



An explicit solution to the generalized Lyapunov matrix inequality

Mario Spirito, Daniele Astolfi

► To cite this version:

Mario Spirito, Daniele Astolfi. An explicit solution to the generalized Lyapunov matrix inequality. *Systems and Control Letters*, 2024, 192, pp.105905. [⟨10.1016/j.sysconle.2024.105905⟩](https://doi.org/10.1016/j.sysconle.2024.105905). [⟨hal-04366789v2⟩](https://hal.science/hal-04366789v2)

HAL Id: hal-04366789

<https://hal.science/hal-04366789v2>

Submitted on 13 Mar 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

An explicit solution to the generalized Lyapunov matrix inequality

Mario Spirito and Daniele Astolfi^a

^a LAGEPP, Université Claude Bernard Lyon 1 (UCBL), 69100 Villeurbanne, France
spirito.mario@gmail.com, astolfi.daniele@univ-lyon1.fr

Abstract

We review the generalized Lyapunov matrix inequality for continuous-time systems and we provide a solution based on a modification of the Jordan normal form. The proposed construction allows to obtain explicit bounds to the decay rate and the inertia of the solution to the generalized Lyapunov matrix inequality.

1. Introduction

In this article, we study the solution to the generalized Lyapunov matrix inequality

$$PA + A^\top P \preceq 2\alpha P,$$

in which A is a given matrix (possibly non-Hurwitz), α is a real number, and P is an unknown symmetric matrix which may be non-positive definite. Such an inequality arises in many control problems and plays a fundamental role in p -dominance [3] and k -contraction theory [13, 1]. The generalized Lyapunov matrix inequality has been deeply studied and it is well-known that the inertia of P , namely the number of eigenvalues with positive, negative, and zero real part, is directly related to the inertia of the matrix $(A - \alpha I)$, see, for instance, [9, 6, 11, 12]. In this work, which extends our preliminary contribution [10], we provide a new explicit solution based on the solely algebraic properties of the matrix A , namely on its eigenvalues and (generalized) eigenvectors, focusing on the special case in which A is a non-diagonalizable matrix, for which it is known the parameter α cannot be arbitrarily chosen. In particular, we show that we cannot select α equal to an eigenvalue with non-unitary geometric multiplicity and that selecting α arbitrarily close to it provides a solution P which is ill-conditioned. These facts are provided thanks to a modified Jordan's normal form. We believe that these new results allow to better understand the decay rate of stable systems possessing nontrivial Jordan blocks (of large dimensions), e.g., when designing a high gain observer characteristic polynomial via a Lyapunov equation design as described in [4, sec. 2.3.1], or in large-scale systems obtained via the spatial discretization of PDEs or in the case of linear first order (nonstrictly) hyperbolic systems as addressed in [2], see also [7, Part II].

The paper is structured as follows. In the next section, i.e., Section 2, we introduce the definition of the inertia of a matrix and some preliminary ingredients to define the modified Jordan normal form. Then, in Section 3, we introduce such a modification of the Jordan block normal form as proposed in [10] for continuous-time systems, and we then analyze a closed-form

solution of the generalized Lyapunov inequality. Some numerical tests to show the effectiveness of the proposed approach are given in Section 4. Finally, conclusions are drawn in Section 5.

2. Notation and Preliminaries

We denote with \mathbb{R} (resp. \mathbb{C}) the set of real (resp. complex) numbers, and with \mathbf{i} the imaginary unit. Given $s \in \mathbb{C}$, $s = a + \mathbf{i}b$, we denote with $\Re\{s\} = a$ its real part. Given a matrix $P \in \mathbb{R}^{n \times n}$, we denote by $\sigma_{\max}(P)$, resp. $\sigma_{\min}(P)$, its largest, resp. smallest, eigenvalue. We recall the following definitions of inertia of a matrix, see [11].

Definition 1. The inertia of a matrix P with respect to the imaginary axis (c -inertia) is defined by the triplet of integers

$$\text{In}_c(A) = \{\pi_-(P), \pi_0(P), \pi_+(P)\}$$

where $\pi_-(P)$, $\pi_0(P)$ and $\pi_+(P)$, denote the numbers of eigenvalues of P , counted with their algebraic multiplicities, with negative, zero, and positive real part, respectively.

