gamma_k(W) = 0$ for gas-liquid flows.

Source terms are interfacial transfer terms, such that :

$$\sum_k S_{E_k}(W) = 0$$

together with:

$$\sum_k \mathbf{S_{Q_k}}(\mathbf{W}) = \mathbf{0}$$

Moreover, for liquid-vapour flows, mass transfer terms agree with:

$$\sum_k \Gamma_k(W) = 0$$

We assume that the interfacial velocity takes the form:

$$\mathcal{V}_I(W) = \beta(W)U_l + (1-\beta(W))U_{v,g}$$

(with some abuse of notation), in order to satisfy Galilean invariance, and also that the scalar function $\beta(W)$ guarantees unique field by field jump conditions (see below).

If $\eta, F_\eta$ denotes the mixture entropy - entropy flux pair:

$$\eta = \sum_k m_k S_k(P_k, \rho_k), \qquad \mathcal{F}_\eta = \sum_k m_k S_k(P_k, \rho_k) U_k, \tag{1.2}$$

for given phasic entropies $S_k(P_k, \rho_k)$ such that:

$$c_k^2 \partial_{P_k}(S_k(P_k, \rho_k)) + \partial_{\rho_k}(S_k(P_k, \rho_k)) = 0 \ ,$$

where:

$$\rho_k c_k^2 = \left(\frac{P_k}{\rho_k} - \rho_k \partial_{\rho_k}(\epsilon_k(P_k, \rho_k))\right) / \left(\partial_{P_k}(\epsilon_k(P_k, \rho_k))\right)$$

we recall that there exists a **unique** pressure of the form:

$$\Pi_I(W) = \mu(W)P_l + (1-\mu(W))P_{v,g} \tag{1.3}$$

with $\mu(W) \in [0,1]$, such that the following entropy balance holds:

$$\partial_t(\eta) + \nabla.(\mathcal{F}_\eta) = RHS_\eta(W) \geq 0, \tag{1.4}$$

The latter function reads:

$$\mu(W) = \frac{(1-\beta(W))T_{v,g}}{(1-\beta(W))T_{v,g} + \beta(W)T_l} \tag{1.5}$$

setting:

$$\frac{1}{T_k} = \frac{\partial_{P_k}(S_k(P_k,\rho_k))}{\partial_{P_k}(\epsilon_k(P_k,\rho_k))} \ .$$

Suitable closure laws for all source terms $\Gamma_l(W)$, $\phi_l(W)$, $S_{Q_l}(W)$ and $S_{E_l}(W)$, complying with the entropy inequality (1.4) , will be recalled in the sequel. Thus the function $\mu(W)$ is **totally defined** once $\beta(W)$ is given. Eventually, we note that the closure law (1.3) satisfies the Realizable Interfacial Pressure condition (see **appendix A**).

Specific forms of the function $\beta(W)$ are recalled below, while focusing on the sole convective part associated with system (1.1). We restrict to the one-dimensional framework, and we first recall the following classical results concerning the structure of the one-dimensional convective subset of (1.1) :

- *The convective system arising from* (1.1) *is hyperbolic. Its eigenvalues are real:*

$$\lambda_0(W) = \mathcal{V}_I(W) \ ;$$

$$\lambda_1(W) = U_l - c_l \quad ; \quad \lambda_2(W) = U_l \quad ; \quad \lambda_3(W) = U_l + c_l \ ; \tag{1.6}$$

$$\lambda_4(W) = U_{v,g} - c_{v,g} \quad ; \quad \lambda_5(W) = U_{v,g} \quad ; \quad \lambda_6(W) = U_{v,g} + c_{v,g} \ ,$$

  *and the set of associated right eigenvectors spans the whole space away from the resonance state:*

$$|U_k - \mathcal{V}_I(W)| = c_k \tag{1.7}$$

- *Waves associated with eigenvalues* $U_k \pm c_k$ *(with* $k \in (l,v)$ *or* $k \in (l,g)$*) are genuinely non linear ; besides, waves associated with eigenvalues* $U_k$ *are linearly degenerate.*
- *The wave associated with* $\lambda_0(W)$ *is linearly degenerate when:*

$$\beta(W) = 0 \quad , \qquad or: \qquad \beta(W) = 1 \quad , \tag{1.8}$$

  *or when:*

$$\beta(W) = \frac{m_l}{m_l + m_{v,g}} \ . \tag{1.9}$$

- *System* (1.1) *can be symmetrized away from resonant states* (1.7) *in the one-dimensional framework.*

Detailed proofs of the first three properties can be found in [7,11] and in [8,9,31] for the latter feature, in the 1D-case. Moreover, a proof of symmetry for a class of multiphase hybrid models has been pointed out in the recent paper [6], still in the 1D-case. We must note at this stage that a fully different proof of symetrization can also be found in [16] in the one-dimensional framework for hyperbolic models. The latter point has major consequences (see [29, 35]). Eventually we emphasize that one straightforward consequence of the particular choices (1.8), (1.9) of $\beta(W)$ is that shock solutions are well defined (see in particular [19] for a detailed analysis of that feature).

## 2. Gas-liquid two-phase flow models

We first focus on gas-liquid two-phase flow models, thus omitting mass transfer.

