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# Some explicit solutions and bounds to the generalized Lyapunov matrix inequality

Mario Spirito and Daniele Astolfi

## Abstract

We review the generalized Lyapunov matrix inequality for both continuous-time and discrete-time cases and we provide a solution based on a modification of the Jordan normal form. The proposed construction allows to obtain explicit bounds to the decay rate and the inertia of the solution to the generalized Lyapunov matrix inequality.

## 1. Introduction

In this article, we study the solution to the generalized Lyapunov matrix inequality

$$PA + A^\top P \preceq 2\alpha P,$$

in which  $A$  is a given matrix (possibly non-Hurwitz),  $\alpha$  is a real number, and  $P$  is an unknown symmetric matrix which may be non-positive definite. Such an inequality arises in many control problems and plays a fundamental role in  $p$ -dominance [3] and  $k$ -contraction theory [11, 2]. The generalized Lyapunov matrix inequality has been deeply studied and it is well-known that the inertia of  $P$ , namely the number of eigenvalues with positive, negative, and zero real part, is directly related to the inertia of the matrix  $(A - \alpha I)$ , see, for instance, [7, 5, 9, 10]. In this work, which has been inspired by the approach proposed in [8], we provide a new explicit solution based on the solely algebraic properties of the matrix  $A$ , namely on its eigenvalues and (generalized) eigenvectors. The focus of this work is to show that in the case of non-diagonalizable matrices, we have limitations in the choices of  $\alpha$ . In particular, we show that we cannot select  $\alpha$  equal to an eigenvalue with non-unitary geometric multiplicity and that selecting  $\alpha$  arbitrarily close to it provides a solution  $P$  which is ill-conditioned. These facts are provided thanks to a modified Jordan's normal form. A similar analysis is then developed in the case of the discrete generalized Lyapunov matrix inequality

$$A^\top P A \preceq \rho P.$$

We believe that these new results allow to better understand the decay rate of stable systems possessing Jordan blocks of very large dimensions, e.g., in large-scale systems obtained via the spatial discretization of PDEs.

The paper is structured as follows. In the next section, we introduce the preliminary concepts related to Jordan's normal form and the definition of the inertia of a matrix. Then, in Section 3, we introduce the modification of the Jordan block normal form as proposed in [8] for continuous and discrete time. We then analyze in Section 4 the generalized Lyapunov inequality for continuous-time systems. In Section 5 we extend the

analysis to generalized Lyapunov inequalities in discrete-time. The conclusions are then given in Section 6.

## 2. Preliminaries

### 2.1. Notation

We denote with  $\mathbb{R}$  (resp.  $\mathbb{C}$ ) the set of real (resp. complex) numbers, and with  $i$  the imaginary unit. Given  $s \in \mathbb{C}$ ,  $s = a + ib$ , we denote with  $\Re\{s\} = a$  its real part. Given a matrix  $P \in \mathbb{R}^{n \times n}$ , we denote by  $\sigma(P)$  its spectrum, by  $\sigma_{\max}(P)$  the largest eigenvalue and  $\sigma_{\min}(P)$  its smallest, while  $\sigma_i(P)$  is the  $i$ -th eigenvalue of the matrix  $P$ . We recall the following definitions of inertia of a matrix, see [9].

**Definition 1.** *The inertia of a matrix  $P$  with respect to the imaginary axis ( $c$ -inertia) is defined by the triplet of integers*

$$\text{In}_c(A) = \{\pi_-(P), \pi_0(P), \pi_+(P)\}$$

where  $\pi_-(P)$ ,  $\pi_0(P)$  and  $\pi_+(P)$ , denote the numbers of eigenvalues of  $P$ , counted with their algebraic multiplicities, with negative, zero, and positive real part, respectively.

**Definition 2.** *The inertia of a matrix  $P$  with respect to the unit circle ( $d$ -inertia) is defined by the triplet of integers*

$$\text{In}_d(A) = \{\pi_{<1}(P), \pi_1(P), \pi_{>1}(P)\}$$

where  $\pi_{<1}(P)$ ,  $\pi_0(P)$ , and  $\pi_{>1}(P)$ , denote the numbers of eigenvalues of  $P$ , counted with their algebraic multiplicities, which are inside, on, and outside the unit circle.

Given any  $n \in \mathbb{N}$  we define the following matrices

$$\mathbb{J}_n := I_n + N_n = \begin{bmatrix} 1 & 1 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & & 1 \end{bmatrix}, \quad N_n = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad (1)$$

where  $I_n$  is the identity matrix of dimension  $n$  and  $N_n$  is the matrix with ones on the upper diagonal and zeros everywhere

else. Given a number  $\varepsilon \in \mathbb{R}$  and a natural number  $n \in \mathbb{N}$ , we denote

$$D_n(\varepsilon) := \begin{cases} \text{diag}(1, \varepsilon, \dots, \varepsilon^{n-1}), & \text{if } \varepsilon \neq 0, \\ I_n, & \text{if } \varepsilon = 0. \end{cases} \quad (2)$$

Then, with the previous definition, the following identity

$$N_n D_n(\varepsilon) = \varepsilon D_n(\varepsilon) N_n \quad (3)$$

holds for any  $\varepsilon \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Similarly, for a matrix  $A \in \mathbb{R}^{r \times r}$  we can define

$$D_n(A) := \begin{cases} \text{diag}(I_r, A, \dots, A^{n-1}), & \text{if } A \neq 0, \\ I_m, & \text{if } A = 0, \end{cases} \quad (4)$$

and in this case, the following identity can be verified

$$D_n(A)^{-1} (N_n \otimes I_r) D_n(A) = N_n \otimes A. \quad (5)$$

## 2.2. Some explicit eigenvalues

From [6, Ex. 7.2.5], we recall that given a matrix  $A \in \mathbb{R}^{n \times n}$ , with  $a \neq 0$  and  $c \neq 0$ , of the form

$$A = \begin{bmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{bmatrix} = bI_n + aN_n + cN_n^\top, \quad (6)$$

an explicit formula for its eigenvalues  $\lambda_i$  is given by

$$\lambda_i = b + 2a \sqrt{\frac{c}{a}} \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}.$$

For this paper, we only deal with the case  $c = a$ ; leading to the explicit eigenvalues form, as also exploited in [1], given by

$$\lambda_i = b + 2a \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}. \quad (7)$$

## 2.3. Jordan normal form

Given a matrix  $A$  of dimensions  $n \times n$  we suppose, without loss of generality, that its eigenvalues are ordered in decreasing order with respect to the real part, namely

$$\Re\{\lambda_1\} \geq \Re\{\lambda_2\} \geq \dots \geq \Re\{\lambda_n\}.$$

Let  $m \leq n$  be the total number of linearly independent (non-generalized) eigenvectors  $T_i^1 \neq 0$  associated with an eigenvalue  $\lambda_i \in \sigma(A)$ ,  $i = 1, \dots, m$ , such that

$$AT_i^1 = \bar{\lambda}_i T_i^1 \quad \forall i = 1, \dots, m.$$

**Definition 3** (Jordan blocks dimension). *For each  $i \in \{1, \dots, m\}$ , we define the values  $g_i \geq 1$  satisfying  $\sum_{i=1}^m g_i = n$ , such that there exist  $g_i - 1$  linearly independent generalized eigenvectors  $T_i^k \neq 0$ , for  $k = 2, \dots, g_i$ , associated to the corresponding eigenvalue  $\bar{\lambda}_i$  and satisfying*

$$(A - \bar{\lambda}_i I) T_i^k = T_i^{k-1} \quad \forall k = 2, \dots, g_i.$$

The introduced notation allows us to determine in advance the number of distinct Jordan blocks  $m$  and their relative dimensions  $g_i$ , when the matrix  $A$  is transformed into its Jordan form  $J$ , see, e.g. [5].