Definition 2. The inertia of a matrix P with respect to the unit circle (d -inertia) is defined by the triplet of integers

$$\text{In}_d(A) = \{\pi_{<1}(P), \pi_1(P), \pi_{>1}(P)\}$$

where $\pi_{<1}(P)$, $\pi_1(P)$, and $\pi_{>1}(P)$, denote the numbers of eigenvalues of P , counted with their algebraic multiplicities, which are inside, on, and outside the unit circle.

Given any $n \in \mathbb{N}$, we define the following matrices

$$\mathbb{J}_n := I_n + N_n = \begin{bmatrix} 1 & 1 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & & 1 \end{bmatrix}, \quad N_n = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad (1)$$

where I_n is the identity matrix of dimension n and N_n is the matrix with ones on the upper diagonal and zeros elsewhere. Given a number $\varepsilon \in \mathbb{R}$ and a natural number $n \in \mathbb{N}$, we denote

$$D_n(\varepsilon) := \begin{cases} \text{diag}(1, \varepsilon, \dots, \varepsilon^{n-1}), & \text{if } \varepsilon \neq 0, \\ I_n, & \text{if } \varepsilon = 0. \end{cases} \quad (2)$$

Then, with the previous definition, the following identity

$$N_n D_n(\varepsilon) = \varepsilon D_n(\varepsilon) N_n \quad (3)$$

holds for any $\varepsilon \in \mathbb{R}/\{0\}$ and $n \in \mathbb{N}$. Similarly, for a matrix $A \in \mathbb{R}^{r \times r}$ we can define

$$D_n(A) := \begin{cases} \text{diag}(I_r, A, \dots, A^{n-1}), & \text{if } A \neq 0, \\ I_m, & \text{if } A = 0, \end{cases} \quad (4)$$

and in this case, the following identity can be verified

$$D_n(A)^{-1} (N_n \otimes I_r) D_n(A) = N_n \otimes A. \quad (5)$$

Finally, we have the following properties on the matrix \mathbb{J}_n .

Lemma 1. *For any $n \geq 1$, the matrix \mathbb{J}_n satisfies the following properties.*

1. *The matrix $\mathbb{J}_n + \mathbb{J}_n^\top$ satisfies*

$$2 \left[1 + \cos\left(\frac{\pi n}{n+1}\right) \right] I_n \preceq \mathbb{J}_n + \mathbb{J}_n^\top \preceq 2 \left[1 + \cos\left(\frac{\pi}{n+1}\right) \right] I_n.$$

2. *For any real number $\varepsilon \in (0, 1]$, the following holds*

$$2 \left(1 + \varepsilon \cos\left(\frac{\pi n}{n+1}\right) \right) \preceq D_n(\varepsilon)^{-1} \left[\mathbb{J}_n + \mathbb{J}_n^\top \right] D_n(\varepsilon) \preceq 2 \left(1 + \varepsilon \cos\left(\frac{\pi}{n+1}\right) \right).$$

Proof. From [8, Ex. 7.2.5], we recall that given a matrix $A \in \mathbb{R}^{n \times n}$, with $a \neq 0$ and $c \neq 0$, of the form

$$A = \begin{bmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{bmatrix} = bI_n + aN_n + cN_n^\top, \quad (6)$$

an explicit formula for its eigenvalues λ_i is given by

$$\lambda_i = b + 2a \sqrt{\frac{c}{a}} \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}. \quad (7)$$

As a consequence, applying such a formula to \mathbb{J}_n one directly shows item 1. We show now item 2. By applying (3), we have

$$D_n(\varepsilon)^{-1} \left[\mathbb{J}_n + \mathbb{J}_n^\top \right] D_n(\varepsilon) = 2I_n + \varepsilon(N_n + N_n^\top),$$

and therefore, by using again (7), we obtain

$$2 \left(1 + \varepsilon \cos\left(\pi \frac{n}{n+1}\right) \right) \preceq D_n(\varepsilon)^{-1} \left[\mathbb{J}_n + \mathbb{J}_n^\top \right] D_n(\varepsilon) \preceq 2 \left(1 + \varepsilon \cos\left(\frac{\pi}{n+1}\right) \right)$$

concluding the proof. \square

3. Main Result

Given a matrix A of c -inertia $\{n-p, 0, p\}$, we are interested in studying the solutions to the generalized Lyapunov matrix inequality

$$PA + A^\top P \preceq 2\alpha P, \quad \alpha \in \mathbb{R}, \quad P = P^\top, \quad (8)$$

in which $\alpha \in \mathbb{R}$ may be a positive or negative real number and P is a symmetric matrix of c -inertia $\{p, 0, n-p\}$, as show in [9, Lemma 1, Section 3] and [11, Theorem 2.5].