In the following, we consider an homogeneous situation, where:

$$\nabla\psi = 0$$

whatever $\psi$ stands for. Hence system (1.1) reduces to :

$$\begin{cases} \partial_t(\alpha_l) = \phi_l(W) \ ; \\ \partial_t(m_k) = 0 \ ; \\ \partial_t(m_k \mathbf{U}_k) = \mathbf{S_{Q_k}}(W) \ ; \\ \partial_t(\alpha_k E_k) + \Pi_I(W)\partial_t(\alpha_k) = S_{E_k}(W) \ . \end{cases} \tag{2.1}$$

since the interfacial mass transfer $\Gamma_k(W)$ is null. The latter may be rewritten as:

$$\begin{cases} \partial_t(\alpha_l) = \phi_l(W) \ ; \\ \partial_t(\alpha_l E_l) + \Pi_I(W)\partial_t(\alpha_l) = S_{E_l}(W) \ ; \\ \partial_t(m_l \mathbf{U}_l) = \mathbf{S_{Q_l}}(W) \ ; \\ \partial_t(m_l) = \partial_t(m_g) = 0 \ ; \\ \partial_t(m_l \mathbf{U}_l + m_g \mathbf{U}_g) = 0 \ ; \\ \partial_t(\alpha_l E_l + \alpha_g E_g) = 0 \ . \end{cases} \tag{2.2}$$

We emphasize that the entropy increases throughout step (2.2), since:

$$\partial_t(\eta) \geq 0 \ . \tag{2.3}$$

**2.1. Entropy consistent gas-liquid closure laws.** A convenient way to discuss closure laws consists in translating unkown quantities $\mathbf{S_{Q_k}}(W)$ and $S_{E_k}(W)$, thus setting:

$$\begin{cases} D_k(W) = \mathbf{S_{Q_k}}(W) \ ; \\ \psi_k(W) = S_{E_k}(W) - V_I^E(W) D_k(W) \ ; \end{cases} \tag{2.4}$$

Since the latter terms represent interfacial transfer terms, this implies:

$$D_l(W) + D_g(W) = 0$$

and also:

$$\psi_l(W) + \psi_g(W) = 0$$

Using the same Galilean invariance argument as before, the interfacial velocity $V_I^E(W)$ is written as:

$$V_I^E(W) = \beta^E(W) U_l + (1 - \beta^E(W)) U_g$$

and the unknown function $\beta^E(W)$ will be assumed to lie in $[0,1]$. Hence we may recover the mixture entropy budget in step (2.2):

$$\begin{aligned}\partial_t(\eta) = & \psi_l(W)\left(\frac{1}{T_l}-\frac{1}{T_g}\right)\\ & +D_l(W)\left(\frac{V_I^E(W)-U_l}{T_l}-\frac{V_I^E(W)-U_g}{T_g}\right)\\ & -\phi_l(W)\left(\frac{\Pi_I(W)-P_l}{T_l}-\frac{\Pi_I(W)-P_g}{T_g}\right)\end{aligned} \tag{2.5}$$

or alternatively:

$$\begin{aligned}\partial_t(\eta) = & \frac{1}{T_lT_g}\psi_l(W)(T_g-T_l)\\ & +\left(\frac{1-\beta^E(W)}{T_l}+\frac{\beta^E(W)}{T_g}\right)D_l(W)(U_g-U_l)\\ & +\left(\frac{1-\mu(W)}{T_l}+\frac{\mu(W)}{T_g}\right)\phi_l(W)(P_l-P_g)\end{aligned} \tag{2.6}$$

Hence, noting:

$$\left\{\begin{array}{l}\Delta\mathcal{P}=P_l-P_g\ ,\\ \Delta\mathcal{U}=U_l-U_g\ ,\\ \Delta\mathcal{T}=T_l-T_g\ ,\end{array}\right. \tag{2.7}$$

we end up with the classical entropy consistent closure laws for source terms.

**Property 1:** (Entropy consistent source terms for gas-liquid flow models)
*Assume that source terms are such that:*

$$\left\{\begin{array}{l}\phi_l(W)=K(W)\Delta\mathcal{P}\ ,\\ D_l(W)=-d(W)\Delta\mathcal{U}\ ,\\ \psi_l(W)=-q(W)\Delta\mathcal{T}\ ,\end{array}\right. \tag{2.8}$$

*with positive functions $K(W),d(W),q(W)$. Then solutions of system (2.2) guarantee:*

$$\partial_t(\eta)\geq 0$$

□

Owing to (2.7), (2.6) and (2.8), this result is obtained in a straightforward way.

Pressure, temperature and velocity relaxation time scales respectively involved in functions $K(W),q(W),d(W)$ will be noted $\tau_P(W),\tau_T(W),\tau_U(W)$. We obviously retrieve expected forms for drag terms and heat transfer terms $d(W)$ and $q(W)$. These may be taken from [27], when focusing on temperature and velocity relaxation time scales. Of particular interest, references [4, 5, 13] discuss closure laws for the pressure relaxation time scales. Actually, more complex entropy consistent closure laws than (2.8) might be considered, but this will not be discussed herein for sake of clarity.

**2.2. Pressure-velocity-temperature relaxation process.** Starting with system (2.2), the governing equations of $P_k,T_k,U_k$ can be derived in a straightforward manner. These are:

$$m_k\partial_t(U_k)=D_k(W)\ ,$$

$$m_k\partial_{P_k}(\epsilon_k)|_{\rho_k}\partial_t(P_k)=\psi_k(W)+(V_I^E(W)-U_k)D_k(W)-(\Pi_I(W)-\rho_k^2\partial_{\rho_k}(\epsilon_k)|_{P_k})\phi_k(W)\ ,$$

and:

$$m_k\partial_{T_k}(\epsilon_k)|_{\rho_k}\partial_t(T_k)=\psi_k(W)+(V_I^E(W)-U_k)D_k(W)-(\Pi_I(W)-\rho_k^2\partial_{\rho_k}(\epsilon_k)|_{T_k})\phi_k(W)\ .$$

Hence we get:

$$\partial_t\left(\underline{\Delta}^{lg}\right)=-\mathcal{R}^{lg}(W)\underline{\Delta}^{lg} \tag{2.9}$$

noting:

$$\underline{\Delta}^{lg}=(\Delta\mathcal{U},\Delta\mathcal{T},\Delta\mathcal{P})^T$$

and:

$$\mathcal{R}^{lg}(W)=\begin{pmatrix} a_{UU}^{lg}(W) & 0 & 0\\ a_{TU}^{lg}(W) & a_{TT}^{lg}(W) & a_{TP}^{lg}(W)\\ a_{PU}^{lg}(W) & a_{PT}^{lg}(W) & a_{PP}^{lg}(W)\end{pmatrix} \tag{2.10}$$