**Lemma 1.** *Let  $A$  be a  $n \times n$  matrix, let  $m$  be the number of distinct<sup>1</sup> eigenvalues  $\bar{\lambda}_i$  with associated Jordan block dimension  $g_i$  so that  $\sum_{i=1}^m g_i = n$ . Then, there exists a  $T \in \mathbb{R}^{n \times n}$  such that*

$$T^{-1}AT = J, \quad \begin{cases} J := \text{blkdiag}(J_{\bar{\lambda}_1}, \dots, J_{\bar{\lambda}_m}), \\ J_{\bar{\lambda}_i} := \bar{\lambda}_i I_{g_i} + N_{g_i}, \quad i = 1, \dots, m. \end{cases} \quad (8)$$

For complex eigenvalues, one can always consider a real Jordan form in which a Jordan block associated with a complex pair of eigenvalues  $\bar{\lambda}_{i,1/2} = a_i \pm \mathbf{i}b_i$ , with  $g_i - 1$  pairs of associated generalized eigenvectors, is given by

$$J_{\bar{\lambda}_i} := I_{g_i} \otimes \bar{\Lambda}_i + N_{g_i} \otimes I_2 = \begin{bmatrix} \bar{\Lambda}_i & I_2 & 0 & \dots & 0 \\ 0 & \bar{\Lambda}_i & I_2 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & & \bar{\Lambda}_i & I_2 \\ 0 & \dots & & 0 & \bar{\Lambda}_i \end{bmatrix}_{2g_i \times 2g_i} \quad (9)$$

where  $\bar{\Lambda}_i$  is defined as

$$\bar{\Lambda}_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$$

and one can verify that  $\sigma(\bar{\Lambda}_i) = \{a + \mathbf{i}b, a - \mathbf{i}b\}$ . Note also that

$$\bar{\Lambda}_i + \bar{\Lambda}_i^\top = 2a_i I_2 = 2\Re\{\bar{\lambda}_{i,1/2}\} I_2. \quad (10)$$

## 3. Modified Jordan Form

In this section, we present two modifications of the standard normal Jordan form that are instrumental for the following analysis. The two proposed forms refer to continuous-time and to the discrete-time dynamics, respectively.

### 3.1. Modified continuous-time Jordan normal form

Given a Hurwitz matrix  $A$ , one may ask if an identity Lyapunov matrix, i.e.,  $P = I_n$ , in the Jordan coordinates (8) is a solution to the Lyapunov matrix inequality, namely if  $J + J^\top \preceq -\alpha I$  for some  $\alpha > 0$ . The answer is in general negative. Indeed, one can compute the eigenvalues of  $J + J^\top$  for each Jordan block form  $J_{\bar{\lambda}_i}$  applying (7), which gives

$$\sigma_{\max}(J_{\bar{\lambda}_i} + J_{\bar{\lambda}_i}^\top) = 2\Re\{\bar{\lambda}_i\} + 2 \cos\left(\frac{\pi}{g_i + 1}\right)$$

which is negative only for  $\Re\{\lambda_i\} \leq -\cos(\pi/(g_i + 1)) \leq -1$ .

As a consequence, we introduce now a modified Jordan form, denoted in this article as  $\mathbb{J}$ . The main idea is to re-scale the

<sup>1</sup>Distinct in the sense that they have distinct Jordan blocks. Eigenvalues in the standard sense are always denoted as  $\lambda$ .

Jordan blocks  $J_{\bar{\lambda}_i}$  using the property (3). In particular, for a non-zero eigenvalue  $\bar{\lambda} \neq 0$  with associate Jordan block dimension  $g$ , we can use the matrix<sup>2</sup>  $D_g(\bar{\lambda})$  defined in (2) to obtain

$$D_g^{-1}(\bar{\lambda})J_{\bar{\lambda}}D_g(\bar{\lambda}) = \bar{\lambda}\mathbb{J}_g \quad (11)$$

with the matrix  $\mathbb{J}_i$  defined according to (1). Hence, by defining the matrix

$$\mathbb{D} := \text{blkdiag}\left(D_{g_1}(\bar{\lambda}_1), \dots, D_{g_m}(\bar{\lambda}_m)\right) \quad (12)$$

we introduce the matrix  $\mathbb{J}$  defined as

$$\mathbb{J} := \mathbb{T}^{-1}A\mathbb{T}, \quad \mathbb{T} := T\mathbb{D}, \quad (13)$$

in which  $T$  satisfies (8). One can verify, using the property (11) that  $\mathbb{J}$  as the following form

$$\mathbb{J} = \text{blkdiag}\left(\bar{\lambda}_1\mathbb{J}_{g_1}, \dots, \bar{\lambda}_m\mathbb{J}_{g_m}\right), \quad (14)$$

with the matrix  $\mathbb{J}_{g_i}$  defined as in (1), for each  $i = 1, \dots, m$ . As we shall see in the next section, the matrix  $\mathbb{J}$  is now in a more convenient form to study Lyapunov matrix inequalities. Note that in the case of complex eigenvalues, one has to define a slightly different change of coordinates in order to deal with the generalized Jordan block (9). In particular, by using the definition (4) one can verify the following identity

$$\mathbb{J}_{g_i} \otimes \bar{\Lambda}_i = (I_{g_i} + N_{g_i}) \otimes \bar{\Lambda}_i = D_{g_i}^{-1}(\bar{\Lambda}_i)J_{g_i}(\bar{\Lambda}_i)D_{g_i}(\bar{\Lambda}_i)$$

which gives

$$\mathbb{J}_{g_i} \otimes \bar{\Lambda}_i = \begin{bmatrix} \bar{\Lambda}_i & \bar{\Lambda}_i & 0 & \dots & 0 \\ 0 & \bar{\Lambda}_i & \bar{\Lambda}_i & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & & \bar{\Lambda}_i & \bar{\Lambda}_i \\ 0 & \dots & & 0 & \bar{\Lambda}_i \end{bmatrix}. \quad (15)$$

We conclude this section by highlighting a certain number of properties of the matrix  $\mathbb{J}_n$ . To this end, recall that

$$\cos(x\pi) = -\cos((1-x)\pi), \quad \forall x \in [0, \frac{\pi}{2}].$$

**Lemma 2.** *For any  $n \geq 1$ , the matrix  $\mathbb{J}_n$  satisfies the following properties.*

1. *The eigenvalues of  $\mathbb{J}_n + \mathbb{J}_n^\top$  satisfies*

$$\sigma_i(\mathbb{J}_n + \mathbb{J}_n^\top) = 2 \left[ 1 + \cos\left(\pi \frac{i}{n+1}\right) \right], \quad i = \{1, \dots, n\}.$$

2. *The matrix  $\mathbb{J}_n + \mathbb{J}_n^\top$  satisfies*

$$2 \left[ 1 + \cos\left(\pi \frac{n}{n+1}\right) \right] I_n \preceq \mathbb{J}_n + \mathbb{J}_n^\top \preceq 2 \left[ 1 + \cos\left(\frac{\pi}{n+1}\right) \right] I_n.$$

<sup>2</sup>We remark that the change of coordinates (11) has the same form as in high-gain observers theory, see, e.g., [4].

3. *For any real number  $\varepsilon \in (0, 1]$ , the following holds*

$$\begin{aligned} & 2 \left( 1 + \varepsilon \cos\left(\pi \frac{n}{n+1}\right) \right) \\ & \preceq D_n(\varepsilon)^{-1} \left[ \mathbb{J}_n + \mathbb{J}_n^\top \right] D_n(\varepsilon) \preceq \\ & 2 \left( 1 + \varepsilon \cos\left(\frac{\pi}{n+1}\right) \right) \end{aligned}$$

4. *The spectrum of  $\mathbb{J}_n^\top \mathbb{J}_n$  satisfies*

$$\sigma(\mathbb{J}_n^\top \mathbb{J}_n) \subset \sigma(\mathbb{J}_{2n} + \mathbb{J}_{2n}^\top).$$

5.  $\mathbb{J}_n^\top \mathbb{J}_n \preceq \mathbb{J}_n + \mathbb{J}_n$ .

*Proof.* The explicit formula of the eigenvalues of  $\mathbb{J} + \mathbb{J}^\top$  follows from (7), showing items 1 and 2.

