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Beta distribution and associated Stirling numbers of the second kind

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Abstract: This article gives a formula for *associated Stirling numbers of the second kind* based on the moment of a sum of independent random variables having a beta distribution. From this formula we deduce, using probabilistic approaches, lower and upper bounds for these numbers.

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1. Introduction

Classical Stirling numbers of the second kind $S(p, m)$ counts the number of all partitions of $\{1, \dots, p\}$ into m nonempty subsets, for $p \in \mathbb{N}_{>0}$ and $m \in \mathbb{N}$. More generally, the r -associated Stirling number $S_r(p, m)$, with $r \in \mathbb{N}_{>0}$, is the number of all partitions of $\{1, \dots, p\}$, into m subsets where each subset contains at least r elements [3, p. 221]. Obviously $S(p, m) = S_1(p, m)$.

There are well-known connections between Stirling numbers of the second kind and probability theory. For example, sequences $S_1(p, m)$ and $S_2(p, m)$ are asymptotically normal when p tends to $+\infty$ [4, 7]. More precisely, when $r \in \{1, 2\}$, the following convergence in distribution occurs

$$\frac{Y_p - \mathbb{E}(Y_p)}{\sqrt{\text{var}(Y_p)}} \underset{p \rightarrow +\infty}{\overset{d}{\rightarrow}} \mathcal{N}(0, 1) \text{ where } \mathbb{P}(Y_p = m) = \frac{S_r(p, m)}{\sum_{k=1}^p S_r(p, k)}, \text{ for all } m \in \mathbb{N}_{>0}.$$

Otherwise, according to Dobinski's formula, the moment of order p of a Poisson distribution with parameter $\lambda \geq 0$ is $\sum_{m=1}^p S_1(p, m) \lambda^m$ (see, e.g. [3, p. 211]). However, to our knowledge, there is no close formula in the literature for $S_r(k, m)$ based on moments of a sum of independent and identically distributed (i.i.d) random variables. The main result in this article is Theorem 2.1 providing the following new identity

$$S_r(p, m) = \frac{p!}{m!(r!)^m(p-rm)!} \mathbb{E} \left[(X_1 + \dots + X_m)^{p-rm} \right], \quad (1)$$

where X_1, \dots, X_p are i.i.d random variables having beta distribution with parameter $(1, r)$. Note that a beta distribution with parameter $(1, 1)$ is a uniform distribution on $[0, 1]$. Thus, when $r = 1$, the above formula is quite simple:

$$S_1(p, m) = \binom{p}{m} \mathbb{E} (Z^{p-m}), \text{ where } Z = \sum_{i=1}^m X_i \text{ has the Irwin-Hall distribution on } [0, m].$$

Propositions 3.1, 3.2 and 3.3 give upper and lower bounds for $\mathbb{E} \left[(X_1 + \dots + X_m)^{p-rm} \right]$. These bounds are sharp when one parameter m, r , or $p - rm$ tends to $+\infty$ and provides thus accurate approximations of r -associated Stirling numbers.

2. Closed formula for Stirling numbers and moments of random variables

The density g_r of a beta distribution with parameters $(1, r)$ where $r \in \mathbb{N}_{>0}$ is

$$g_r(x) = \begin{cases} r(1-x)^{r-1} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Let X_1, \dots, X_m be independent random variables having the same beta $(1, r)$ distribution. The moment of order $k \in \mathbb{N}$ of the sum of these variables is defined as follows

$$\mathcal{M}_r(k, m) = \mathbb{E} \left[(X_1 + \dots + X_m)^k \right]. \quad (3)$$

Theorem 2.1 provides a closed formula for the Stirling numbers of the second kind based on the moment $\mathcal{M}_r(k, m)$.

Theorem 2.1. *Let $m, r \in \mathbb{N}_{>0}$ and $p \in \mathbb{N}$ where $p \geq rm$. The Stirling numbers of the second kind satisfy the following identity*

$$S_r(p, m) = \frac{p!}{m!(r!)^m(p-rm)!} \mathcal{M}_r(p-rm, m).$$

From Theorem 2.1, one may deduce that $\mathbb{E}(Z^k) = S_1(m+k, m) / \binom{m+k}{m}$ where Z has an Irwin-Hall's distribution on $[0, m]$ [6, 8]. Note that the moment generating function of Z is $\sum_{k \geq 0} \mathbb{E}(Z^k) t^k / k! = ((\exp(t) - 1)/t)^m$ therefore we recover the well-known exponential generating function of the Stirling numbers of the second kind $\sum_{p \geq m} S_1(p, m) t^p / p! = (\exp(t) - 1)^m / m!$ (see Theorem 3.3 page 52 in [9]). The above expression of $S_r(p, m)$ is explicit up to the computation of the moment $\mathcal{M}_r(k, m)$. Whereas computing explicitly $\mathcal{M}_r(k, m)$ might be technical, lower bounds, upper bounds and approximations of $\mathcal{M}_r(k, m)$ are tractable as illustrated in the following section.