Coefficients in matrix $\mathcal{R}^{lg}(W)\in R_{3\times 3}$ read:

$$a_{UU}^{lg}(W)=d(W)\left(\frac{1}{m_l}+\frac{1}{m_g}\right)\ ,$$

together with:

$$\begin{cases} a_{TU}^{lg}(W)=d(W)\Delta\mathcal{U}\left(\frac{\beta^E(W)-1}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{\beta^E(W)}{m_g\partial_{T_g}(\epsilon_g)|_{\rho_g}}\right)\ ,\\ a_{TT}^{lg}(W)=q(W)\left(\frac{1}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{1}{m_g\partial_{T_g}(\epsilon_g)|_{\rho_g}}\right)\ ,\\ a_{TP}^{lg}(W)=K(W)\left(\frac{\Pi_I-\rho_l^2\partial_{\rho_l}(\epsilon_l)|_{T_l}}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{\Pi_I-\rho_g^2\partial_{\rho_g}(\epsilon_g)|_{T_g}}{m_g\partial_{T_g}(\epsilon_g)|_{\rho_g}}\right)\ ,\end{cases} \tag{2.11}$$

and:

$$\begin{cases} a_{PU}^{lg}(W)=-d(W)\Delta\mathcal{U}\left(\frac{1-\beta^E(W)}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}-\frac{\beta^E(W)}{m_g\partial_{P_g}(\epsilon_g)|_{\rho_g}}\right)\ ,\\ a_{PT}^{lg}(W)=q(W)\left(\frac{1}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}+\frac{1}{m_g\partial_{P_g}(\epsilon_g)|_{\rho_g}}\right)\ ,\\ a_{PP}^{lg}(W)=K(W)\left(A_l+A_g+(\frac{\mu(W)-1}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}+\frac{\mu(W)}{m_g\partial_{P_g}(\epsilon_g)|_{\rho_g}})\Delta\mathcal{P}\right)\ .\end{cases} \tag{2.12}$$

We have used the following notation here for $k=l,g$:

$$A_k=\frac{\rho_k c_k^2}{\alpha_k}$$

Obviously, at least two points should be emphasized. First of all, we note that the matrix $\mathcal{R}^{lg}(W)$ is block triangular, and this also means that retaining the "flow in a box" assumption with null velocities, is actually relevant. Moreover, it may be noted that both coefficients $a_{PU}^{lg}(W)$ and $a_{TU}^{lg}(W)$ vanish when the expected velocity equilibrium is reached. We now can summarize results in the following:

**Property 2:** (Relaxation effects in a class of immiscible two-phase flow models)
*Consider EOS within each phase such that, for $k=l,g$:*

$$0\leq\partial_{T_k}(\epsilon_k)|_{\rho_k}. \tag{2.13}$$

*Then the relaxation process is ensured for solutions of (2.9), when using positive functions $K(W), q(W), d(W)$, if eigenvalues of matrix $\mathcal{R}^{lg}(W)$ have a positive real part. This is guaranteed if $\Delta\mathcal{P}$ is small enough in the following sense:*

$$|\Delta\mathcal{P}| \times |\frac{(\mu(W)-1)\alpha_g}{\rho_l \partial_{P_l}(\epsilon_l)|_{\rho_l}} + \frac{\mu(W)\alpha_l}{\rho_g \partial_{P_g}(\epsilon_g)|_{\rho_g}}| \leq \alpha_g \rho_l c_l^2 + \alpha_l \rho_g c_g^2 \ , \tag{2.14}$$

*and if:*

$$a_{TT}^{lg}(W) a_{PP}^{lg}(W) - a_{TP}^{lg}(W) a_{PT}^{lg}(W) > 0 \ . \tag{2.15}$$

□

**Proof**:

- First we may check that $\lambda = a_{UU}^{lg}(W)$ is a real positive eigenvalue of matrix $\mathcal{R}^{lg}(W)$.
- The other two eigenvalues of $\mathcal{R}^{lg}(W)$ are solutions $\lambda_\pm$ of polynomial:

$$q(\lambda) = \lambda^2 - (a_{TT}^{lg}(W) + a_{PP}^{lg}(W))\lambda + a_{TT}^{lg}(W) a_{PP}^{lg}(W) - a_{TP}^{lg}(W) a_{PT}^{lg}(W)$$

  Considering the two conditions (2.13) and (2.14), we may conclude that the sum $a_{TT}^{lg}(W) + a_{PP}^{lg}(W)$ is positive, thus:
  - If the two eigenvalues are complex conjugate, then their real part $(a_{TT}^{lg}(W) + a_{PP}^{lg}(W))/2$ is positive ;
  - Suppose now that the two eigenvalues $\lambda_\pm$ lie in $\mathcal{R}$, then: $\lambda_+ + \lambda_- = a_{TT}^{lg}(W) + a_{PP}^{lg}(W)$. Thus the sum of eigenvalues is positive and their product reads :

$$\lambda_+ \lambda_- = a_{TT}^{lg}(W) a_{PP}^{lg}(W) - a_{TP}^{lg}(W) a_{PT}^{lg}(W)$$

    Hence both $\lambda_+$ and $\lambda_-$ are positive, owing to (2.15).

□

**Remark 1:**

- The **first** condition (2.13) is satified in many EOS. Among these we may at least mention generalized stiffened gas EOS:

$$(\gamma_k - 1)\rho_k \epsilon_k = P_k + \gamma_k \Pi_k$$

- The threshold effect on $\Delta\mathcal{P}$ arising in **second** condition (2.14), which comes out when taking the energy budget equation into account, and which does not exist when restricting to the barotropic case (see [22]), was first mentionned in [2]. Simple practical computations enable to show that it can be hardly violated in an industrial framework. Eventually we note that the following sufficient condition:

$$|\Delta\mathcal{P}|(|\frac{1}{\rho_l \partial_{P_l}(\epsilon_l)|_{\rho_l}}| + |\frac{1}{\rho_g \partial_{P_g}(\epsilon_g)|_{\rho_g}}|) \leq min(\rho_l c_l^2, \rho_g c_g^2)$$

  guarantees that the condition (2.14) holds.