We show now item 3. By property 3, we have

$$D_n(\varepsilon)^{-1} \left[ \mathbb{J}_n + \mathbb{J}_n^\top \right] D_n(\varepsilon) = 2I_n + \varepsilon(N_n + N_n^\top),$$

and by (7) we have

$$\begin{aligned} & 2 \left( 1 + \varepsilon \cos\left(\pi \frac{n}{n+1}\right) \right) \\ & \preceq D_n(\varepsilon)^{-1} \left[ \mathbb{J}_n + \mathbb{J}_n^\top \right] D_n(\varepsilon) \preceq \\ & 2 \left( 1 + \varepsilon \cos\left(\frac{\pi}{n+1}\right) \right). \end{aligned}$$

We show now item 4. Let  $W_n$  be the anti-diagonal matrix of dimension  $n$ , namely

$$W_n = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}.$$

Note that  $W_n^{-1} = W_n$ . Then, the following identity can be verified from the definition of  $\mathbb{J}_i$ :

$$\mathbb{J}_{2n} + \mathbb{J}_{2n}^\top = \begin{pmatrix} \mathbb{J}_n + \mathbb{J}_n^\top & W_n(I_n - N_n^\top N_n) \\ (I_n - N_n N_n^\top)W_n^\top & \mathbb{J}_n + \mathbb{J}_n^\top \end{pmatrix}.$$

Applying the change of coordinates

$$R = \begin{pmatrix} W_n & 0 \\ W_n^\top & I_n \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} W_n & 0 \\ -W_n^\top & I_n \end{pmatrix}$$

one can verify that

$$R(\mathbb{J}_{2n} + \mathbb{J}_{2n}^\top)R^{-1} = \begin{pmatrix} \mathbb{J}_n + \mathbb{J}_n^\top - N_n^\top N_n & E_1 \\ 0 & E_2 \end{pmatrix}$$

for some matrices  $E_1, E_2$ . Using the identity

$$\mathbb{J}_n + \mathbb{J}_n^\top - N_n^\top N_n = \mathbb{J}_n^\top \mathbb{J}_n$$

and the previous block diagonal form we immediately obtain

$$\sigma(\mathbb{J}_{2n} + \mathbb{J}_{2n}^\top) = \sigma(\mathbb{J}_n^\top \mathbb{J}_n) \cup \sigma(E_2)$$

concluding the proof.

Finally, for item 5 we note that  $N_n^\top N_n \leq I_n$ . Hence

$$\begin{aligned} (I_n + N_n)^\top (I_n + N_n) & \preceq I_n + N_n^\top + N_n + N_n^\top N_n \\ & \preceq 2I_n + N_n^\top + N_n = \mathbb{J}_n^\top + \mathbb{J}_n. \end{aligned}$$

□

### 3.2. Modified discrete-time Jordan normal form

Similarly to the continuous-time case, one can ask if the identity matrix is a suitable solution to the discrete-time Lyapunov matrix inequality for a Schur matrix  $A$  in the Jordan normal form (or in the new modified Jordan normal form), namely if one the following  $J^\top J - I \preceq \rho I$  or  $\mathbb{J}^\top \mathbb{J} - I \preceq \rho I$  holds for some  $\rho \in (0, 1)$ . And in both cases, the answer is in general negative, see the later discussion in Section 5. Actually, it can be verified that the second inequality (i.e. concerning the modified Jordan normal form  $\mathbb{J}$ ) holds if all the eigenvalues have a modulo smaller or equal<sup>3</sup> to  $\frac{1}{2}$ . For this reason, we introduce a second modification of the Jordan form. In particular, for a block  $J_g(\lambda)$  associated with an eigenvalue  $\lambda$  with associate Jordan block dimension  $g$ , we use the following identity

$$D_g \left( \frac{1-|\lambda|}{3} \right)^{-1} J_g(\lambda) D_g \left( \frac{1-|\lambda|}{3} \right) = \lambda I_g + \frac{1-|\lambda|}{3} N_g.$$

So by defining the matrix

$$\mathcal{D} := \text{blkdiag} \left( D_{g_1} \left( \frac{1-|\bar{\lambda}_1|}{3} \right), \dots, D_{g_m} \left( \frac{1-|\bar{\lambda}_m|}{3} \right) \right)$$

we give the second modified Jordan form

$$\mathcal{J} := \mathcal{T}^{-1} A \mathcal{T}, \quad \mathcal{T} := T \mathcal{D}, \quad (16)$$

in which  $T$  satisfies (8). The resulting matrix  $\mathcal{J}$  has the following form

$$\begin{aligned} \mathcal{J} &= \text{blkdiag} \left( \mathcal{J}_{g_1}(\bar{\lambda}_1), \dots, \mathcal{J}_{g_m}(\bar{\lambda}_m) \right), \\ \mathcal{J}_{g_i}(\bar{\lambda}_i) &= \bar{\lambda}_i I_{g_i} + \frac{1}{3} (1 - |\bar{\lambda}_i|^2) N_{g_i}. \end{aligned} \quad (17)$$

Equivalently for the real matrix realization in the case of complex conjugate eigenvalues, considering for  $\bar{\lambda}_{i,2} = a_i \pm i b_i$  we define

$$\bar{\Lambda}_i = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}, \quad \bar{\Delta}_i = \frac{1}{3} (1 - a_i^2 - b_i^2) I_2, \quad (18)$$

then the change of coordinates modifies by defining

$$D_{g_i}(\bar{\Lambda}_i) = \text{blkdiag} \left( I_2, \bar{\Delta}_i, \dots, \bar{\Delta}_i^{g_i-1} \right)$$

and thus the blocks of  $\mathcal{J}$  associated with complex conjugate eigenvalues will read as

$$\begin{aligned} \mathcal{J}_{g_i}(\bar{\Lambda}_i) &= I_{g_i} \otimes \bar{\Lambda}_i + N_{g_i} \otimes \bar{\Delta}_i \\ &= \begin{bmatrix} \bar{\Lambda}_i & \bar{\Delta}_i & 0 & \dots & 0 \\ 0 & \bar{\Lambda}_i & \bar{\Delta}_i & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & & \bar{\Lambda}_i & \bar{\Delta}_i \\ 0 & \dots & & 0 & \bar{\Lambda}_i \end{bmatrix}. \end{aligned} \quad (19)$$

In Section 5 we will show how this new form can be employed to find suitable solutions to the generalized discrete Lyapunov matrix inequality.

<sup>3</sup>It is somehow bizarre to notice the parallelism with the continuous-time case in which  $J + J^\top \preceq 0$  if  $\Re\{\lambda_i\} \leq -1$ ,  $i = 1, \dots, n$ .

## 4. Continuous-time generalized Lyapunov inequality

Given a matrix  $A$  of  $c$ -inertia  $\{n - q, 0, q\}$ , we are interested in studying the solutions to the generalized Lyapunov matrix inequality

$$PA + A^\top P \preceq 2\alpha P, \quad \alpha \in \mathbb{R}, \quad P = P^\top, \quad (20)$$

in which  $\alpha \in \mathbb{R}$  may be a positive or negative real number and  $P$  is a symmetric matrix of some  $c$ -inertia. We recall the following result.

**Lemma 3.** *Given a matrix  $A \in \mathbb{R}^{n \times n}$ , a real constant  $\mu \in \mathbb{R}$ , and an integer  $q \in \{0, \dots, n\}$ , the following statements are equivalent.*

1. *A has  $q$  eigenvalues with real part larger than  $\mu$  and  $n - q$  eigenvalues with real part smaller than  $\mu$ .*
2. *The matrix  $A - \mu I$  has inertia  $\{n - q, 0, q\}$ .*
3. *There exists a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with  $c$ -inertia  $\{q, 0, n - q\}$  satisfying*

$$A^\top P + PA \prec 2\mu P \quad (21)$$

4. *Given a symmetric positive definite matrix  $Q \succ 0$  there exists a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with  $c$ -inertia  $\{q, 0, n - q\}$  satisfying*

$$(A - \mu I)^\top P + P(A - \mu I) = -Q.$$

*Proof.* The eigenvalues of  $A - \alpha I$  are the shifted eigenvalues  $\lambda_1 - \mu, \dots, \lambda_n - \mu$ , where we recall that  $\lambda_i$  are order such that

$$\Re(\lambda_1) \geq \Re(\lambda_2) \geq \dots \geq \Re(\lambda_n).$$

If  $A$  has inertia  $\{n - q, 0, q\}$ , it implies that  $\Re(\lambda_{q+1}) - \alpha < 0$ . This shows that (1)  $\Leftrightarrow$  (2). The implication (3)  $\Leftrightarrow$  (1) is due to [7, Lemma 1, Section 3]. The implication (2)  $\Leftrightarrow$  (4) is due to [9, Theorem 2.5]. Finally, we have that (4)  $\Rightarrow$  (3). Indeed, (4) implies

$$A^\top P + PA = -2\mu P - Q \prec -2\mu P$$

because  $Q$  is positive definite.  $\square$

The objective of this article is to better characterize the admissible set of solution to (21), characterizing both  $c$ -inertia of  $P$  and admissible values of  $\mu$  based on the solely algebraic properties of  $A$ . Furthermore, quasi-explicit solutions based on the aforementioned Jordan forms will be provided. We highlight that our results are of particular interest in the presence of non-trivial Jordan blocks, where the characterization of  $\mu$  is more difficult due to polynomial behaviour of solutions.