3. Upper and lower bounds

Hereafter, we will use probabilistic approaches to derive upper and lower bounds for the moment $\mathcal{M}_r(k, m)$.

3.1. Sharp upper and lower bounds when m is large

Let $\bar{X}_m = (X_1 + \dots + X_m)/m$, the Jensen's inequality provides the following lower bound

$$\mathcal{M}_r(k, m) = m^k \mathbb{E} \left(\bar{X}_m^k \right) \geq m^k \mathbb{E} \left(\bar{X}_m \right)^k = \frac{m^k}{(r+1)^k}. \quad (4)$$

This inequality relies on the linearization of the function $q(x) = x^k$ at $x_0 = \mathbb{E}(\bar{X}_m) = \frac{1}{r+1}$. Specifically, the following inequality holds for all $x \in [0, 1]$

$$x^k = q(x) \geq q(x_0) + q'(x_0)(x - x_0) = \frac{1}{(r+1)^k} + \frac{k}{(r+1)^{k-1}} \left(x - \frac{1}{r+1} \right). \quad (5)$$

Moreover one may choose $c \geq 0$ for which the following inequality is true for all $x \in [0, 1]$ (see Lemma 5.2)

$$x^k \leq q(x_0) + q'(x_0)(x - x_0) + c(x - x_0)^2. \quad (6)$$

Proposition 3.1 below is a consequence of inequalities (5) and (6).

Proposition 3.1. *Let $r, m \in \mathbb{N}_{>0}$ and $k \in \mathbb{N}$. The following inequality holds*

$$\frac{m^k}{(r+1)^k} \leq \mathcal{M}_r(k, m) \leq \frac{m^k}{(r+1)^k} + m^{k-1} \frac{(r+1)^k - 1 - kr}{(r+1)^{kr}(r+2)}.$$

The leading term when m is large in both the lower and upper bounds is $m^k / (r+1)^k$. Therefore, lower and upper bounds are asymptotically equivalent when $k \in \mathbb{N}_{\geq 0}$ and $r \in \mathbb{N}_{>0}$ are fixed and m tends to $+\infty$. These bounds are accurate when m is large since \bar{X}_m converges to $\mathbb{E}(\bar{X}_m)$ and both inequalities (5) and (6) are accurate on the neighbourhood of $\mathbb{E}(\bar{X}_m)$.

3.2. Sharp upper and lower bounds when k is large

Asymptotic behaviour of moments, when k is large, depends on the density of $X_1 + \dots + X_m$ on the tail, *i.e.* on the neighbourhood of m . This motivates us to introduce the following inequality proved in Corollary 5.4:

$$g_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1}, \text{ for all } x \in [0, m],$$

where g_r is given by (2) and g_r^{*m} is m -th convolution of g_r . Moreover, this inequality is an equality for $x \in [m-1, m]$. We derive from this fact a lower and upper bounds for $\mathcal{M}_r(k, m)$ given in Proposition 3.2 below.

Proposition 3.2. For any $r \in \mathbb{N}_{>0}$, for any $k \in \mathbb{N}$ for any $m \in \mathbb{N}_{>0}$ the following inequalities hold.

$$\begin{aligned} \text{Upper bound: } \mathcal{M}_r(k, m) &\leq \frac{(r!)^m}{(mr-1)!} \int_0^m x^k (m-x)^{mr-1} dx = \frac{k!(r!)^m m^{k+rm}}{(k+mr)!}. \\ \text{Lower bound: } \mathcal{M}_r(k, m) &\geq \frac{(r!)^m}{(mr-1)!} \int_{m-1}^m x^k (m-x)^{mr-1} dx \\ &\geq \frac{k!(r!)^m m^{k+mr}}{(k+mr)!} \left(1 - \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i \right). \end{aligned}$$

These bounds are accurate when k is large since $\lim_{k \rightarrow +\infty} \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i = 0$.

As a consequence of Proposition 3.2, we observe that $S_r(p, m) \leq m^p/m!$. Moreover, lower and upper bounds are asymptotically equivalent when k tends to $+\infty$ therefore $S_r(p, m) \sim m^p/m!$ when p is large. This approximation, well known when $r = 1$ (see [2]), remains true when $r > 1$.

