



**HAL**  
open science

# Silent sources on a surface for the Helmholtz equation and decomposition of $L^2$ vector fields

Laurent Baratchart, Housseem Haddar, Cristóbal Villalobos Guillén

► **To cite this version:**

Laurent Baratchart, Housseem Haddar, Cristóbal Villalobos Guillén. Silent sources on a surface for the Helmholtz equation and decomposition of  $L^2$  vector fields. 2024. hal-04367726v2

**HAL Id: hal-04367726**

**<https://hal.science/hal-04367726v2>**

Preprint submitted on 20 Dec 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# SILENT SOURCES ON A SURFACE FOR THE HELMHOLTZ EQUATION AND DECOMPOSITION OF $L^2$ VECTOR FIELDS

L. BARATCHART\*, H. HADDAR†, AND C. VILLALOBOS GUILLÉN‡

**Abstract.** We present a decomposition of  $\mathbb{R}^3$ -valued vector fields of  $L^2$ -class on the boundary of a Lipschitz domain in  $\mathbb{R}^3$ , that relates to an inverse source problem with source term in divergence form for the Helmholtz equation. Applications thereof include weak scattering from thin interfaces. The inverse problem is not uniquely solvable, as the forward operator has infinite-dimensional kernel. The proposed decomposition brings out constraints that can be used to restore uniqueness. This work subsumes the one in [3] dealing with the Laplace equation, and shines a light on new ties that arise in the Helmholtz case between solutions on each side of the surface. Our approach uses properties of the Calderón projectors on the boundary of Lipschitz domains, that we carry out in the  $L^2 \times H^{-1}$  setting to handle  $L^2$  vector fields in the style of [29], dwelling on results from [18]; it applies for any complex wave number.

**1. Introduction.** Inverse source problems for the wave equation are classical issues stemming from various applications, notably medical imaging; *e.g.*, ultrasound imaging, microwave imaging, or multimodal imaging techniques such as photoacoustics [19]. This work is concerned with source terms in divergence form which arise naturally, for instance when modelling anisotropy in the medium response [13] or when a static electromagnetic setting is used as in Electro-Encephalography [27] and Paleomagnetism [8]. The corresponding inverse problems are extremely ill-posed, since the forward operator is not even injective and the solution is thus subject to fundamental uncertainty that can only be resolved by making additional assumptions. Our aim is to contribute to this analysis in the time-harmonic case, by bringing out the structure of this uncertainty when sources are supported on a surface.

Specifically, the model problem we are interested in is governed by an equation of the form

$$(1.1) \quad \Delta u + k^2 u = \nabla \cdot \mathbf{M} \quad \text{in } \mathbb{R}^3,$$

where the scalar-valued potential  $u$  satisfies a Sommerfeld radiation condition at infinity. The left-hand side of (1.1) is the Helmholtz operator with wave number  $k$ , while the right-hand side is a source term which is the divergence on  $\mathbb{R}^3$  of a  $\mathbb{R}^3$ -valued distribution  $\mathbf{M}$  supported on some compact Lipschitz surface  $\Gamma$ ; the latter is the boundary of a bounded domain  $\Omega \subset \mathbb{R}^3$ , assumed to be known. When  $k$  is real, Equation (1.1) can be construed as an approximate model for scattering from thin interfaces, see Remark 3.1 and references [13, 9]. In the particular case where  $k = 0$ , it yields the law of the magnetic potential generated by a magnetization distribution  $\mathbf{M}$  [7]. A typical inverse problem associated with (1.1) is to recover  $\mathbf{M}$  from knowledge of the field  $\nabla u$  away from the surface, usually either inside or outside  $\Omega$ . In many applications,  $\Gamma$  is connected hence  $\Omega$  has connected complement, but this does not simplify things much and we shall avoid this extra-assumption. If  $\mathbf{M}$  is such that the field  $\nabla u$  vanishes inside (resp. outside)  $\Omega$ , it is said to be *silent* inside (resp. outside). The existence of non-trivial silent  $\mathbf{M}$  makes the forward operator non-injective, which is one of the big issues facing such inverse problems. Hereafter, we shall investigate silent sources  $\mathbf{M}$  lying in  $(L^2(\Gamma))^3$ , the space of  $\mathbb{R}^3$ -valued (not necessarily tangent) vector fields defined on  $\Gamma$  with square summable components there.

When  $k = 0$  and  $\Gamma$  is a compact, connected Lipschitz surface, so that (1.1) reduces to a Poisson equation and  $\Gamma$  is the boundary of a domain with connected complement, then a direct sum decomposition of  $(L^2(\Gamma))^3$  into sources silent inside (but not outside) whose tangent component is a gradient, sources silent outside (but not inside) whose tangent component is a gradient, and tangent divergence-free sources (which are silent on both sides) was obtained in [3]; see also [6] for the planar case and weaker regularity assumptions.

---

\*Projet FACTAS, INRIA, 2004 route des Lucioles, BP 93, Sophia-Antipolis, 06902 Cedex, FRANCE ([Laurent.Baratchart@inria.fr](mailto:Laurent.Baratchart@inria.fr)).

†Project IDEFIX, INRIA, UMA, ENSTA Paris, Institut Polytechnique de Paris, Palaiseau, France ([houssem.haddar@inria.fr](mailto:houssem.haddar@inria.fr)).

‡Computational Science Center Faculty of Mathematics University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria ([crisobal.villalobos.guillen@univie.ac.at](mailto:crisobal.villalobos.guillen@univie.ac.at)).

In the present paper, we generalize this decomposition to non-zero (possibly complex)  $k$ , and show that in this case a fourth, finite-dimensional summand is generally required. More precisely, letting  $\Gamma := \partial\Omega$  be the boundary of a Lipschitz domain  $\Omega \subset \mathbb{R}^3$ , we establish in Theorem 4.3 the following orthogonal decomposition for  $\mathbb{R}^3$ -valued square summable vector fields on  $\Gamma$ :

$$L^2(\Gamma)^3 = \mathcal{M}_\nu \oplus \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0,$$

where  $\mathcal{M}_0$  consists of tangent, divergence-free fields in  $L^2(\Gamma)$  (these are silent both inside and outside  $\Gamma$ ), while  $\mathcal{M}_+$  (resp.  $\mathcal{M}_-$ ) comprises vector fields silent inside (resp. outside)  $\Gamma$  whose tangential component is a gradient, and  $\mathcal{M}_\nu$  is a finite-dimensional space of normal vector fields  $\mathbf{M}$  generating *via* (1.1) a potential  $u$  which is locally constant on  $\Gamma$  (equivalently: such that the restriction to  $\Gamma$  of the field  $\nabla u$  is also normal). When  $k = 0$  (the case of the Laplacian), no such nonzero field  $\mathbf{M}$  exists, and we recover the results of [3]. In any case the dimension of  $\mathcal{M}_\nu$  cannot exceed the number of connected components of  $\Gamma$ , and it is equal to the latter in the generic case where  $k^2 \neq 0$  is not a Neumann eigenvalue of (minus) the Laplacian in  $\Omega$  nor in  $\mathbb{R}^3 \setminus \bar{\Omega}$  (note that the exterior Neumann eigenvalue problem is not trivial if  $\Gamma$  is not connected).

Beyond its own, differential-geometric interest, this decomposition sheds light on the structure of the solutions of the inverse problem and allows one to *ansatz* them, so as to gain insight on how much information can be recovered from the given data.

Our approach departs from [3] and makes systematic use of Calderón projectors, in particular it is connected with the data completion algorithm proposed in [1]. Properties of Calderón projectors that we need are standard on the product  $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$  of fractional Sobolev spaces [24], but we must carry them over to  $L^2(\Gamma) \times H^{-1}(\Gamma)$  in order to handle the case where  $\mathbf{M} \in (L^2(\Gamma))^3$  in (1.1); for this, we dwell on the theory of singular integrals on Lipschitz (and even more general) surfaces expounded in [18], but several results can also be found in [29]. While it would be possible, and in fact simpler to derive similar results for vector fields  $\mathbf{M}$  whose tangential and normal components belong to  $H^{1/2}(\Gamma)$ , we feel that the  $L^2$  theory is not only more general but also more natural and better suited for numerical implementation: it is more natural because the membership  $\mathbf{M} \in (L^2(\Gamma))^3$  naturally splits into a product membership of both the normal and tangent component to  $L^2(\Gamma)$ , whereas fields whose tangential and normal components belong to  $H^{1/2}(\Gamma)$  cannot be characterized independently of the embedding  $\Gamma \rightarrow \mathbb{R}^3$ , making them difficult to describe intrinsically; and the  $L^2$  theory is also better suited for numerical implementation, because convergent discretization in  $H^{1/2}$  is known to be hard to handle. We work in  $\mathbb{R}^3$  throughout, even though the generalization to  $\mathbb{R}^n$  is straightforward.

The paper is organized as follows. In Section 2, we set up notation and conventions used for function spaces and operators in Euclidean space and on a surface. Section 3 states the problem and characterizes silent sources in terms of Calderón projectors. The main result of the paper, namely the decomposition of  $(L^2(\Gamma))^3$  in terms related to silent sources, is stated and proven in Section 4. Finally, this decomposition is illustrated by explicit calculations in the case of spheres. The paper also contains a technical appendix, gathering some results on surface potentials and elliptic regularity that we could not find in the literature; several of them are adaptations to the case  $k \neq 0$  of material from [18] or reproduction of results from [29].

**2. Preliminaries and Notation.** If  $V$  is a topological vector space over  $\mathbb{R}$  or  $\mathbb{C}$ , we denote by  $V^*$  its dual space and write  $\mathcal{G}^t : W^* \rightarrow V^*$  for the transpose of an operator  $\mathcal{G} : V \rightarrow W$ ; also,  $\text{Ker } \mathcal{G}$  denote its kernel and  $\text{Im } \mathcal{G}$  its image. For  $v \in V$  and  $\omega \in V^*$ , we let  $\langle \omega, v \rangle$  indicate the duality product. If  $V$  is equipped with a conjugation, say  $v \mapsto \bar{v}$ , we define a sesquilinear form on  $V^* \times V$  by  $\langle\langle \omega, v \rangle\rangle := \langle \omega, \bar{v} \rangle$ , and when  $V$  is a Hilbert space we denote the (Hermitian) inner product of  $v, u \in V$  by  $\langle\langle v, u \rangle\rangle_V$ . Notice our convention that such products are linear on the *second* entry. In the Hilbertian case, one can identify  $V$  with  $V^*$  *via* the *linear* isometry  $v \mapsto \langle\langle \bar{v}, \cdot \rangle\rangle_V$ . Then, the adjoint  $\mathcal{G}^* : W \rightarrow V$  of an operator  $\mathcal{G} : V \rightarrow W$  is defined by  $\langle\langle \mathcal{G}^* w, v \rangle\rangle_V = \langle\langle w, \mathcal{G} v \rangle\rangle_W$ .

When  $\Omega \subset \mathbb{R}^n$  is open, we put  $C^\infty(\Omega)$  for the space of infinitely differentiable functions on  $\Omega$ , and  $C_c^\infty(\Omega)$  for the subspace of those having compact support. We denote by  $\mathcal{E}(\Omega)$  the space  $C^\infty(\Omega)$  endowed with the topology of uniform convergence of all derivatives on compact sets, and by  $\mathcal{D}(\Omega)$  the space  $C_c^\infty(\Omega)$  equipped with the inductive topology of subspaces with support in a fixed compact set [28, Chapter I, Section

2]. Then,  $\mathcal{D}^*(\Omega)$  is the space of distributions on  $\Omega$ , and  $\mathcal{E}^*(\Omega)$  the subspace of distributions with compact support. Recall that the support of  $\omega \in \mathcal{D}^*(\mathbb{R}^n)$  is the relatively closed set  $\text{supp}(\omega) \subset \Omega$ , no point of which has a neighborhood where  $\omega$  restricts to the zero distribution. Throughout, we let  $\partial_j \omega$  denote the (distributional) partial derivative of  $\omega$  with respect to the  $j$ -th coordinate in  $\mathbb{R}^n$ .

For  $1 \leq p \leq \infty$  and  $Q$  a Borel set in  $\mathbb{R}^3$  with  $\rho$  a positive Borel measure on  $Q$ , we let  $L^p(Q, \rho)$  denote the familiar Lebesgue space of  $p$ -summable functions on  $Q$  (essentially bounded if  $p = \infty$ ). When  $\rho$  is Lebesgue measure, we simply write  $L^p(Q)$ . When  $E \subset \mathbb{R}^n$ , a map  $f : E \rightarrow \mathbb{R}^m$  is Lipschitz if  $|f(x) - f(y)| \leq k|x - y|$  for  $x, y \in E$ , and the smallest  $k$  for which this holds is the *Lipschitz constant of  $f$* , denoted as  $k_f$ . We set  $\text{Lip}(E, \mathbb{R}^m)$  to be the space of Lipschitz functions  $E \rightarrow \mathbb{R}^m$  endowed with the norm  $\|f\|_{L^\infty(E)} + k_f$ . Members of  $\text{Lip}(E, \mathbb{R}^m)$  extend to Lipschitz functions on  $\mathbb{R}^3$  with no increase of the Lipschitz constant [23, Theorem 7.2]; clearly, the extension can be chosen to have compact support if  $E$  is bounded.

For  $\Omega \subset \mathbb{R}^3$  an open set and  $s \in \mathbb{R}$ , let  $H^s(\Omega)$  denote the Bessel potential space of order  $s$  (with index 2); the latter consists of restrictions to  $\Omega$  of tempered distributions  $T$  on  $\mathbb{R}^3$  whose Fourier transform  $\hat{T}$  is such that  $(1 + |\xi|^2)^{s/2} \hat{T} \in L^2(\mathbb{R}^3)$ . On  $H^s(\Omega)$  one puts the norm  $\|(1 + |\xi|^2)^{s/2} \hat{T}\|_{L^2(\mathbb{R}^3)}$ , and if  $s \geq 0$  then  $H^s(\Omega)$  is a space of functions [24, ch. 3]. Clearly,  $H^0(\Omega) = L^2(\Omega)$  and  $H^t(\Omega) \subset H^s(\Omega)$  for  $s < t$ , with dense inclusion. In particular,  $H^s(\mathbb{R}^3)$  densely contains Lipschitz functions with compact support for  $s \leq 1$ .

A Lipschitz domain  $\Omega \subset \mathbb{R}^3$  is one whose boundary is locally isometric to the graph of a Lipschitz function. If  $\Omega$  is Lipschitz then  $H^1(\Omega)$  coincides with functions in  $L^2(\Omega)$  whose distributional derivatives again lie in  $L^2(\Omega)$ , moreover  $H^s(\Omega)$  is the real interpolation space  $[L^2(\Omega), H^1(\Omega)]_s$  for  $s \geq 0$ , while  $H^{-s}(\Omega) = (H_0^s(\Omega))^*$ , where  $H_0^s(\Omega)$  is the closure of  $\mathcal{D}(\Omega)$  in  $H^s(\Omega)$ , see [24, Theorems 3.18 & 3.30 & 3.33]. In particular  $H^s(\Omega) \subset (\text{Lip}(\Omega))^*$  for  $s \geq -1$  as soon as  $\Omega$  is Lipschitz and bounded, and in this range of  $s$  a member of  $H^s(\Omega)$  is completely determined by its action on compactly supported Lipschitz functions in  $\Omega$ . Still in the case that  $\Omega$  is bounded and Lipschitz, we also define for  $s \geq 0$ :

$$(2.1) \quad H_\ell^s(\mathbb{R}^3 \setminus \overline{\Omega}) := \{\omega \in \mathcal{D}^*(\mathbb{R}^3 \setminus \overline{\Omega}) : \omega|_{\mathbb{B}_r \setminus \overline{\Omega}} \in H^s(\mathbb{B}_r \setminus \overline{\Omega}), \text{ for each } r > 0 \text{ such that } \overline{\Omega} \subset \mathbb{B}_r\},$$

where  $\mathbb{B}_r \subset \mathbb{R}^3$  denotes the open ball of radius  $r$  centered at 0. This space is denoted as  $H_{\text{loc}}^s(\mathbb{R}^3 \setminus \overline{\Omega})$  in [24], which conflicts with standard notation and is why we adopt a subscript  $\ell$  (for ‘‘local’’). For convenience, we also (redundantly) define  $H_\ell^s(\Omega) := H^s(\Omega)$  to streamline notation at places. This is consistent with (2.1), in that  $H_\ell^s(\Omega)$  is comprised in any case of functions lying in  $H^s(\Omega \cap B_r)$  for all  $r$  large enough.

For a compact Lipschitz surface  $M \subset \mathbb{R}^3$  which is the boundary of a Lipschitz open set, we let  $\sigma$  indicate surface measure on  $M$ ; hence  $\sigma = \mathcal{H}_{[M]}^2$ , the restriction to  $M$  of 2-dimensional Hausdorff measure [32, Remark 5.8.3]. We write  $L^2(M)$  rather than  $L^2(M, \sigma)$ , and for  $\phi, \tilde{\phi} \in L^2(M)^n$  with  $n \geq 1$  we let

$$\langle \phi, \tilde{\phi} \rangle_{L^2(M)^n} := \int_M \phi \cdot \tilde{\phi} \, d\sigma \quad \text{and} \quad \langle\langle \phi, \tilde{\phi} \rangle\rangle_{L^2(M)^n} := \int_M \overline{\phi} \cdot \tilde{\phi} \, d\sigma,$$

where  $\overline{\phi}$  denotes the complex conjugate of  $\phi$ . For the remaining definitions, we fix a particular compact Lipschitz surface  $M \subset \mathbb{R}^3$  with atlas  $\{(\theta_j, U_j)\}_{j \in I}$ , in such a way that  $\theta_j(U_j)$  is a ball  $B_j \subset \mathbb{R}^2$  for each  $j$  and, for some rigid motion  $R_j$  of  $\mathbb{R}^3$ , the map  $\theta_j^{-1} : B_j \rightarrow \mathbb{R}^3$  is of the form  $R_j \circ (I_2 \times \psi_j)$  where  $I_2$  is the identity operator on  $\mathbb{R}^2$  and  $\psi_j : B_j \rightarrow \mathbb{R}$  is Lipschitz-smooth. Without loss of generality, we assume that the charts are finitely many. A point  $x \in M$  such that  $\theta_j^{-1}$  is differentiable at  $\theta_j(x)$  for all  $j$  such that  $x \in U_j$  is called *regular*. By Rademacher’s theorem,  $\sigma$ -a.e.  $x \in M$  is regular. Defined this way regular points depend on the atlas, but this is unimportant to us; see [32, Section 5.8] for a more intrinsic definition. Given a regular point  $x \in M$ , we let  $T_x M \subset \mathbb{R}^3$  denote the tangent space of  $M$  at  $x$ . The latter is defined as the image of the derivative  $D\theta_j^{-1}(\theta_j(x))$ , and by the chain rule this definition is independent of  $j$  such that  $x \in U_j$ . For a function  $f : M \rightarrow \mathbb{C}$  and a point  $x \in U_i$  such that  $f \circ \theta_j^{-1}$  is differentiable at  $\theta_j(x)$ , we let  $\nabla_{\text{T}} f(x) \in T_x M$  denote the surface gradient of  $f$  at the point  $x$ . Note that if  $f : M \rightarrow \mathbb{C}$  is Lipschitz then, for  $\sigma$ -a.e.  $x \in M$ ,  $\nabla_{\text{T}} f(x)$  is well defined. We endow  $\text{Lip}(M)$  with the norm  $\|f\|_\infty + \|\nabla_{\text{T}} f\|_\infty$ , which is consistent with the norm previously defined on  $\text{Lip}(E)$ . Lipschitz partitions of unity subordinated to an open cover exist as in the smooth case. Using  $\text{Lip}(M)$  and Lipschitz partitions of unity, we can quickly define Sobolev

spaces of index  $s \in [-1, 1]$  on  $M$ , which is all we shall need (more general cases may be found in [24, 15]). Indeed, if  $\text{Lip}_c(U_j)$  denotes the space of Lipschitz functions compactly supported in  $U_j$ , we see on using partitions of unity that a member of  $\text{Lip}(M)^*$  is completely determined by its effect on  $\text{Lip}_c(U_j)$  for each  $j$ . In addition,  $\text{Lip}_c(B_j)$  and  $\text{Lip}_c(U_j)$  are isomorphic under the map  $\text{Lip}_c(B_j) \ni f \mapsto f \circ \theta_j \in \text{Lip}_c(U_j)$ . Now, letting  $\tilde{g}$  denote the extension by zero to all of  $M$  of a function initially defined on a subset of  $M$ , we put for  $s \in [-1, 1]$ :

$$(2.2) \quad H^s(M) := \{ \psi \in \text{Lip}(M)^* : \forall j \in I, \text{ the map } \text{Lip}_c(B_j) \ni f \mapsto \langle \psi, \widetilde{f \circ \theta_j} \rangle \text{ belongs to } H^s(B_j) \}.$$

Writing  $\psi^{\theta_j} : \text{Lip}_c(B_j) \rightarrow \mathbb{R}$  for the map  $\psi^{\theta_j}(f) := \langle \psi, \widetilde{f \circ \theta_j} \rangle$  above, we define the convergence of a sequence  $(\psi_n)_n \subset H^s(M)$  to  $\psi \in H^s(M)$  as the convergence  $\psi_n^{\theta_j} \rightarrow \psi^{\theta_j}$  in  $H^s(B_j)$  for any  $j \in I$ . This notion is independent of the atlas and the  $H^s(M)$  are Hilbert spaces. Again, for  $s < t$  we have that  $H^t(M) \subset H^s(M)$  and  $H^0(M) = L^2(M)$ , furthermore  $H^{-s}(M)$  identifies with  $(H^s(M))^*$ . Note that  $\text{Lip}(M)$  is dense in  $H^s(M)$  for  $s \in [-1, 1]$ . This allows one to define the (weak) tangential gradient  $\nabla_T f$  of  $f \in H^1(M)$  as the limit in  $(L^2(M))^3$  of the vector fields  $\nabla_T g_n$ , where  $g_n$  is any sequence of Lipschitz functions converging to  $f$  in  $H^1(M)$ . When  $f \in H^s(M)$  with  $0 \leq s < 1$ , the tangential gradient  $\nabla_T f$  can still be defined as a current acting on tangent vector fields of class  $H^{1-s}(M)$  (viewed as 1-forms upon pairing them with tangent vector fields using Euclidean scalar product); see for instance [4].