### 4.1. Solutions fixing the inertia

In this section, we are interested in providing a constructive solution and explicit bounds for the inequality (20) characterizing the admissible values of  $\alpha$  and  $c$ -inertia of  $P$ . In particular, we show that if the matrix  $A$  possesses a  $c$ -inertia  $\{n - q, 0, q\}$ , then the inequality (20) admits a solution  $P$  of inertia  $\{q, 0, n - q\}$  and we characterize the set of admissible values for  $\alpha$ . The first result is based on the modified continuous-time Jordan normal form introduced in Section 3.1.

**Proposition 1.** Let  $A$  be a matrix with  $c$ -inertia  $\{n - q, 0, q\}$ , where  $q \in \mathbb{N}$  is the number of eigenvalues with positive real part, namely

$$\Re\{\lambda_1\} \geq \dots \geq \Re\{\lambda_q\} > 0 > \Re\{\lambda_{q+1}\} \geq \dots \geq \Re\{\lambda_n\},$$

and define  $m_q \in \{0, \dots, m\}$  the number of eigenvalues  $\bar{\lambda}_i$  with positive real part, i.e.  $\Re\{\bar{\lambda}_i\} > 0$  for all  $i = 1, \dots, m_q$ . Then, the matrix  $P$  with  $c$ -inertia  $\{q, 0, n - q\}$  defined as

$$P := \mathbb{T}^{-\top} \mathbb{I} \mathbb{T}^{-1}, \quad \mathbb{I} := \text{blkdiag}(-I_q, I_{n-q}),$$

with  $\mathbb{T}$  defined as in (13), is a solution to the generalized Lyapunov matrix inequality (20) for any  $\alpha \in [\underline{\alpha}, \bar{\alpha}]$  with

$$\begin{aligned} \bar{\alpha} &:= \min_{i \in \{1, \dots, m_q\}} \left\{ \Re\{\bar{\lambda}_i\} \left[ 1 + \cos\left(\pi \frac{g_i}{g_i + 1}\right) \right] \right\}, \\ \underline{\alpha} &:= \max_{i \in \{m_q + 1, \dots, m\}} \left\{ \Re\{\bar{\lambda}_i\} \left[ 1 + \cos\left(\pi \frac{g_i}{g_i + 1}\right) \right] \right\}, \end{aligned} \quad (22)$$

where  $g_i$  are the associate Jordan block dimension, of the eigenvalue  $\bar{\lambda}_i$ .

*Proof.* First, note that  $P$  has the same inertia of  $\mathbb{I}$ . For simplicity, this proof we consider only real eigenvalues, but the proof can be easily generalized using the modified real Jordan block as explained in Section 3.1. Consider inequality (20) and pre and post-multiply by  $\mathbb{T}^\top$  and  $\mathbb{T}$ . We obtain

$$\begin{aligned} \mathbb{T}^\top (PA + A^\top P - 2\alpha P) \mathbb{T} &\leq 0, \\ \mathbb{T}^\top \mathbb{T}^{-\top} \mathbb{I} \mathbb{T}^{-1} A \mathbb{T} + \mathbb{T}^\top A^\top \mathbb{T}^{-\top} \mathbb{I} \mathbb{T}^{-\top} \mathbb{T} - 2\alpha \mathbb{I} &\leq 0, \\ \mathbb{I} \mathbb{J} + \mathbb{J}^\top \mathbb{I} - 2\alpha \mathbb{I} &\leq 0. \end{aligned}$$

Due to the block diagonal structure of  $\mathbb{J}$  and  $\mathbb{I}$ , the previous inequality gives a set of  $m$  independent inequalities

$$\begin{aligned} -\bar{\lambda}_i (\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top) + 2\alpha \mathbb{I}_{g_i} &\leq 0, \quad i = 1, \dots, m_q, \\ \bar{\lambda}_i (\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top) - 2\alpha \mathbb{I}_{g_i} &\leq 0, \quad i = m_q + 1, \dots, m, \end{aligned} \quad (23)$$

where  $m \leq n$  is the number of Jordan blocks, and  $0 \leq m_q \leq m$  is the number of Jordan blocks associated to a positive eigenvalue. For the first  $m_q$  inequalities in (23), positiveness of  $\bar{\lambda}_i$  yields

$$2\alpha \mathbb{I}_{g_i} \leq \bar{\lambda}_i (\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top) \Rightarrow \alpha \leq \frac{1}{2} \Re\{\bar{\lambda}_i\} \sigma_{\min}(\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top),$$

for  $i = 1, \dots, m_q$ , which gives, by using item 1 of Lemma 2,  $\alpha \leq \bar{\alpha}$ , with  $\bar{\alpha}$  defined in (22). Conversely, since  $\Re\{\bar{\lambda}_i\} < 0$  for  $i = m_q + 1, \dots, m$ , the remaining  $m - m_q$  inequalities in (23) are verified with

$$\bar{\lambda}_i (\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top) \leq 2\alpha \mathbb{I}_{g_i} \Rightarrow \alpha \geq \frac{1}{2} \Re\{\bar{\lambda}_i\} \sigma_{\min}(\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top)$$

which gives, by using again item 1 of Lemma 2,  $\alpha \geq \underline{\alpha}$ , with  $\underline{\alpha}$  defined in (22) thus concluding the proof. Note that for a complex pair of eigenvalues we can use the definition of the complex Jordan block in (15). As a consequence, the corresponding inequality in (23) is modified as  $2\alpha \mathbb{I}_{2g_i} \leq \bar{\Lambda}_i \otimes (\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top)$  or  $\bar{\Lambda}_i \otimes (\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top) \leq 2\alpha \mathbb{I}_{2g_i}$  which gives the same bounds in view of (10).  $\square$

Note that in the case of a matrix  $A$  with inertia  $\{n, 0, 0\}$ , namely with all eigenvalues with negative real part, inequality (20) boils down to the standard Lyapunov matrix inequality which has a solution  $P$  positive definite if and only if  $(A - \alpha I_n)$  is Hurwitz. Thanks to the modified continuous-time Jordan form defined in (13), it can be shown that the Lyapunov matrix admits as a solution of the form  $P = \mathbb{T}^{-\top} \mathbb{T}^{-1}$ .

**Corollary 1.** Suppose that  $A$  is Hurwitz and let  $m$  be the number of distinct Jordan blocks. Then, inequality (20) is satisfied with a  $P$  of  $c$ -inertia  $\{0, 0, n\}$  defined as

$$P := \mathbb{T}^{-\top} \mathbb{T}^{-1}, \quad \alpha := \min_{i \in \{1, \dots, m\}} \left\{ \Re\{\bar{\lambda}_i\} \left[ 1 + \cos\left(\pi \frac{g_i}{g_i + 1}\right) \right] \right\} \quad (24)$$

in which  $\mathbb{T}$  is defined in (13).

We remark that Proposition 1 (or equivalently Corollary 1 for Hurwitz matrices) establishes that a quasi-identity matrix (namely containing only  $+1$  or  $-1$  values in the diagonal) is always a solution to the generalized Lyapunov matrix inequality for a matrix expressed in the modified Jordan normal form.

However, such a choice doesn't necessarily optimize the selection of  $\alpha$ . In particular, if one wants to select  $\alpha$  as close as possible to the convergence rate, e.g. as close as possible to the slowest stable eigenvalue of the matrix  $A$  in the Hurwitz case, one needs to modify the previous selection of  $P$ . In particular, it is shown that  $\alpha$  can be chosen arbitrarily close to  $\lambda_q$  or  $\lambda_{q+1}$  at the price of obtaining ill-conditioned solutions  $P$ . The following theorem mostly relies on the property of item 3 in Lemma 2.