In this short note, we focus on the case in which A is a $n \times n$ non-diagonalizable matrix and we look for an explicit solution to the inequality (8). We denote by T the transformation putting A in its canonical (real) Jordan form [6], namely

$$T^{-1}AT = J, \quad J := \text{blkdiag}(J_{\lambda_1}, \dots, J_{\lambda_m}), \quad (9)$$

where $m \leq n$ is the number of Jordan blocks, J_{λ_i} is the Jordan block corresponding to the eigenvalue λ_i with dimension $g_i \in \mathbb{N}$, with the convention that if λ_i is real then

$$J_{\lambda_i} := \lambda_i I_{g_i} + N_{g_i}, \quad (10)$$

while, if λ_i is related to a pair of complex conjugate eigenvalues of the form $a_i \pm \mathbf{i}b_i$, then and its corresponding Jordan block is described as

$$J_{\lambda_i} := I_{g_i} \otimes \Lambda_i + N_{g_i} \otimes I_2 = \begin{bmatrix} \Lambda_i & I_2 & 0 & \dots & 0 \\ 0 & \Lambda_i & I_2 & \dots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & \dots & & \Lambda_i & I_2 \\ & & & 0 & \Lambda_i \end{bmatrix}_{2g_i \times 2g_i} \quad (11)$$

where Λ_i is defined as

$$\Lambda_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$$

and one can verify that $\sigma(\Lambda_i) = \{a + \mathbf{i}b, a - \mathbf{i}b\}$. Note also that $\Lambda_i + \Lambda_i^\top = 2a_i I_2 = 2\Re\{\lambda_i\} I_2$. With such a notation, we also have $n = \sum_{i=1}^m g_i$. Without loss of generality, the eigenvalues $\lambda_1, \dots, \lambda_m$ are ordered as

$$\Re\{\lambda_1\} \geq \dots \geq \Re\{\lambda_q\} > 0 > \Re\{\lambda_{q+1}\} \geq \dots \geq \Re\{\lambda_m\}, \quad (12)$$

where $q \in [0, n]$ is the number of eigenvalues (associated with a Jordan block) with positive real part and we let $p = \sum_{i=1}^q g_i$. The first observation is that the matrix T could be exploited to design a solution to the Lyapunov matrix inequality (8) if A is diagonalizable, otherwise this might not be the case. To simplify the explanation, consider the case in which A is a Hurwitz matrix, with only real eigenvalues and we look for a P and α satisfying (8). By taking $P = T^{-\top} T^{-1}$, simple computations yields

$$J + J^\top \preceq 2\alpha I.$$

Due to the block-diagonal structure of J , one can verify that it is possible to find a solution with a negative α only if $\Re\{\lambda_i\} < -1$

for any $i = 1, \dots, m$. This can be easily computed using (7). Thus, for a matrix in Jordan form, the identity is in general not a good candidate solution of a Lyapunov matrix inequality.

As a consequence, in order to construct an explicit solution to the inequality (8) and provide the main result, we first need to introduce the following two sets of matrices which allow us to modify the standard Jordan canonical form. In particular, for any Jordan block J_{λ_i} related to a real eigenvalue, we define the corresponding matrices $D_{g_i}(\lambda_i)$ and $D_{g_i}(\varepsilon)$ following the definition given in (2), while, for a complex pair of eigenvalues λ_i , we define, with some abuse of notation $D_{g_i}(\lambda_i) = D_{g_i}(\Lambda_i)$ and $D_{g_i}(\varepsilon) = D_{g_i}(\varepsilon I_2)$, with D_{g_i} defined as in (4). With such a definition, we construct the matrices \mathbb{D} and \mathbb{D}_ε as follows:

$$\begin{aligned}\mathbb{D} &:= \text{blkdiag}(D_{g_1}(\lambda_1), \dots, D_{g_m}(\lambda_m)), \\ \mathbb{D}_\varepsilon &:= \text{blkdiag}(D_{g_1}(\varepsilon), \dots, D_{g_m}(\varepsilon)).\end{aligned}\quad (13)$$