We refer on several occasions to results from [18] that uses a more general definition of Sobolev spaces, discussed for example in [16]; in the present context, the latter reduces to the one just described.

We say that  $\mathbf{f} \in \text{Lip}(M)^3$  (resp.  $L^2(M)^3$ ,  $H^1(M)^3$ ...) belongs to  $\text{Lip}_T(M)$  (resp.  $L_T^2(M)$ ,  $H_T^1(M)$ ...) if, for  $\sigma$ -a.e.  $x \in M$ , there holds  $\mathbf{f}(x) \in T_x M$ . Now, for a  $\phi \in L_T^2(M)$ , one can define by duality the surface divergence of  $\phi$ , as a member of  $H^{-1}(M)$  denoted by  $\nabla_T \cdot \phi$ ; i.e. for each  $f \in \text{Lip}(M)$ , it is required that

$$\langle \nabla_T \cdot \phi, f \rangle := -\langle \phi, \nabla_T f \rangle_{L^2(M)^3}.$$

By density, we get for  $\varphi \in H^1(M)$ ,  $\phi \in L_T^2(M)$ ,  $\phi \in L^2(M)$  and  $\varphi \in H_T^1(M)$  that

$$\langle \nabla_T \cdot \phi, \varphi \rangle = -\langle \phi, \nabla_T \varphi \rangle_{L^2(M)^3}, \quad \text{and} \quad \langle \nabla_T \cdot \varphi, \phi \rangle_{L^2(M)} = -\langle \varphi, \nabla_T \phi \rangle.$$

In this paper, we consider a bounded Lipschitz domain  $\Omega_+$  with boundary  $\Gamma$ , and we let  $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega_+}$ . This choice of signs, where a “-” is attached to the unbounded complement of the bounded domain (itself denoted with a “+”), is as in [18] but departs from [24]; we implicitly take this discrepancy into account when quoting results from [24]. Note that  $\Gamma$  is a Lipschitz surface, that needs not be connected in general. As a short hand, unless otherwise stated, we use the symbol  $\pm$  to mean *both + and -*, and we employ the symbol  $\mp$  to designate the opposite sign to  $\pm$ .

Using [18, Theorem 4.3.6] together with Lemma A.1 and its proof (see equation (A.1)), we get that

$$H^1(\Gamma) = \{ \varphi \in L^2(\Gamma) : \nabla_T \varphi \in L^2(\Gamma)^3 \}.$$

For  $\varphi, \tilde{\varphi} \in H^1(\Gamma)$ , one has  $\nabla_T \cdot \nabla_T \varphi \in H^{-1}(\Gamma) = H^1(\Gamma)^*$  and  $\langle \nabla_T \cdot \nabla_T \varphi, \tilde{\varphi} \rangle = -\langle \nabla_T \varphi, \nabla_T \tilde{\varphi} \rangle_{L^2(\Gamma)^3}$ . We put  $\Delta_T := \nabla_T \cdot \nabla_T$  which is the so-called Laplace-Beltrami operator on  $\Gamma$ . We also use the Hermitian form:

$$\langle\langle \varphi, \tilde{\varphi} \rangle\rangle_{H^1(\Gamma)} := \langle\langle \varphi, \tilde{\varphi} \rangle\rangle_{L^2(\Gamma)} + \langle\langle \nabla_T \varphi, \nabla_T \tilde{\varphi} \rangle\rangle_{L^2(\Gamma)^3}$$

which generates the same topology on  $H^1(\Gamma)$  as the one defined after (2.2), by invariance of Sobolev functions under composition with Lipschitz maps [32, Theorem 2.2.2]. We denote by  $\|\cdot\|_{H^1(\Gamma)}$  the corresponding norm. Also, we denote by  $\|\cdot\|_{H^{-1}(\Gamma)}$  the dual norm in  $H^{-1}(\Gamma)$ , that arises from a Hermitian product written as  $\langle\langle \cdot, \cdot \rangle\rangle_{H^{-1}(\Gamma)}$ . In [18], a different norm is used which is equivalent to the present one.

We denote the classical trace on  $\Gamma$  from  $\Omega_{\pm}$  by  $\gamma^{\pm} : H^1(\Omega_{\pm}) \rightarrow H^{1/2}(\Gamma)$ ; it is a bounded linear operator. If for  $\phi \in H_{\ell}^1(\mathbb{R}^3)$  it holds that  $\gamma^+ \phi = \gamma^- \phi$ , we simply write  $\gamma \phi := \gamma^{\pm} \phi$ .

We also use nontangential limits on  $\Gamma$ . That is, given  $\alpha > 0$ , we define a nontangential domain of approach to  $x \in \Gamma$  from  $\Omega_{\pm}$  by

$$\mathfrak{C}_{\alpha}^{\pm}(x) := \{y \in \Omega_{\pm} : |x - y| \leq (\alpha + 1)\text{dist}(y, \Gamma)\},$$

where  $\text{dist}$  indicates the Euclidean distance between a point and a set. Subsequently, for  $\psi$  a measurable function on  $\Omega_{\pm}$  and  $x \in \Gamma$ , we put

$$\gamma_{\alpha}^{\pm}\psi(x) := \lim_{\substack{y \rightarrow x \\ y \in \mathfrak{C}_{\alpha}^{\pm}(x)}} \psi(y)$$

whenever this limit exists. From [18, Proposition 3.3.1] it follows that  $x$  lies in  $\overline{\mathfrak{C}_{\alpha}^{\pm}(x)}$  for  $\sigma$ -a.e.  $x \in \Gamma$ , hence this definition is meaningful  $\sigma$ -a.e. If the limit exists for every  $\alpha > 0$ , we say that the nontangential limit of  $\psi$  from  $\Omega_{\pm}$  exists at  $x$ . In the case that the nontangential limit of  $\psi$  exists for  $\sigma$ -a.e.  $x \in \Gamma$ , we denote the resulting function by  $\gamma^{\pm}\psi$  (same notation as the trace of a Sobolev function), and in case  $\gamma^{+}\psi = \gamma^{-}\psi$  we likewise drop the subscript and write  $\gamma\psi$ .

*Remark 2.1.* The slight abuse of notation assigning the same symbol to the trace and the nontangential limit should cause no confusion, since for Lipschitz domains and  $H_{\ell}^1$  functions the trace coincides  $\sigma$ -a.e. with the nontangential limits whenever the latter exist. One way to show this is to prove the result locally, when the boundary is a Lipschitz graph above a plane  $A$ , in which case one can invoke the absolute continuity of Sobolev functions on a.e. line perpendicular to  $A$  [32, Section 2.1].

Note that the restriction mapping  $\gamma : \mathcal{E}(\mathbb{R}^3) \rightarrow \text{Lip}(\Gamma)$ , is continuous; we will use the symbol  $\gamma^*$  to denote the adjoint operator of this particular version of the trace.

### 3. Statement of the problem and layer potentials.

**3.1. Statement of the problem.** We fix throughout some  $k \in \mathbb{C}$  in (1.1), together with a bounded Lipschitz domain  $\Omega_{+} \subset \mathbb{R}^3$  with boundary  $\Gamma$ , surface measure  $\sigma$  and outward-pointing unit normal  $\nu(x)$  at  $\sigma$ -a.e.  $x \in \Gamma$ . In applications we have in mind,  $k$  is real and may be taken positive, but the decomposition of vector fields that we establish in Theorem 4.3 holds for complex  $k$  as well.

Set  $G(x) := -\frac{e^{ik|x|}}{4\pi|x|}$ , which is a fundamental solution for the Helmholtz equation (1.1). We denote with  $\mathcal{G}$  its potential operator, that is:

$$\begin{aligned} \mathcal{G} & : \mathcal{E}^*(\mathbb{R}^n) & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & d & \longmapsto & G * d, \end{aligned}$$

where  $G * d$  indicates convolution. By [30, Theorem 27.6], the map  $\mathcal{G}$  is continuous and injective. For  $\mathbf{M} \in L^2(\Gamma)^3$ , let us decompose the normal and tangential components as  $\mathbf{M} = \nu M_{\nu} + \mathbf{M}_T$  with  $M_{\nu} := \mathbf{M} \cdot \nu$  and  $\mathbf{M}_T := \mathbf{M} - \nu M_{\nu}$ . Clearly,  $\mathbf{M}_T \in L_T^2(\Gamma)$ , therefore one can define  $\nabla_T \cdot \mathbf{M}_T \in H^{-1}(\Gamma)$ . We now introduce the so-called *forward operator* of the inverse source problem:

$$\begin{aligned} \mathcal{F} & : L^2(\Gamma)^3 & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & \mathbf{M} & \longmapsto & \mathcal{G}[\nabla \cdot (\mathbf{M}\sigma)], \end{aligned}$$

where  $\mathbf{M}\sigma$  designates the measure on  $\mathbb{R}^3$  such that  $d(\mathbf{M}\sigma) = \mathbf{M}d\sigma = \mathbf{M}d\mathcal{H}_{\Gamma}^1$ , and  $\nabla \cdot (\mathbf{M}\sigma)$  is the (weak) Euclidean divergence of  $\mathbf{M}\sigma$  in  $\mathbb{R}^3$ . Note that  $\mathcal{F}(\mathbf{M}) = \nabla G * (\mathbf{M}\sigma)$  and thus,  $\mathcal{F}(\mathbf{M})$  is a locally integrable function on  $\mathbb{R}^3$  which is real analytic on  $\mathbb{R}^3 \setminus \Gamma$ . If we set  $u = \mathcal{F}(\mathbf{M})$ , we see then that  $u$  satisfies the Helmholtz equation:

$$(3.1) \quad \Delta u + k^2 u = \nabla \cdot (\mathbf{M}\sigma) \quad \text{on } \mathbb{R}^3,$$

as well as the Sommerfeld radiation condition:

$$(3.2) \quad \lim_{|x| \rightarrow \infty} |x| \left( \frac{\partial}{\partial |x|} - ik \right) u(x) = 0.$$

Since  $\mathcal{G}$  is injective, the kernel of  $\mathcal{F}$  consists of those  $\mathbf{M} \in L^2(\Gamma)^3$  such that  $\nabla \cdot (\mathbf{M}\sigma) = 0$ . Also, as  $\mathcal{F}(M)$  is continuous off  $\Gamma$  and lies in  $L^1_{loc}(\mathbb{R}^3)$ , membership of  $M$  in that kernel is tantamount to  $\mathcal{F}(M)$  being identically zero on  $\mathbb{R}^3 \setminus \Gamma$ . In other words, we have the following chain of equivalences:

$$(3.3) \quad \mathcal{F}(\mathbf{M})(x) = 0 \text{ for all } x \in \mathbb{R}^3 \setminus \Gamma \iff \mathbf{M} \in \text{Ker } \mathcal{F} \iff \nabla \cdot (\mathbf{M}\sigma) = 0.$$

We say that  $\mathbf{M}$  is *silent inside* (resp. *silent outside*) if  $(\mathcal{F}(\mathbf{M}))|_{\Omega_+} = 0$  (resp.  $(\mathcal{F}(\mathbf{M}))|_{\Omega_-} = 0$ ). When  $\mathbf{M}$  is silent both inside and outside, we say that it is *silent everywhere* (or simply *silent*), in which case it lies in the kernel of  $\mathcal{F}$ , by (3.3). If  $\mathbf{M}$  is neither silent inside nor outside, we say it is *silent nowhere*. The existence of silent distributions on either side is a major difficulty facing the inverse problem of recovering  $\mathbf{M}$ , and the issue that we raise is to describe the vector fields in  $L^2(\Gamma)^3$  that correspond to these various notions of silence. Note that we only distinguish between silence inside and outside  $\Omega^+$ , and do not consider diverse qualifications of silence in a prescribed subset of components of  $\mathbb{R}^3 \setminus \Omega_+$  arising when  $\Gamma$  is not connected. While the present approach could be adapted for that purpose, the case under study is entirely typical already, and the results are simpler to state.

Note that a temperate distribution  $u$  and a vector field  $\mathbf{M} \in L^2(\Gamma)^3$  satisfy (3.1) and (3.2) if and only if  $u = \mathcal{F}(\mathbf{M})$ . Indeed, the Fourier transform of a temperate solution  $T$  to  $\Delta T + k^2 T = 0$  on  $\mathbb{R}^n$  has compact support, hence  $T \in H^s(\mathbb{R}^n)$  for some  $s$ . So, if in addition  $T$  satisfies (3.2), then we can appeal to [24, Theorems 7.12 & 9.6] to conclude that  $T \equiv 0$ .

*Remark 3.1.* Besides inverse magnetisation or EEG problems studied for example in [6, 3, 27] that correspond to the case  $k = 0$ , equation (3.1) can serve as a model for scattering from thin films [13, 9]. Indeed, consider a thin layer of constant width  $\epsilon \ll 1$  coating  $\partial\Omega$  with some material characterised by a coefficient  $\beta$ , so that the total field  $u_\epsilon$  generated by a source  $f$  (compactly supported outside the thin layer) satisfies

$$\nabla \cdot (1 + \beta_\epsilon) \nabla u_\epsilon + k^2 u_\epsilon = f \quad \text{in } \mathbb{R}^3$$

together with Sommerfeld radiation condition, where  $\beta_\epsilon = \beta$  inside the thin layer and  $\beta_\epsilon = 0$  outside. Then, formally at least,

$$u_\epsilon = u_0 + \epsilon u_1 + o(\epsilon)$$

where the incident field  $u_0$  satisfies  $\Delta u_0 + k^2 u_0 = f$  in  $\mathbb{R}^3$  and  $u_1$  is a solution of (3.1) with  $\mathbf{M} = -A\nabla u_0$ , the (anisotropic) matrix field  $A$  being defined on  $\partial\Omega$  by

$$A\boldsymbol{\nu} = \frac{\beta}{1 + \beta} \boldsymbol{\nu} \quad \text{and} \quad A\boldsymbol{\tau} = \beta\boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \text{ tangent to } \Gamma.$$

The scattered field  $u_\epsilon - u_0$  can then be approximated to the first order by  $\epsilon u_1$ , see [13] for a rigorous justification of this type of model in the case of thin interfaces with constant width  $\epsilon$ .

**3.2. Layer potentials and Green identities.** We recall below classical tools such as layer potentials and Calderón projectors to express the solutions to the Helmholtz equation in  $\Omega_\pm$ . We refer to [24] for the  $H^1$ -theory, where the densities of single and double layer potentials lie in  $H^{1/2}(\Gamma)$  and  $H^{-1/2}(\Gamma)$  respectively. However, to deal with  $L^2(\Gamma)$  and  $H^{-1}(\Gamma)$  densities as is necessary to handle the case that  $M \in L^2(\Gamma)^3$ , we need to extend the domain of definition of the operators under consideration, and for this we appeal to the work in [18]. The results of [18] are derived for the case  $k = 0$  only, but we adapt them to  $k \neq 0$  (possibly complex) in Appendix A.1. Regarding reference [24], we warn the reader that the Helmholtz equation there is minus ours, hence the fundamental solution there differs by a sign from ours; we implicitly take into account this discrepancy when quoting formulas from [24].

We write  $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$  for the coordinates of the unit outer normal of  $\Gamma$ , pointing into  $\Omega_-$ . For  $u \in H^1_\ell(\Omega_\pm)$  such that  $\Delta u \in L^2_\ell(\Omega_\pm)$ , we let  $\partial^\pm_\boldsymbol{\nu} u \in H^{-1/2}(\Gamma)$  be the interior and exterior co-normal derivatives for the Helmholtz differential operator [24, Chapter 4]). These are well-known extensions, based on the first Green formula, of the natural definition:

$$\partial^+_\boldsymbol{\nu} u = \boldsymbol{\nu} \cdot \boldsymbol{\gamma}^+(\nabla u) \text{ for } u \in H^2(\Omega_+) \quad \text{and} \quad \partial^-_\boldsymbol{\nu} u = \boldsymbol{\nu} \cdot \boldsymbol{\gamma}^-(\nabla u) \text{ for } u \in H^2_\ell(\Omega_-).$$

As with the trace, if  $\partial_{\nu}^- u = \partial_{\nu}^+ u$  for  $u \in H_{\ell}^1(\mathbb{R}^3)$  we simply write  $\partial_{\nu} u := \partial_{\nu}^{\pm} u$ .

We denote the single and double layer potentials associated to (3.1) by  $SL$  and  $DL$ . Recall that formally  $SL = \mathcal{G} \circ \gamma^*$  and  $DL = \mathcal{G} \circ \partial_{\nu}^*$  [24, Chapter 6], both being continuous and injective from  $\text{Lip}(\Gamma)^*$  to  $\mathcal{D}^*(\mathbb{R}^3)$  by density of traces of  $C_c^{\infty}(\mathbb{R}^3)$ -functions in  $\text{Lip}(\Gamma)$  and the injectivity of  $\mathcal{G}$  already pointed out. Unwinding the definitions, we find for  $x \in \mathbb{R}^3 \setminus \Gamma$  and  $\phi \in L^2(\Gamma)$  that

$$(3.4) \quad SL\phi(x) = \int_{\Gamma} G(x-y)\phi(y)d\sigma(y), \quad DL\phi(x) = \int_{\Gamma} (\partial_{\nu,y} G(x-y))\phi(y)d\sigma(y),$$

where  $\partial_{\nu,y}$  indicates the normal derivative with respect to the variable  $y$ ; indeed, (3.4) follows from [24, Equations (6.16) and (6.17)] on taking into account that  $\partial_{\nu,y}$  commutes with complex conjugation. It holds the mapping properties [24, Theorem 6.11]:

$$(3.5) \quad SL : H^{-1/2}(\Gamma) \longrightarrow H_{\ell}^1(\mathbb{R}^3) \quad \text{and} \quad DL : H^{1/2}(\Gamma) \longrightarrow H_{\ell}^1(\Omega_{\pm}).$$

*Remark 3.2.* Note that, for any  $\mathbf{M} \in L^2(\Gamma)^3$ , one has  $\mathcal{F}(\mathbf{M}) = \sum_j \partial_j SL(M_j)$ . Hence, in view of Lemma A.2 and the corresponding result for harmonic functions (namely, the case  $k = 0$  that follows at once from [31, Theorem 3.3 (i) & Corollary 3.5 (i)]), we get that  $\mathcal{F}(\mathbf{M}) \in L_{\text{loc}}^2(\mathbb{R}^3)^3$ .