3.3. Sharp upper bound when r is large

Proposition 3.3 proves that the moment $\mathcal{M}_r(k, m)$ is bounded, up to an explicit expression, by the moment of a sum of independent random variables having the same standard exponential distribution.

Proposition 3.3. Let $k \in \mathbb{N}$, $m \in \mathbb{N}_{>0}$, $r \in \mathbb{N}_{>1}$ and $\mathcal{E}_1, \dots, \mathcal{E}_m$ be i.i.d random variables having standard exponential distribution with density $\exp(-x)$.

i) The following inequality holds

$$r^k \mathcal{M}_r(k, m) \leq \left(\frac{r}{r-1} \right)^{2k} \mathbb{E}(\mathcal{E}_1 + \dots + \mathcal{E}_m)^k = \left(\frac{r}{r-1} \right)^{2k} \frac{(m-1+k)!}{(m-1)!}.$$

ii) The upper bound given in i) is sharp since the following limit holds

$$\lim_{r \rightarrow +\infty} r^k \mathcal{M}_r(k, m) = \mathbb{E}(\mathcal{E}_1 + \dots + \mathcal{E}_m)^k = \frac{(m-1+k)!}{(m-1)!}.$$

It seems difficult for authors to find lower bounds which are sharp when r tends to $+\infty$. Finally, we recap hereafter lower and upper bounds for r -associated Stirling number of the second kind:

- Proposition 3.1 provides the following lower and upper bounds

$$\begin{cases} S_r(p, m) \geq \frac{p! m^{p-rm}}{m!(r!)^m (p-rm)!(r+1)^{p-rm}} \\ S_r(p, m) \leq \frac{p! m^{p-rm}}{m!(r!)^m (p-rm)!(r+1)^{p-rm}} \left(1 + \frac{(r+1)^{p-rm} - 1 - r(p-rm)}{mr(r+2)} \right) \end{cases}.$$

These bound are equivalent when $p - rm$, r are fixed and when m tends to $+\infty$.

- Proposition 3.2 provides the following lower and upper bounds

$$\begin{cases} S_r(p, m) \geq \frac{m^p}{m!} - \frac{(m-1)^{p-rm}}{m!} \sum_{i=1}^{mr} \binom{p}{p-rm+i} (m-1)^i \\ S_r(p, m) \leq \frac{m^p}{m!} \end{cases}.$$

These bound are equivalent when m, r are fixed and when p tends to $+\infty$.

- Proposition 3.3 provides the following upper bound when $r \geq 2$

$$S_r(p, m) \leq \frac{p! r^{2(p-rm)} (m-1+p-rm)!}{m!(r!)^m (p-rm)!(r-1)^{2(p-rm)} (m-1)!}.$$

This upper bound is equivalent to $S_r(p, m)$ when $p - rm, m$ are fixed and when r tends to $+\infty$.

4. Numerical experiments

Upper and lower bounds of Stirling numbers of the second kind

According to Propositions 3.1 and 3.2, for all $m \in \mathbb{N}$ and all $p \in \mathbb{N}_{>0}$, Stirling numbers of the second kind do satisfy the following inequalities:

$$S_1(p, m) \leq \underbrace{\min \left\{ \frac{m^p}{m!}, \binom{p}{m} \left(\frac{m}{2} \right)^{p-m} \left(1 + \frac{2^{p-m} + m - p - 1}{3m} \right) \right\}}_{U(p,m)},$$

$$S_1(p, m) \geq \underbrace{\max \left\{ \frac{m^p}{m!} - \frac{(m-1)^{p-m}}{m!} \sum_{i=1}^m \binom{p}{m-i} (m-1)^i, \binom{p}{m} \left(\frac{m}{2} \right)^{p-m} \right\}}_{L(p,m)}.$$

First of all we are going to compare these bounds $U(m, p)$ and $L(p, m)$ to bounds given in Rennie and Dobson [10] reported below:

$$\underbrace{\frac{1}{2}(m^2 + m + 2)m^{p-m-1} - 1}_{L_{rd}(p,m)} \leq S_1(p, m) \leq \underbrace{\frac{1}{2} \binom{p}{m} m^{p-m}}_{U_{rd}(p,m)}.$$

Numerical comparison between $U(p, m)$ and $U_{rd}(p, m)$ is not relevant since $\frac{1}{2} \binom{p}{m} m^{p-m} \geq \binom{p}{m} \left(\frac{m}{2} \right)^{p-m} \left(1 + \frac{2^{p-m} + m - p - 1}{3m} \right)$ holds for $m < p$. Contrarily to the upper bound, the lower bound $L(p, m)$ is not uniformly larger than the one given by Rennie and Dobson; for instance, $31 = L_{rd}(6, 2) > L(6, 2) = 28.5$. Numerical experiments in Figure 1 illustrate that for most integers p, m , $L(p, m)$ is a better approximation of $S_1(p, m)$ than $L_{rd}(p, m)$.