The two set of matrices \mathbb{D} and \mathbb{D}_ε have different interpretations. First of all, the effect of the matrix $D_g(\lambda)$ is to rescale the corresponding Jordan block as follows:

$$D_g^{-1}(\lambda) J_\lambda D_g(\lambda) = \lambda \mathbb{J}_g.$$

In particular, with the new rescaling factor, one obtains that for any λ with negative real part, $\lambda \mathbb{J}_g + \lambda \mathbb{J}_g^\top \prec 0$ in view of (7). Therefore, for this new modified Jordan form (in which the value of the eigenvalues multiplies also the upper-diagonal matrix), the identity is a good candidate solution to a Lyapunov matrix inequality (8) for α in a certain interval, as expressed by the result in Corollary 1 below. It is then readily seen that by applying to the matrix A the transformation $\mathbb{T} = T\mathbb{D}$, defined in (16) and (13), one obtains

$$\mathbb{T} = T\mathbb{D}, \quad \mathbb{T}^{-1}A\mathbb{T} = \mathbb{J} := \text{blkdiag}(\mathbb{J}_{\lambda_1}, \dots, \mathbb{J}_{\lambda_m}) \quad (14)$$

where, for $i = 1, \dots, m$, \mathbb{J}_{λ_i} is defined as $\mathbb{J}_{\lambda_i} = \lambda_i \mathbb{J}_{g_i}$ if λ_i is real, or as $\mathbb{J}_{\lambda_i} = \mathbb{J}_{g_i} \otimes \Lambda_i$ if λ_i corresponds to a pair of complex eigenvalues, with \mathbb{J}_{g_i} defined as in (1), for $g_i \in \mathbb{N}$, that is

$$\mathbb{J}_{\lambda_i} = \begin{cases} \lambda_i \mathbb{J}_{g_i} & \text{if } \lambda_i \text{ is real,} \\ \mathbb{J}_{g_i} \otimes \Lambda_i & \text{if } \lambda_i \text{ is complex,} \end{cases}$$

Then, the second transformation matrix, i.e., \mathbb{D}_ε , is used to further modify the structural matrix \mathbb{J}_g of each block, by imposing an ε value on its off-diagonal terms. This later coefficient is instrumental in getting some degree of freedom on the convergence rate, so to select an α to be as close as possible to λ , as one can observe from item 2 of Lemma 1.

To summarize, the matrix P solution to the inequality (8) that we propose, is obtained by composing the matrices T , \mathbb{D} and \mathbb{D}_ε back in the original basis of A . We remark that the selection of these matrices is similar to the change of coordinates which is used in high-gain observers theory, see, e.g., [5]. Finally, given

$\delta \in (0, 1)$, we let ε_δ be defined as

$$\begin{aligned}\varepsilon_\delta &= \min\{\varepsilon_+^*, \varepsilon_-^*\}, \quad r_{j,i} = \frac{|\Re\{\lambda_j\}|}{|\Re\{\lambda_i\}|} \\ \varepsilon_+^* &= \min_{\substack{i \in \{1, \dots, q\} \\ \text{s.t. } g_i > 1}} \left\{ \cos\left(\frac{\pi}{g_i+1}\right)^{-1} \left[1 - (1-\delta)r_{q,i} \right] \right\}, \\ \varepsilon_-^* &= \min_{\substack{i \in \{q+1, \dots, m\} \\ \text{s.t. } g_i > 1}} \left\{ \cos\left(\frac{\pi}{g_i+1}\right)^{-1} \left[1 - (1-\delta)r_{q+1,i} \right] \right\},\end{aligned}\quad (15)$$

Note that ε_+^* is always nonnegative for any $\delta \in (0, 1)$ because, $\cos(\pi/(g_i+1))$ is positive for any $g_i > 1$, and the ratio $r_{q,i}$ is always less than or equal to one, $i = 1, \dots, q$. Similar arguments hold for ε_-^* . We then have the following theorem providing an explicit solution to the Lyapunov matrix inequality (8) based on solely knowledge of T and λ_i in (9), although the Jordan form transformation T might not be trivial to obtain.