**Theorem 1.** Let  $A$  be a matrix with  $c$ -inertia  $\{n - q, 0, q\}$ , and let  $q \in \mathbb{N}$  the number of eigenvalues with positive real part, namely

$$\Re\{\lambda_1\} \geq \dots \geq \Re\{\lambda_q\} > 0 > \Re\{\lambda_{q+1}\} \geq \dots \geq \Re\{\lambda_n\},$$

and define  $m_q \in \{0, \dots, m\}$  the number of eigenvalues  $\bar{\lambda}_i$  with positive real part, i.e.  $\Re\{\bar{\lambda}_i\} > 0$  for all  $i = 1, \dots, m_q$ . For any arbitrarily small  $\delta \in (0, 1)$  there exists  $\varepsilon^* > 0$  such that, for any  $\varepsilon \in (0, \varepsilon^*)$ , the following  $P$  with  $c$ -inertia  $\{q, 0, n - q\}$

$$\begin{aligned} P &:= \mathbb{T}^{-\top} \mathbb{I} \mathbb{D}_\varepsilon^{-2} \mathbb{T}^{-1}, \\ \mathbb{I} &:= \text{blkdiag}(-I_q, I_{n-q}), \\ \mathbb{D}_\varepsilon &:= \text{blkdiag}(D_{g_1}(\varepsilon), \dots, D_{g_m}(\varepsilon)), \end{aligned}$$

where  $\mathbb{T}$  is defined as in (13) and  $g_i$  is the associate Jordan block dimension of the eigenvalue  $\bar{\lambda}_i$ , is a solution to the generalized Lyapunov matrix inequality (20) for any  $\alpha \in \mathbb{R}$  satisfying

$$(1 - \delta) \Re\{\lambda_{q+1}\} \leq \alpha \leq (1 - \delta) \Re\{\lambda_q\}. \quad (25)$$

*Proof.* Following the same steps performed in the proof of Theorem 1 one obtains from

$$\mathbb{D}_\varepsilon \mathbb{T}^\top (PA + A^\top P - 2\alpha P) \mathbb{T} \mathbb{D}_\varepsilon \leq 0$$

the following set of inequalities

$$\begin{aligned} -\bar{\lambda}_i D_{g_i}^{-1}(\varepsilon) \left[ \mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top \right] D_{g_i}(\varepsilon) + 2\alpha I_{g_i} &\leq 0, \\ i &= 1, \dots, m_q, \\ \bar{\lambda}_i D_{g_i}^{-1}(\varepsilon) \left[ \mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top \right] D_{g_i}(\varepsilon) - 2\alpha I_{g_i} &\leq 0, \\ i &= m_q + 1, \dots, m. \end{aligned} \quad (26)$$

Using item 3 of Lemma 2, it is readily seen that the inequalities (26) are satisfied for all  $i$  if  $\alpha$  is chosen such that  $\underline{\alpha}_\varepsilon \leq \alpha \leq \bar{\alpha}_\varepsilon$  with

$$\begin{aligned} \bar{\alpha}_\varepsilon &\leq \min_{i \in \{1, \dots, m_q\}} \left\{ \left( 1 + \varepsilon \cos \left( \pi \frac{g_i}{g_i+1} \right) \right) \Re \{ \bar{\lambda}_i \} \right\}, \\ \underline{\alpha}_\varepsilon &\leq \max_{i \in \{m_q+1, \dots, m\}} \left\{ \left( 1 + \varepsilon \cos \left( \pi \frac{g_i}{g_i+1} \right) \right) \Re \{ \bar{\lambda}_i \} \right\}. \end{aligned}$$

As a consequence, by selecting  $\varepsilon^* = \min \{ \varepsilon_+^*, \varepsilon_-^* \}$ , with

$$\begin{aligned} \varepsilon_+^* &= \min_{i \in \{1, \dots, m_q\}} \left\{ \cos \left( \pi \frac{g_i}{g_i+1} \right)^{-1} \left[ \frac{|\Re \{ \lambda_q \}|}{|\Re \{ \bar{\lambda}_i \}|} (1 - \delta) - 1 \right] \right\}, \\ \varepsilon_-^* &= \min_{i \in \{m_q+1, \dots, m\}} \left\{ \cos \left( \pi \frac{g_i}{g_i+1} \right)^{-1} \left[ \frac{|\Re \{ \lambda_{q+1} \}|}{|\Re \{ \bar{\lambda}_i \}|} (1 - \delta) - 1 \right] \right\}, \end{aligned}$$

one can verify that

$$\bar{\alpha}_\varepsilon \leq (1 - \delta) \Re \{ \lambda_q \}, \quad \underline{\alpha}_\varepsilon \leq (1 - \delta) \Re \{ \lambda_{q+1} \},$$

for any  $\varepsilon \in (0, \varepsilon^*]$  thus concluding the proof. For complex eigenvalues, similar bounds are obtained following the same steps of the proof of Theorem 1.  $\square$

The previous theorem provides an explicit solution to the generalized Lyapunov matrix inequality which can be optimized in the set of admissible values for  $\alpha$ . We don't claim however that our choice is the optimal one in terms of conditioning number  $\sigma_{\max}(P)/\sigma_{\min}(P)$  of  $P$ , because of the presence of  $\varepsilon$  that may be very small. For instance, one could improve the previous design by selecting a different  $\varepsilon_i$  for each Jordan block. In Section 4.3, numerical tests compare the proposed solution with the one obtained by a standard optimization algorithm.

#### 4.2. Solutions fixing the rate of convergence

In the previous section we characterized a solution to the inequality (20) by fixing the  $c$ -inertia of the matrix  $P$  and providing a set of admissibility for  $\alpha$ . Now, we investigate the opposite, namely we fix a value for  $\alpha$  and we characterize the admissible values of the  $c$ -inertia for  $P$  focusing on a degenerate case in which  $\alpha$  is selected exactly equal to the real part of an eigenvalue of  $A$ . To this end, we focus to the case in which  $A$  is Hurwitz and  $\alpha$  is selected equal to real part of the first eigenvalue. The result can be easily generalized by shifting the matrix  $A$ . Now, recalling that the eigenvalues are ordered in decreasing order, namely

$$0 \geq \Re \{ \lambda_1 \} \geq \dots \geq \Re \{ \lambda_n \}.$$

we denote with  $\mathcal{S}_1$  the set of eigenvalues  $\bar{\lambda}_i$  with real part equal to  $\Re \{ \lambda_1 \}$  and with  $k_1$  the cardinality of  $\mathcal{S}_1$ . Then, we denote with  $g_i$  the dimension of the  $i$ -th Jordan block associated with any (complex) eigenvalue in  $\mathcal{S}_1$ , and we let  $k$  be the sum of all dimensions of the relative Jordan blocks, i.e.,  $k = \sum_{i=1}^{k_1} g_i$ .

**Proposition 2.** *Let  $A \in \mathbb{R}^{n \times n}$  be a Hurwitz matrix. Then, selecting  $\alpha = \Re \{ \lambda_1 \}$ , inequality (20) admits solutions  $P$  of any  $c$ -inertia of the form  $\{ \ell - j, k - \ell, n - (k - \ell) - (\ell - j) \}$ , with  $\ell = \sum_{i=1}^{k_1} \ell_i$  and  $j = \sum_{i=1}^{k_1} j_i$  where  $\ell_i \in \{0, 1\}$  and  $\ell_i - j_i \geq 0$ ,  $i = 1, \dots, k_1$ .*