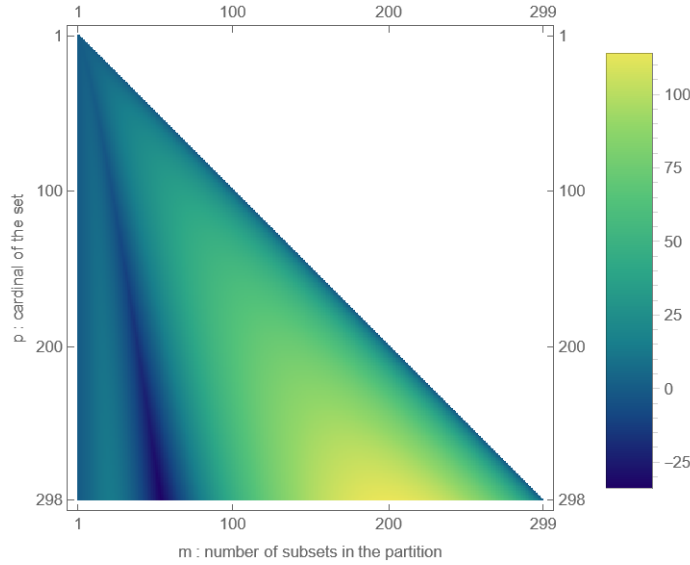


FIG 1. This figure report $\ln(L(p, m)) - \ln(L_{rd}(p, m))$ as a function of m (on the x -axis) and p (on the y -axis). One may observe that for most integers the lower bound $L(p, m)$ is a better approximation of $S_1(p, m)$ than $L_{rd}(p, m)$ (as $\ln(L(p, m)) - \ln(L_{rd}(p, m)) > 0$).

Figure 2 provides a comparison between $L(p, m)$, $U(p, m)$ and $S_1(p, m)$.

Upper bounds of Bell numbers

The Bell number $B(p)$, where $p \in \mathbb{N}_{>0}$, represents the number of partition of $\{1, \dots, p\}$. Since a Bell number is a sum of Stirling numbers of the second kind $B(p) = \sum_{m=1}^p S_1(p, m)$ then, the following inequality occurs:

$$B(p) \leq \underbrace{\sum_{m=0}^p \min \left\{ \frac{m^p}{m!}, \binom{p}{m} \left(\frac{m}{2} \right)^{p-m} \left(1 + \frac{2^{p-m} + m - p - 1}{3m} \right) \right\}}_{=U(p)} \quad \forall p \in \mathbb{N}_{>0}.$$

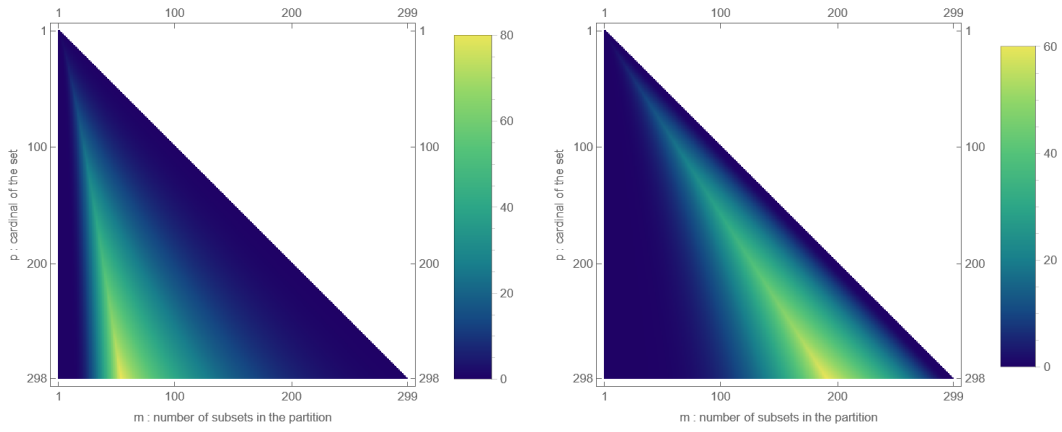


FIG 2. This figure report $\ln(S_1(p, m)) - \ln(L(p, m))$ (on the left) and $\ln(U(p, m)) - \ln(S_1(p, m))$ (on the right) as a function of m and p . These numerical experiments comply with Propositions 3.1 and 3.2 since both lower and upper bounds accurately approximate $S_1(p, m)$ when p is large and m is small or when m is large and $p - m$ is small.