Theorem 1. *Let A be a matrix with c -inertia $\{n-p, 0, p\}$, satisfying (9). For any $\delta \in (0, 1)$ there exists $\varepsilon_\delta > 0$ defined in (15), such that, for any $\varepsilon \in (0, \varepsilon_\delta]$, the following P with c -inertia $\{p, 0, n-p\}$*

$$\begin{aligned}P &:= \mathbb{T}^{-\top} \mathbb{D}_\varepsilon^{-2} \mathbb{T}^{-1}, \quad \mathbb{T} := T\mathbb{D}, \\ \mathbb{I} &:= \text{blkdiag}(-I_p, I_{n-p}),\end{aligned}\quad (16)$$

with T , \mathbb{D} and \mathbb{D}_ε defined as in (9) and (13), is a solution to the generalized Lyapunov matrix inequality (8) with $\alpha \in \mathbb{R}$ satisfying

$$(1-\delta)\Re\{\lambda_{q+1}\} \leq \alpha \leq (1-\delta)\Re\{\lambda_q\}. \quad (17)$$

Note that, for diagonalizable matrices A , i.e., $g_i = 1$, for all $i \in \{1, \dots, n\}$, Theorem 1 boils down to the forthcoming Corollary 1 below since, when $g_i = 1$ for all i , then by definition one obtains $\mathbb{D}_\varepsilon = I$. See also the remark at the end of this section.

Proof. First, note that P has the same inertia of \mathbb{I} . Take inequality (8) and pre-multiply (post-multiply) by $\mathbb{D}_\varepsilon \mathbb{T}^\top$ ($\mathbb{T} \mathbb{D}_\varepsilon$). Using the definition of P and recalling (9) we obtain

$$\begin{aligned}\mathbb{D}_\varepsilon \mathbb{T}^\top (PA + A^\top P - 2\alpha P) \mathbb{T} \mathbb{D}_\varepsilon &\preceq 0 \\ \mathbb{D}_\varepsilon^{-1} (\mathbb{D}^{-1} J \mathbb{D}) \mathbb{D}_\varepsilon + \mathbb{D}_\varepsilon (\mathbb{D}^{-1} J \mathbb{D})^\top \mathbb{D}_\varepsilon^{-1} \mathbb{I} - 2\alpha \mathbb{I} &\preceq 0 \\ \mathbb{D}_\varepsilon^{-1} J \mathbb{D}_\varepsilon + \mathbb{D}_\varepsilon J^\top \mathbb{D}_\varepsilon^{-1} \mathbb{I} - 2\alpha \mathbb{I} &\preceq 0,\end{aligned}$$

where J is a modified Jordan form defined in (14). As a consequence, due to the block-diagonal structure, the previous inequality corresponds to the following set of inequalities

$$\begin{aligned}-\Re\{\lambda_i\} D_{g_i}^{-1}(\varepsilon) \left[\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top \right] D_{g_i}(\varepsilon) + 2\alpha I_{g_i} &\preceq 0, \\ i &= 1, \dots, q, \\ \Re\{\lambda_i\} D_{g_i}^{-1}(\varepsilon) \left[\mathbb{J}_{g_i} + \mathbb{J}_{g_i}^\top \right] D_{g_i}(\varepsilon) - 2\alpha I_{g_i} &\preceq 0, \\ i &= q+1, \dots, m.\end{aligned}\quad (18)$$

It is readily seen that the inequalities (18) are satisfied for all $i \in \{1, \dots, m\}$ if α is chosen such that $\underline{\alpha}_\varepsilon \leq \alpha \leq \bar{\alpha}_\varepsilon$ with

$$\begin{aligned}\bar{\alpha}_\varepsilon &\leq \min_{i \in \{1, \dots, q\}} \left\{ \left(1 + \varepsilon \cos \left(\pi \frac{g_i}{g_i+1} \right) \right) \Re\{\lambda_i\} \right\}, \\ \underline{\alpha}_\varepsilon &\leq \max_{i \in \{q+1, \dots, m\}} \left\{ \left(1 + \varepsilon \cos \left(\pi \frac{g_i}{g_i+1} \right) \right) \Re\{\lambda_i\} \right\}.\end{aligned}$$