Recall the three *Green Identities*: for  $u, v \in H^1(\Omega_{\pm})$  with  $\Delta u \in L^2(\Omega_{\pm})$  and for  $\pm$  to mean  $+$  or  $-$ , one has by [24, Theorem 4.4 (i)]:

$$(3.6a) \quad \langle \nabla u, \nabla v \rangle_{L^2(\Omega_{\pm})^3} = - \langle \Delta u, v \rangle_{L^2(\Omega_{\pm})} \pm \langle \partial_{\nu}^{\pm} u, \gamma^{\pm} v \rangle;$$

if moreover  $\Delta v \in L^2(\Omega_{\pm})$ , then it holds in view of [24, Theorem 4.4 (iii)] that

$$(3.6b) \quad \langle \Delta u + k^2 u, v \rangle_{L^2(\Omega_{\pm})} - \langle u, \Delta v + k^2 v \rangle_{L^2(\Omega_{\pm})} = \mp \langle \gamma^{\pm} u, \partial_{\nu}^{\pm} v \rangle \pm \langle \partial_{\nu}^{\pm} u, \gamma^{\pm} v \rangle$$

and, for  $u \in L_{\text{loc}}^2(\mathbb{R}^3)$  with  $u|_{\Omega_{\pm}} \in H_{\ell}^1(\Omega_{\pm})$  satisfying

$$(3.6c) \quad \Delta u|_{\Omega_{\pm}} + k^2 u|_{\Omega_{\pm}} = 0 \quad \text{in } \Omega_{\pm},$$

as well as (3.2) in the case of  $\Omega_-$ , we get on applying [24, Theorem 6.10] to  $\Phi_{\rho} u$  for arbitrary large  $\rho$ , with  $\Phi_{\rho}$  a member of  $C_c^{\infty}(\mathbb{R}^n)$  which is 1 on  $B_{\rho}$ , that

$$(3.6d) \quad u = DL(\gamma^+ u - \gamma^- u) - SL(\partial_{\nu}^+ u - \partial_{\nu}^- u).$$

The boundary version of layer potentials are bounded linear operators, with the mapping properties

$$S : H^{s-1}(\Gamma) \longrightarrow H^s(\Gamma) \quad \text{and} \quad K : H^s(\Gamma) \longrightarrow H^s(\Gamma)$$

for  $s \in \{1, 1/2, 0\}$  (these are the only cases we need)<sup>1</sup>. They have for  $\phi \in L^2(\Gamma)$  and  $\sigma$ -a.e.  $x \in \Gamma$  the integral representations

$$(3.7) \quad S\phi(x) = \int_{\partial D} G(x-y)\phi(y)d\sigma(y), \quad K\phi(x) = \text{p.v.} \int_{\partial D} \partial_{\nu,y} G(x-y)\phi(y)d\sigma(y),$$

which connect to jump relations for layer potentials through the following identities for traces, valid for  $\phi \in L^2(\Gamma)$  and  $\psi \in H^{-1}(\Gamma)$ :

$$(3.8) \quad (SL\psi)|_{\Gamma} = S\psi \quad \text{and} \quad \gamma^{\pm}(DL\phi) = \left( \pm \frac{1}{2} Id + K \right) \phi,$$

<sup>1</sup>For the case  $s = 1/2$  these operators are defined in [24, Chapter 7, Eq. (7.3)] by  $\gamma SL\psi$  and  $\gamma^+(DL\phi) + \gamma^-(DL\phi)$  (and so their “ $K$ ” which they call  $T$  differs by (minus) a factor 2 from ours); there, Equation (3.7) is proven for  $\phi$  Lipschitz while Equation (3.8) is proven for  $\psi \in H^{-1/2}(\Gamma)$  and  $\phi \in H^{1/2}(\Gamma)$ . When  $s = 1, 0$ , the case  $k = 0$  is treated in [18, Proposition 3.3.2, Corollary 3.6.3, Proposition 3.6.2 and Proposition 3.6.4], and adaptation to  $k \neq 0$  is made through Propositions A.3, A.4, A.5 and A.6 in the Appendix.



where  $Id$  represents the identity operator, see [24, Equation (7.5)]. In (3.8), the first relation means that  $S\psi$  is well defined a.e. on  $\Gamma$ . We will also use the operator  $\overline{DL} := \overline{\mathcal{G}} \circ \partial_\nu^*$ , where  $\overline{\mathcal{G}}$  is simply convolution by  $\overline{G}$ . In view of [24, Eqn. (7.3)], one can define a bounded linear endomorphism of  $H^{1/2}(\Gamma)$  by

$$\overline{K} := \frac{1}{2}(\gamma^+ \overline{DL} + \gamma^- \overline{DL}),$$

and clearly  $\overline{K}$ ,  $\overline{DL}$  have the same mapping properties as  $K$ ,  $DL$ , respectively. We also have a jump relation for the normal derivative of  $SL\phi$  when  $\phi \in H^{-1/2}(\Gamma)$  [24, Eqns. (7.3)-(7.5)], namely:

$$(3.9) \quad \partial_\nu^\pm(SL\phi) = \left( \mp \frac{1}{2} Id + \overline{K}^* \right) \phi.$$

In another connection, the following operators are well-defined and bounded for  $s \in \{1, 1/2, 0\}$ <sup>2</sup>:

$$(3.10) \quad T := \partial_\nu DL : H^s(\Gamma) \longrightarrow H^{s-1}(\Gamma).$$

*Remark 3.3.* Note that [18, Proposition 3.6.2] and Proposition A.5 further imply that the linear operators  $\varphi \mapsto \gamma^\pm(\nabla DL\varphi)$  are bounded from  $H^1(\Gamma)$  into  $L^2(\Gamma)$ <sup>3</sup>.

Finally, let us introduce the *Calderón projectors*:  $P^\pm : H^s(\Gamma) \times H^{s-1}(\Gamma) \longrightarrow H^s(\Gamma) \times H^{s-1}(\Gamma)$ , defined for  $s \in \{0, 1/2, 1\}$  as block-matrix multiplication by

$$(3.11) \quad P^\pm(\phi, \psi) := \begin{pmatrix} \frac{1}{2} Id \pm K & \mp S \\ \pm T & \frac{1}{2} Id \mp \overline{K}^* \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

where, in the case  $s = 0$ , the operator  $K^*$  is adjoint to  $K : H^1(\Gamma) \longrightarrow H^1(\Gamma)$ . These operators are bounded on the indicated spaces by what precedes, and clearly  $P^+ + P^- = Id$ . When  $s = 1/2$  it is known that  $P^\pm$  are projections [24, Ex. 7.6], so by density and continuity we deduce they are projections in the case  $s = 0$  as well. The case  $s = 1$  follows by restriction of the case  $s = 1/2$  to  $H^1(\Gamma) \times L^2(\Gamma)$ . Hereafter, we let  $P_j^\pm(\phi, \psi)$  denote the  $j$ -th component of  $P^\pm(\phi, \psi)$ , for  $j = 1, 2$ .

Note that if  $\phi \in L^2(\Gamma)$  then  $\gamma^*\phi$  is in fact a measure, absolutely continuous with respect to  $\sigma$ , such that  $d(\gamma^*\phi) = \phi d\sigma$ . It entails in view of the discussion before (3.4) that

$$\mathcal{F}(\mathbf{M}) = -DL(M_\nu) + SL(\nabla_T \mathbf{M}_T),$$

which justifies the following definition of a new, related operator:

$$\begin{aligned} \tilde{\mathcal{F}} & : L^2(\Gamma) \times H^{-1}(\Gamma) & \longrightarrow & \mathcal{D}^*(\mathbb{R}^3) \\ & (\phi, \psi) & \mapsto & -DL(\phi) + SL(\psi). \end{aligned}$$

*Remark 3.4.* Note that  $u = \tilde{\mathcal{F}}(\phi, \psi)|_{\mathbb{R}^3 \setminus \Gamma}$  belongs to  $C^\infty(\mathbb{R}^3 \setminus \Gamma)$  and that

$$\Delta u + k^2 u = 0 \text{ on } \Omega_\pm.$$

Now,  $\Gamma$  has finitely many components, say  $\Gamma_1, \dots, \Gamma_l$  (see Lemma A.12), and for  $j = 1, \dots, l$  we let  $1_{\Gamma_j}$  be the piecewise constant function on  $\Gamma$  with value 1 on  $\Gamma_j$  and 0 on  $\Gamma_k$  when  $k \neq j$ . For  $\psi \in H^{-1}(\Gamma)$ , let us define  $c_\psi := \sum_{j=1}^l \langle \psi, 1_{\Gamma_j} \rangle 1_{\Gamma_j}$  and consider  $\varphi_{\psi-c_\psi} \in H^1(\Gamma)$  such that  $\Delta_T \varphi_{\psi-c_\psi} = \psi - c_\psi$ ; such a  $\varphi_{\psi-c_\psi}$  exists by Lemma A.10. Then, we can write

$$\tilde{\mathcal{F}}(\phi, \psi) = \tilde{\mathcal{F}}(\phi, \Delta_T \varphi_{\psi-c_\psi} + c_\psi) = \mathcal{F}(\phi \nu + \nabla_T \varphi_{\psi-c_\psi}) + SL(c_\psi),$$

and thus, by (3.7) and Remark 3.2, the image of  $\tilde{\mathcal{F}}$  is included in  $L_{\text{loc}}^2(\mathbb{R}^3)$ .

<sup>2</sup>The case  $s = 1/2$  is part of [24, Theorem 7.1]. When  $s = 1$ , the result for  $k = 0$  follows from [18, Theorem 3.2.8, Proposition 3.6.2] together with equation (3.6b), and Proposition A.5 adapts it to the case  $k \neq 0$ . To deal with  $s = 0$ , let  $\mathcal{C}$  indicate the complex conjugation operator and observe from [24, Eqns. (7.3)-(7.5)] (in that reference  $T$  is called  $-R$ ) that  $T = \mathcal{C} \circ T^* \circ \mathcal{C} : H^{1/2} \rightarrow H^{-1/2}$ ; so, as  $T$  maps  $H^1$  into  $H^0$ , we can use  $\mathcal{C} \circ T|_{H^0(\Gamma)}^* \circ \mathcal{C}$  to extend  $T : H^0 \rightarrow H^{-1}$ .

To conclude this section, we address that  $\partial_{\nu}^{\pm}u$  has been defined for those  $u \in H^1(\Omega_{\pm})$  such that  $\Delta u \in L^2_{\ell}(\Omega_{\pm})$  only, whereas we need a definition which is valid for any function in the image of  $\tilde{\mathcal{F}}$ . To this end, we will use the following facts and the proceeding lemma.

First, by Equation (3.8), the nontangential limits  $\gamma^{\pm}u$  of  $u = \tilde{\mathcal{F}}(\phi, \psi)$  are well-defined and belong to  $L^2(\Gamma)$ . In view of (3.11), they satisfy for any  $(\tilde{\phi}, \tilde{\psi}) \in L^2(\Gamma) \times H^{-1}(\Gamma)$  such that  $u = \tilde{\mathcal{F}}(\tilde{\phi}, \tilde{\psi})$ , the relation

$$(3.12) \quad \gamma^{\pm}u = \left( \mp \frac{1}{2} Id - K \right) (\tilde{\phi}) + S(\tilde{\psi}) = \mp P_1^{\pm}(\tilde{\phi}, \tilde{\psi}).$$

Second, for  $(\phi, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ , we get from (3.11), (3.8), (3.9) and (3.10) that

$$(3.13) \quad \text{if } u = \tilde{\mathcal{F}}(\phi, \psi) \text{ then } (\gamma^{\pm}u, \partial_{\nu}^{\pm}u)^t = \mp P^{\pm}(\phi, \psi) \text{ and } \Delta u + k^2u = 0 \text{ on } \Omega_{\pm},$$

where the superscript “ $t$ ” means “transpose”. Third, extending  $u \in H^1_{\ell}(\Omega_{\pm})$  by zero on  $\Omega_{\mp}$  while using (3.8), (3.9), (3.10) and (3.6d) as well as (3.13) and  $P^+ + P^- = Id$ , we get (with  $\pm$  to mean + or -) that

$$(3.14) \quad \Delta u + k^2u = 0 \text{ on } \Omega_{\pm} \implies u = -\tilde{\mathcal{F}}(\pm\gamma^{\pm}u, \pm\partial_{\nu}^{\pm}u) \text{ and } (\gamma^{\pm}u, \partial_{\nu}^{\pm}u)^t = P^{\pm}(\gamma^{\pm}u, \partial_{\nu}^{\pm}u).$$

Finally, the following lemma holds.

LEMMA 3.5. *Let  $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$  and  $u = \tilde{\mathcal{F}}(\phi, \psi)$ . For any fixed choice of sign  $\pm$  holds the equivalence:*

$$u|_{\Omega_{\pm}} = 0 \iff P^{\pm}(\phi, \psi) = 0.$$

*Proof.* Assume first that  $(\phi, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ . Then, (3.13) gives us  $\Delta u + k^2u = 0$  on  $\Omega_{\pm}$  and  $(\gamma^{\pm}u, \partial_{\nu}^{\pm}u)^t = \mp P^{\pm}(\phi, \psi)$ . If  $u|_{\Omega_{\pm}} = 0$ , then clearly  $0 = (\gamma^{\pm}u, \partial_{\nu}^{\pm}u)$  whence  $P^{\pm}(\phi, \psi) = 0$ . Conversely, suppose that  $P^{\pm}(\phi, \psi) = 0$  so that  $(\gamma^{\pm}u, \partial_{\nu}^{\pm}u) = 0$ . From (3.5) we see that  $u|_{\Omega_{\pm}} \in H^1_{\ell}(\Omega_{\pm})$ , and by Remark 3.4 we know that  $u \in L^2_{\text{loc}}(\mathbb{R}^3)$ . Thus, letting  $\tilde{u}$  be the extension by zero of  $u|_{\Omega_{\pm}}$  to  $\Omega_{\mp}$ , Implication (3.14) gives us  $\tilde{u} = -\tilde{\mathcal{F}}(\pm\gamma^{\pm}u, \pm\partial_{\nu}^{\pm}u) = 0$ . Therefore, it holds indeed that  $u|_{\Omega_{\pm}} = 0$ .

Next, assume that  $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$  and suppose that  $P^{\pm}(\phi, \psi) = 0$ , hence  $P^{\mp}(\phi, \psi) = (\phi, \psi)$ . By density, there exist a sequence,  $((\phi_n, \psi_n))_n \subset H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$  such that  $(\phi_n, \psi_n) \rightarrow (\phi, \psi)$  in  $L^2(\Gamma) \times H^{-1}(\Gamma)$ . On the one hand,  $P^{\mp}(\phi_n, \psi_n)$  converges to  $(\phi, \psi)$  in  $L^2(\Gamma) \times H^{-1}(\Gamma)$  by the continuity of  $P^{\mp}$ . On the other hand, as  $P^{\mp}(\phi_n, \psi_n) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$  and the equality  $P^{\pm}P^{\mp}(\phi_n, \psi_n) = 0$  mechanically holds because  $(P^{\pm})^2 = P^{\pm} = P^{\pm}(P^{\pm} + P^{\mp})$ , we get by the first part of the proof that  $\tilde{\mathcal{F}}(P^{\mp}(\phi_n, \psi_n))|_{\Omega_{\pm}} = 0$ . Noticing that  $\tilde{\mathcal{F}}(P^{\mp}(\cdot, \cdot))|_{\Omega_{\pm}}$  is continuous from  $L^2(\Gamma) \times H^{-1}(\Gamma)$  into  $\mathcal{D}^*(\Omega_{\pm})$ , we conclude that  $\tilde{\mathcal{F}}(P^{\mp}(\phi_n, \psi_n))|_{\Omega_{\pm}} \rightarrow u|_{\Omega_{\pm}}$  in  $\mathcal{D}^*(\Omega_{\pm})$  and therefore  $u|_{\Omega_{\pm}} = 0$ .

Conversely, assume that  $u|_{\Omega_{\pm}} = 0$  and define  $(\tilde{\phi}, \tilde{\psi}) := P^{\pm}(\phi, \psi)$ . Then, Equation (3.12) implies that

$$(3.15) \quad \tilde{\phi} = P_1^{\pm}(\phi, \psi) = \mp \gamma^{\pm}u = 0.$$

Besides,  $P^{\mp}(\tilde{\phi}, \tilde{\psi}) = P^{\mp}P^{\pm}(\phi, \psi) = 0 = P^{\pm}P^{\mp}(\phi, \psi)$  and thus, by the implication already proven, we get

$$\tilde{\mathcal{F}}(P^{\mp}(\phi, \psi))|_{\Omega_{\pm}} = 0 \quad \text{and} \quad 0 = \tilde{\mathcal{F}}(P^{\pm}(\phi, \psi))|_{\Omega_{\mp}} = \tilde{\mathcal{F}}(0, \tilde{\psi})|_{\Omega_{\mp}} = SL(\tilde{\psi})|_{\Omega_{\mp}}$$

where, in next to last equality, we used (3.15). Furthermore, by the linearity of  $\tilde{\mathcal{F}}$  and (3.15) again, it also holds since  $Id = P^{\mp} + P^{\pm}$  and we assumed  $u|_{\Omega_{\pm}} = 0$  that

$$0 = u|_{\Omega_{\pm}} = \tilde{\mathcal{F}}(\phi, \psi)|_{\Omega_{\pm}} = \tilde{\mathcal{F}}(P^{\mp}(\phi, \psi))|_{\Omega_{\pm}} + \tilde{\mathcal{F}}(P^{\pm}(\phi, \psi))|_{\Omega_{\pm}} = \tilde{\mathcal{F}}(0, \tilde{\psi})|_{\Omega_{\pm}} = SL(\tilde{\psi})|_{\Omega_{\pm}}.$$

Thus, one has  $SL(\tilde{\psi})|_{\Omega_{+}} = SL(\tilde{\psi})|_{\Omega_{-}} = 0$ , and since  $SL$  is injective from  $\text{Lip}(\Gamma)^*$  to  $\mathcal{D}^*(\mathbb{R}^3)$  while  $SL(\tilde{\psi}) = \tilde{\mathcal{F}}(P^{\pm}(\phi, \psi))$  by what precedes and the latter is locally integrable by Remark 3.4, it follows that  $\tilde{\psi} = 0$  whence  $P^{\pm}(\phi, \psi) = 0$ , as was to be shown.  $\square$

From Lemma 3.5 it is clear that, for  $(\phi, \psi), (\tilde{\phi}, \tilde{\psi}) \in L^2(\Gamma) \times H^{-1}(\Gamma)$ , one has

$$\tilde{\mathcal{F}}(\phi, \psi)|_{\Omega_{\pm}} = \tilde{\mathcal{F}}(\tilde{\phi}, \tilde{\psi})|_{\Omega_{\pm}} \quad \text{if and only if} \quad P^{\pm}(\phi, \psi) = P^{\pm}(\tilde{\phi}, \tilde{\psi}).$$

Now, based on (3.13), for  $u = \tilde{\mathcal{F}}(\phi, \psi)$  with  $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$  and  $u^{\pm} = u|_{\Omega_{\pm}}$  we define:

$$\partial_{\nu}^{\pm} u = \partial_{\nu}^{\pm} u^{\pm} := \mp P_2^{\pm}(\phi, \psi) = -T(\phi) + \left( \mp \frac{1}{2} Id + \overline{K}^* \right) (\psi),$$

which extends the classical definition of normal derivatives. Altogether, it holds in this case as well that

$$(3.16) \quad P^{\pm}(\gamma^{\pm} u, \partial_{\nu}^{\pm} u) = (\gamma^{\pm} u, \partial_{\nu}^{\pm} u)^t = \mp P^{\pm}(\phi, \psi) \quad \text{and} \quad u|_{\Omega_{\pm}} = \tilde{\mathcal{F}}(\gamma^{\pm} u, \partial_{\nu}^{\pm} u).$$

*Remark 3.6.* Using once more [18, Proposition 3.6.2] together with Proposition A.5 and Lemma A.9, we get for any  $u = \tilde{\mathcal{F}}(\varphi, \phi)$  with  $(\varphi, \phi) \in H^1(\Gamma) \times L^2(\Gamma)$  that

$$\partial_{\nu}^{\pm} u = \gamma^{\pm}(\nabla u) \cdot \nu$$

and, by an argument similar to the one in Remark 2.1,

$$\gamma^{\pm}(\nabla u) = \partial_{\nu}^{\pm} u \nu + \nabla_{\Gamma} \gamma^{\pm} u.$$

**4. Decomposition of  $L^2(\Gamma)^3$ .** We start by introducing the spaces that we will use to decompose  $L^2(\Gamma)^3$ . First, let us define

$$\mathcal{M}_0 := \{ \mathbf{M} \in L^2(\Gamma)^3 : \mathbf{M} \text{ is silent everywhere} \}$$

and let  $\mathcal{M}_0^{\perp}$  denote the subspace perpendicular to  $\mathcal{M}_0$  in  $L^2(\Gamma)^3$ . Next, let us introduce the following subspaces of  $\mathcal{M}_0^{\perp}$ :

$$\begin{aligned} \mathcal{M}_- &= \{ \mathbf{M} \in \mathcal{M}_0^{\perp} : \mathbf{M} \text{ is silent outside} \}, \\ \mathcal{M}_+ &= \{ \mathbf{M} \in \mathcal{M}_0^{\perp} : \mathbf{M} \text{ is silent inside} \}. \end{aligned}$$