- It also seems worth examining the **third** condition (2.15) in some specific conditions of EOS. For instance, if we restrict to perfect gas EOS:

$$\rho_k \epsilon_k = P_k/(\gamma_k - 1)$$

it may be checked that it will be automaticaly satisfied.

□

## 3. Liquid-vapour flow models

We still consider an homogeneous situation, and focus now on system (1.1) with mass transfer terms $\Gamma_k(W)$, for $k = l, v$, thus:

$$\begin{cases} \partial_t(\alpha_l) = \phi_l(W) \ ; \\ \partial_t(\alpha_l E_l) + \Pi_I(W)\partial_t(\alpha_l) = S_{E_l}(W) \ ; \\ \partial_t(m_l \mathbf{U}_l) = \mathbf{S_{Q_l}}(W) \ ; \\ \partial_t(m_l) = \Gamma_l(W) \ ; \\ \partial_t(m_l + m_v) = 0 \ ; \\ \partial_t(m_l \mathbf{U}_l + m_v \mathbf{U}_v) = 0 \ ; \\ \partial_t(\alpha_l E_l + \alpha_v E_v) = 0 \ . \end{cases} \tag{3.1}$$

**3.1. Entropy consistent liquid-vapour closure laws.** In a similar way, we rewrite unknown source terms in order to focus on the following variables $D_k(W)$, $\psi_k(W)$ such that:

$$\begin{cases} D_k(W) = \mathbf{S_{Q_k}}(W) - \frac{U_l + U_v}{2}\Gamma_k(W) \ ; \\ \psi_k(W) = S_{E_k}(W) - V_I^E(W) D_k(W) - H_I(W)\Gamma_k(W) \ ; \end{cases} \tag{3.2}$$

Recall that interfacial mass, momentum, and heat transfer terms, are such that:

$$\Gamma_l(W) + \Gamma_v(W) = 0 \ ,$$

$$D_l(W) + D_v(W) = 0 \ ,$$

and also:

$$\psi_l(W) + \psi_v(W) = 0 \ .$$

We have set (see [10, 23, 32]):

$$H_I(W) = h_I(W) + U_l U_v/2 \ ,$$

and we still consider the Galilean invariant formulation:

$$V_I^E(W) = \beta^E(W) U_l + (1 - \beta^E(W)) U_v \ .$$

where $\beta^E(W) \in [0,1]$.

We note in the sequel:

$$\mu_k = h_k - T_k S_k$$

where the free enthalpy $h_k$ writes:

$$h_k = \epsilon_k(P_k,\rho_k) + \frac{P_k}{\rho_k}$$

The governing equation for the mixture entropy $\eta$ now writes:

$$\begin{array}{l} \partial_t(\eta) = \frac{1}{T_l T_v}\left(\psi_l(W) + h_I(W)\Gamma_l(W)\right)(T_v - T_l) \\ \quad + \left(\frac{1-\beta^E(W)}{T_l} + \frac{\beta^E(W)}{T_v}\right) D_l(W)(U_v - U_l) \\ \quad + \left(\frac{1-\mu(W)}{T_l} + \frac{\mu(W)}{T_v}\right)\phi_l(W)(P_l - P_v) \\ \quad + (\frac{\mu_v}{T_v} - \frac{\mu_l}{T_l})\Gamma_l(W) \end{array} \tag{3.3}$$

We still note:

$$\left\{ \begin{array}{l} \Delta\mathcal{P} = P_l - P_v \ , \\ \Delta\mathcal{U} = U_l - U_v \ , \\ \Delta\mathcal{T} = T_l - T_v \ , \\ \Delta\mu = \frac{\mu_l}{T_l} - \frac{\mu_v}{T_v} \ , \end{array} \right. \tag{3.4}$$

Hence we get:

**Property 3:** (Entropy consistent source terms for liquid-vapor flow models)
*The following source terms:*

$$\left\{ \begin{array}{l} \phi_l(W) = K(W)\Delta\mathcal{P} \ , \\ D_l(W) = -d(W)\Delta\mathcal{U} \ , \\ \psi_l(W) + h_I(W)\Gamma_l(W) = -q(W)\Delta\mathcal{T} \ , \\ \Gamma_l(W) = -\Lambda(W)\Delta\mu \ , \end{array} \right. \tag{3.5}$$

*are entropy consistent, assuming that functions $K(W), d(W), q(W), \Lambda(W)$ are positive, which means that solutions of (3.1) comply with:*

$$\partial_t(\eta) \geq 0$$

□

In the sequel, we will retain the closure law:

$$h_I(W) = 0 \ .$$

**3.2. The relaxation process in liquid-vapour two-fluid models.** Using system (3.1), we may first derive the governing equations of $U_k$, which read:

$$m_k\partial_t(U_k) = D_k(W) + (\frac{U_l + U_v}{2} - U_k)\Gamma_k(W) \ ,$$

Thus we get:

$$\partial_t(\Delta\mathcal{U}) = -\left((\frac{1}{m_l} + \frac{1}{m_v})d(W) + (\frac{1}{2m_l} - \frac{1}{2m_v})\Gamma_l(W)\right)\Delta\mathcal{U} = -a^{lv}_{UU}(W)\Delta\mathcal{U} \ .$$

Again, starting with the total phasic energy equation,substracting the phasic kinetic energy governing equation, the time variation of the internal energy follows. This in turn enables to get :

$$\begin{array}{l} m_k\partial_{P_k}(\epsilon_k)|_{\rho_k}\partial_t(P_k) = \psi_k(W) + (V_I^E(W) - U_k)D_k(W) - (\Pi_I(W) - \rho_k^2\partial_{\rho_k}(\epsilon_k)|_{P_k})\phi_k(W) \\ \qquad -(\epsilon_k + \rho_k\partial_{\rho_k}(\epsilon_k)|_{P_k})\Gamma_k(W) \ . \end{array} \tag{3.6}$$