In Figure 3 we compare $U(p)$ with the upper bound: $B(p) \leq U_{bt}(p) = \left(\frac{0.792p}{\ln(p+1)}\right)^p$ given in Berend and Tassa [1].

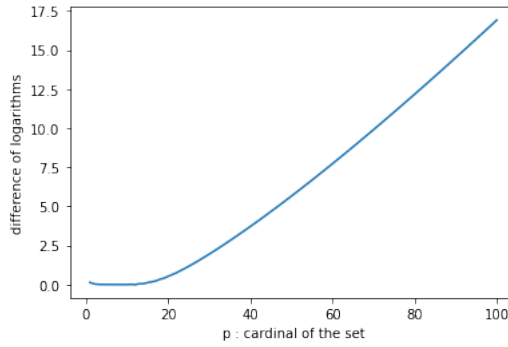


FIG 3. This figure report $\ln(U_{bt}(p)) - \ln(U(p))$ as a function of p . One may observe that when $p \geq 13$, $U(p)$ is more accurate upper bound for $S_1(p, m)$ than $U_{bt}(p)$ (as $\ln(U_{bt}(p)) - \ln(U(p)) > 0$ for $p \geq 13$).

Note that $U(p) \leq \sum_{m=0}^{+\infty} m^p/m! = eB(p)$ (the last equality is due to the Dobiński formula). In Figure 4 we shows that $U(p)/B(p)$ is very close to e when p is large.

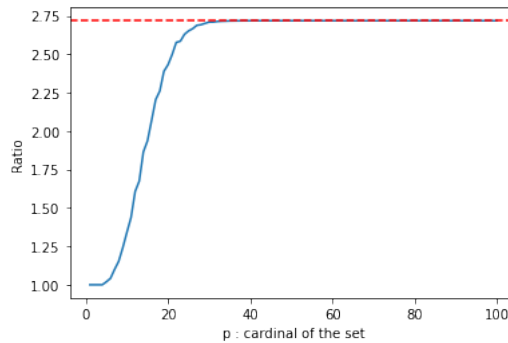


FIG 4. This figure report $U(p)/B(p)$ as a function of p . One may observe that $U(p)/B(p)$ is approximately equal to e when p is large.

5. Proofs

5.1. Proof of Theorem 2.1

The identity given in Lemma 5.1 below, combined with the multinomial formula, allows us to complete the proof of Theorem 2.1.

Lemma 5.1. *Let $m, r \in \mathbb{N}_{>0}$ and $p \in \mathbb{N}$ where $p \geq rm$. The r -associated Stirling numbers of the second kind satisfy the following equality*

$$S_r(p, m) = \frac{p!}{m!} \sum_{i_1 + \dots + i_m = p - rm} \frac{1}{(r + i_1)! \times \dots \times (r + i_m)!},$$

where the sum is computed over all the integers $i_1, \dots, i_m \in \{0, \dots, p - rm\}$ satisfying $i_1 + \dots + i_m = p - rm$.

Proof. Given $i_1, \dots, i_m \in \mathbb{N}$ such that $i_1 + \dots + i_m = p - rm$, let us count the number of ordered partitions of $\{1, \dots, p\}$ in m parts where the first part has $r + i_1$ elements, the second part has $r + i_2$ elements and so on.

There are $\binom{p}{r+i_1}$ possibilities for the first part. There are $\binom{p-r-i_1}{r+i_2}$ possibilities for the second part and so on. Therefore the number of ordered partitions where the first part has $r + i_1$ elements, the second part has $r + i_2$ elements and so on is

$$\frac{p!}{(r + i_1)! \times \dots \times (r + i_m)!}.$$

Consequently, the number of ordered partitions of $\{1, \dots, p\}$ in m parts having at least r elements is

$$\sum_{i_1 + \dots + i_m = p - rm} \frac{p!}{(r + i_1)! \times \dots \times (r + i_m)!}.$$

Finally, when the order is not taken into account, by dividing by $m!$, one may deduce that

$$S_r(p, m) = \frac{p!}{m!} \sum_{i_1 + \dots + i_m = p - rm} \frac{1}{(r + i_1)! \times \dots \times (r + i_m)!}. \quad \square$$

Let us recall the multinomial formula. Given $x_1, \dots, x_m \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$(x_1 + \dots + x_m)^k = \sum_{i_1 + \dots + i_m = k} \frac{k!}{i_1! \times \dots \times i_m!} x_1^{i_1} \times \dots \times x_m^{i_m}.$$