*Remark 4.1.* It follows from the definition that  $\mathcal{M}_+ \cap \mathcal{M}_- = \{0\}$ , since this intersection consists of fields silent everywhere whereas both spaces belong to  $\mathcal{M}_0^{\perp}$ . Also, thanks to Lemma 3.5, it holds that

$$(4.1) \quad \mathcal{M}_0 := \{ \mathbf{M} \in L^2(\Gamma)^3 : \nabla_{\Gamma} \cdot \mathbf{M}_T = 0 \text{ and } M_{\nu} = 0 \},$$

$$(4.2) \quad \mathcal{M}_{\pm} = \{ \mathbf{M} \in \mathcal{M}_0^{\perp} : P^{\pm}(M_{\nu}, \nabla_{\Gamma} \cdot \mathbf{M}_T) = 0 \},$$

and it follows easily from Lemma A.10 that every tangent field in  $L^2(\Gamma)^3$  is the sum of a gradient and a divergence-free one (the Helmholtz decomposition), hence

$$(4.3) \quad \mathcal{M}_0^{\perp} = \{ \mathbf{M} \in L^2(\Gamma)^3 : \mathbf{M}_T = \nabla_{\Gamma} U_{M_T}, \text{ for some } U_{M_T} \in H^1(\Gamma) \}.$$

When  $k \neq 0$ ,  $\mathcal{M}_-$ ,  $\mathcal{M}_+$  and  $\mathcal{M}_0$  are not enough to decompose  $L^2(\Gamma)^3$ . That is, for  $k \neq 0$  there exists a bounded Lipschitz domain  $\Omega_+$  with boundary  $\Gamma$  carrying a nonzero  $\mathbf{M} \in L^2(\Gamma)^3 \setminus (\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$ , which is thus silent nowhere and whose potential in  $\Omega_{\pm}$  is not generated by a distribution silent in  $\Omega_{\mp}$ ; this cannot happen when  $k = 0$  [3]. At the end of this section we will describe the space perpendicular to  $(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$ , but prior to this we shall introduce a space  $\mathcal{M}_{\nu} \subset L^2(\Gamma)^3$ , whose elements are purely normal to  $\Gamma$ , that satisfies

$$\mathcal{M}_{\nu} \oplus \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0 = L^2(\Gamma)^3.$$

Let  $\{\Gamma_j\}_{j \in J}$  be the family of connected components of  $\Gamma$ . The fact that  $\Omega_+$  is a bounded Lipschitz domain implies that  $J$  must be finite and each  $\Gamma_j$  has finite strictly positive area (see for example Lemma A.12). We can index the connected components of  $\Omega_-$  by  $\Omega_-^j$  for  $j \in J$ , and assume that

- $J = \{1, \dots, n_\Gamma\}$ , so that  $n_\Gamma$  is the number of connected components of  $\Gamma$ ,
- $\Omega_-^1$  is unbounded,
- for each  $j > 1$ , the set  $\Omega_-^j$  is bounded,
- for each  $j \in J$ , the set  $\Gamma_j$  is the boundary of  $\Omega_-^j$ .

For  $\Sigma \subset \Gamma$ , we let  $1_\Sigma : \Gamma \rightarrow \mathbb{R}$  denote the characteristic function of  $\Sigma$ . Also, for  $V$  a vector space and  $\{v_\ell\}_{\ell \in L}$  a family of vectors in  $V$ , we indicate by  $\langle v_\ell \rangle_{\ell \in L}$  their linear span. In order to study the dimension of  $\mathcal{M}_\nu$ , we introduce the space  $\mathcal{O} := \langle 1_{\Gamma_j} \rangle_{j \in J} \subset H^1(\Gamma)$ , and the spaces  $\mathcal{N}_\pm$  defined in the lemma below:

LEMMA 4.2. *For a fixed sign  $\pm$ , the following subspaces of  $H^{1/2}(\Gamma)$  coincide:*

$$\begin{aligned} \mathcal{N}_\pm^1 &:= \{ \gamma^\pm u : u \in H_\ell^1(\Omega_\pm) \text{ satisfies (3.2), } \Delta u + k^2 u = 0 \text{ on } \Omega_\pm, \text{ and } \partial_\nu^\pm u = 0 \text{ on } \Gamma \} \\ \mathcal{N}_\pm^2 &:= \{ \phi \in H^{1/2}(\Gamma) : P^\pm(\phi, 0) = (\phi, 0) \} \\ \mathcal{N}_\pm^3 &:= \{ \phi \in H^{1/2}(\Gamma) : \phi \nu \in \mathcal{M}_\mp \}. \end{aligned}$$

We denote them by  $\mathcal{N}_\pm$  and there holds  $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$ , moreover these spaces are finite-dimensional.

Of course, condition (3.2) is void if one chooses the “+” sign.

*Proof.* Remark 4.1 and the identity  $P^+ + P^- = Id$  together imply that  $\mathcal{N}_\pm^2 = \mathcal{N}_\pm^3$ . Take now a  $\gamma^\pm u \in \mathcal{N}_\pm^1$ . By Implication (3.14), we have that  $(\gamma^\pm u, 0) = (\gamma^\pm u, \partial_\nu^\pm u) = P^\pm(\gamma^\pm u, \partial_\nu^\pm u) = P^\pm(\gamma^\pm u, 0)$  and thus  $\gamma^\pm u \in \mathcal{N}_\pm^2$ . Conversely, if  $\phi \in \mathcal{N}_\pm^2$  and we let  $u = -DL(\mp\phi)$ , then  $u|_{\Omega_\pm} \in H_\ell^1(\Omega_\pm)$  by (3.5) and it follows from Implication (3.13) that  $\Delta u + k^2 u = 0$  on  $\Omega_\pm$ , and  $(\gamma^\pm u, \partial_\nu^\pm u)^t = \mp P^\pm(\mp\phi, 0) = (\phi, 0)$ . Hence,  $\phi \in \mathcal{N}_\pm^1$  and therefore,  $\mathcal{N}_\pm^1 = \mathcal{N}_\pm^2$ . We now see that all three definitions are equivalent.

If  $\Omega_-$  is connected; *i.e.* if  $\Omega_- = \Omega_-^1$ , then by uniqueness of the exterior Neumann problem for the Helmholtz equation when the Sommerfeld condition (3.2) is satisfied [24, Ex. 9.5], we obtain that  $\{0\} = \mathcal{N}_-^1 = \mathcal{N}_-$ . Otherwise, for either choice of sign  $\pm$ , the sets  $\Omega_\pm \setminus \Omega_\pm^1$  are bounded and there exist Neumann eigenvalues  $\{\xi_j^\pm\}_{j=1}^\infty$  for  $-\Delta$ , with  $0 \leq \xi_1^\pm \leq \xi_2^\pm \leq \dots$  and  $\xi_j^\pm \rightarrow \infty$  as  $j \rightarrow \infty$ , as well as corresponding eigenfunctions  $\{u_j\}_{j=1}^\infty \subset H^1(\Omega_\pm \setminus \Omega_\pm^1)$ , satisfying

$$(4.4) \quad \begin{cases} -\Delta u_j = \xi_j^\pm u_j & \text{in } \Omega_\pm \setminus \Omega_\pm^1 \quad (\text{of course } \Omega_+ \setminus \Omega_-^1 = \Omega_+), \\ \partial_\nu^\pm u_j = 0 & \text{on } \Gamma, \end{cases}$$

where the  $\{u_j\}_{j=1}^\infty$  form a complete orthonormal system in  $L^2(\Omega_\pm \setminus \Omega_\pm^1)$ ; see [24, Theorem 4.12] for the case of a bounded and connected open set. To deal with the case where  $\Omega_-$  is not connected, one may apply the previous reference to each  $\Omega_-^j$  in order to get eigenvalues  $\xi_1^{j,-}, \xi_2^{j,-}, \dots$  and eigenvectors  $u_1^j, u_2^j, \dots$  defined on  $\Omega_-^j$  for each  $j \in \{2, \dots, n_\Gamma\}$ , then extend every  $u_k^j$  by zero on  $\Omega_-^m$  for  $m \neq j$  so as to get an orthonormal basis of  $L^2(\Omega_- \setminus \Omega_-^1) = \sum \Omega_-^j$  where the last sum is orthogonal, and arrange the  $\xi_k^{j,-}$  in nondecreasing order for  $2 \leq j \leq n_\Gamma$  and  $k \in \mathbb{N}$ . That  $\mathcal{N}_\pm$  is finite-dimensional now comes from the fact that there can only be finitely many  $j$  for which  $k^2 = \xi_j^\pm$ , and of necessity each  $\phi \in \mathcal{N}_\pm$  is a linear combination of the corresponding  $\gamma^\pm u_j$ . Finally, that  $\mathcal{N}_+ \cap \mathcal{N}_- = \{0\}$  follows from the definition of  $\mathcal{N}_\pm^3$  and Remark 4.1.  $\square$

Continuing towards the definition of the space  $\mathcal{M}_\nu$ , fix an orthonormal basis  $\{\omega_j\}_{j \in J}$  of  $\mathcal{O}$  for the  $L^2(\Gamma)$  scalar product, that contains a basis of  $\mathcal{O} \cap (\mathcal{N}_+ \oplus \mathcal{N}_-)$  if the latter does not reduce to zero. Define

$$\tilde{J} := \{j \in J : \omega_j \notin \mathcal{N}_+ \oplus \mathcal{N}_-\},$$

and put  $\tilde{n}_\Gamma$  for the cardinality of  $\tilde{J}$ .

For each  $j \in \tilde{J}$ , let  $\tilde{\Lambda}_j$  be the orthogonal projection of  $\omega_j$  onto  $V_j := (\mathcal{N}_+ \oplus \mathcal{N}_- \oplus \langle \omega_\ell \rangle_{\ell \in \tilde{J}})^\perp$ , so that  $0 \neq \tilde{\Lambda}_j \in V_j$  while  $\omega_j - \tilde{\Lambda}_j \in \mathcal{N}_+ \oplus \mathcal{N}_- \oplus \langle \omega_\ell \rangle_{\ell \in \tilde{J}}$ . Then,  $\Lambda_j := \tilde{\Lambda}_j / \|\tilde{\Lambda}_j\|_{L^2(\Gamma)}^2$  satisfies,  $\langle \Lambda_j, \omega_l \rangle_{L^2(\Gamma)} = \delta_l^j$  for  $l \in J$ , and  $\langle \Lambda_j, \phi \rangle_{L^2(\Gamma)} = 0$  for each  $\phi \in \mathcal{N}_+ \oplus \mathcal{N}_-$ . Therefore, by Lemma 4.2 and the Fredholm alternative for coercive Dirichlet-Neumann problems [24, Theorem 4.10], we can define for each  $j \in \tilde{J}$  a function

$u_j^+ \in H^1(\Omega_+)$  satisfying

$$\begin{cases} \Delta u_j^+ + k^2 u_j^+ = 0 & \text{in } \Omega_+ \\ \partial_{\nu}^+ u_j^+ = \Lambda_j & \text{on } \Gamma, \end{cases}$$

as well as a function  $u_j^- \in H_\ell^1(\Omega_-)$  satisfying

$$\begin{cases} \Delta u_j^- + k^2 u_j^- = 0 & \text{in } \Omega_- \\ \partial_{\nu}^- u_j^- = \Lambda_j & \text{on } \Gamma, \end{cases}$$

together with the Sommerfeld radiation condition (3.2). Let us define functions  $\phi_j^\pm$  and  $\phi_j$  in  $H^{1/2}(\Gamma)$  by

$$\phi_j^+ := \gamma^- u_j^-, \quad \phi_j^- := \gamma^+ u_j^+ \quad \text{and} \quad \phi_j := \phi_j^- - \phi_j^+,$$

to finally put  $\mathcal{M}_{\nu} := \langle \phi_j \nu \rangle_{j \in \tilde{J}}$ . The theorem below is the main result of the paper; for the proof, we will use  $n_\Gamma$ ,  $\tilde{n}_\Gamma$ ,  $\mathcal{O}$ ,  $\omega_j$ ,  $\phi_j$ ,  $\phi_j^\pm$  and  $\Lambda_j$  as defined above.

**THEOREM 4.3.** *There holds the direct sum decomposition,*

$$(4.5) \quad L^2(\Gamma)^3 = \mathcal{M}_{\nu} \oplus \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0,$$

where the orthogonal  $\mathcal{M}_{\nu}$  to  $\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0$  in  $L^2(\Gamma)^3$  is finite-dimensional, consists of purely normal vector fields and can be rewritten as

$$(4.6) \quad (\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp = \left\langle \gamma \left[ \overline{\nabla \mathcal{F}(\omega_j \nu)} \right] \right\rangle_{j \in \tilde{J}}.$$

If  $k = 0$ , then  $\mathcal{M}_{\nu} = \{0\}$  and therefore the decomposition reduces to:

$$(4.7) \quad L^2(\Gamma)^3 = \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0.$$

In contrast, if  $k^2 \neq 0$  is not an eigenvalue for one of the problems in (4.4) then

$$\text{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0) = \dim(\mathcal{M}_{\nu}) = n_\Gamma;$$

in particular, this is the case when  $k \in \mathbb{C} \setminus \mathbb{R}$ . In general, one has  $\dim(\mathcal{M}_{\nu}) = \tilde{n}_\Gamma \leq n_\Gamma$ .

*Proof.* We first show that the space  $\mathcal{M}_{\nu}$ , which is clearly contained in  $\mathcal{M}_0^\perp$ , intersects  $\mathcal{M}_- \oplus \mathcal{M}_+$  in 0 only, and that the  $\phi_j \nu$  are linearly independent for  $j \in \tilde{J}$ . Indeed, assume for a contradiction that there exists coefficients  $c_j \in \mathbb{C}$  together with  $\mathbf{M}^+ \in \mathcal{M}_+$  and  $\mathbf{M}^- \in \mathcal{M}_-$  such that  $\sum_j c_j \phi_j \nu = \mathbf{M}^+ + \mathbf{M}^-$ . By definition of the  $\phi_j^\pm$  and Implication (3.14), one has for each  $j$  since  $P^+ + P^- = Id$  that

$$P^+(\phi_j^+, \Lambda_j) = 0 \quad \text{and} \quad P^-(\phi_j^-, \Lambda_j) = 0,$$

hence also

$$(4.8) \quad P^-(\phi_j, 0) = P^-(\phi_j^-, \Lambda_j) - P^-(\phi_j^+, \Lambda_j) = -(\phi_j^+, \Lambda_j).$$

However, since  $\mathbf{M}_T^+ + \mathbf{M}_T^- = 0$  because  $\mathbf{M}^+ + \mathbf{M}^-$  is a normal vector field by assumption, (4.2) implies:

$$P^-\left(\sum_j c_j \phi_j, 0\right) = P^-(M_\nu^-, \nabla_T \cdot \mathbf{M}_T^-) + P^-(M_\nu^+, \nabla_T \cdot \mathbf{M}_T^+) = (M_\nu^+, \nabla_T \cdot \mathbf{M}_T^+),$$

whence  $c_j = 0$  for all  $j$  in view of (4.8) because  $\langle \Lambda_j, \omega_k \rangle_{L^2(\Gamma)} = \delta_{j,k}$  whereas  $\langle \nabla_T \cdot \mathbf{M}_T^+, \omega_k \rangle_{L^2(\Gamma)} = 0$ , as the  $\omega_k$  are locally constant on  $\Gamma$ . Therefore the sum in (4.5) is indeed direct and the  $\phi_j \nu$ ,  $j \in \tilde{J}$ , form a basis of  $\mathcal{M}_{\nu}$ . Moreover, since  $\mathcal{M}_{\nu} \cap (\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0) = \{0\}$ , one has the inequality

$$(4.9) \quad \tilde{n}_\Gamma = \dim \mathcal{M}_{\nu} \leq \text{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$$

and thus, for Equation (4.5) to hold it is enough to show that  $\tilde{n}_\Gamma \geq \text{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$ .

Define two continuous linear operators by

$$\begin{aligned} \pi &: \mathcal{M}_0^\perp \longrightarrow L^2(\Gamma) \times H^{-1}(\Gamma) & \text{and} & & \eta &: H^{-1}(\Gamma) \longrightarrow \mathbb{C}^{n_\Gamma} \\ \mathbf{M} &\mapsto (M_\nu, \nabla_{\mathbf{T}} \cdot \mathbf{M}_T) & & & \psi &\mapsto (\langle \psi, 1_{\Gamma_j} \rangle)^t. \end{aligned}$$

Note that  $\pi$  is injective, by (4.1). From Lemma A.10, we further get for  $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$  that

$$(4.10) \quad (\phi, \psi) \in \text{Im } \pi \quad \text{if and only if} \quad \psi \in \text{Ker } \eta.$$

Since  $P^+P^- = P^-P^+ = 0$ , we get if  $\mathbf{M}_1, \mathbf{M}_2 \in \mathcal{M}_0^\perp$  satisfy  $\pi(\mathbf{M}_1) = P^\mp(\pi(\mathbf{M}_2))$  that  $P^\pm(\pi(\mathbf{M}_1)) = 0$ , which entails in view of (4.2) that  $\mathbf{M}_1 \in \mathcal{M}_\pm$ . As  $P^+ + P^- = Id$ , it ensues readily that

$$(4.11) \quad \pi(\mathcal{M}_\pm) = \text{Im}(P^\mp \pi) \cap \text{Im } \pi.$$

Moreover, still using the identity  $P^+ + P^- = Id$ , we get for any  $\mathbf{M} \in \mathcal{M}_0^\perp$  that

$$P^+(\pi(\mathbf{M})) \in \text{Im } \pi \quad \text{if and only if} \quad P^-(\pi(\mathbf{M})) \in \text{Im } \pi.$$

Hence, by (4.10), the following four memberships are equivalent for  $\mathbf{M} \in \mathcal{M}_0^\perp$ :

$$P_2^+(\pi(\mathbf{M})) \in \text{Ker } \eta \Leftrightarrow P^+(\pi(\mathbf{M})) \in \text{Im } \pi \Leftrightarrow P^-(\pi(\mathbf{M})) \in \text{Im } \pi \Leftrightarrow P_2^-(\pi(\mathbf{M})) \in \text{Ker } \eta.$$

Subsequently, we define

$$(4.12) \quad \begin{aligned} \Pi &:= \text{Ker}(\eta P_2^+ \pi) = \{\mathbf{M} \in \mathcal{M}_0^\perp : P^+(\pi(\mathbf{M})) \in \text{Im } \pi\} \\ &= \text{Ker}(\eta P_2^- \pi) = \{\mathbf{M} \in \mathcal{M}_0^\perp : P^-(\pi(\mathbf{M})) \in \text{Im } \pi\}. \end{aligned}$$

From (4.2) it is clear that  $\mathcal{M}_+ \oplus \mathcal{M}_- \subset \Pi$ , hence  $\pi(\mathcal{M}_\pm) = [P^\mp \circ \pi](\Pi)$  by (4.11). Thus, on using that  $P^+ + P^- = Id$  and  $P^\pm P^\mp = 0$ , we obtain  $\pi(\mathcal{M}_+ \oplus \mathcal{M}_-) = \pi(\Pi)$  and so  $\mathcal{M}_+ \oplus \mathcal{M}_- = \Pi$ , since  $\pi$  is injective.