Moreover we have:

$$m_k\partial_{T_k}(\epsilon_k)|_{\rho_k}\partial_t(T_k)=\psi_k(W)+(V_I^E(W)-U_k)D_k(W)-(\Pi_I(W)-\rho_k^2\partial_{\rho_k}(\epsilon_k)|_{T_k})\phi_k(W) \\ -(\epsilon_k+\rho_k\partial_{\rho_k}(\epsilon_k)|_{T_k})\Gamma_k(W)\ . \tag{3.7}$$

Besides, the entropy variations agree with:

$$m_kT_k\partial_t(S_k)=\psi_k(W)+(V_I^E(W)-U_k)D_k(W)-(\Pi_I(W)-P_k)\phi_k(W)-h_k\Gamma_k(W)\ . \tag{3.8}$$

By introducing the vector of unknowns $\underline{\Delta}^{lv}\in R^4$:

$$\underline{\Delta}^{lv}=(\Delta\mathcal{U},\Delta\mathcal{T},\Delta\mathcal{P},\Delta\mu)^T$$

we may write its time evolution:

$$\partial_t\left(\underline{\Delta}^{lv}\right)=-\mathcal{R}^{lv}(W)\underline{\Delta}^{lv} \tag{3.9}$$

with real entries for the matrix $\mathcal{R}^{lv}(W)\in R_{4\times4}$:

$$\mathcal{R}^{lv}(W)=\begin{pmatrix} a_{UU}^{lv}(W) & 0 & 0 & 0\\ a_{TU}^{lv}(W) & a_{TT}^{lv}(W) & a_{TP}^{lv}(W) & a_{T\mu}^{lv}(W)\\ a_{PU}^{lv}(W) & a_{PT}^{lv}(W) & a_{PP}^{lv}(W) & a_{P\mu}^{lv}(W)\\ a_{\mu U}^{lv}(W) & a_{\mu T}^{lv}(W) & a_{\mu P}^{lv}(W) & a_{\mu\mu}^{lv}(W)\end{pmatrix} \tag{3.10}$$

The above mentionned coefficients are as follows:

$$\begin{cases} a_{TU}^{lv}(W)=d(W)\Delta\mathcal{U}\left(\frac{\beta^E(W)-1}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{\beta^E(W)}{m_v\partial_{T_v}(\epsilon_v)|_{\rho_v}}\right)\ ,\\ a_{TT}^{lv}(W)=q(W)\left(\frac{1}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{1}{m_v\partial_{T_v}(\epsilon_v)|_{\rho_v}}\right)\ ,\\ a_{TP}^{lv}(W)=K(W)\left(\frac{\Pi_I(W)-\rho_l^2\partial_{\rho_l}(\epsilon_l)|_{T_l}}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{\Pi_I(W)-\rho_v^2\partial_{\rho_v}(\epsilon_v)|_{T_v}}{m_v\partial_{T_v}(\epsilon_v)|_{\rho_v}}\right)\ ,\\ a_{T\mu}^{lv}(W)=\Lambda(W)\left(\frac{-\epsilon_l-\rho_l\partial_{\rho_l}(\epsilon_l)|_{T_l}}{m_l\partial_{T_l}(\epsilon_l)|_{\rho_l}}+\frac{-\epsilon_v-\rho_v\partial_{\rho_v}(\epsilon_v)|_{T_v}}{m_v\partial_{T_v}(\epsilon_v)|_{\rho_v}}\right)\end{cases} \tag{3.11}$$

and:

$$\begin{cases} a_{PU}^{lv}(W)=d(W)\Delta\mathcal{U}\left(\frac{\beta^E(W)-1}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}+\frac{\beta^E(W)}{m_v\partial_{P_v}(\epsilon_v)|_{\rho_v}}\right)\ ,\\ a_{PT}^{lv}(W)=q(W)\left(\frac{1}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}+\frac{1}{m_v\partial_{P_v}(\epsilon_v)|_{\rho_v}}\right)\ ,\\ a_{PP}^{lv}(W)=K(W)\left(A_l+A_g+(\frac{\mu(W)-1}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}+\frac{\mu(W)}{m_v\partial_{P_v}(\epsilon_v)|_{\rho_v}})\Delta\mathcal{P}\right)\ ,\\ a_{P\mu}^{lv}(W)=\Lambda(W)\left(-\frac{\epsilon_l+\rho_l\partial_{\rho_l}(\epsilon_l)|_{P_l}}{m_l\partial_{P_l}(\epsilon_l)|_{\rho_l}}-\frac{\epsilon_v+\rho_v\partial_{\rho_v}(\epsilon_v)|_{P_v}}{m_v\partial_{P_v}(\epsilon_v)|_{\rho_v}}\right)\end{cases} \tag{3.12}$$

Eventually, setting:

$$F_k=(\frac{1}{\rho_k\partial_{P_k}(\epsilon_k)|_{\rho_k}}-\frac{h_k}{T_k\partial_{T_k}(\epsilon_k)|_{\rho_k}})$$

$$G_k=\frac{h_k}{T_k\partial_{T_k}(\epsilon_k)|_{\rho_k}}(\epsilon_k+\rho_k\partial_{\rho_k}(\epsilon_k)|_{T_k})-\frac{1}{\rho_k\partial_{P_k}(\epsilon_k)|_{\rho_k}}(\epsilon_k+\rho_k\partial_{\rho_k}(\epsilon_k)|_{P_k})$$

$$H_k=\frac{h_k}{T_k\partial_{T_k}(\epsilon_k)|_{\rho_k}}(-\Pi_I(W)+\rho_k^2\partial_{\rho_k}(\epsilon_k)|_{T_k})-\frac{1}{\rho_k\partial_{P_k}(\epsilon_k)|_{\rho_k}}(-\Pi_I(W)+\rho_k^2\partial_{\rho_k}(\epsilon_k)|_{P_k})$$