Let $k = p - rm$. Since $\mathbb{E}(X_1^s) = \frac{s!r!}{(s+r)!}$, the multinomial formula and Lemma 5.1 give

$$\begin{aligned} \mathbb{E}[(X_1 + \dots + X_m)^k] &= \sum_{i_1 + \dots + i_m = k} \frac{k!}{i_1! \times \dots \times i_m!} \mathbb{E}(X_1^{i_1}) \times \dots \times \mathbb{E}(X_m^{i_m}), \\ &= (r!)^m k! \sum_{i_1 + \dots + i_m = k} \frac{1}{(r + i_1)! \times \dots \times (r + i_m)!}, \\ &= \frac{m!(r!)^m k!}{(k + rm)!} S_r(k + rm, m) = \frac{m!(r!)^m (p - rm)!}{p!} S_r(p, m), \end{aligned}$$

which finishes the proof of Theorem 2.1.

5.2. Proof of Proposition 3.1

Proposition 3.1 is a consequence of the following lemma.

Lemma 5.2. *Let $k \geq 2$, $a \in (0, 1)$ and $f: x \in [0, 1] \rightarrow a^k + ka^{k-1}(x - a) + c(x - a)^2$ where $c \geq 0$ is such that $f(1) = 1$ (namely $c = \frac{a^{k-1}(ak - a - k) + 1}{(1-a)^2}$). Then $f(x) \geq x^k$ for all $x \in [0, 1]$.*

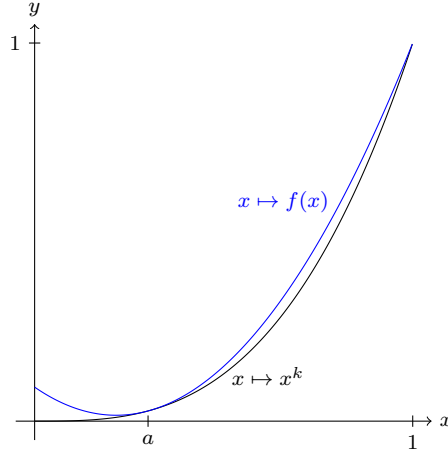


FIG 5. Illustration of the inequality given in Lemma 5.2.

Proof. First note that for all $x \in [0, 1]$ the condition $f(x) \geq x^k$ is equivalent to

$$ka^{k-1}(x-a) + c(x-a)^2 \geq x^k - a^k = (x-a)(x^{k-1} + ax^{k-2} + \dots + a^{k-1}).$$

Note that this inequality holds if and only if

$$\begin{aligned} ka^{k-1} + c(x-a) &\geq x^{k-1} + ax^{k-2} + \dots + a^{k-1} \text{ for all } x \in [a, 1] \text{ and,} \\ ka^{k-1} + c(x-a) &\leq x^{k-1} + ax^{k-2} + \dots + a^{k-1} \text{ for all } x \in [0, a]. \end{aligned}$$

Let $d(x) = ka^{k-1} + c(x-a)$ and $p(x) = x^{k-1} + ax^{k-2} + \dots + a^{k-1}$. Thus $p(a) = d(a)$ and $p(1) = d(1)$, by construction of c . Because p is convex and d is affine, one may deduce that $d(x) \leq p(x)$ once $x \in [0, a]$ and $d(x) \geq p(x)$ once $x \in [a, 1]$, which completes the proof. \square

By Lemma 5.2, for $a = \mathbb{E}(\bar{X}_m) = \frac{1}{r+1}$, $r > 0$, for all $x \in [0, 1]$ we get that

$$\begin{aligned} x^k &\leq \mathbb{E}(\bar{X}_m)^k + k\mathbb{E}(\bar{X}_m)^{k-1}(x - \mathbb{E}(\bar{X}_m)) + c(x - \mathbb{E}(\bar{X}_m))^2 \\ &\leq \frac{1}{(r+1)^k} + \frac{k}{(r+1)^{k-1}} \left(x - \frac{1}{r+1}\right) + c \left(x - \frac{1}{r+1}\right)^2, \end{aligned}$$

where $c = (1 + \frac{1}{r})^2 \left(1 - \frac{kr+1}{(r+1)^k}\right)$. This inequality implies that

$$\begin{aligned} \mathcal{M}_r(k, m) = m^k \mathbb{E}(\bar{X}_m^k) &\leq m^k (\mathbb{E}(\bar{X}_m)^k + \text{cvar}(\bar{X}_m)) = \frac{m^k}{(1+r)^k} + c \frac{rm^{k-1}}{(1+r)^2(2+r)} \\ &= \frac{m^k}{(r+1)^k} + \frac{(r+1)^k - 1 - kr}{(r+1)^k r(r+2)} m^{k-1}. \end{aligned}$$

5.3. Proof of Proposition 3.2

We use in the proof a well-known beta integral (see, for example, [5]): let $a, b \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$ then we have

$$\int_0^x (x-t)^a t^b dt = \frac{a!b!}{(a+b+1)!} x^{a+b+1}. \quad (7)$$

To compute explicitly the density of $X_1 + \dots + X_m$ on the tail $[m-1, m]$, we use the following technical lemma. Let h_r be the following density

$$h_r(x) = \begin{cases} rx^{r-1} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Actually, h_r is the density of $1 - X$ where the density X is g_r (2). Convolution computations are slightly easier to handle with h_r than g_r .