Now, for  $V$  a closed subspace of  $\mathcal{M}_0^\perp \subset L^2(\Gamma)^3$ , let us denote by  $V^{\perp 0}$  its orthogonal complement in  $\mathcal{M}_0^\perp$ . From (4.12) and the continuity of  $\eta P_2^\pm \pi : \mathcal{M}_0^\perp \rightarrow \mathbb{C}^{n_\Gamma}$ , we get that

$$(4.13) \quad (\mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{M}_0)^\perp = (\mathcal{M}_+ \oplus \mathcal{M}_-)^\perp = \Pi^{\perp 0} = (\text{Ker}(\eta P_2^\pm \pi))^{\perp 0} = \overline{(\pi^*(P_2^\pm)^* \eta^*)}^{clos} = \text{Im}(\pi^*(P_2^\pm)^* \eta^*),$$

where we used a superscript *clos* to distinguish topological closure from complex conjugation, and the last equality uses that  $\text{Im}(\pi^*(P_2^\pm)^* \eta^*)$  is finite-dimensional (for  $\eta$  is  $\mathbb{C}^{n_\Gamma}$ -valued), whence it is closed in  $\mathcal{M}_0^\perp$ . Taking  $\mathbf{c} = (c_1, \dots, c_{n_\Gamma})^t \in \mathbb{C}^{n_\Gamma}$  together with  $\psi \in H^{-1}(\Gamma)$ , we have for  $\mathbf{M} \in \mathcal{M}_0^\perp$  that

$$\langle \eta^* \mathbf{c}, \psi \rangle = \sum_j \bar{c}_j \langle \psi, 1_{\Gamma_j} \rangle = \left\langle \sum_j \bar{c}_j 1_{\Gamma_j}, \psi \right\rangle = \left\langle \sum_j c_j 1_{\Gamma_j}, \psi \right\rangle.$$

Besides, given a pair  $(\phi, \varphi) \in L^2(\Gamma) \times H^1(\Gamma)$ , it holds that

$$(4.14) \quad \langle \pi^*(\phi, \varphi), \mathbf{M} \rangle = \langle (\phi, \varphi), (M_\nu, \nabla_{\mathbf{T}} \cdot \mathbf{M}_T) \rangle = \langle \phi, M_\nu \rangle - \langle \nabla_{\mathbf{T}} \varphi, \mathbf{M}_T \rangle = \langle \phi \boldsymbol{\nu} - \nabla_{\mathbf{T}} \varphi, \mathbf{M} \rangle_{L^2(\mathcal{M})^3}.$$

So, by (3.11) and Remark 3.6, we get on using that  $T^* = \mathcal{C} \circ T \circ \mathcal{C}$  on  $H^{1/2}(\Gamma)$  (see footnote 2 before (3.10)):

$$\begin{aligned} \mp \pi^*(P_2^\pm)^* \eta^*(\mathbf{c}) &= \mp \sum_{j \in J} c_j \left[ \pm (T^* 1_{\Gamma_j}) \boldsymbol{\nu} - \nabla_{\mathbf{T}} \left( \frac{1}{2} 1_{\Gamma_j} \mp \bar{K} 1_{\Gamma_j} \right) \right] \\ &= \sum_{j \in J} c_j \left[ -(\overline{T 1_{\Gamma_j}}) \boldsymbol{\nu} - \nabla_{\mathbf{T}} (\overline{K 1_{\Gamma_j}}) \right] \\ &= \sum_{j \in J} c_j \left[ -(\overline{T 1_{\Gamma_j}}) \boldsymbol{\nu} - \nabla_{\mathbf{T}} \left( \overline{\pm \frac{1}{2} 1_{\Gamma_j} + K 1_{\Gamma_j}} \right) \right] = \gamma^\pm \left( \overline{\nabla \mathcal{F} \left( \sum_{j \in J} c_j 1_{\Gamma_j} \boldsymbol{\nu} \right)} \right). \end{aligned}$$

Thus, the  $\gamma [\overline{\nabla \mathcal{F}(\omega_j \boldsymbol{\nu})}]$  are well-defined in  $L^2(\Gamma)^3$  for any  $j \in J$ ; *i.e.*, their traces from either side of  $\Gamma$  coincide, and in light of (4.13) together with the third definition of Lemma 4.2 one has  $\gamma [\overline{\nabla \mathcal{F}(\omega_j \boldsymbol{\nu})}] = 0$  whenever  $j \notin \tilde{J}$ . Hence, Equation (4.6) is satisfied and consequently

$$\tilde{n}_\Gamma \geq \dim((\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp) = \text{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0).$$

Therefore, it follows from (4.9) that  $\tilde{n}_\Gamma = \text{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)$ , and so (4.5) holds. In addition, we obtain that the set  $\{\gamma [\overline{\nabla \mathcal{F}(\omega_j \boldsymbol{\nu})}]\}_{j \in \tilde{J}}$  consists of linearly independent functions.

Clearly, if  $k^2 \neq 0$  is not an eigenvalue for the problem (4.4) then  $\tilde{J} = J$ , and hence

$$n_\Gamma = \tilde{n}_\Gamma = \dim(\mathcal{M}_\nu) = \text{codim}(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0).$$

Finally, when  $k = 0$ , we notice that

$$\mathcal{F}(1_{\Gamma_j} \boldsymbol{\nu}) = \begin{cases} -\chi_{\mathbb{R}^3 \setminus \Omega^j} & \text{if } j = 1 \\ \chi_{\Omega^j} & \text{otherwise,} \end{cases}$$

entailing that  $\gamma [\overline{\nabla \mathcal{F}(\omega_j \boldsymbol{\nu})}] = 0$  for every  $j \in J$ . Then,  $\tilde{n}_\Gamma = 0$  and  $\mathcal{M}_\nu = \{0\}$ , so that (4.7) is satisfied.  $\square$

To conclude this section, let us characterize the orthogonal space to silent sources on either side:

COROLLARY 4.4. *For a fixed choice of sign  $\pm$ , it holds that*

$$(\mathcal{M}_\pm \oplus \mathcal{M}_0)^\perp = \left\{ \gamma^\mp \left( \overline{\nabla \tilde{\mathcal{F}}(\varphi, \phi)} \right) : (\varphi, \phi) \in H^1(\Gamma) \times L^2(\Gamma) \right\}.$$

*Proof.* Take  $\eta$ ,  $\pi$  and  $\Pi$  as in the proof of Theorem 4.3, so that  $\Pi = \mathcal{M}_+ \oplus \mathcal{M}_-$ , and let  $\pi_\Pi$  be the restriction of  $\pi$  to  $\Pi$  onto its image:

$$\begin{aligned} \pi_\Pi &: \Pi &\longrightarrow & \pi(\Pi) \\ & \mathbf{M} &\longmapsto & \pi(\mathbf{M}). \end{aligned}$$

Observing that  $\pi_\Pi$  is a bijective operator and recalling from the discussion after (4.12) the identity  $[P^\mp \circ \pi](\Pi) = \pi(\mathcal{M}_\pm)$ , we get since the  $\pi_\Pi^{-1} P^\mp \pi_\Pi$  are projections adding up to the identity that

$$(4.15) \quad \mathcal{M}_\pm = \text{Im} \left( \pi_\Pi^{-1} P^\mp \pi_\Pi \right), \quad \mathcal{M}_\pm = \text{Ker} \left( \pi_\Pi^{-1} P^\pm \pi_\Pi \right).$$

In particular (4.15) implies that  $\pi_\Pi^{-1} P^\mp \pi_\Pi$  has closed range, hence so does the range of its adjoint [24, Theorem 2.13]. Now, for a  $V \subset \Pi$  a subspace, let  $V^{\perp \Pi}$ , denote the orthogonal space to  $V$  in  $\Pi \subset L^2(\Gamma)^3$ . From (4.15) and the closedness of the range of  $\pi_\Pi^*(P^\pm)^* (\pi_\Pi^{-1})^*$ , we deduce that

$$(4.16) \quad \mathcal{M}_\pm^{\perp \Pi} = \text{Im} \left[ \pi_\Pi^*(P^\pm)^* (\pi_\Pi^{-1})^* \right].$$

Let  $f_j^\pm$  designate, for  $1 \leq j \leq n_\Gamma$ , the linear form on  $L^2(\Gamma) \times H^{-1}(\Gamma)$  given by the  $j$ -th coordinate of  $\eta P_2^\pm$ . It follows from (4.12) that  $\pi(\Pi) = \cap_{j=1}^m \text{Ker} \varphi_j^+ \cap \text{Im} \pi = \cap_{j=1}^m \text{Ker} \varphi_j^- \cap \text{Im} \pi$  and hence, by the Hahn-Banach theorem, the dual  $(\pi(\Pi))^*$  is the quotient space  $(\text{Im} \pi)^* / \langle \varphi_j^+ \rangle_{j \in J} = (\text{Im} \pi)^* / \langle \varphi_j^- \rangle_{j \in J}$ . Now, given  $\mathbf{M} \in \Pi \sim \Pi^*$  and  $(\phi, \psi) \in \pi(\Pi)$ , we get from (4.10) while using Remark 4.1, Lemma A.10 and (4.3) that

$$\begin{aligned} \left\| \left( \pi_\Pi^{-1} \right)^* \mathbf{M}, (\phi, \psi) \right\| &= \left\| \mathbf{M}, \pi_\Pi^{-1}(\phi, \psi) \right\| = \left\| \mathbf{M}, \phi \boldsymbol{\nu} + \nabla_T \varphi_\psi \right\| = \left\| M_\nu, \phi \right\| - \left\| \nabla_T U_{M_T}, \nabla_T \varphi_\psi \right\| \\ &= \left\| M_\nu, \phi \right\| - \left\| U_{M_T}, \Delta_T \varphi_\psi \right\| = \left\| M_\nu, \phi \right\| - \left\| U_{M_T}, \psi \right\|. \end{aligned}$$

Thus, it holds that  $(\pi_\Pi^{-1})^* \mathbf{M} = (M_\nu, -U_{M_T}) \in L^2(\Gamma) \times H^1(\Gamma)$  and therefore, by Equation (4.16),

$$(4.17) \quad \mathcal{M}_\pm^{\perp \Pi} \subset \left[ \pi_\Pi^* \circ (P^\pm)^* \right] \left( L^2(\Gamma) \times H^1(\Gamma) \right).$$

In another connection, in view of (3.11), we get for  $(\phi, \varphi) \in L^2(\Gamma) \times H^1(\Gamma)$  that

$$(P^\pm)^*(\phi, \varphi) = \begin{pmatrix} \frac{1}{2}Id \pm K^* & \pm T^* \\ \mp S^* & \frac{1}{2}Id \mp \bar{K} \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix}.$$

Consequently, using Equation (4.14) and Remark 3.6, we obtain for  $(\phi, \varphi) \in L^2(\Gamma) \times H^1(\Gamma)$ ,

$$\begin{aligned} \pm [\pi_\Pi^* \circ (P^\pm)^*](\bar{\phi}, -\bar{\varphi}) &= \pi_\Pi^* \left( \left( \pm \frac{1}{2}Id + K^* \right) \bar{\phi} - T^* \bar{\varphi}, -S^* \bar{\phi} + \left( \mp \frac{1}{2}Id + \bar{K} \right) \bar{\varphi} \right) \\ &= \left( -T^* \bar{\varphi} + \left( \pm \frac{1}{2} + K^* \right) \bar{\phi} \right) \boldsymbol{\nu} + \nabla_\Gamma \left( - \left( \mp \frac{1}{2}Id + \bar{K} \right) \bar{\varphi} + S^* \bar{\phi} \right) \\ &= \overline{\left( -T\varphi + \left( \pm \frac{1}{2} + \bar{K}^* \right) \phi \right) \boldsymbol{\nu} + \nabla_\Gamma \left( - \left( \mp \frac{1}{2}Id + K \right) \varphi + S\phi \right)} \\ &= \partial_\nu^\mp \left( \overline{-DL(\varphi) + SL(\phi)} \right) + \nabla_\Gamma \gamma^\mp \left( \overline{-DL(\varphi) + SL(\phi)} \right) \\ &= \gamma^\mp \left( \overline{\nabla \tilde{\mathcal{F}}(\varphi, \phi)} \right). \end{aligned}$$

Then, noticing that  $(\mathcal{M}_\pm \oplus \mathcal{M}_0)^\perp = \mathcal{M}_\pm^{\perp \Pi} \oplus (\Pi \oplus \mathcal{M}_0)^\perp = \mathcal{M}_\pm^{\perp \Pi} \oplus (\mathcal{M}_+ \oplus \mathcal{M}_- \oplus \mathcal{M}_0)^\perp$  and, recalling Equations (4.17) and (4.6), we get from what precedes the inclusion

$$(\mathcal{M}_\pm \oplus \mathcal{M}_0)^\perp \subset \left\{ \gamma^\mp \left( \overline{\nabla \tilde{\mathcal{F}}(\varphi, \phi)} \right) : (\varphi, \phi) \in H^1(\Gamma) \times L^2(\Gamma) \right\}.$$

To finish the proof, it remains to show the opposite inclusion. Take any  $(\varphi, \phi) \in H^1(\Gamma) \times L^2(\Gamma)$  and set  $\mathbf{M} := \gamma^\mp \left( \overline{\nabla \tilde{\mathcal{F}}(\varphi, \phi)} \right)$ , together with  $w = \tilde{\mathcal{F}}(\varphi, \phi)$ . Using Remark 4.1 and the fact that  $\mathbf{M}_T$  is a tangential gradient, it follows the orthogonality relation  $\mathbf{M} \perp \mathcal{M}_0$ . Besides, if we pick  $\mathbf{M}^\pm \in \mathcal{M}_\pm$  and  $w^\pm = \mathcal{F}(\mathbf{M}^\pm)$ , then we note by Implication 3.13 and (4.2) that

$$\pm(\gamma^\mp w^\pm, \partial_\nu^\mp w^\pm) = P^\mp(\mathbf{M}_\nu^\pm, \nabla_\Gamma \cdot \mathbf{M}_T^\pm) = (\mathbf{M}_\nu^\pm, \nabla_\Gamma \cdot \mathbf{M}_T^\pm).$$

Then, since  $\nabla_\Gamma \cdot \mathbf{M}_T^\pm = \pm \partial_\nu^\mp w^\pm$  and  $\mathbf{M}_\nu^\pm = \pm \gamma^\mp w^\pm$  by (3.16), (4.2) and the fact that  $P^+ + P^- = Id$ , it ensues

$$\begin{aligned} \langle \mathbf{M}, \mathbf{M}^\pm \rangle_{L^2(\Gamma)} &= \langle \nabla_\Gamma(\gamma^\mp w), \mathbf{M}_T^\pm \rangle + \langle \partial_\nu^\mp w, \mathbf{M}_\nu^\pm \rangle = -\langle \gamma^\mp w, \nabla_\Gamma \cdot \mathbf{M}_T^\pm \rangle + \langle \partial_\nu^\mp w, \mathbf{M}_\nu^\pm \rangle \\ &= \mp \langle \gamma^\mp w, \partial_\nu^\mp w^\pm \rangle \pm \langle \partial_\nu^\mp w, \gamma^\mp w^\pm \rangle \\ &= -\langle \Delta w, w^\pm \rangle_{L^2(\Omega_\mp)} + \langle w, \Delta w^\pm \rangle_{L^2(\Omega_\mp)} \\ &= \left\langle \bar{k}^2 w, w^\pm \right\rangle_{L^2(\Omega_\mp)} - \left\langle w, k^2 w^\pm \right\rangle_{L^2(\Omega_\mp)} = 0 \\ &= -\langle \Delta w + k^2 w, w^\pm \rangle_{L^2(\Omega_\mp)} + \langle w, \Delta w^\pm + k^2 w^\pm \rangle_{L^2(\Omega_\mp)} \\ &= -\left\langle \left( k^2 - \bar{k}^2 \right) w, w^\pm \right\rangle_{L^2(\Omega_\mp)} \end{aligned}$$

where the fourth equality follows from Equation 3.6b and the fifth uses that  $w$  satisfies (3.6c) with  $k$  replaced by  $\bar{k}$ . Therefore  $\mathbf{M} \perp \mathcal{M}_\pm$  as well, and the corollary follows.  $\square$

**4.1. Spherical case.** In this subsection we assume that  $\Gamma = \mathbb{S}^2$ , the unit sphere on  $\mathbb{R}^3$ , and that  $k > 0$ . In this case, some calculations from the previous subsection can be made explicit using the Addition Theorem.

Recall that, if we let  $P_n^m$  denote the associated Legendre function of order  $m$ , then the following define a complete orthonormal system in  $L^2(\mathbb{S})$  [10, Theorem 2.8] and a complete orthogonal system in  $H^1(\mathbb{S})$  [26,



Theorem 2.4.4]:

$$Y_n^m(x) := \sqrt{\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{im\varphi} \quad \text{for } m = -n, \dots, n, \text{ and } n = 0, 1, 2, \dots,$$

where  $x = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$ . Note that  $\overline{(Y_n^m(x))} = Y_n^{-m}(x)$ . These functions also satisfy

$$(4.18) \quad \Delta_{\mathbb{T}} Y_n^m = -n(n+1) Y_n^m.$$

Note that this implies that  $\langle\langle Y_n^m, Y_n^m \rangle\rangle_{H^1(\mathbb{S})} = 1 + n(n+1)$ . Given  $\phi \in L^2(\mathbb{S})$  and  $\mathbf{M} \in L^2(\mathbb{S})^3$  define the coefficients:

$$c_n^m(\phi) := \langle\langle Y_n^m, \phi \rangle\rangle_{L^2(\mathbb{S})} = \frac{\langle\langle Y_n^m, \phi \rangle\rangle_{H^1(\mathbb{S})}}{\langle\langle Y_n^m, Y_n^m \rangle\rangle_{H^1(\mathbb{S})}} \quad \text{for } m = -n, \dots, n, \text{ and } n = 0, 1, 2, \dots,$$

and, for  $m = -n, \dots, n$  and  $n = 1, 2, 3, \dots$ ,

$$g_n^m(\mathbf{M}) := \frac{\langle\langle \nabla_{\mathbb{T}} Y_n^m, \mathbf{M} \rangle\rangle_{L^2(\mathbb{S})^3}}{n(n+1)}, \quad r_n^m(\mathbf{M}) := \frac{\langle\langle \boldsymbol{\nu} \times \nabla_{\mathbb{T}} Y_n^m, \mathbf{M} \rangle\rangle_{L^2(\mathbb{S})^3}}{n(n+1)},$$

$$g_0^0(\mathbf{M}) := 0 \quad \text{and} \quad r_0^0(\mathbf{M}) := 0.$$

Then,  $\phi = \sum c_n^m(\phi) Y_n^m$  in  $L^2(\mathbb{S})$ . Note that, for any  $n$  and  $m$ ,  $g_n^m(\mathbf{M}) = g_n^m(\mathbf{M}_T)$  and  $r_n^m(\mathbf{M}) = r_n^m(\mathbf{M}_T)$ . Additionally, if  $u \in H^1(\mathbb{S})$  then  $u = \sum c_n^m(u) Y_n^m$  in  $H^1(\mathbb{S})$  and we have:

$$g_n^m(\nabla_{\mathbb{T}} u) = c_n^m(u), \quad r_n^m(\nabla_{\mathbb{T}} u) = 0, \quad g_n^m(\boldsymbol{\nu} \times \nabla_{\mathbb{T}} u) = 0 \quad \text{and} \quad r_n^m(\boldsymbol{\nu} \times \nabla_{\mathbb{T}} u) = c_n^m(u).$$

By Hodge decomposition, there exist  $u, v \in H^1(\mathbb{S})$  such that  $\mathbf{M}_T = \nabla_{\mathbb{T}} u + \boldsymbol{\nu} \times \nabla_{\mathbb{T}} v$ , and hence,

$$\begin{aligned} \mathbf{M}_T &= \nabla_{\mathbb{T}} \sum c_n^m(u) Y_n^m + \boldsymbol{\nu} \times \nabla_{\mathbb{T}} \sum c_n^m(v) Y_n^m \\ &= \sum g_n^m(\mathbf{M}) \nabla_{\mathbb{T}} Y_n^m + \sum r_n^m(\mathbf{M}) (\boldsymbol{\nu} \times \nabla_{\mathbb{T}} Y_n^m) \end{aligned}$$

in  $L^2(\mathbb{S})^3$ . Therefore,  $\nabla_{\mathbb{T}} \cdot \mathbf{M}_T = -n(n+1) \sum g_n^m(\mathbf{M}) Y_n^m$  in  $H^{-1}(\mathbb{S})$ .

Further define, for a non-negative integer  $n$ ,  $h_n^{(1)}$  as the spherical Hankel function of the first kind of order  $n$ , and let  $j_n$  denote the spherical Bessel function of order  $n$ . Then, since we are assuming that  $k \neq 0$ , we have the following Addition Theorem [10, Theorem 2.11]

$$G(x-y) = -ik \sum_{n=0}^{\infty} \sum_{m=-n}^n h_n^{(1)}(k|x|) Y_n^m \left( \frac{x}{|x|} \right) j_n(k|y|) \overline{\left( Y_n^m \left( \frac{y}{|y|} \right) \right)} \quad \text{for } |x| > |y|,$$

where the series and its term by term first derivatives with respect to  $|x|$  and  $|y|$  are absolutely and uniformly convergent on compact subsets of  $|x| > |y|$ .