for $k=l,v$, we have:

$$\begin{cases} a^{lv}_{\mu U}(W)=d(W)\Delta\mathcal{U}\left((\beta^E(W)-1)\frac{F_l}{m_lT_l}+\beta^E(W)\frac{F_v}{m_vT_v}\right) , \\ a^{lv}_{\mu T}(W)=q(W)\left(\frac{F_l}{m_lT_l}+\frac{F_v}{m_vT_v}\right) , \\ a^{lv}_{\mu P}(W)=K(W)\left(\frac{H_l}{m_lT_l}+\frac{H_v}{m_vT_v}\right) , \\ a^{lv}_{\mu\mu}(W)=\Lambda(W)\left(\frac{G_l}{m_lT_l}+\frac{G_v}{m_vT_v}\right) \end{cases} \tag{3.13}$$

□

**Remark 2:** In some specific situations, for instance when considering stiffened gas EOS:

$$P_k+\gamma_k\Pi_k=(\gamma_k-1)\rho_k\epsilon_k$$

we may have:

$$a^{lv}_{P\mu}(W)=0$$

and also:

$$a^{lv}_{\mu\mu}(W)>0$$

□

Now, we may give the following general result:

**Property 4:** (Relaxation effects in a class of liquid-vapour flow models)
*We still assume that EOS within each phase are such that, for $k=l,v$:*

$$0\leq\partial_{T_k}(\epsilon_k)|_{\rho_k} \tag{3.14}$$

*and also that functions $K(W),q(W),d(W),\Lambda(W)$ remain positive.*

*In order to guarantee the relaxation process, four conditions are required.*

- *The velocity relaxation process is ensured in (3.9), if and only if:*

$$\frac{(m_v-m_l)}{(m_l+m_v)}\Delta\mu<2\frac{d(W)}{\Lambda(W)} \tag{3.15}$$

- *In order to guarantee the pressure-temperature-potential relaxation process in (3.9), the following three conditions are mandatory:*

$$\begin{cases} I^{lv}>0 \\ II^{lv}>0 \\ III^{lv}>0 \end{cases} \tag{3.16}$$

*noting:*

$$I^{lv}=a^{lv}_{TT}(W)+a^{lv}_{PP}(W)+a^{lv}_{\mu\mu}(W)$$

*but also:*

$$II^{lv}=a^{lv}_{TT}a^{lv}_{\mu\mu}+a^{lv}_{PP}a^{lv}_{\mu\mu}+a^{lv}_{PP}a^{lv}_{TT}-a^{lv}_{TP}a^{lv}_{PT}-a^{lv}_{P\mu}a^{lv}_{\mu P}-a^{lv}_{T\mu}a^{lv}_{\mu T}$$

*and:*

$$III^{lv} = a^{lv}_{TT}a^{lv}_{\mu\mu}a^{lv}_{PP} - a^{lv}_{TT}a^{lv}_{P\mu}a^{lv}_{\mu P} - a^{lv}_{PP}a^{lv}_{T\mu}a^{lv}_{\mu T} - a^{lv}_{\mu\mu}a^{lv}_{PT}a^{lv}_{TP} + a^{lv}_{T\mu}a^{lv}_{\mu P}a^{lv}_{PT} + a^{lv}_{\mu T}a^{lv}_{TP}a^{lv}_{P\mu}$$

□

**Proof**:
Note first that the relaxation process will be effective if eigenvalues of matrix $\mathcal{R}^{lv}(W)$ are either real positive, or imaginary with a positive real part. The first obvious eigenvalue is real and equal to:

$$\lambda_0 = a^{lv}_{UU}(W)$$

The other three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are those of matrix

$$\begin{pmatrix} a^{lv}_{TT}(W) & a^{lv}_{TP}(W) & a^{lv}_{T\mu}(W) \\ a^{lv}_{PT}(W) & a^{lv}_{PP}(W) & a^{lv}_{P\mu}(W) \\ a^{lv}_{\mu T}(W) & a^{lv}_{\mu P}(W) & a^{lv}_{\mu\mu}(W) \end{pmatrix} \tag{3.17}$$

a) Actually the first condition (3.15) simply corresponds to the constraint :

$$a^{lv}_{UU}(W) > 0$$

which implies that $\lambda_0$ is real positive

b) Without any loss of generality, we may assume that $\lambda_1$ is real positive, and that $\lambda_2, \lambda_3$ are either real positive, or complex conjugate with a positive real part.

- First note that :
  - The function:

    $$I^{lv} = \lambda_1 + \lambda_2 + \lambda_3$$

    is always positive. It is obvious in the real case, but also in the imaginary case since:

    $$I^{lv} = \lambda_1 + 2Real(\lambda_2) > 0$$

  - A similar result holds for :

    $$II^{lv} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

    since $II^{lv}$ may be rewritten as:

    $$II^{lv} = 2\lambda_1 Real(\lambda_2) + ||\lambda_2||^2 > 0$$

    when $\lambda_2$ is imaginary.
  - This also holds true for:

    $$III^{lv} = \lambda_1\lambda_2\lambda_3$$

    which writes : $III^{lv} = \lambda_1||\lambda_2||^2 > 0$ when $\lambda_2$ is not real.
- The characteristic polynomial associated with matrix arising in (3.17) is:

  $$p_3(\lambda) = -\lambda^3 + I^{lv}\lambda^2 - II^{lv}\lambda + III^{lv}$$

  where the functions $I^{lv}, II^{lv}, III^{lv}$ match with expressions arising in Property 4. This completes the proof.

□

**Remark 3:**

- As it occured in the gas-liquid framework, the three coefficients $a^{lv}_{TU}$ , $a^{lv}_{PU}$ and $a^{lv}_{\mu U}$ vanish when $\Delta\mathcal{U}$ tends to zero, that is when the flow is close to velocity equilibrium.