As a consequence, by selecting for any $\varepsilon \in (0, \varepsilon_\delta]$ one can verify that

$$\bar{\alpha}_\varepsilon \leq (1 - \delta) \Re\{\lambda_q\}, \quad \underline{\alpha}_\varepsilon \leq (1 - \delta) \Re\{\lambda_{q+1}\}, \quad (19)$$

thus concluding the proof. \square

In the previous theorem, one can thus select δ in order to obtain a value α in (8) which is as close as possible to the real part of the eigenvalues λ_q and λ_{q+1} , see equation (19). However, as explained before Theorem 1, one can still consider the solution with $\varepsilon = 1$ but obtaining more conservative values of α , as shown in the next corollary.

Corollary 1. *Let A be a matrix with c -inertia $\{n - p, 0, p\}$, satisfying (9). Then, the matrix P with c -inertia $\{p, 0, n - p\}$ defined as*

$$P := \mathbb{T}^{-\top} \mathbb{I} \mathbb{T}^{-1}, \quad \mathbb{T} := T \mathbb{D}, \quad \mathbb{I} := \text{blkdiag}(-I_q, I_{n-q}),$$

with T satisfying (9), is a solution to the generalized Lyapunov matrix inequality (8) for any $\alpha \in [\underline{\alpha}, \bar{\alpha}]$ with

$$\begin{aligned}\bar{\alpha} &:= \min_{i \in \{1, \dots, q\}} \left\{ \Re\{\lambda_i\} \left[1 + \cos \left(\pi \frac{g_i}{g_i+1} \right) \right] \right\}, \\ \underline{\alpha} &:= \max_{i \in \{q+1, \dots, m\}} \left\{ \Re\{\lambda_i\} \left[1 + \cos \left(\pi \frac{g_i}{g_i+1} \right) \right] \right\}.\end{aligned} \quad (20)$$

In particular, the previous corollary can be derived from Theorem 1 in which $\varepsilon = 1$.

Remark. *It is worth noticing that in case A is Hurwitz, i.e., $p = q = 0$, we obtain a closed-form solution to the standard Lyapunov inequality, for more details refer to [10].*

Remark. *In case A is diagonalizable, i.e., for $i = 1, \dots, m$ we have $g_i = 1$, then $\mathbb{D}_\varepsilon = I$ and the solution P given in Corollary 1 is valid for $\alpha \in [\underline{\alpha}, \bar{\alpha}]$ with $\underline{\alpha} = \Re\{\lambda_q\}$ and $\bar{\alpha} = \Re\{\lambda_{q+1}\}$.*

4. Numerical tests

The main theorem of this article provides an explicit solution to the generalized Lyapunov matrix inequality which can be optimized in the set of admissible values for α . We don't claim however that our choice is always the optimal one in terms of conditioning number $\sigma_{\max}(P)/\sigma_{\min}(P)$ of P , because of the presence of ε that may be very small. For instance, one could improve the previous design by selecting a different ε_i for each block in (13) correlated to each Jordan block.

As a numerical test, we fix a Jordan block matrix J with dimension 9 with 3 distinct eigenvalues, i.e., $\{-0.5, -2, -4\}$ with

associated block dimensions $(g_1, g_2, g_3) = (3, 5, 1)$ and we construct the matrix A by taking an *invertible* random matrix transformation T . We compare the analytic solution P to (8), proposed in Corollary 1 and Theorem 1, with a numerical solution P_{LMI} obtained employing a standard LMI solver such as the one implemented in Matlab. We repeat this experiment 5000 times and compute the relative average condition numbers to compare the two solutions P and P_{LMI} .

We thus consider an application of Corollary 1, and accordingly, from the convergence rate formula (20) we have

$$\underline{\alpha} = -0.5 \left(1 + \cos \left(\frac{3}{4} \pi \right) \right) \approx -0.1464, \quad \bar{\alpha} = 0$$

with $P = \mathbb{T}^{-\top} \mathbb{T}^{-1}$, thus the condition number of P is equal to the condition number of \mathbb{T} . By comparing the solution P_{LMI} of LMI solver (getlmis()) implemented in the Robust Control ToolboxTM of Matlab version R2022a with the given $\alpha = \bar{\alpha}$, we get approximatively, over 5000 comparisons, 99% of the times a better condition number of \mathbb{T} than that of P_{LMI} . The numerical testing details are summarized in Table 1.