Then, using Fubini-Tonelli theorem we obtain:

$$SL(Y_n^m)(x) = \begin{cases} -ik h_n^{(1)}(k|x|) j_n(k) Y_n^m \left( \frac{x}{|x|} \right) & \text{for } |x| > 1 \\ -ik h_n^{(1)}(k) j_n(k|x|) Y_n^m \left( \frac{x}{|x|} \right) & \text{for } |x| < 1 \end{cases},$$

$$DL(Y_n^m)(x) = \begin{cases} -ik^2 h_n^{(1)}(k|x|) j_n'(k) Y_n^m \left( \frac{x}{|x|} \right) & \text{for } |x| > 1 \\ -ik^2 (h_n^{(1)})'(k) j_n(k|x|) Y_n^m \left( \frac{x}{|x|} \right) & \text{for } |x| < 1 \end{cases},$$

thus,

$$K(Y_n^m) = -\frac{1}{2} ik^2 (h_n^{(1)}(k) j_n'(k) + (h_n^{(1)})'(k) j_n(k)) Y_n^m \quad T(Y_n^m) = -ik^3 (h_n^{(1)})'(k) j_n'(k) Y_n^m$$

$$S(Y_n^m) = -ik h_n^{(1)}(k) j_n(k) Y_n^m \quad K^*(Y_n^m) = K(Y_n^m),$$

Since  $K$ ,  $T$ ,  $S$ ,  $K^*$  and  $\nabla_{\mathbb{T}^*} : L^2(\mathbb{S})^3 \rightarrow H^{-1}(\mathbb{S})$  are continuous, and  $h_n^{(1)}(k)j_n'(k) - (h_n^{(1)})'(k)j_n(k) = 1/(ik^2)$ , using (4.18) we get

$$\begin{aligned} P^-(M_\nu, \nabla_{\mathbb{T}^*} \mathbf{M}_T) &= \left( \sum_{n,m} [(1/2 - K(Y_n^m))c_n^m(M_\nu) + S(Y_n^m)(-n(n+1)g_n^m(\mathbf{M}_T))] Y_n^m, \right. \\ &\quad \left. \sum_{n,m} [-T(Y_n^m)c_n^m(M_\nu) + (1/2 + K(Y_n^m))(-n(n+1)g_n^m(\mathbf{M}_T))] Y_n^m \right) \\ &= \left( \sum_{n,m} ikh_n^{(1)}(k) [kj_n'(k)c_n^m(M_\nu) + j_n(k)n(n+1)g_n^m(\mathbf{M}_T)] Y_n^m, \right. \\ &\quad \left. \sum_{n,m} ik^2(h_n^{(1)})'(k) [kj_n'(k)c_n^m(M_\nu) + j_n(k)n(n+1)g_n^m(\mathbf{M}_T)] Y_n^m \right), \end{aligned}$$

and

$$\begin{aligned} P^+(M_\nu, \nabla_{\mathbb{T}^*} \mathbf{M}_T) &= \left( \sum_{n,m} -ikj_n(k) [k(h_n^{(1)})'(k)c_n^m(M_\nu) + h_n^{(1)}(k)n(n+1)g_n^m(\mathbf{M}_T)] Y_n^m, \right. \\ &\quad \left. \sum_{n,m} -ik^2j_n'(k) [k(h_n^{(1)})'(k)c_n^m(M_\nu) + h_n^{(1)}(k)n(n+1)g_n^m(\mathbf{M}_T)] Y_n^m \right), \end{aligned}$$

Recall that for all  $n$ ,  $h_n^{(1)}(k) \neq 0 \neq (h_n^{(1)})'(k)$  for  $k$  is real and positive. Therefore,

$$(4.19) \quad \mathcal{M}_- = \{ \mathbf{M} \in L^2(\mathbb{S})^3 : kj_n'(k)c_n^m(M_\nu) = -j_n(k)n(n+1)g_n^m(\mathbf{M}_T) \text{ for } m = -n, \dots, n, \text{ and } n = 0, 1, 2, \dots, \},$$

and

$$(4.20) \quad \mathcal{M}_+ = \{ \mathbf{M} \in L^2(\mathbb{S})^3 : k(h_n^{(1)})'(k)c_n^m(M_\nu) = -h_n^{(1)}(k)n(n+1)g_n^m(\mathbf{M}_T) \text{ for } m = -n, \dots, n, \text{ and } n = 0, 1, 2, \dots, \text{ such that } j_n(k) \neq 0 \text{ or } j_n'(k) \neq 0 \}.$$

Since  $j_0(k) = \sin(k)/k$ , for no real  $k$  do we get  $j_0(k) = 0 = j_0'(k)$ . Hence, for  $\mathbf{M} \in \mathcal{M}_+ + \mathcal{M}_-$  and  $j_0'(k) \neq 0$ , we have that  $c_0^0(M_\nu) = 0$  and thus,  $\langle \mathbf{M}, Y_0^0 \nu \rangle_{L^2(\mathbb{M})^3} = 0$ . Otherwise, when  $j_0'(k) = 0$  and  $k > 0$ , we obtain  $P^-(Y_0^0, 0) = 0$ . Therefore, using Theorem 4.3 we deduce the following result.

**THEOREM 4.5.** *For a  $k > 0$  such that  $j_0'(k) \neq 0$  (which happens for almost every  $k$ ), holds in  $L^2(\mathbb{S})^3$  that*

$$(\mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0)^\perp = \{ \mathbf{M} \in L^2(\mathbb{S})^3 : \mathbf{M}_T = 0 \text{ and } M_\nu \text{ is constant} \};$$

on the other hand, if  $j_0'(k) = 0$  (this happens in particular when  $k = 0$ ), then

$$L^2(\mathbb{S})^3 = \mathcal{M}_- \oplus \mathcal{M}_+ \oplus \mathcal{M}_0.$$

It is worth mentioning that when  $k = 0$ , the subspaces  $\mathcal{M}_\mp$  of  $L^2(\mathbb{S})^3$  coincide with the Hardy spaces  $H^\pm$  of harmonic gradients inside and outside  $\mathbb{S}$  introduced by Stein and Weiss. This follows easily from (4.19) and (4.20); compare [2].

## Appendix A. .

### A.1. $L^2$ behaviour of layer potentials on Lipschitz domains for the Helmholtz equation.

The statements in this section are adapted to the case  $k \neq 0$  from [18], or mere reproductions of material from that reference. For each of them, we write in parenthesis where in [18] they can be found. We also want to stress that, as pointed out to the authors by a referee, many results of this subsection can be found in sections 4 and 5 of [29]. In particular this is true of Propositions A.3 and A.4, of the second part of Proposition A.5 and of the first equality in Proposition A.7.

For convenience, throughout this section, we will denote the function  $G$  and the operators  $SL$ ,  $DL$ ,  $S$ , and  $K$  by  $G_k$ ,  $SL_k$ ,  $DL_k$ ,  $S_k$  and  $K_k$  respectively. For the operators  $S_k$  and  $K_k$  we will use the definitions given in [24] (without the  $1/2$  for  $K_k$  and  $K_k^*$ ) and then show that they can be extended with the required properties.

Recall  $\mathfrak{C}_\alpha^\pm(x)$  from the definition of the nontangential limit. For a vector valued measurable function  $\psi$  on  $\Omega_\pm$ , we define the function  $\mathfrak{N}_\alpha^\pm \psi$ , on  $\Gamma$ , such that, for  $x \in \Gamma$ ,

$$\mathfrak{N}_\alpha^\pm \psi(x) := \sup \{ |\psi(y)| : y \in \mathfrak{C}_\alpha^\pm(x) \},$$

taking the convention that  $\mathfrak{N}_\alpha^\pm \psi(x) = 0$  when  $\mathfrak{C}_\alpha^\pm(x) = \emptyset$ .

In [18, section 3.6], the Sobolev space  $L_1^2(\Gamma, d\sigma)$  is defined as the subspace of  $L^2(\Gamma)$  comprised of those functions  $\varphi$  such that  $|\langle \varphi, \nu_j \gamma(\partial_l f) - \nu_l \gamma(\partial_j f) \rangle_{L^2(\Gamma)}| \leq C \|f\|_{L^2(\Gamma)}$  for all  $f \in C^1(\mathbb{R}^3)$ , any  $l, j \in \{1, 2, 3\}$  and some constant  $C = C(\varphi)$ , with  $\nu_j$  to mean the  $j$ -th coordinate of the unit normal field on  $\Gamma$ . That is, if one puts as in [18]  $\partial_{\tau_{i,j}} f := \nu_j \gamma(\partial_i f) - \nu_i \gamma(\partial_j f)$  for  $f \in C^1(\mathbb{R}^3)$  then  $\partial_{\tau_{i,j}} f$  depends only on the restriction  $f|_\Gamma$  and members of  $L_1^2(\Gamma)$  are those  $\varphi \in L^2(\Gamma)$  whose distributional  $\partial_{\tau_{i,j}} \varphi$  is an  $L^2(\Gamma)$ -function for each  $j, l$ . To justify quoting certain results from [18], we will show in the next lemma that this definition agrees with the one of the Sobolev space  $H^1(\Gamma)$  made in Section 2.

LEMMA A.1. *Given  $j, l \in \{1, 2, 3\}$ , one can define a bounded linear operator  $\partial_{\tau_{i,j}} : H^1(\Gamma) \rightarrow L^2(\Gamma)$  on letting, for any  $\varphi \in H^1(\Gamma)$  and  $f \in C^1(\mathbb{R}^3)$ :*

$$\langle \partial_{\tau_{i,j}} \varphi, \gamma(f) \rangle := - \langle \varphi, \nu_j \gamma(\partial_i f) - \nu_i \gamma(\partial_j f) \rangle_{L^2(\Gamma)}.$$

Moreover, a function  $\varphi \in L^2(\Gamma)$  lies in  $H^1(\Gamma)$  if and only if the operators  $\partial_{\tau_{i,j}}$  defined above (in the weak sense) correspond to scalar product with  $L^2$ -functions.

*Proof.* Note that a tangent vector field on  $\Gamma$  can be regarded as a 1-form, defined by taking the scalar product in the tangent space at regular points. For  $\{(\theta_j, U_j)\}_{j \in I}$  ( $I$  finite) a Lipschitz atlas on  $\Gamma$ , we say that a  $k$ -form  $\omega$  is of  $L^2$ -class (here  $k \in \{0, 1, 2\}$ ) if its expression in local coordinates (pullback of  $\omega$  under the Lipschitz map  $\theta_j^{-1}$ ), say

$$(\theta_j^{-1})^*(\omega)(y) = \sum_{i_1 < i_2, \dots, < i_k} a_{i_1, \dots, i_k}^{\{\phi_j\}}(y) dy_{i_1} \wedge \dots \wedge dy_{i_k}$$

has coefficients  $a_{i_1, \dots, i_k}^{\{\phi_j\}}$  that are  $L^2$  functions on  $\theta_j(U_j)$ . This notion is independent of the atlas. Now, for  $f \in C_c^\infty(\mathbb{R}^3)$ , it holds that

$$(A.1) \quad (\partial_{\tau_{2,3}} f, \partial_{\tau_{3,1}} f, \partial_{\tau_{1,2}} f)^t = \nabla f \times \nu$$

where “ $\times$ ” indicates the vector product and the superscript “ $t$ ” means “transpose”. Thus, observing that  $\nu = \partial_{y_1} \theta_j^{-1} \times \partial_{y_2} \theta_j^{-1} / |\partial_{y_1} \theta_j^{-1} \times \partial_{y_2} \theta_j^{-1}|$  on  $\theta_j(U_j)$ , we get from the double vector product formula that the 1-form associated with  $\nabla f \times \nu$  is given in local coordinates  $(y_1, y_2)$  on  $\theta_j(U_j)$  by

$$(A.2) \quad \left( g_{1,1} \partial_{y_2} (f \circ \theta_j^{-1}) - g_{2,1} \partial_{y_1} (f \circ \theta_j^{-1}) \right) dy_1 + \left( g_{1,2} \partial_{y_2} (f \circ \theta_j^{-1}) - g_{2,2} \partial_{y_1} (f \circ \theta_j^{-1}) \right) dy_2$$

where  $(g_{i_1, i_2})$  is the metric tensor (the Gram matrix of  $\partial_{y_1} \theta_j^{-1}, \partial_{y_2} \theta_j^{-1}$ ). Since the latter is uniformly boundedly invertible on compact manifold that are local Lipschitz graphs, the fact that (A.2) is of  $L^2$ -class amounts to say that  $\nabla f \circ \theta_j^{-1}$  lies in  $(L^2(\theta_j(U_j)))^3$ . By density of traces of  $C_c^\infty(\mathbb{R}^3)$ -functions in  $L^2(\Gamma)$ , we conclude what we want.  $\square$

Then, we have a lemma that was just stated on [18] since it was proven in [14]. However, we add a proof for convenience of the reader.

LEMMA A.2 (Lemma 6.4.2). *For each fixed  $R > 0$  and  $k > 0$ , there exists a constant  $C > 0$  such that, for  $1 \leq j \leq 3$  the following estimates are uniformly satisfied for  $0 < |x| < R$ :*

$$\begin{aligned} |G_k(x) - G_0(x)| &\leq C \\ |\partial_j G_k(x) - \partial_j G_0(x)| &\leq C \\ |\partial_\ell \partial_j G(x) - \partial_\ell \partial_j G_0(x)| |x| &\leq C \end{aligned}$$

*Proof.* Since  $G_k - G_0$  is  $C^\infty(\mathbb{R}^3 \setminus \{0\})$ , it is enough to show that the limsup when  $x \rightarrow 0$  in all of the left hand sides of the equations of the lemma are bounded by a constant depending only on  $k$ :

$$\limsup_{x \rightarrow 0} |G_k(x) - G_0(x)| = \lim_{x \rightarrow 0} \frac{|-1 + e^{ik|x|}|}{4\pi|x|} = \frac{k}{4\pi},$$

and

$$\limsup_{x \rightarrow 0} |\partial_j G_k(x) - \partial_j G_0(x)| = \limsup_{x \rightarrow 0} \left| x_j \frac{e^{ik|x|} k|x| + ie^{ik|x|} - i}{4\pi|x|^3} \right| \leq \lim_{x \rightarrow 0} \frac{|x_j e^{ik|x|} k|x| + ie^{ik|x|} - i|}{4\pi|x|^2} = \frac{k^2}{8\pi}.$$

$$\begin{aligned} \limsup_{x \rightarrow 0} |\partial_j \partial_j G(x) - \partial_j \partial_j G_0(x)| |x| &= \limsup_{x \rightarrow 0} \frac{|e^{ik|x|} (ik|x|^3 - |x|^2 - k^2 x_j^2 |x|^2 - 3ikx_j^2 |x| + 3x_j^2) + |x|^2 - 3x_j^2|}{4\pi|x|^4} \\ &\leq \limsup_{x \rightarrow 0} \frac{|e^{ik|x|} (ik|x|^3 - |x|^2) + |x|^2|}{4\pi|x|^4} + \limsup_{x \rightarrow 0} \frac{|e^{ik|x|} (-k^2 x_j^2 |x|^2 - 3ikx_j^2 |x| + 3x_j^2) - 3x_j^2|}{4\pi|x|^4} \\ &\leq \lim_{x \rightarrow 0} \frac{|e^{ik|x|} (ik|x| - 1) + 1|}{4\pi|x|^2} + \lim_{x \rightarrow 0} \frac{|e^{ik|x|} (-k^2|x|^2 - 3ik|x| + 3) - 3|}{4\pi|x|^2} = \frac{k^2}{4\pi}, \end{aligned}$$

and, for  $j \neq \ell$ ,

$$\begin{aligned} \limsup_{x \rightarrow 0} |\partial_\ell \partial_j G(x) - \partial_\ell \partial_j G_0(x)| |x| &= \limsup_{x \rightarrow 0} \frac{|x_j x_\ell (3 + e^{ik|x|} (k^2|x|^2 + 3ik|x| - 3))|}{4\pi|x|^4} \\ &\leq \lim_{x \rightarrow 0} \frac{|3 + e^{ik|x|} (k^2|x|^2 + 3ik|x| - 3)|}{4\pi|x|^2} = \frac{k^2}{8\pi}. \quad \square \end{aligned}$$

Then, we continue with a generalization of a relatively basic result that is just partly stated on [18] and whose proof, for the  $k = 0$  case, can be found as part of [5, Theorem 4.5].

PROPOSITION A.3 (Partly stated on equation (3.6.27) and Corollary 3.6.3). *Given a  $\phi \in L^2(\Gamma)$ , it is satisfied in the nontangential sense that  $\gamma^\pm SL_k \phi = S_k \phi$ ,  $\sigma$ -a.e. and, for every  $\alpha > 0$ , there exists a constant  $\tilde{C}_\alpha$  such that  $\|\mathfrak{N}_\alpha^\pm(SL_k \phi)\|_2 \leq \tilde{C}_\alpha \|\phi\|_2$ . Also, the left equation of (3.7) is satisfied and we have the mapping property,*

$$(A.3) \quad S_k : L^2(\Gamma) \longrightarrow H^1(\Gamma).$$

*Proof.* Note that for any  $x \in \Gamma$  and  $\phi \in L^2(\Gamma)$ , using the  $k = 0$  result,

$$\left| \int_\Gamma G_k(x-y) \phi(y) d\sigma(y) \right| \leq \int_\Gamma |G_k(x-y)| |\phi(y)| d\sigma(y) = \int_\Gamma G_0(x-y) |\phi(y)| d\sigma(y) = S_0 |\phi|(x),$$

and thus, we have that in general, for  $\sigma$ -a.e.  $x \in \Gamma$ , the integral in the left equation of (3.7) defines a bounded linear operator from  $L^2(\Gamma)$  to itself. Let's call this operator  $\tilde{S}_k$ . Now, notice the following facts;  $\text{Lip}(\Gamma)$

is dense in  $L^2(\Gamma)$ ; both  $\text{Lip}(\Gamma)$  and  $L^2(\Gamma)$  are dense in  $H^{-1/2}(\Gamma)$ ;  $S_k$  and  $\tilde{S}_k$  coincide in  $\text{Lip}(\Gamma)$ ; and the image of  $\text{Lip}(\Gamma)$  over  $\tilde{S}_k$  belongs to  $L^2(\Gamma)$ . Then,  $S_k$  and  $\tilde{S}_k$  must also coincide in  $L^2(\Gamma)$ . Thus, as a small abuse of notation we will refer to  $\tilde{S}_k$  as simply  $S_k$ . Next, if  $\|\phi\|_2 = 1$ , and we take  $C$  from Lemma A.2

$$\begin{aligned} \|\nabla_{\Gamma} S_k \phi - \nabla_{\Gamma} S_0 \phi\|_2 &= \sup_{\substack{f \in \text{Lip}_T(\Gamma) \\ \|f\|_{\infty} \leq 1}} \int_{\Gamma} \left( \int_{\Gamma} (G_k - G_0)(x-y) \phi(y) \, d\sigma(y) \right) \nabla_{\Gamma} \mathbf{f}(x) \, d\sigma(x) \\ &= \sup_{\substack{f \in \text{Lip}_T(\Gamma) \\ \|f\|_{\infty} \leq 1}} \int_{\Gamma} \left( \int_{\Gamma} (\nabla G_k - \nabla G_0)(x-y) \cdot \mathbf{f}(x) \, d\sigma(x) \right) \phi(y) \, d\sigma(y) \\ &= \int_{\Gamma} \left( \int_{\Gamma} |(\nabla G_k - \nabla G_0)(x-y)| \, d\sigma(x) \right) |\phi(y)| \, d\sigma(y) \\ &\leq \sqrt{3} C \sigma(\Gamma) \|\phi\|_1 \leq \sqrt{3} C \sigma(\Gamma) (\|\phi\|_2^2 + \sigma(\Gamma)) = \sqrt{3} C \sigma(\Gamma) (1 + \sigma(\Gamma)), \end{aligned}$$

Then, as  $S_0 : L^2(\Gamma) \rightarrow H^1(\Gamma)$  is bounded and  $\|S_k \phi\|_{H^1(\Gamma)} \leq \|S_k \phi - S_0 \phi\|_{H^1(\Gamma)} + \|S_0 \phi\|_{H^1(\Gamma)}$ , we obtain that  $S_k$  is also a bounded linear operator from  $L^2(\Gamma)$  to  $H^1(\Gamma)$ .