Lemma 5.3. *Let $m \in \mathbb{N}_{>0}$ then the following equality and inequality hold*

$$h_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} x^{mr-1}, \text{ for all } x \in [0, 1], \quad (9)$$

$$h_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} x^{mr-1}, \text{ for all } x \in \mathbb{R}_{\geq 0}. \quad (10)$$

Proof. Let us prove (9) by induction. When $m = 1$, one may notice that whatever $x \in [0, 1]$ we have $h_r^{*m}(x) = h_r(x)$. Let $m \in \mathbb{N}_{>0}$ such that

$$h_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \forall x \in [0, 1].$$

Therefore for $x \in [0, 1]$ we have that

$$\begin{aligned} h_r^{*(m+1)}(x) &= \int_{\mathbb{R}} h_r^{*m}(x-t) h_r(t) dt = \int_0^x h_r^{*m}(x-t) r t^{r-1} dt, \\ &= \frac{r(r!)^m}{(mr-1)!} \int_0^x (x-t)^{mr-1} t^{r-1} dt, \\ &= \frac{r(r!)^m}{(mr-1)!} \frac{(mr-1)!(r-1)!}{((m+1)r-1)!} x^{(m+1)r-1}, \\ &= \frac{(r!)^{m+1}}{((m+1)r-1)!} x^{(m+1)r-1}. \end{aligned}$$

The proof of (10) by induction is quite similar than the proof of (9). When $m = 1$, the result is straightforward. Let $m \in \mathbb{N}_{>0}$ such that

$$h_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \forall x \in \mathbb{R}_{\geq 0}.$$

Therefore for $x \in \mathbb{R}_{\geq 0}$ we have

$$\begin{aligned} h_r^{*(m+1)}(x) &= \int_{\mathbb{R}} h_r^{*m}(x-t) h_r(t) dt = \int_0^x h_r^{*m}(x-t) h_r(t) dt, \\ &\leq \frac{r(r!)^m}{(mr-1)!} \int_0^x (x-t)^{mr-1} t^{r-1} dt, \\ &\leq \frac{r(r!)^m}{(mr-1)!} \frac{(mr-1)!(r-1)!}{((m+1)r-1)!} x^{(m+1)r-1}, \\ &\leq \frac{(r!)^{m+1}}{((m+1)r-1)!} x^{(m+1)r-1}. \end{aligned}$$

□

Note that the only difference between the proof of (9) and (10) is the majorization of $h_r(t)$ by rt^{r-1} . We are now ready to prove the explicit formula for the m -th convolution g_r^{*m} on $[m-1, m]$ and an upper bound on $[0, m]$.

Corollary 5.4. *For all $x \in [0, m]$ we have that*

$$g_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1}.$$

Moreover if $x \in [m-1, m]$ then

$$g_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1}.$$

Proof. By Lemma 5.3, it suffices to prove that $g_r^{*m}(x) = h_r^{*m}(m-x)$ for all $x \in [0, m]$. Let $x \in [0, m]$ and set $z = m-x$, because the density of $(1-X_1) + \dots + (1-X_m)$ is h_r^{*m} then the following equalities occur

$$\begin{aligned} \frac{\partial}{\partial z} \mathbb{P}((1-X_1) + \dots + (1-X_m) \leq z) &= h_r^{*m}(z), \\ \frac{\partial}{\partial z} \mathbb{P}(X_1 + \dots + X_m \geq m-z) &= h_r^{*m}(z), \\ g_r^{*m}(m-z) &= h_r^{*m}(z), \\ g_r^{*m}(x) &= h_r^{*m}(m-x). \end{aligned}$$

□

One may notice that if $r = 1$ and $x \in [m - 1, m]$, then $g_r^{*m}(x) = (m - x)^{m-1}/(m - 1)!$ which is the density in the tail of the Irvin-Hall distribution (see [6, 8]).