*Proof.* For simplicity in the proof, we suppose that  $A$  has only real eigenvalues and moreover that the associate Jordan block dimension of the first eigenvalue  $\bar{\lambda}_1$  is  $g_1 = n$ . The general proof follows by similar arguments. Let  $T$  be the transformation defined in (8) associated with the Jordan form. Pre and post-multiplying inequality (20) by  $T^\top$  and  $T$  respectively, yields

$$\begin{aligned} T^\top P T T^{-1} A T + T^\top A^\top T^{-1} T^\top P T &\preceq 2\alpha T^\top P T \\ P'_{g_1} J + J^\top P'_{g_1} &\preceq 2\alpha P' \end{aligned}$$

with  $P'_{g_1} = T^\top P T \in \mathbb{R}^{g_1 \times g_1}$  being a positive definite symmetric matrix. Recalling that  $J_{g_1} = \lambda_1 I_{g_1} + N_{g_1}$ , if one selects  $\alpha = -\lambda_1$  one obtains

$$P'_{g_1} N_{g_1} + N_{g_1}^\top P'_{g_1} \preceq 0 \quad (27)$$

which has solution  $P'_{g_1}$  with inertia  $\{ \ell_1 - j_1, g_1 - \ell_1, j_1 \}$ , with  $\ell_1 \in \{0, 1\}$  being a degree of freedom for this block and  $j_1$  is chosen accordingly, satisfying the inequality  $\ell_1 - j_1 \geq 0$ . This inertia choice is also valid for  $g_1 = 1$ , in particular, any  $\ell_1 \in \{0, 1\}$  allows us to choose any desired inertia for the relative  $P'$  block, i.e., because  $P'_{g_1}$  is multiplied by zero it can have any inertia of the form  $\{1, 0, 0\}$ ,  $\{0, 1, 0\}$  or  $\{0, 0, 1\}$ , and the inequality is preserved. The generalization to any other particular case is just a simple extension of the above reasoning and some simple algebra provides the formulae in the proposition.  $\square$

Note that whenever  $A$  is a diagonalizable matrix, we have the following trivial result.

**Corollary 2.** *Let  $A$  be a diagonalizable Hurwitz matrix. Let  $\mathcal{S}_1$  be the set of eigenvalues with real part equal to  $\Re \{ \lambda_1 \}$  and  $k_1$  the cardinality of  $\mathcal{S}_1$ . Then, selecting  $\alpha = \Re \{ \lambda_1 \}$ , inequality (20) admits solutions  $P$  of any  $c$ -inertia of the form  $\{ \ell - j, k_1 - \ell, n - k_1 + j \}$ , with  $\ell = \sum_{i=1}^{k_1} \ell_i$  and  $j = \sum_{i=1}^{k_1} j_i$  where  $\ell_i \in \{0, 1\}$  and  $\ell_i - j_i \geq 0$ ,  $i = 1, \dots, k_1$ .*

*Proof.* The proof follows from a variation of the previous Proposition 2. In particular, following the same steps, the inequality in (27) boils down to  $0 * P'_{g_1} \preceq 0$ , where  $P'_{g_1}$  is a scalar value, varying the overall inertia according to such a choice. Such a value can be either positive, negative, or zero. The same arguments apply to any other eigenvalue with the same real part.  $\square$

Note that, when  $A$  is Hurwitz, and its largest real part eigenvalue  $\lambda_1$  has an associated Jordan Block which is non-trivial,

one cannot obtain  $\alpha = \lambda_1$ , if its associate Jordan block dimension is strictly larger than 1, as shown in the following “trivial” lemma.

**Corollary 3.** *Let  $A$  be a Hurwitz matrix and suppose that at least one of the eigenvalues with the largest real part has different algebraic and geometric multiplicities<sup>4</sup>. Then, the solution  $P$  to (20) has  $c$ -inertia  $\{0, 0, n\}$  if and only if  $\alpha > \Re\{\lambda_1\}$ .*

#### 4.3. Numerical tests

We consider as numerical test a matrix  $A$  with dimension 9 with 3 distinct eigenvalues  $\bar{\lambda}$ , i.e.,  $\{-0.5, -2, -4\}$  with associated Jordan block dimensions  $(g_1, g_2, g_3) = (3, 5, 1)$ . That is, according to the value convergence rate formula (24) we have

$$\bar{\alpha} = -0.5 \left( 1 + \cos \left( \frac{3}{4} \pi \right) \right) \approx -0.1464$$

with  $P = \mathbb{T}^{-\top} \mathbb{T}^{-1}$ , thus the condition number of  $P$  is equivalent to the condition number of  $\mathbb{T}$ . By comparing the solution  $P_{LMI}$  of LMI solver (getlmis()) implemented in the Robust Control Toolbox<sup>TM</sup> of Matlab version R2022a with the given  $\alpha = \bar{\alpha}$ , we get approximatively, over 5000 comparisons, 99% of the times a better condition number of  $\mathbb{T}$  than that of  $P_{LMI}$ . The numerical testing details are summarized in Table 1.

# tests	%	average $\mu(\mathbb{T}^{\top} \mathbb{T})$	average $\mu(P_{LMI})$
5000	99%	$1.4839e+03$	$1.0417e+06$

Moreover, with  $\alpha = -0.45$ , we get, over 5000 comparisons, 100% of the times a better condition number of  $\mathbb{T}D_{\varepsilon}$  than that of  $P_{LMI}$ . The numerical testing details are summarized in Table 2.

# tests	%	average $\mu(D_{\varepsilon} \mathbb{T}^{\top} \mathbb{T} D_{\varepsilon})$	average $\mu(P_{LMI})$
5000	100%	$6.2556e+03$	$1.3463e+16$

## 5. Discrete-time generalized Lyapunov inequality

Given a matrix  $A$  with  $d$ -inertia  $\{n - q, 0, q\}$ , we consider in this section the generalized discrete-time Lyapunov matrix inequality

$$A^{\top} P A \preceq \rho P, \quad \rho \in (0, \infty), \quad (28)$$

where the matrix  $P$  has  $c$ -inertia  $\{q, 0, n - q\}$ . One may think of using the same kind of modified Jordan Block form  $\mathbb{J}$  approach exploited for the continuous time case. In the following lemma, we show that actually, this form has strong limitations for the discrete-time case and that we need to exploit a different modified Jordan Block form as introduced in Section 3.2.

<sup>4</sup>This implies that (at least) one of the Jordan blocks corresponding to these eigenvalues has dimension larger than one. Note that this statement includes the case of multiple complex eigenvalues with the same real part and different imaginary parts.

**Lemma 4.** *Let  $A$  be a Schur stable matrix and  $m$  the total number of Jordan blocks, associated with eigenvalues  $\bar{\lambda}$  ordered in decreasing modulo order, and suppose in addition that all the eigenvalues have modulo smaller than  $1/2$ , namely*

$$\frac{1}{2} \geq |\lambda_1| \geq \dots \geq |\lambda_m| > 0.$$

Then, inequality (28) is satisfied with

$$P := \mathbb{T}^{-\top} \mathbb{T}^{-1}, \quad \rho := \max_{i \in \{1, \dots, m\}} \left\{ |\bar{\lambda}_i|^2 \left[ 1 + \cos \left( \frac{\pi}{2g_i + 1} \right) \right] \right\} \quad (29)$$

with  $\mathbb{T}$  defined as in (13), and moreover  $\rho < 1$ .

*Proof.* With the choice of  $P = \mathbb{T}^{-\top} \mathbb{T}^{-1}$ , inequality (28) gives

$$\mathbb{T}^{\top} A \mathbb{T}^{-\top} \mathbb{T}^{-1} A \mathbb{T} \leq \rho I_n, \quad \mathbb{J}^{\top} \mathbb{J} \leq \rho I_n,$$

further yielding to

$$|\bar{\lambda}_i|^2 \mathbb{J}_{g_i}^{\top} \mathbb{J}_{g_i} \leq \rho I_{g_i}, \quad i = 1, \dots, m.$$

Applying item 5 and 2 of Lemma 2 show that the previous set of inequality is always satisfied with  $\rho$  selected as in (29). Next, noting that

$$2|\bar{\lambda}_i|^2 \left[ 1 + \cos \left( \frac{\pi}{2g_i + 1} \right) \right] < 4|\bar{\lambda}_i|^2$$

for any  $g_i \geq 1$ , we are guaranteed to obtain  $\rho < 1$  only if  $|\bar{\lambda}_i| < \frac{1}{2}$  for any  $i = 1, \dots, m$ .  $\square$

Motivated, thus, by the restrictive conditions of the previous lemma, we develop the modified Jordan form for the discrete-time case  $\mathcal{J}$  that can be obtained via the transformation matrix  $\mathcal{T}$ , previously introduced. We state the following general result.