Take  $\alpha > 0$ ,  $x \in \Gamma$  and  $y \in \mathfrak{C}_{\alpha}^{\pm}(x)$ . Then for any  $z \in \Gamma$

$$(A.4) \quad |y - z| \geq \text{dist}(y, \Gamma) \geq \frac{|x - y|}{\alpha + 1} \quad \text{so,} \quad |y - z|(\alpha + 2) \geq |x - y| + |y - z| \geq |x - z|.$$

Thus,

$$|SL_k \phi(y)| = \left| \int_{\Gamma} G_k(y-z) \phi(z) \, d\sigma(z) \right| \leq \int_{\Gamma} \frac{(\alpha + 2) |\phi(z)|}{4\pi |x - z|} \, d\sigma(z) = (\alpha + 2) S_0 |\phi|(x).$$

Hence, we can use Dominated convergence and the result for  $k = 0$ , to obtain for  $\sigma$ -a.e.  $x \in \Gamma$ , that it is satisfied in the nontangential sense  $\gamma^{\pm} SL_k \phi = S_k \phi$ . Also, taking  $\tilde{C}_{\alpha}$  to be the operator norm of  $S_0$  times  $\alpha + 2$  we obtain that  $\|\mathfrak{N}_{\alpha}^{\pm}(SL_k \phi)\|_2 \leq \tilde{C}_{\alpha} \|\phi\|_2$ .  $\square$

PROPOSITION A.4 (Proposition 3.3.2). *Take a  $\phi \in L^2(\Gamma)$ . For  $f = \phi$ , the principal value of equation (3.7) exists for  $\sigma$ -a.e.  $x \in \Gamma$  and it can be used to extend the operator  $K_k$  to*

$$K_k : L^2(\Gamma) \rightarrow L^2(\Gamma)$$

which is bounded. Furthermore, the right equation of (3.8) is satisfied in the nontangential limit sense and for every  $\alpha > 0$ , we have that  $\|\mathfrak{N}_{\alpha}^{\pm}(DL_k \phi)\|_2 \leq \tilde{C}_{\alpha} \|\phi\|_2$  for some  $\tilde{C}_{\alpha} > 0$  depending only on  $\Gamma$ ,  $k$  and  $\alpha$ .

*Proof.* By [18, Proposition 3.3.2] the result is valid for  $k = 0$ . Take any  $\phi \in L^2(\Gamma)$  and  $x \in \Gamma$  such that  $K_0 \phi(x)$  is well-defined, which is  $\sigma$ -a.e. Define for any  $\varepsilon > 0$

$$K_k^{\varepsilon} \phi(x) := \int_{\substack{y \in \Gamma \\ |x - y| > \varepsilon}} \partial_{\nu, y} G_k(x-y) \phi(y) \, d\sigma(y) = - \int_{\substack{y \in \Gamma \\ |x - y| > \varepsilon}} (\nabla G_k)(x-y) \cdot \nu(y) \phi(y) \, d\sigma(y).$$

Then,  $K_0^{\varepsilon}$  defines a bounded linear operator from  $L^2(\Gamma)$  to itself and, whenever  $K_0 \phi(x)$  is well defined,  $K_0^{\varepsilon} \phi(x) \rightarrow K_0 \phi(x)$  as  $\varepsilon \rightarrow 0$ . Thus, for any sequence  $(\varepsilon_n)_n$  such that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , the sequence  $(K_0^{\varepsilon_n} \phi(x))_n$  is Cauchy, whenever  $K_0 \phi(x)$  is well-defined. Hence, showing that the principal value of equation (3.7) exists for  $x$  is equivalent to showing that the sequence  $(K_k^{\varepsilon_n} \phi(x))_n$  is Cauchy as well. Take  $m > n$

and see that,

$$\begin{aligned}
|K_k^{\varepsilon_n} \phi(x) - K_k^{\varepsilon_m} \phi(x)| &\leq |K_k^{\varepsilon_n} \phi(x) - K_k^{\varepsilon_m} \phi(x) - K_0^{\varepsilon_n} \phi(x) + K_0^{\varepsilon_m} \phi(x)| + |K_0^{\varepsilon_n} \phi(x) - K_0^{\varepsilon_m} \phi(x)| \\
&\leq \int_{\substack{y \in \Gamma \\ \varepsilon_n > |x-y| > \varepsilon_m}} |(\nabla G_k - \nabla G_0)(x-y)| |\phi(y)| \, d\sigma(y) + |K_0^{\varepsilon_n} \phi(x) - K_0^{\varepsilon_m} \phi(x)| \\
&\leq \int_{\substack{y \in \Gamma \\ \varepsilon_n > |x-y| > \varepsilon_m}} \sqrt{3}C |\phi(y)| \, d\sigma(y) + |K_0^{\varepsilon_n} \phi(x) - K_0^{\varepsilon_m} \phi(x)|
\end{aligned}$$

where the constant  $C$ , taken from Lemma A.2, depends only on  $k$  and the size of the bounded set  $\Gamma$ . Thus, the integrability of  $\phi$  and the fact that  $(K_0^{\varepsilon_n} \phi(x))_{n=1}^{\infty}$  is Cauchy imply that  $(K_k^{\varepsilon_n} \phi(x))_{n=1}^{\infty}$  is Cauchy as well. Since the result is valid for  $k = 0$ , the value  $K_0 \phi(x)$  is well defined for  $\sigma$ -a.e.  $x \in \Gamma$ . Then,  $\overline{K}_k \phi(x) := \lim_{\varepsilon \rightarrow 0} K_k^{\varepsilon} \phi(x)$  is also well defined for  $\sigma$ -a.e.  $x \in \Gamma$ , and it defines a measurable function since it is the point-wise limit of  $L^2$  functions.

Note that  $\overline{K}_k$  defines a linear operator on  $L^2(\Gamma)$ . Take now any  $\phi \in L^2(\Gamma)$  with  $\|\phi\|_2 = 1$ . Then using, Fatou's lemma we get

$$\begin{aligned}
\|\overline{K}_k \phi - K_0 \phi\|_2^2 &\leq \liminf_{\varepsilon \rightarrow 0} \int_{x \in \Gamma} \left| \int_{\substack{y \in \Gamma \\ |x-y| > \varepsilon}} (\nabla G_k - \nabla G_0)(x-y) \cdot \nu(y) \phi(y) \, d\sigma(y) \right|^2 d\sigma(x) \\
&\leq \liminf_{\varepsilon \rightarrow 0} \int_{x \in \Gamma} \left( \int_{\substack{y \in \Gamma \\ |x-y| > \varepsilon}} |(\nabla G_k - \nabla G_0)(x-y)| |\phi(y)| \, d\sigma(y) \right)^2 d\sigma(x) \\
&\leq 3C^2 \sigma(\Gamma) \|\phi\|_1^2 \leq 3C^2 \sigma(\Gamma) (\|\phi\|_2^2 + \sigma(\Gamma))^2 = 3C^2 \sigma(\Gamma) (1 + \sigma(\Gamma))^2,
\end{aligned}$$

with the same constant  $C$  as before. Then, as  $K_0$  is bounded and  $\|\overline{K}_k \phi\|_2 \leq \|\overline{K}_k \phi - K_0 \phi\|_2 + \|K_0 \phi\|_2$ , we obtain that  $\overline{K}_k$  is bounded having  $L^2(\Gamma)$  as its image. Now, using an argument analogous to the one in Lemma A.3 for  $\tilde{S}_k$ , we can show that  $\overline{K}_k$  coincides with  $K_k$  in  $L^2(\Gamma)$  and thus, as a small abuse of notation we will refer to  $\overline{K}_k$  as just  $K_k$ .

Fix a  $\phi \in L^2(\Gamma)$  such that  $\|\phi\|_2 = 1$ . By [18, equation (3.3.6)], for any  $\alpha > 0$  there exists a constant  $C_\alpha$  such that

$$\|\mathfrak{N}_\alpha^\pm(DL_0 \phi)\|_2 \leq C_\alpha.$$

On the other hand,

$$\begin{aligned}
|\mathfrak{N}_\alpha^\pm(DL_k \phi)(x) - \mathfrak{N}_\alpha^\pm(DL_0 \phi)(x)| &\leq \mathfrak{N}_\alpha^\pm(DL_k \phi - DL_0 \phi)(x) \\
&= \sup_{z \in \mathfrak{C}_\alpha^\pm(x)} \left| \int_{\Gamma} (\nabla G_k - \nabla G_0)(z-y) \cdot \nu(y) \phi(y) \, d\sigma(y) \right| \\
&\leq \sup_{z \in \mathfrak{C}_\alpha^\pm(x)} \int_{\Gamma} |(\nabla G_k - \nabla G_0)(z-y)| |\phi(y)| \, d\sigma(y) \\
&\leq \sqrt{3}C \|\phi\|_1 \leq \sqrt{3}C(1 + \sigma(\Gamma)).
\end{aligned}$$

Then,

$$\begin{aligned}
\|\mathfrak{N}_\alpha^\pm(DL_k \phi)\|_2 &\leq \|\mathfrak{N}_\alpha^\pm(DL_k \phi) - \mathfrak{N}_\alpha^\pm(DL_0 \phi)\|_2 + \|\mathfrak{N}_\alpha^\pm(DL_0 \phi)\|_2 \\
&\leq \sqrt{3\sigma(\Gamma)}C(1 + \sigma(\Gamma)) + C_\alpha =: \tilde{C}_\alpha.
\end{aligned}$$

Therefore, for a general  $\phi \in L^2(\Gamma)$  we get

$$(A.5) \quad \|\mathfrak{N}_\alpha^\pm(DL_k \phi)\|_2 \leq \tilde{C}_\alpha \|\phi\|_2.$$

With a slightly modified argument to the one of the proof of [18, Proposition 3.3.2], it follows that for all  $f \in \text{Lip}(\Gamma)$ , the nontangential limit  $\gamma^\pm DL_k f$  exists and satisfies the right equation of (3.8).

We will prove that the nontangential limits  $\gamma^\pm DL\phi(x)$  exists for  $\sigma$ -a.e.  $x \in \Gamma$  for real valued functions  $\phi$  but the result for the complex valued ones follows immediately by linearity. Take now any real-valued  $\phi \in L^2(\Gamma)$  and, using the density of  $\text{Lip}(\Gamma)$  in  $L^2(\Gamma)$ , we can take a sequence  $(f_n)_n \subset \text{Lip}(\Gamma)$  of real value functions that converge to  $\phi$  in  $L^2(\Gamma)$ . Then define, for any real-valued measurable function  $\psi$  on  $\Omega_\pm$  and for any  $x \in \Gamma$  such that  $x \in \mathfrak{C}_\alpha^\pm(x)$  (which by [18, Proposition 3.3.1], happens for  $\sigma$ -a.e.  $x \in \Gamma$ ),

$$(A.6) \quad \gamma_{\alpha,\text{inf}}^\pm \psi(x) := \liminf_{\substack{y \rightarrow x \\ y \in \mathfrak{C}_\alpha^\pm(x)}} \psi(x) \quad \text{and} \quad \gamma_{\alpha,\text{sup}}^\pm \psi(x) := \limsup_{\substack{y \rightarrow x \\ y \in \mathfrak{C}_\alpha^\pm(x)}} \psi(x),$$

and denote the resulting function on  $\Gamma$  by  $\gamma_{\alpha,\text{inf}}^\pm \psi$  and  $\gamma_{\alpha,\text{sup}}^\pm \psi$ , respectively. Then, using Equation (A.5)

$$\begin{aligned} \|\gamma_{\alpha,\text{inf}}^\pm DL_k \phi - \gamma_{\alpha,\text{inf}}^\pm DL_k f_n\|_2 &= \|\gamma_{\alpha,\text{inf}}^\pm DL_k \phi - \gamma_{\alpha,\text{sup}}^\pm DL_k f_n\|_2 \leq \|\gamma_{\alpha,\text{inf}}^\pm DL_k(\phi - f_n)\|_2 \\ &\leq \|\mathfrak{N}_\alpha^\pm DL_k(\phi - f_n)\|_2 \leq \tilde{C}_\alpha \|(\phi - f_n)\|_2 \end{aligned}$$

and

$$\begin{aligned} \|\gamma_{\alpha,\text{sup}}^\pm DL_k \phi - \gamma_{\alpha,\text{sup}}^\pm DL_k f_n\|_2 &= \|\gamma_{\alpha,\text{sup}}^\pm DL_k \phi - \gamma_{\alpha,\text{sup}}^\pm DL_k f_n\|_2 \leq \|\gamma_{\alpha,\text{sup}}^\pm DL_k(\phi - f_n)\|_2 \\ &\leq \|\mathfrak{N}_\alpha^\pm DL_k(\phi - f_n)\|_2 \leq \tilde{C}_\alpha \|(\phi - f_n)\|_2. \end{aligned}$$

This implies, by the convergence of  $(f_n)_n$  to  $\phi$  in  $L^2(\Gamma)$ , that for any  $\alpha > 0$  it is satisfied that  $\gamma_{\alpha,\text{inf}}^\pm \psi(x) = \gamma_{\alpha,\text{sup}}^\pm \psi(x)$  for  $\sigma$ -a.e.  $x \in \Gamma$ . Hence, for any  $\alpha > 0$  the limit  $\gamma_\alpha^\pm DL\phi(x)$  exists for  $\sigma$ -a.e.  $x \in \Gamma$ . Next, note that for any  $x \in \Gamma$  and  $\alpha > \beta > 0$ , if  $\gamma_\alpha^\pm DL\phi(x)$  exists then  $\gamma_\beta^\pm DL\phi(x)$  also exists and is equal to  $\gamma_\alpha^\pm DL\phi(x)$ . Thus, by taking a sequence of  $\alpha_n \rightarrow \infty$ , we obtain that for  $\sigma$ -a.e.  $x \in \Gamma$ , the nontangential limit  $\gamma^\pm DL\phi(x)$  exists.

Finally, by Remark 2.1, the nontangential limit  $\gamma^\pm DL\phi(x)$  is equal to the classical trace and therefore, by the density of  $\text{Lip}(\Gamma)$  in  $L^2(\Gamma)$ , the continuity of operator  $K_k : L^2(\Gamma) \rightarrow L^2(\Gamma)$  and  $K_k : H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ , we obtain that  $\gamma^\pm DL\phi(x)$  satisfies the right equation of (3.8) in the nontangential sense.  $\square$

PROPOSITION A.5 (Proposition 3.6.2). *For each  $\varphi \in H^1(\Gamma)$ , the nontangential limit  $\gamma^\pm \partial_j DL_k \varphi$  exists  $\sigma$ -a.e. on  $\Gamma$ , for each  $j = 1, 2, 3$ . Also,  $\tilde{C}_\alpha > 0$  can be taken such that,*

$$(A.7) \quad \|\mathfrak{N}_\alpha^\pm(\nabla DL_k \varphi)\|_2 \leq \tilde{C}_\alpha \|\varphi\|_{H^1(\Gamma)}.$$

Finally, the restriction of  $K_k$  to  $H^1(\Gamma)$  is bounded as an operator on  $H^1(\Gamma)$  and we get the mapping property,

$$K_k : H^1(\Gamma) \rightarrow H^1(\Gamma).$$

*Proof.* Adapting the proof of [18, Proposition 3.6.2], take any  $x \in \Omega_\pm$  and  $j = 1, 2, 3$ . Then,

$$\begin{aligned} \partial_j DL_k \varphi(x) &= - \int_\Gamma \sum_{l=1}^3 [\partial_j \partial_l G_k](x-y) \nu_l(y) \varphi(y) d\sigma(y) \\ &= \int_\Gamma \varphi(y) \left( k^2 G_k(x-y) \nu_j(y) + \sum_{l \neq j} [\partial_l \partial_l G_k](x-y) \nu_j(y) - [\partial_j \partial_l G_k](x-y) \nu_l(y) \right) d\sigma(y) \\ (A.8) \quad &= k^2 SL_k(\varphi \nu_j) + \sum_{l \neq j} \int_\Gamma \partial_{\tau_j, l} \varphi(y) \partial_l G_k(x-y) d\sigma(y), \end{aligned}$$

where the second inequality uses the fact that  $\Delta G + k^2 G = 0$  on  $\mathbb{R}^3 \setminus \{0\}$  and the third uses Lemma A.1. The first term in (A.8) is only weakly singular and can be handled as in Lemma A.3. As for the second term, recalling that the result is known for the case  $k = 0$  [31, Lemma 5.7], we are left to prove: (i) the existence of the nontangential limit a.e. on  $\Gamma$  and (ii) the domination of the  $L^2$ -norm of the nontangential maximal function by  $C\|\varphi\|_{H^1(\Gamma)}$ , this time for the quantity

$$\sum_{l \neq j} \int_{\Gamma} \partial_{\tau_{j,l}} \varphi(y) \left( \partial_l G_k(x-y) - \partial_l G_0(x-y) \right) d\sigma(y).$$

Now, both (i) and (ii) follow by dominated convergence from the second inequality in Lemma A.2.  $\square$

PROPOSITION A.6 (Proposition 3.6.4). *The operator  $S_k : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$  can be extended to the bounded linear operator*

$$S_k : H^{-1}(\Gamma) \rightarrow L^2(\Gamma),$$

which is the dual of  $(S_k)|_{H^1(\Gamma)}$ . Also, it is satisfied in the nontangential sense that

$$\gamma^\pm SL_k \psi = S_k \psi, \quad \sigma\text{-a.e. for every } \psi \in H^{-1}(\Gamma)$$

and, for every  $\alpha > 0$ , there exists a constant  $\tilde{C}_\alpha$  such that,

$$\|\mathfrak{N}_\alpha^\pm(SL_k \psi)\|_2 \leq \tilde{C}_\alpha \|\psi\|_{H^{-1}(\Gamma)}.$$

*Proof.* Note that by Lemma A.3 and Equation (3.7), for every  $\phi \in L^2(\Gamma)$ , we get  $(S_k)|_{H^1(\Gamma)}^*(\phi) = S_k(\phi)$ . Then, using again density of  $H^s(\Gamma)$  in  $H^t(\Gamma)$  for  $t < s$ , and Lemma A.3, we obtain that  $(S_k)|_{H^1(\Gamma)}^*$  is indeed an extension of  $S_k : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ . The rest of the proof follows from similar arguments to Proposition A.4.  $\square$

PROPOSITION A.7 (Proposition 6.3.1). *For any  $\phi \in L^2(\Gamma)$  we get*

$$\begin{aligned} \partial_\nu^\pm SL_k \phi &= \left( \mp \frac{1}{2} Id + K_k^* \right) \phi \\ &= \nu \cdot \gamma^\pm(\nabla SL_k \phi) \end{aligned}$$

*Proof.* The first equality is just the classical result [24, Equation (7.5)]. For the second equality, it can be shown, similarly as in the previous lemmas, that  $\gamma^\pm \circ \partial_j SL_k - \gamma^\pm \circ \partial_j SL_0$  defines a bounded linear operator from  $L^2(\partial)$  to itself, so that, by [18, Proposition 6.3.1],  $\gamma^\pm \circ \partial_j SL_k$  is as well bounded. Finally, we can show the result for Lipschitz functions, dividing the integral as in the proof of [18, Proposition 3.3.2] and also integrating against a test function; and finish the proof by a density argument.  $\square$

**A.2. Auxiliary regularity results.** In this section, we state and prove a couple of lemmas which are folklore but not easy to find in the literature.