- The first condition $I^{lv} > 0$ arising in (3.16) is more or less equivalent to the condition (2.14) arising in the gas-liquid case, while cancelling the last term $a^{lv}_{\mu\mu}(W)$. This is in fact exact in the limit case where $\tau_P \to 0$.

  We also note that when condition (3.14) is satisfied for all phasic EOS, the quantity $a^{lv}_{TT}(W)$ is positive.

  Eventually, it seems worth mentioning that for stiffened gas EOS - see remark 2 -, the coefficient $a^{lv}_{\mu\mu}(W)$ remains positive. Actually a simplified form of $G_k$ writes:

$$G_k = \frac{h_k}{T_k \partial_{T_k}(\epsilon_k)|_{\rho_k}} (\epsilon_k + \rho_k \partial_{\rho_k}(\epsilon_k)|_{T_k})$$

  since:

$$\epsilon_k + \rho_k \partial_{\rho_k}(\epsilon_k)|_{P_k} = 0$$

  in that case. Moreover, we have : $\epsilon_k + \rho_k \partial_{\rho_k}(\epsilon_k)|_{T_k} = C_{v,k} T_k > 0$.
- When enforcing $a^{lv}_{\mu\mu}(W), a^{lv}_{\mu T}(W)$ and $a^{lv}_{\mu P}(W)$ to zero, the second condition $II^{lv} > 0$ degenerates, and we obviously retrieve the counterpart of the second condition:

$$a_{TT}(W) a_{PP}(W) - a_{TP}(W) a_{PT}(W) > 0$$

  in Property 2 associated to the gas-liquid framework.

- It must also be emphasized that $\frac{III^{lv}}{q(W)K(W)\Lambda(W)}$ no longer includes any pressure / temperature / potential relaxation time scale $\tau_P, \tau_T, \tau_\mu$. This latter condition must be checked by the computer code.

□

## 4. Concluding remarks

We first recall that the present investigation is the sequel of a detailed analysis of the sole pressure relaxation process in various multiphase flow models (see [25]).

Some conditions pertaining to initial conditions or phasic equations of state have been pointed out, focusing on liquid-gas models first, and turning then to liquid-vapour flows including mass transfer. Detailed expressions of relaxation time scales have not been discussed herein. We refer to classical literature [27] for that purpose, while restricting to velocity, temperature and potential relaxation time scales. Turning to the pressure relaxation time scale $\tau_P$, the reader may for instance examine [13], and

also [4,5]. The latter time scales rely on a simple combination of phasic viscosities and statistical fractions.

Eventually we would like to give emphasis on the following:

- Quite obviously, the relaxation process is much more intricated when considering the liquid-vapour case, as it was expected. As it has been emphasized in the text, the velocity relaxation is in some sense decoupled from the thermodynamical effects, but this is actually less obvious when mass transfer occurs. The exact phasic EOS also play a crucial role when investigating the relaxation process. An important point is that the four relaxation time scales do not interfer in the relaxation conditions arising in properties $1-3$. Conditions must be implemented in computer codes in order to check the relaxation process.

- As far as numerical approximations of solutions are concerned, at least two different strategies may be considered. A first one, which is classicaly applied [10,12,24,33], consists in using the fractional step method in order to account for source terms, thus accounting for various relaxation time scales successively. One immediate advantage of this strategy dwells in its great simplicity. Moreover, this approach is clearly consistent with the overall entropy inequality. However, it may have some weaknesses when the time scales are very different from one another (see [2] for instance), and thus it motivates some comparison with more coupled numerical strategies. Actually, the present analysis may be used for numerical purposes. This precisely corresponds to the algorithm which is detailed in **appendix B**. Some first results are provided in [26], while using first-order time schemes in the relaxation step. The latter strategy can be extended to higher-order time schemes.

- We also recall that the present analysis has been conducted with the particular choice $h_I(W)=0$. At this stage, it also seems worth mentionning another particular case where:

$$h_I(W)=\omega(W)\mu_l+(1-\omega(W))\mu_v$$

  where $\omega(W)$ lies in $[0,1]$. which was recently pointed out in [32]. The corresponding analysis remains to be done in that case. This in some sense represents a little change of paradigm, as far as the mass transfer closure law is concerned.

**Appendix A : The RIP condition.** We consider some gas-liquid flow model as discussed in section 2, thus focusing on immiscible components. We assume some initial condition $W^0(x) = W(x, t=0)$, such that:

$$U_k(x, t=0) = 0 \ ,$$

and with uniform pressure and temperature fields:

$$P_k(x, t=0) = P_0 \quad ; \quad T_k(x, t=0) = T_0.$$

The flow will remain steady, if:

$$\partial_t(\psi)(x, t=0) = 0$$

whatever $\psi$ is.

Obviously, source terms vanish, and we get:

$$\left\{ \begin{array}{l} \partial_t(\alpha_k)(x, t=0) = 0 \ ; \\ \partial_t(m_k)(x, t=0) = 0 \ ; \\ \partial_t(m_k \mathbf{U}_k)(x, t=0) + P_0 \nabla \alpha_k^0 - \Pi_I(W^0) \nabla \alpha_k^0 = 0 \ ; \\ \partial_t(\alpha_k E_k)(x, t=0) = 0 \ . \end{array} \right. \tag{4.1}$$

Thus the flow will remain steady if:

$$\left(P_0 - \Pi_I(W^0)\right) \nabla \alpha_l^0 = 0$$

whatever the initial profile of $\alpha_l^0$ is, or in other words:

$$\Pi_I(W^0) = P_0$$

This is refered to as the Realizable Interfacial Pressure (RIP) condition.