Upper and lower bounds given in Proposition 3.2 are straightforward consequences of Corollary 5.4. Indeed, the upper bound is just the beta integral of

$$\int_0^m \frac{(r!)^m}{(mr - 1)!} (m - x)^{mr-1} x^k dx.$$

The lower bound is derived from the following computations

$$\begin{aligned} \frac{(r!)^m}{(mr - 1)!} \int_0^{m-1} x^k (m - x)^{mr-1} dx &= \frac{(r!)^m}{(mr - 1)!} \int_0^{m-1} x^k (1 + (m - 1) - x)^{mr-1} dx, \\ &= \frac{(r!)^m}{(mr - 1)!} \int_0^{m-1} x^k \sum_{i=0}^{mr-1} \binom{mr-1}{i} (m - 1 - x)^i dx, \\ &= \frac{(r!)^m}{(mr - 1)!} \sum_{i=0}^{mr-1} \binom{mr-1}{i} \frac{k! i! (m - 1)^{k+i+1}}{(k + i + 1)!}, \\ &= \frac{(r!)^m k!}{(k + mr)!} \sum_{i=1}^{mr} \binom{k + mr}{k + i} (m - 1)^{k+i}. \end{aligned}$$

Because $\mathcal{M}_r(k, m) \geq \frac{(r!)^m}{(mr-1)!} \int_{m-1}^m x^k (m - x)^{mr-1} dx$ one may deduce the following inequality

$$\mathcal{M}_r(k, m) \geq \frac{k!(r!)^m m^{k+mr}}{(k + mr)!} \left(1 - \frac{(m - 1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k + mr}{k + i} (m - 1)^i \right).$$

Let us assume that $k > mr$, then $\binom{k+mr}{k+i} \leq \binom{k+mr}{k}$, for $1 \leq i \leq mr$, hence

$$\begin{aligned} \frac{(m - 1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k + mr}{k + i} (m - 1)^i &\leq \frac{1}{m^{mr}} \left(1 - \frac{1}{m} \right)^k \sum_{i=1}^{mr} \binom{k + mr}{k} (m - 1)^i \\ &\leq \left(1 - \frac{1}{m} \right)^k \binom{k + mr}{k} \frac{((m - 1)^{mr+1} - (m - 1))}{m - 2} \\ &\leq \left(1 - \frac{1}{m} \right)^k (k + mr)^{mr} \frac{(m - 1)^{2mr}}{(mr)!} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

5.4. Proof of Proposition 3.3

Proof. i) Note that the density of the random variable $Y_i = (r - 1)X_i$ on $[0, r - 1]$ is

$$f(x) = \frac{r}{r - 1} \left(1 - \frac{x}{r - 1} \right)^{r-1}.$$

Due to the following inequality for all $x \in [0, r - 1]$

$$\frac{r}{r - 1} \left(1 - \frac{x}{r - 1} \right)^{r-1} \leq \frac{r}{r - 1} \exp(-x),$$

one may deduce that

$$(r - 1)^k \mathbb{E} \left(\sum_{i=1}^m X_i \right)^k = \mathbb{E} \left(\sum_{i=1}^m Y_i \right)^k \leq \left(\frac{r}{r - 1} \right)^k \mathbb{E} \left(\sum_{i=1}^m \mathcal{E}_i \right)^k = \left(\frac{r}{r - 1} \right)^k \frac{(m - 1 + k)!}{(m - 1)!},$$

since $\sum_{i=1}^m \mathcal{E}_i$ has an Erlang distribution (the density of $\sum_{i=1}^m \mathcal{E}_i$ is $h(x) = \frac{x^{m-1} \exp(-x)}{(m-1)!}$) whose moment of order k is $\frac{(m-1+k)!}{(m-1)!}$. This inequality completes the proof of i).

ii) The density of the random variable $(r-1)X_1$ converges pointwise to the density of \mathcal{E}_1 namely $\lim_{r \rightarrow +\infty} f(x) = \exp(-x)$, for all $x \in [0, +\infty)$. Therefore, the following limit holds

$$\lim_{r \rightarrow +\infty} x^k f^{*m}(x) = \frac{x^{m+k-1} \exp(-x)}{(m-1)!}.$$

Finally, the Dominated Convergence Theorem gives that

$$\begin{aligned} \lim_{r \rightarrow +\infty} (r-1)^k \mathbb{E} \left(\sum_{i=1}^m X_i \right)^k &= \lim_{r \rightarrow +\infty} \int_0^{+\infty} \frac{x^{m+k-1} \exp(-x)}{(m-1)!} dx, \\ &= \mathbb{E} \left(\sum_{i=1}^m \mathcal{E}_i \right)^k = \frac{(m-1+k)!}{(m-1)!}. \end{aligned}$$

□

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