LEMMA A.8. *For  $\Omega_+ \subset \mathbb{R}^3$  a bounded Lipschitz domain, the map  $S_0 : L^2(\Gamma) \rightarrow H^1(\Gamma)$  is an isomorphism. Moreover, for each  $f \in L^2(\Gamma)$ , the harmonic function  $SL_0 f$  has gradient with nontangential maximal function  $\mathfrak{N}_\alpha^\pm(\nabla SL_0 f) \in L^2(\Gamma)$ . In addition,  $SL_0 f$  lies in  $H^{3/2}(\Omega)$ .*

*Proof.* We adopt the notation of Lemma A.12:  $\Gamma_1, \dots, \Gamma_l$  are the components of  $\Gamma$  ordered so that the connected components  $O_1, \dots, O_l$  of  $\mathbb{R}^3 \setminus \bar{\Omega}$  satisfy  $O_1 = \text{Ext}(\Gamma_1)$  and  $O_j = \text{Int}(\Gamma_j)$  for  $j \neq 1$ . When  $l = 1$ , the lemma follows from [31, Theorem 3.3 & Corollary 3.5], except for the last statement. The latter is made in [21, Remark (b)], but that part of the argument based on interpolation which is given there is wrong. Instead, one can observe like these authors that  $x \mapsto |\partial_i \partial_j SL_0 f(x)| \text{dist}(x, \Gamma)^{1/2} \in L^2(\Omega)$  for  $1 \leq i, j \leq n$  (this follows from [12, Theorem 1] using Fubini's theorem), and appeal to [20, Theorem 4.1] to obtain that



$SL_0 f \in H^{3/2}(\Omega)$ . In the general case, let us write  $S_0(f_j)$  (resp.  $SL_0(f_j)$ ) for the single layer potential of  $f_j \in L^2(\Gamma_j)$  on  $\Gamma_j$  (resp. on  $\mathbb{R}^3 \setminus \Gamma_j$ ), and consider the map  $F : \Pi_j L^2(\Gamma_j) \rightarrow \Pi_j H^1(\Gamma_j)$  given by

$$F(f_1, \dots, f_l) := \left( S_0(f_j) + \sum_{k \neq j} \gamma_{\Gamma_j} SL_0(f_k) \right)_{j=1}^l.$$

Clearly, by the case  $l = 1$ , this map is of the form  $J + K$  where  $J(f_1, \dots, f_l) = (S_0(f_j))_{j=1}^l$  is invertible and  $K$  is a compact operator. Moreover  $F$  is injective, for if  $F(f_1, \dots, f_l) = 0$  then the harmonic function  $\sum_j SL_0(f_j)$  is identically zero in  $\Omega_{\pm}$  as it has vanishing nontangential limit a.e on  $\Gamma$  and  $L^2(\Gamma)$ -nontangential maximal function by the case  $l = 1$  and the smoothness of  $SL_0 f_j$  across  $\Gamma_k$  for  $k \neq j$ , so that we can apply [11, Theorems 1 & 3] (note that  $\sum_j SL_0(f_j)$  is zero at infinity by construction); taking the Laplacian, we conclude that all  $f_j$  are zero, thereby proving the announced injectivity. Thus, by a well-known theorem of F. Riesz,  $F$  is an isomorphism, and since  $S_0 f = \sum_j S_0(f_j)$  when we put  $f_j = f|_{\Gamma_j}$  the fact that  $\mathfrak{N}_{\alpha} |\nabla SL_0(f)|$  lies in  $L^2(\Gamma)$  and that  $SL_0 f \in H^{3/2}(\Omega)$  now follows immediately from the case  $l = 1$ .  $\square$

LEMMA A.9. *Let  $\Omega_+ \subset \mathbb{R}^3$  be a bounded Lipschitz domain,  $(\phi, \psi) \in L^2(\Gamma) \times H^{-1}(\Gamma)$  and  $u = \tilde{\mathcal{F}}(\phi, \psi)$ . If  $\gamma^+ u \in H^1(\Gamma)$  then  $u \in H^{3/2}(\Omega_+)$ , and if  $\gamma^- u \in H^1(\Gamma)$  then  $u \in H^{3/2}(\Omega_-)$ .*

*Proof.* We only prove the statement for  $\gamma^- u$ , as the case of  $\gamma^+ u$  is analogous but simpler. Let  $\mathbb{B} \subset \mathbb{R}^3$  be an open ball centered at 0 containing  $\overline{\Omega_+}$ , and let  $u' = u|_{\mathbb{B} \setminus \overline{\Omega_+}}$  which is square integrable by remark 3.2. By [20, Theorem B], there is a  $w \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$  such that  $\Delta w = -k^2 u'$  and  $\gamma_{\mathbb{B} \setminus \overline{\Omega_+}} w = 0$ . Note that  $\gamma_{\mathbb{B} \setminus \overline{\Omega_+}} u' \in H^1(\partial(\mathbb{B} \setminus \overline{\Omega_+}))$ , since  $\gamma^- u \in H^1(\Gamma)$  by assumption and  $u$  is analytic on  $\Omega_-$ . So, by Lemma A.8, there is a harmonic function  $v \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$  whose gradient has  $\mathfrak{N}_{\alpha} |\nabla SL_0(f)| \in L^2(\Gamma)$ , and whose nontangential limit a.e. on  $\Gamma$  is  $\gamma_{\mathbb{B} \setminus \overline{\Omega_+}} u'$ . Hence, as  $v + w \in H^{3/2}(\mathbb{B} \setminus \overline{\Omega_+})$ , it is enough to show that  $h := u' - v - w$  is the zero function. For this we shall prove that it lies in  $H^1(\mathbb{B} \setminus \overline{\Omega_+})$  and has zero trace; since it is harmonic by construction, this will achieve the proof. Now,  $h \in H^1(\mathbb{B} \setminus \overline{\Omega_+})$  if and only if  $u'$  does, and remark 3.2 together with the third inequality in Lemma A.2 entail in view of Lemma A.8 that  $u'$  is the sum of a harmonic function of the form  $SL_0 f$  with  $f \in L^2(\Gamma)$  (that lies in  $H^1(\mathbb{B} \setminus \overline{\Omega_+})$  plus a function with nontangentially bounded derivative (because  $x \mapsto 1/|x|$  is locally integrable in dimension 2). Altogether,  $u' \in H^1(\mathbb{B} \setminus \overline{\Omega_+})$ , and  $h \in H^1(\mathbb{B} \setminus \overline{\Omega_+})$  as well. Finally, the trace of  $w$  is zero and the nontangential limit of  $v$ , which is also its trace, is  $\gamma_{\mathbb{B} \setminus \overline{\Omega_+}} u'$ . Hence  $h$  has zero trace, as wanted.  $\square$

LEMMA A.10. *Let  $\Gamma \subset \mathbb{R}^3$  be the boundary of a bounded Lipschitz domain and let  $\{\Gamma_j\}_{j \in J}$  be its connected components. If  $\psi \in H^{-1}(\Gamma)$  is such that for every  $j \in J$ ,  $\langle \psi, 1_{\Gamma_j} \rangle = 0$ , then there exists a  $\varphi_{\psi} \in H^1(\Gamma)$  such that  $\Delta_{\Gamma} \varphi_{\psi} = \psi$ .*

*Proof.* Let  $Z$  denote the space  $\{\varphi \in H^1(\Gamma) : \text{for every } j \in J, \langle \varphi, 1_{\Gamma_j} \rangle = 0\}$  together with the inner product  $\langle \varphi, \tilde{\varphi} \rangle_Z := \langle \nabla_{\Gamma} \varphi, \nabla_{\Gamma} \tilde{\varphi} \rangle_{L^2(\Gamma)^3}$ . By the Poincaré inequality (obtained from its Euclidean version applied in a minimal system of finitely many charts  $(V_j, \Phi_j)$  with Lipschitz smooth image that cover  $\Gamma$  to bound  $\|\varphi - \int_{V_j \setminus (\cup_{k \neq j} V_k)} \varphi\|_{L^2(V_j)}$  by  $K_j \|\nabla \varphi\|_{L^2(V_j)}$  for each  $j$ ), one checks that  $Z$  is a Hilbert space. Pick  $\psi \in H^{-1}(\Gamma)$  such that, for every  $j \in J$ ,  $\langle \psi, 1_{\Gamma_j} \rangle = 0$ . Using the Poincaré inequality again, the function  $\varphi \mapsto -\langle \psi, \varphi \rangle$  belongs to the dual of  $Z$ . Thus there exists a  $\varphi_{\psi} \in Z$  such that, for every  $\varphi \in Z$ ,  $\langle \psi, \varphi \rangle = -\langle \overline{\varphi_{\psi}}, \varphi \rangle_Z$ . Take now any  $\varphi \in H^1(\Gamma)$  and let, for any  $j \in J$ ,  $\alpha_j = \sigma(\Gamma_j)^{-1} \langle \varphi, 1_{\Gamma_j} \rangle$ . Then,

$$\begin{aligned} \langle \psi, \varphi \rangle &= \left\langle \psi, \varphi - \sum_{j \in J} \alpha_j 1_{\Gamma_j} + \sum_{j \in J} \alpha_j 1_{\Gamma_j} \right\rangle = \left\langle \psi, \varphi - \sum_{j \in J} \alpha_j 1_{\Gamma_j} \right\rangle = - \left\langle \overline{\varphi_{\psi}}, \varphi - \sum_{j \in J} \alpha_j 1_{\Gamma_j} \right\rangle_Z \\ &= - \left\langle \nabla_{\Gamma} \varphi_{\psi}, \nabla_{\Gamma} \left( \varphi - \sum_{j \in J} \alpha_j 1_{\Gamma_j} \right) \right\rangle_{L^2(\Gamma)^3} = - \langle \nabla_{\Gamma} \varphi_{\psi}, \nabla_{\Gamma} \varphi \rangle_{L^2(\Gamma)^3} = \langle \Delta_{\Gamma} \varphi_{\psi}, \varphi \rangle, \end{aligned}$$

and hence  $\Delta_{\Gamma} \varphi_{\psi} = \psi$ .  $\square$

**A.3. Basic topological facts.** Using the fact that all surfaces embedded in  $\mathbb{R}^3$  are triangulable [22, Theorem 5.12], the following lemma can be found in [25, Corollary 74.2]. This is generally true for any connected compact hypersurface on  $\mathbb{R}^n$  and follows as a consequence of *Alexander duality* [17, Corollary 3.45], but the proof is more involved.

LEMMA A.11. *Take a connected surface  $\Gamma \subset \mathbb{R}^3$  which is compact as a topological space.*

*Then the set  $\mathbb{R}^3 \setminus \Gamma$  has two connected components; one bounded, which we will denote by  $\text{Int}(\Gamma)$ , and another unbounded, which we will denote by  $\text{Ext}(\Gamma)$ .*

*Furthermore,  $\partial(\text{Int}(\Gamma)) = \Gamma = \partial(\text{Ext}(\Gamma))$ .*

We say that a set  $\Gamma \subset \mathbb{R}^3$  is *locally a Lipschitz graph* if for every  $x \in \Gamma$  there exists an open ball  $\mathbb{B} \subset \mathbb{R}^3$ , a  $h > 0$ , a plane  $H \subset \mathbb{R}^3$  passing through  $s$  and with a normal unit vector  $\nu$ , and a real-valued Lipschitz continuous function  $g$  on  $H$  such that the set defined as

$$C := \{x + t\nu : x \in \mathbb{B} \cap H, -h < t < h\},$$

satisfies:

$$C \cap \Gamma = \{x + t\nu : x \in \mathbb{B} \cap H, t = g(x)\}.$$

LEMMA A.12. *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain. Then,  $\Gamma$  has finitely many connected components, say  $\Gamma_1, \dots, \Gamma_l$ , each of which is locally a Lipschitz graph in  $\mathbb{R}^3$ .*

*Moreover, the connected components of  $\mathbb{R}^3 \setminus \overline{\Omega}$  consist of  $l$  Lipschitz domains  $O_1, \dots, O_l$ , and with a suitable ordering  $O_1 = \text{Ext}(\Gamma_1)$  while  $O_j = \text{Int}(\Gamma_j)$  for  $j \neq 1$ .*

*Proof.* The connected components  $\Omega \subset \mathbb{R}^3$  are finite in number; otherwise indeed, there would exist a sequence  $(\Omega_k)_k$  of such components, with  $\Omega_k \cap \Omega_j = \emptyset$  for  $k \neq j$ . Then, we could construct a sequence  $(x_k)_k \in \Omega_k$  such that  $x_k$  remains at bounded distance from  $\Gamma_k \subset \Gamma$ , hence  $x_k$  would be bounded and extracting a subsequence if necessary we might assume that  $x_k$  converges in  $\mathbb{R}^3$  to some  $y$ . However, this is impossible for  $y$  cannot lie in  $\Omega$  since the connected components of the latter are open, nor can it lie in  $\mathbb{R}^3 \setminus \overline{\Omega}$ , and it cannot belong to  $\Gamma$  either because, as  $\Gamma$  is a compact Lipschitz manifold which is locally a Lipschitz graph, each  $x \in \Gamma$  has a neighborhood whose intersections with both  $\Omega$  and  $\Gamma$  are connected. Consequently, by compactness,  $\Gamma$  has finitely many connected components, say  $\Gamma_1, \dots, \Gamma_l$ , and each  $\Gamma_j$  is locally a Lipschitz graph in  $\mathbb{R}^3$ .

As  $\Omega$  is connected by assumption, for each  $j \in \{1, \dots, l\}$  one of the following is true; either  $\Omega \subset \text{Int}(\Gamma_j)$ , so that  $\overline{\Omega} \subset \overline{\text{Int}(\Gamma_j)}$  and then, using Lemma A.11,  $\text{Ext}(\Gamma_j) \subset \mathbb{R}^3 \setminus \overline{\Omega}$ ; or else  $\Omega \subset \text{Ext}(\Gamma_j)$  and then, analogously,  $\text{Int}(\Gamma_j) \subset \mathbb{R}^3 \setminus \overline{\Omega}$ . Since there is exactly one unbounded connected component of  $\mathbb{R}^3 \setminus \overline{\Omega}$ , say  $O_1$ , it must contain  $\text{Ext}(\Gamma_j)$  for all  $j$  such that  $\Omega \subset \text{Int}(\Gamma_j)$ ; let us enumerate these  $j$  as  $j_1, \dots, j_m$ . For  $1 \leq i, k \leq m$ , it holds that  $\text{Int}(\Gamma_{j_i}) \cap \text{Int}(\Gamma_{j_k}) \neq \emptyset$  because  $\Omega$  lies in this intersection, and since the  $\Gamma_j$  are disjoint one of these interiors is included in the other, say  $\text{Int}(\Gamma_{j_i}) \subset \text{Int}(\Gamma_{j_k})$ . But if  $j_i \neq j_k$ , then  $\Gamma_{j_k} \subset \text{Ext}(\Gamma_{j_i})$  and the latter is contained in  $O_1$ , a contradiction. Consequently,  $m = 1$  and  $\Omega$  lies interior to exactly one of the  $\Gamma_j$ , say  $\Gamma_1$ . Necessarily then,  $O_1 = \text{Ext}(\Gamma_1)$  because  $O_1$  cannot strictly contain  $\text{Ext}(\Gamma_1)$  without containing a point of  $\Gamma_1$ , which is impossible. Likewise,  $\Omega \subset \text{Ext}(\Gamma_j)$  for  $j \neq 1$  and then  $\text{Int}(\Gamma_j)$  is a connected component of  $\mathbb{R}^3 \setminus \overline{\Omega}$ . Next, the closure of every bounded connected component of  $\mathbb{R}^3 \setminus \overline{\Omega}$  must meet some  $\Gamma_j$ , and necessarily  $j \neq 1$  for each point of  $\Gamma_1$  has a neighborhood included in  $\overline{O_1} \cup \Omega$ , by the local Lipschitz graph property. Hence, this connected component meets  $\text{Int}(\Gamma_j)$  for some  $j \neq 1$ , therefore it must coincide with  $\text{Int}(\Gamma_j)$ . Finally, due to Lemma A.11 and the definition of locally Lipschitz graphs, for each  $j \in \{1, \dots, l\}$  both  $\text{Int}(\Gamma_j)$  and  $\text{Ext}(\Gamma_j)$  are Lipschitz domains.  $\square$

**Acknowledgement** The authors are grateful to an anonymous referee for bringing up reference [29] to their attention.

- [1] M. AUSSAL, Y. BOUKARI, AND H. HADDAR, *Data completion method for the helmholtz equation via surface potentials for partial cauchy data*, Inverse Problems, 36 (2020), p. 055012, <https://doi.org/10.1088/1361-6420/ab730c>, <https://dx.doi.org/10.1088/1361-6420/ab730c>.
- [2] L. BARATCHART AND C. GERHARDS, *On the recovery of core and crustal components of geomagnetic potential fields*, SIAM J. Appl. Math., 77 (2017), pp. 1756–1780.
- [3] L. BARATCHART, C. GERHARDS, AND A. KEGELES, *Decomposition of  $l^2$ -vector fields on lipschitz surfaces: characterization via null-spaces of the scalar potential*, SIAM Journal on Mathematical Analysis, (2021).
- [4] L. BARATCHART, J. LEBLOND, AND M. NEMAIRE, *Silent sources and equivalent  $l^p$ -magnetizations*. <https://hal.science/hal-03915548v1>, 2022.
- [5] L. BARATCHART, D. PEI, AND T. QIAN, *Hardy-Hodge decomposition of vector fields on compact Lipschitz hypersurfaces*. working paper or preprint, Sept. 2020, <https://inria.hal.science/hal-02936934>.
- [6] L. BARATCHART, C. VILLALOBOS GUILLÉN, D. P. HARDIN, M. C. NORTHINGTON, AND E. B. SAFF, *Inverse potential problems for divergence of measures with total variation regularization*, Foundations of Computational Mathematics, (2019).
- [7] R. J. BLAKELY, *Potential Theory in Gravity & Magnetic Applications*, Cambridge University Press, 1996.
- [8] L. E. F. B.P. WEISS, E. A. LIMA AND F. BAUDENBACHER, *Paleomagnetic analysis using squid microscopy*, J. Geophys Res., (2007).
- [9] F. CAKONI, I. DE TERESA, H. HADDAR, AND P. MONK, *Nondestructive testing of the delaminated interface between two materials*, SIAM Journal on Applied Mathematics, 76 (2016), pp. 2306–2332, <https://doi.org/10.1137/16M1064167>, <https://arxiv.org/abs/https://doi.org/10.1137/16M1064167>.
- [10] D. COLTON AND R. KRESS, *Inverse acoustic and electromagnetic scattering theory*, Springer, third ed., 2012.
- [11] B. E. DAHLBERG, *On the poisson integral for lipschitz and  $c^1$  domains*, Studia Math., 66 (1979), pp. 13–24.
- [12] B. E. DAHLBERG, *Weighted norm inequalities for the lusin area integral and the nontangential maximal functions for functions harmonic in a lipschitz domain*, Studia Math., (1980), pp. 279–314.
- [13] B. DELOURME, H. HADDAR, AND P. JOLY, *Approximate models for wave propagation across thin periodic interfaces*, Journal de Mathématiques Pures et Appliquées, 98 (2012), pp. 28–71, <https://doi.org/https://doi.org/10.1016/j.matpur.2012.01.003>, <https://www.sciencedirect.com/science/article/pii/S0021782412000128>.
- [14] F. GESZTESY AND M. MITREA, *Generalized robin boundary conditions, robin-to-dirichlet maps, and krein-type resolvent formulas for schrödinger operators on bounded lipschitz domains*, Perspectives in Partial Differential Equations, Harmonic Analysis and Applications, (2008), p. 105–173, <https://doi.org/10.1090/pspum/079/2500491>.
- [15] P. GRISVARD, *Elliptic problems in nonsmooth domains*, Pitman, 1985.
- [16] P. HAJLASZ, *Sobolev spaces on an arbitrary metric space*, Potential Anal., (1996), p. 403–415, <https://doi.org/10.1007/BF002754754>.
- [17] A. HATCHER, *Algebraic topology*, Cambridge University Press, 2002.
- [18] S. HOFMANN, M. MITREA, AND M. TAYLOR, *Singular integrals and elliptic boundary problems on regular semmes-kenig-toro domains*, International Mathematics Research Notices, (2009), <https://doi.org/10.1093/imrn/rnp214>.
- [19] V. ISAKOV, *Inverse problems for partial differential equations*, vol. 127 of AMS, Springer, 2017.
- [20] D. JERISON AND C. KENIG, *The inhomogeneous dirichlet problem in lipschitz domains*, Journal of Functional Analysis, 130 (1995), p. 161–219, <https://doi.org/10.1006/jfan.1995.1067>.
- [21] D. S. JERISON AND C. E. KENIG, *The Neumann problem on Lipschitz domains*, Bulletin of the American Mathematical Society, 4 (1981), p. 203–207, <https://doi.org/10.1090/s0273-0979-1981-14884-9>.
- [22] J. M. LEE, *Introduction to topological manifolds*, Springer, 2000.
- [23] P. MATTILA, *Geometry of sets and measures in Euclidean spaces*, vol. 44 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.
- [24] W. C. H. MCLEAN, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, 2000.
- [25] J. R. MUNKRES, *Elements Of Algebraic Topology*, CRC Press, 1984.
- [26] J.-C. NEDELEC, *Acoustic and electromagnetic equations: integral representations for harmonic problems*, Springer, 2011.
- [27] J. SARVAS, *Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem*, Physics in Medicine & Biology, 32 (1987).
- [28] L. SCHWARTZ, *Théorie des distributions*, Hermann, 1978.
- [29] R. TORRES AND G. WELLAND, *The helmholtz equation and transmission problems with lipschitz interfaces*, Indiana University Mathematics Journal, 42 (1993), pp. 1457–1485.
- [30] F. TRÈVES, *Topological vector spaces, distributions and kernels*, Academic Press, 1967.
- [31] G. VERCHOTA, *Layer potentials and regularity for the dirichlet problem for laplace's equation in lipschitz domains*, Journal of Functional Analysis, 59 (1984).
- [32] W. P. ZIEMER, *Weakly differentiable functions*, Springer, 1989.