**Theorem 2.** *Let  $A$  be a matrix with  $d$ -inertia  $\{n - q, 0, q\}$  and let  $q \in \mathbb{N}$  the number of eigenvalues outside the unit circle, namely*

$$|\lambda_1| \geq \dots \geq |\lambda_q| > 1 > |\lambda_{q+1}| \geq \dots \geq |\lambda_n| > 0.$$

and define  $m_q$  the number of eigenvalues  $\bar{\lambda}_i$  outside the unit circle, with  $i \in \{1, \dots, m\}$ . Then, the matrix  $P$  with  $c$ -inertia  $\{q, 0, n - q\}$  defined as

$$P = \mathcal{T}^{-\top} \mathbb{I} \mathcal{T}^{-1}, \quad \mathbb{I} = \text{blkdiag}(-I_q, I_{n-q}),$$

with  $\mathcal{T}$  defined as in (16), verifies the generalized Lyapunov matrix inequality (28) for any  $\rho \in [\underline{\rho}, \bar{\rho}]$  with

$$\begin{aligned} \bar{\rho} &= \frac{1}{9} \min_{i \in \{1, \dots, m_q\}} \left\{ 9\bar{\lambda}_i^2 - 6(\bar{\lambda}_i^2 - 1) \cos \left( \frac{\pi}{g_i + 1} \right) \right\} \\ \underline{\rho} &= \frac{1}{9} \max_{i \in \{m_q + 1, \dots, m\}} \left\{ 8\bar{\lambda}_i^2 + 6(1 - \bar{\lambda}_i^2) \cos \left( \frac{\pi}{g_i + 1} \right) \right\}. \end{aligned} \quad (30)$$



*Proof.* First, note that  $P$  has the same inertia of  $\mathbb{I}$ . For simplicity, this proof we consider only real eigenvalues, but the proof can be easily generalized using the modified real Jordan block as explained in Section 3.2. Consider inequality (28) and pre and post-multiply by  $\mathcal{T}^\top$  and  $\mathcal{T}$ . We obtain

$$\begin{aligned} \mathcal{T}^\top A^\top \mathcal{T}^{-\top} \mathbb{I} \mathcal{T}^{-1} A \mathcal{T} &\preceq \rho \mathcal{T}^\top \mathcal{T}^{-\top} \mathbb{I} \mathcal{T}^{-1} \mathcal{T} \\ \mathcal{J}^\top \mathbb{I} \mathcal{J} &\preceq \rho \mathbb{I} \end{aligned}$$

Using the block triangular structure of  $\mathcal{J}$ , which is preserved in the multiplication of the left term, one obtains a set of  $m$  independent inequalities

$$\begin{aligned} -\mathcal{J}_i^\top \mathcal{J}_i + \rho I_{g_i} &\preceq 0, & i = 1, \dots, m_q, \\ \mathcal{J}_i^\top \mathcal{J}_i - \rho I_{g_i} &\preceq 0, & i = m_q + 1, \dots, m. \end{aligned}$$

Thus, for the first case, we have

$$\rho I_{g_i} \preceq \mathcal{J}_i^\top \mathcal{J}_i \Rightarrow \rho \leq \sigma_{\min}(\mathcal{J}_i^\top \mathcal{J}_i)$$

for  $i = 1, \dots, m_q$ , and the generic modified Jordan block associated with  $\bar{\lambda}_i$ , whose  $|\bar{\lambda}_i| > 1$ , for Discrete time system have the form

$$\begin{aligned} \mathcal{J}_i^\top \mathcal{J}_i &= \bar{\lambda}_i^2 I - \frac{\bar{\lambda}_i^2 - 1}{3} (N^\top + N) + \frac{(\lambda^2 - 1)^2}{9} N^\top N \\ &\geq \bar{\lambda}_i^2 I - \frac{\lambda^2 - 1}{3} (N^\top + N) =: M \end{aligned}$$

and the eigenvalues of  $M$  can be compute explicitly as in (7), i.e.,

$$\sigma_k(M) = \bar{\lambda}_i^2 - \frac{2}{3} (\bar{\lambda}_i^2 - 1) \cos\left(\frac{\pi}{g_i + 1}\right), \quad k = 1, \dots, g_i$$

from which we can determine the minimum eigenvalue as considered in the first line of (30).

For the second case, instead, we have

$$\mathcal{J}_i^\top \mathcal{J}_i \preceq \rho I_{g_i} \Rightarrow \rho \geq \sigma_{\max}(\mathcal{J}_i^\top \mathcal{J}_i)$$

for  $i = m_q + 1, \dots, m$ , and the generic modified Jordan block associated with  $\bar{\lambda}_i$ , whose  $|\bar{\lambda}_i| < 1$ , for Discrete time system have the form

$$\begin{aligned} \mathcal{J}_i^\top \mathcal{J}_i &= \bar{\lambda}_i^2 I + \frac{1 - \bar{\lambda}_i^2}{3} (N^\top + N) + \frac{(1 - \bar{\lambda}_i^2)^2}{9} N^\top N \\ &= \bar{\lambda}_i^2 I + \frac{1 - \bar{\lambda}_i^2}{3} (N^\top + N) + \frac{(1 - \bar{\lambda}_i^2)^2}{9} I \\ &\quad - \frac{(1 - \bar{\lambda}_i^2)^2}{9} \text{diag}(1, 0, \dots, 0) \\ &\leq \left(\bar{\lambda}_i^2 + \frac{1 - \bar{\lambda}_i^2}{9}\right) I + \frac{1 - \bar{\lambda}_i^2}{3} (N^\top + N) =: M \end{aligned}$$

and the eigenvalues of  $M$  can be compute explicitly as in (7), i.e.,

$$\sigma_k(M) = \frac{1}{9} (8\bar{\lambda}_i^2 + 1) + \frac{2}{3} (1 - \bar{\lambda}_i^2) \cos\left(\frac{\pi}{g_i + 1}\right), \quad k = 1, \dots, g_i$$

from which we can determine in closed form the maximum eigenvalues as involved in the second line of (30).  $\square$

We can obtain a parallel result as the one presented in Corollary 1. As in the continuous-time case, we show now that it is always possible to construct a matrix  $P$ , by slightly modifying the one given in Theorem 2, so that to obtain  $\bar{\rho}$ , resp.  $\underline{\rho}$ , as close as desired to the slowest unstable eigenvalue  $|\lambda_q|$ , resp. the slowest stable eigenvalue  $|\lambda_{q+1}|$ . For Schur stable system, we have the following corollary.

**Corollary 4.** *Suppose that  $A$  is Shur and let  $m$  be the number of distinct Jordan blocks. Then, inequality (28) is satisfied with*

$$\begin{aligned} P &= \mathcal{T}^{-\top} \mathcal{T}^{-1}, \quad \text{and} \\ \rho &= \min_{i \in \{1, \dots, m\}} \left\{ \frac{8|\bar{\lambda}_i|^2 + 1}{9} + \frac{2}{3} (1 - |\bar{\lambda}_i|^2) \cos\left(\frac{\pi}{g_i + 1}\right) \right\}, \end{aligned} \quad (31)$$

in which  $\mathcal{T}$  is defined in (16), and moreover  $\rho < 1$ .

Note that this corollary is just a particular case of Theorem 2, thus its proof is omitted.