# tests	%	average $\mu(\mathbb{T}^\top \mathbb{T})$	average $\mu(P_{\text{LMI}})$
5000	99%	$1.4839e+03$	$1.0417e+06$

Moreover, when we consider the application of Theorem 1, we fix $\alpha = -0.45$, we get, over 5000 comparisons, 100% of the times a better condition number of $\mathbb{T} \mathbb{D}_\varepsilon$ than that of P_{LMI} . The numerical testing details are summarized in Table 2.

# tests	%	average $\mu(\mathbb{T}^\top \mathbb{D}_\varepsilon^2 \mathbb{T})$	average $\mu(P_{\text{LMI}})$
5000	100%	$6.2556e+03$	$1.3463e+16$

5. Conclusions

In this work, we exploit a modification of the Jordan block normal form to obtain an explicit solution of the generalized Lyapunov inequality for continuous time systems. Moreover, we show that the proposed solution of the inequality preserves its matrix inertia when selecting a desired convergence rate to be arbitrary to either the slowest unstable or the slowest stable eigenvalue, with the price of worsening the condition number of the proposed solution if those eigenvalues have geometric multiplicity larger than one. Some numerical test results are also provided to show that the proposed solution most of the time provides a better-conditioned matrix than the one obtained via numerical optimization. However, the proposed solution is mainly a theoretical result because its numerical implementation might not always be feasible due to the lack of a reliable numerical algorithm for computing the Jordan canonical form.

Acknowledgement

This work is supported by the ANR IMPACT (ANR-21-CE48-0018) and ANR ALLIGATOR (ANR-22-CE48-0009-01) projects.

References

- [1] Andreu Cecilia, Samuele Zoboli, Ulysse Serres, Daniele Astolfi, and Vincent Andrieu. Generalized Lyapunov conditions for k -contraction: analysis and feedback design. Submitted to IEEE Transactions on Automatic Control, preprint available at <https://hal.science/hal-04300588v1>, 2023.
- [2] Francesco Fanelli. A few remarks on hyperbolic systems with Zygmund in time coefficients. In *Shocks, Singularities and Oscillations in Nonlinear Optics and Fluid Mechanics*, Shocks, Singularities and Oscillations in Nonlinear Optics and Fluid Mechanics, 2015.
- [3] Fulvio Forni and Rodolphe Sepulchre. Differential dissipativity theory for dominance analysis. *IEEE Transactions on Automatic Control*, 64(6):2340–2351, 2019.
- [4] Hassan K. Khalil. *High-Gain Observers in Nonlinear Feedback Control*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2017.
- [5] Hassan K Khalil and Laurent Praly. High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control*, 24(6):993–1015, 2014.
- [6] Peter Lancaster and Miron Tismenetsky. *The theory of matrices: with applications*. Elsevier, 1985.
- [7] Andrew Majda and Stanley Osher. Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary. *Communications on Pure and Applied Mathematics*, 28(5):607–675, 1975.
- [8] Carl D Meyer. *Matrix analysis and applied linear algebra*, volume 71. SIAM, 2000.
- [9] Russell A Smith. The Poincaré–Bendixson theorem for certain differential equations of higher order. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 83(1-2):63–79, 1979.
- [10] Mario Spirito and Daniele Astolfi. Explicit convergence rate parameters for linear autonomous systems. 4th IFAC MICNON 2024, preprint available at <https://hal.science/hal-04282568>.
- [11] Tatjana Stykel. Stability and inertia theorems for generalized Lyapunov equations. *Linear Algebra and its Applications*, 355(1-3):297–314, 2002.
- [12] Bart Vandereycken and Stefan Vandewalle. A Riemannian optimization approach for computing low-rank solutions of Lyapunov equations. *SIAM Journal on Matrix Analysis and Applications*, 31(5):2553–2579, 2010.
- [13] Samuele Zoboli, Andreu Cecilia, Ulysse Serres, Daniele Astolfi, and Vincent Andrieu. LMI conditions for k -contraction analysis: a step towards design. In *IEEE 62st Conference on Decision and Control*, 2023.