**Appendix B : a relaxation algorithm for source terms.** We discuss here an algorithm in order to account for source terms in the so-called relaxation step. Four distinct -positive- time scales are involved in this step, which are $\tau_P, \tau_U, \tau_T$ and $\tau_\mu$. We need to get approximations of solutions of the following system of ODE:

$$\left\{ \begin{array}{l} \partial_t(\alpha_l) = \phi_l(W) \ ; \\ \partial_t(\alpha_l E_l) + \Pi_I(W) \partial_t(\alpha_l) = S_{E_l}(W) \ ; \\ \partial_t(m_l \mathbf{U}_l) = \mathbf{S}_{\mathbf{Q}_l}(W) \ ; \\ \partial_t(m_l) = \Gamma_l(W) \ ; \\ \partial_t(m_l + m_v) = 0 \ ; \\ \partial_t(m_l \mathbf{U}_l + m_v \mathbf{U}_v) = 0 \ ; \\ \partial_t(\alpha_l E_l + \alpha_v E_v) = 0 \ . \end{array} \right. \tag{4.2}$$

The algorithm contains two distinct steps.

- **Relaxation step:** first compute approximate solutions of the vector :

$$\underline{\Delta}^{lv} = (\Delta \mathcal{U}, \Delta \mathcal{T}, \Delta \mathcal{P}, \Delta \mu)^T$$

at time $t^{n+1}$, starting with known values at time $t^n$, where $\underline{\Delta}^{lv}$ is solution of :

$$\partial_t\left(\underline{\Delta}^{lv}\right) = -\mathcal{R}^{lv}(W) \underline{\Delta}^{lv}$$

still noting:

$$\begin{cases} \Delta\mathcal{P} = P_l - P_v \ , \\ \Delta\mathcal{U} = U_l - U_v \ , \\ \Delta\mathcal{T} = T_l - T_v \ , \\ \Delta\mu = \frac{\mu_l}{T_l} - \frac{\mu_v}{T_v} \ , \end{cases} \tag{4.3}$$

- **Second step:** for a given value of $\underline{\Delta}^{lv}$ at time $t^{n+1}$ provided by the previous relaxation step, look for values of $T_v^{n+1}$, $P_v^{n+1}$ and $U_v^{n+1}$ that satisfy the three constraints:

$$\begin{cases} \partial_t (m_l + m_v) = 0 \ ; \\ \partial_t (m_l \mathbf{U}_l + m_v \mathbf{U}_v) = 0 \ ; \\ \partial_t (\alpha_l E_l + \alpha_v E_v) = 0 \ ; \end{cases} \tag{4.4}$$

which can be rewritten as:

$$\begin{cases} (m_l + m_v)^{n+1} = M_0 = (m_l + m_v)^n \ ; \\ (m_l \mathbf{U}_l + m_v \mathbf{U}_v)^{n+1} = M_0 U_0 = (m_l \mathbf{U}_l + m_v \mathbf{U}_v)^n \ ; \\ (\alpha_l E_l + \alpha_v E_v)^{n+1} = M_0 E_0 = (\alpha_l E_l + \alpha_v E_v)^n \ . \end{cases} \tag{4.5}$$

Besides, intermediate unknowns $T_l^{n+1}$,$P_l^{n+1}$,$U_l^{n+1}$ and $\mu_l^{n+1}$ are given straightfully by:

$$\begin{cases} P_l^{n+1} = P_v^{n+1} + \Delta\mathcal{P} \ , \\ U_l^{n+1} = U_v^{n+1} + \Delta\mathcal{U} \ , \\ T_l^{n+1} = T_v^{n+1} + \Delta\mathcal{T} \ , \\ (\frac{\mu_l}{T_l})^{n+1} = (\frac{\mu_v}{T_v})^{n+1} + \Delta\mu \ . \end{cases} \tag{4.6}$$

At the discrete level, this turns into the following sequence (getting rid of supscript $\cdot^{n+1}$):

- **Step 1:**
  Compute an approximate solution of the relaxation step, using a first-order Euler time scheme, while frozing the matrix $\mathcal{R}^{lv}$:

$$\underline{\Delta}^{lv} = (I_d + \Delta t \mathcal{R}^{lv}(W^n))^{-1} (\underline{\Delta}^{lv})^n$$

- **Step 2:**
  a) Use the mass conservation law (first equation in (4.4)) to get the liquid statistical fraction $\alpha_l$:

$$\alpha_l = \frac{M_0 - \rho_v(P_v, T_v)}{\rho_l(P_v + \Delta\mathcal{P}, T_v + \Delta\mathcal{T}) - \rho_v(P_v, T_v)} \tag{4.7}$$

  b) Use the momentum conservation law (second equation in (4.4)) to get velocities:

$$U_v = U_0 - \frac{M_0 - m_v}{M_0} \Delta\mathcal{U} \quad ; \quad U_l = U_0 + \frac{m_v}{M_0} \Delta\mathcal{U}$$

  with:

$$m_v = (1 - \alpha_l)\rho_v(P_v, T_v) \quad ; \quad m_l = M_0 - m_v \tag{4.8}$$

c) Use the total energy balance (third equation in (4.4)) and the latter formulas to obtain a first constraint:

$$\begin{aligned} M_0 E_0 = m_v \epsilon_v(P_v, T_v) + (M_0 - m_v)\epsilon_l(P_v + \Delta\mathcal{P}, T_v + \Delta\mathcal{T}) \\ + \tfrac{1}{2}\left( M_0 U_0^2 + \tfrac{m_v(M_0 - m_v)}{M_0}(\Delta\mathcal{U})^2 \right) \end{aligned} \tag{4.9}$$

d) Use the Gibbs potential disequilibrium to obtain the second constraint linking the two unknowns $T_v, P_v$:

$$\frac{\mu_l(P_v + \Delta\mathcal{P}, T_v + \Delta\mathcal{T})}{T_v + \Delta\mathcal{T}} - \frac{\mu_v(P_v, T_v)}{T_v} = \Delta\mu \tag{4.10}$$

- **Step 3:**
  Solve the two-equation system (4.9), (4.10), using (4.7), (4.8), in terms of the two main unknowns $T_v, P_v$, in the admissible range.

- **Step 4:**
  Update all variables.

Of course higher-order schemes may be used in **Step 1** instead of Euler scheme.

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