**Theorem 3.** *Let  $A$  be a matrix with  $d$ -inertia  $\{n - q, 0, q\}$ , let  $q \in \mathbb{N}$  the number of eigenvalues outside the unit circle, namely*

$$|\lambda_1| \geq \dots \geq |\lambda_q| > 1 > |\lambda_{q+1}| \geq \dots \geq |\lambda_n| > 0.$$

and define  $m_q$  the number of eigenvalues  $\bar{\lambda}_i$  outside the unit circle, with  $i \in \{1, \dots, m\}$ . For any  $\delta \in (0, 1)$  there exists  $\varepsilon^* > 0$  such that, for any  $\varepsilon \in (0, \varepsilon^*)$ , the following  $P$  with  $c$ -inertia  $\{q, 0, n - q\}$

$$\begin{aligned} P &= \mathcal{T}^{-\top} \mathbb{D}_\varepsilon^{-2} \mathcal{T}^{-1}, \\ \mathbb{I} &= \text{blkdiag}(-I_q, I_{n-q}), \\ \mathbb{D}_\varepsilon &= \text{blkdiag}(D_{g_1}(\varepsilon), \dots, D_{g_m}(\varepsilon)), \end{aligned}$$

with  $\mathcal{T}$  defined as in (16), is a solution to the generalized Lyapunov matrix inequality (28) for any  $\rho \in (0, \infty)$  satisfying

$$(1 - \delta) |\lambda_{q+1}| \leq \rho \leq (1 - \delta) |\lambda_q|. \quad (32)$$

*Proof.* Select  $\varepsilon^* = \min\{\varepsilon_{>1}^*, \varepsilon_{<1}^*\}$  where

$$\begin{aligned} \varepsilon_{>1}^* &= \frac{3}{2} \min_{i \in \{1, \dots, m_q\}} \left\{ \frac{\bar{\lambda}_i^2 (1 - \delta) - 9\bar{\lambda}_i^2}{(\bar{\lambda}_i^2 - 1) \cos\left(\frac{\pi}{g_i + 1}\right)} \right\}, \\ \varepsilon_{<1}^* &= \frac{4}{3} \min_{i \in \{m_q + 1, \dots, m\}} \left\{ \frac{\frac{9}{8} \bar{\lambda}_i^2 (1 - \delta) - \bar{\lambda}_i^2}{(1 - \bar{\lambda}_i^2) \cos\left(\frac{\pi}{g_i + 1}\right)} \right\}. \end{aligned}$$

Now, following the same steps of the proof of Theorem 2, we obtain the following set of inequalities

$$\begin{aligned} -\mathcal{J}_{g_i}^\top D_{g_i}^{-2}(\varepsilon) \mathcal{J}_{g_i} + \rho^2 D_{g_i}^{-2}(\varepsilon) &\preceq 0, & i = 1, \dots, m_q, \\ \mathcal{J}_{g_i}^\top D_{g_i}^{-2}(\varepsilon) \mathcal{J}_{g_i} - \rho^2 D_{g_i}^{-2}(\varepsilon) &\preceq 0, & i = m_q + 1, \dots, m. \end{aligned} \quad (33)$$

Consider the first set of inequalities in (33). Using the same arguments used in the proof of Theorem 2 we obtain

$$D_{g_i}(\varepsilon) \mathcal{J}_{g_i}^\top D_{g_i}^{-2}(\varepsilon) \mathcal{J}_{g_i} D_{g_i}(\varepsilon) \geq 9\bar{\lambda}_i^2 - \frac{2}{3} \varepsilon (\bar{\lambda}_i^2 - 1) \cos\left(\frac{\pi}{g_i + 1}\right)$$

With the selected  $\varepsilon^*$ , we can verify that

$$9\bar{\lambda}_i^2 - \frac{2}{3}\varepsilon(\bar{\lambda}_i^2 - 1)\cos\left(\frac{\pi}{g_i + 1}\right) \geq \bar{\lambda}_{m_q}^2(1 - \delta)$$

for any  $\varepsilon \in (0, \varepsilon^*]$  because  $|\bar{\lambda}_i| \geq |\bar{\lambda}_{m_q}|$  for any  $i \in \{1, \dots, m_q\}$ . Therefore, the first set of inequalities in (33) is verified for any  $\rho$  satisfying (32). Consider now the second set of inequalities in (33). Using Theorem 2 yields

$$D_{g_i}(\varepsilon)\bar{\lambda}_i^2\mathbb{J}_{g_i}^\top D_{g_i}^{-2}(\varepsilon)\mathbb{J}_{g_i} D_{g_i}(\varepsilon) \leq \frac{8}{9}\bar{\lambda}_i^2 + \frac{2}{3}(1 - \bar{\lambda}_i^2)\varepsilon\cos\left(\frac{\pi}{g_i + 1}\right)$$

and therefore, thanks to the choice of  $\varepsilon^*$ , we have that

$$\frac{8}{9}\bar{\lambda}_i^2 + \frac{2}{3}(1 - \bar{\lambda}_i^2)\varepsilon\cos\left(\frac{\pi}{g_i + 1}\right) \leq \bar{\lambda}_{m_q+1}^2(1 - \delta)$$

for any  $\varepsilon \in (0, \varepsilon^*]$  because  $|\bar{\lambda}_{m_q+1}| \geq |\bar{\lambda}_i|$  for any  $i \in \{m_q + 1, \dots, m\}$ . This shows that the second set of inequalities in (33) is satisfied for any  $\rho$  satisfying (32).  $\square$

As in the continuous-time case, we showed the existence of a solution to the generalized discrete-time Lyapunov matrix inequality in which the bound  $\rho$  can be taken arbitrarily close to the value of the slowest unstable/stable eigenvalue.

## 6. Conclusions

In this work, we use modifications of the Jordan Block normal form to obtain an explicit solution of the generalized Lyapunov inequalities. In particular, with the defined transformations we show that we can obtain concurrently a solution for the inequality and the relative convergence rate. We analyze both continuous and discrete-time cases of this inequality. Moreover, we show that the proposed solution of the inequality preserves its matrix inertia when selecting a desired convergence rate to be arbitrary to either the slowest unstable or the slowest stable eigenvalue, with the price of worsening the condition number of the proposed solution if those eigenvalues have geometric multiplicity larger than one. By exploiting this method, we were also able to show that for a Hurwitz matrix, the inertia of the related inequality solution can always be taken with some degrees of freedom, whenever the chosen convergence rate equals the real part of the slowest eigenvalue.

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## Appendix A. Matlab code related to the example

The following Matlab code, that exploits the LMI Toolbox of Matlab to get the solution of (20), has been exploited to compare the condition numbers of the solution solutions.

```
%block dimensions
n1 = 3;
n2 = 5;
n3 = 1;
n = n1+n2+n3;
%Related eigenvalues
a1 = -0.5;
a2 = -2;
a3 = -4;
%% Modified Jordan blocks
JJ_1 = eye(n1) + diag(ones(1, n1-1), 1);
JJ_2 = eye(n2) + diag(ones(1, n2-1), 1);
JJ_3 = eye(n3) + diag(ones(1, n3-1), 1);
%Modified Jordan form
JJ = blkdiag(a1*JJ_1, a2*JJ_2, a3*JJ_3);
%Convergence rate
alpha=max(eig(JJ+JJ'))/2;
%% Real Jordan form
J = blkdiag(
    a1*eye(n1) + diag(ones(1, n1-1), 1)
    , ...
    a2*eye(n2) + diag(ones(1, n2-1), 1)
    , ...
    a3*eye(n3) + diag(ones(1, n3-1), 1)
);
%% Defining DD
DD = zeros(n, n);
```

```

for k=1:n1
    DD(k,k) = a1^(k-1);
end
for k=1:n2
    DD(k+n1,k+n1) = a2^(k-1);
end
for k=1:n3
    DD(k+n1+n2,k+n1+n2) = a3^(k-1);
end
eigD = eig(DD*DD);
%% Initialization of test variables
n_k_TT=0;
n_k_LMI=0;
k_TT_tot=0;
k_LMI_tot=0;
% setting up the test
n_test=1000;
%% start testing
for i=1:n_test
%Transformation definition
T = randn(n,n);
% The Matrix A under consideration
A=T*J/T;
%% condition number of the transformation
TT=T*DD;
eigTT = eig(TT'*TT);
k_TT=sqrt(max(eigTT)/min(eigTT));
%% LMI part
setlmis([]);
P = lmivar(1, [n,1]);
lmiterm([-1 1 1 P],1,1);
lmiterm([2 1 1 P],(A-2*alpha*eye(n))',1,
    's'); % A'P + PA < 0

% solve the LMI
lyap_lmi=getlmis; %get the LMI system
%check feasibility
[~, sol]=feasp(lyap_lmi);
%calculate the value of P
P=dec2mat(lyap_lmi, sol, P);
% computing the condition number
eigP=eig(P);
k_LMI= max(eigP)/min(eigP);
%% comparison
if k_TT < k_LMI
    n_k_TT= n_k_TT +1;
else
    n_k_LMI=n_k_LMI +1;
end
k_TT_tot=k_TT_tot+ k_TT;
k_LMI_tot=k_LMI_tot+ k_LMI;
%
end
%% Condition numbers comparison
display('average k from TT');
display(k_TT_tot/n_test);
display('average k from LMI');
display(k_LMI_tot/n_test);
display(n_k_TT/n_test*100);

```