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# Beta distribution and associated Stirling numbers of the second kind

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## Abstract

This article gives a formula for associated Stirling numbers of the second kind based on the moment of a sum of independent random variables having a beta distribution. From this formula we deduce, using probabilistic approaches, lower and upper bounds for these numbers.

## 1 Introduction

Classical Stirling numbers of the second kind  $S(p, m)$  counts the number of all partitions of  $\{1, \dots, p\}$  into  $m$  nonempty subsets, for  $p \in \mathbb{N}_{>0}$  and  $m \in \mathbb{N}$ . More generally, the  $r$ -associated Stirling number  $S_r(p, m)$ , with  $r \in \mathbb{N}_{>0}$ , is the number of all partitions of  $\{1, \dots, p\}$ , into  $m$  subsets where each subset contains at least  $r$  elements [3, p. 221]. Obviously  $S(p, m) = S_1(p, m)$ . Some sub-sequences of the multi-sequence  $\{S_r(p, m) : p, m, r \in \mathbb{N}_{>0}, p \geq rm\}$  are in the On-Line Encyclopedia of Integer Sequences (OEIS) [11]. Specifically, arrays  $\{S_1(p, m)\}_{\{p, m\}}$ ,  $\{S_2(p, m)\}_{\{p, m\}}$ ,  $\{S_3(p, m)\}_{\{p, m\}}$  are identified in the OEIS by A008277, A008299, A059022. Moreover, sequences  $\{S_2(k+6, 3)\}_{\{k\}}$ ,  $\{S_2(k+8, 4)\}_{\{k\}}$ , representing the number of ways of placing  $k+6$  or  $k+8$  labelled balls into 3 or 4 indistinguishable boxes with at least 2 balls in each box are identified in the OEIS by A000478, A058844.

There are well-known connections between Stirling numbers of the second kind and probability theory. For example, sequences  $S_1(p, m)$  and  $S_2(p, m)$  are asymptotically normal when  $p$  tends to  $+\infty$  [6, 4]. More precisely, when  $r \in \{1, 2\}$ , the following convergence in distribution occurs

$$\frac{Y_p - \mathbb{E}(Y_p)}{\sqrt{\text{var}(Y_p)}} \xrightarrow{p \rightarrow +\infty} \mathcal{N}(0, 1) \text{ where } \mathbb{P}(Y_p = m) = \frac{S_r(p, m)}{\sum_{k=1}^p S_r(p, k)}, \text{ for all } m \in \mathbb{N}_{>0}.$$

Otherwise, according to Dobiński's formula, the moment of order  $p$  of a Poisson distribution with parameter  $\lambda \geq 0$  is  $\sum_{m=1}^p S_1(p, m) \lambda^m$  (see, e.g. [3, p. 211]). However, to our knowledge, there is no close formula in the literature for  $S_r(k, m)$  based on moments of a sum of independent and identically distributed (i.i.d) random variables. The main result in this article is Theorem 2.1 providing the following new identity

$$S_r(p, m) = \frac{p!}{m!(r!)^m(p-rm)!} \mathbb{E} \left[ (X_1 + \dots + X_m)^{p-rm} \right] \quad (1)$$

where  $X_1, \dots, X_p$  are i.i.d random variables having beta distribution with parameter  $(1, r)$ . Note that the beta distribution with parameter  $(1, 1)$  is a uniform distribution on  $[0, 1]$ . Thus, when  $r = 1$ , the above formula is quite simple:

$$S_1(p, m) = \binom{p}{m} \mathbb{E} (Z^{p-m})$$

where  $Z = \sum_{i=1}^m X_i$  has the Irwin-Hall distribution on  $[0, m]$ . Propositions 3.1, 3.2 and 3.3 give upper and lower bounds for  $\mathbb{E} \left[ (X_1 + \dots + X_m)^{p-rm} \right]$ . These bounds are sharp when one parameter  $m$ ,  $r$ , or  $p - rm$  tends to  $+\infty$  and provides thus accurate approximations of  $r$ -associated Stirling numbers.

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## 2 Closed formula for Stirling numbers and moments of random variables

The density  $g_r$  of a beta distribution with parameters  $(1, r)$  where  $r \in \mathbb{N}_{>0}$  is

$$g_r(x) = \begin{cases} r(1-x)^{r-1} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}. \quad (2)$$

Let  $X_1, \dots, X_m$  be independent random variables having the same beta  $(1, r)$  distribution. The moment of order  $k \in \mathbb{N}$  of the sum of these variables is defined as follows

$$\mathcal{M}_r(k, m) = \mathbb{E} \left[ (X_1 + \dots + X_m)^k \right]. \quad (3)$$

Theorem 2.1 provides a closed formula for the Stirling numbers of the second kind based on the moment  $\mathcal{M}_r(k, m)$ .

**Theorem 2.1.** *Let  $m, r \in \mathbb{N}_{>0}$  and  $p \in \mathbb{N}$  where  $p \geq rm$ . The Stirling numbers of the second kind satisfy the following identity*

$$S_r(p, m) = \frac{p!}{m!(r!)^m (p - rm)!} \mathcal{M}_r(p - rm, m). \quad (4)$$

From Theorem 2.1, one may deduce that  $\mathbb{E}(Z^k) = S_1(m + k, m) / \binom{m+k}{m}$  where  $Z$  has an Irwin-Hall's distribution on  $[0, m]$  [8, 7]. Note that the moment generating function of  $Z$  is  $\sum_{k \geq 0} \mathbb{E}(Z^k) t^k / k! = ((\exp(t) - 1)/t)^m$  therefore we recover the well-known exponential generating function of the Stirling numbers of the second kind  $\sum_{p \geq m} S_1(p, m) t^p / p! = (\exp(t) - 1)^m / m!$  (see Theorem 3.3 page 52 in [9]). The above expression of  $S_r(p, m)$  is explicit up to the computation of the moment  $\mathcal{M}_r(k, m)$ . Whereas computing explicitly  $\mathcal{M}_r(k, m)$  might be technical, lower bounds, upper bounds and approximations of  $\mathcal{M}_r(k, m)$  are tractable as illustrated in the following section.

## 3 Upper and lower bounds

Hereafter, we will use probabilistic approaches to derive upper and lower bounds for the moment  $\mathcal{M}_r(k, m)$ .

### 3.1 Sharp upper and lower bounds when $m$ is large

Let  $\bar{X}_m = (X_1 + \dots + X_m)/m$ , the Jensen's inequality provides the following lower bound

$$\mathcal{M}_r(k, m) = m^k \mathbb{E} \left( \bar{X}_m^k \right) \geq m^k \mathbb{E} \left( \bar{X}_m \right)^k = \frac{m^k}{(r+1)^k}. \quad (5)$$

This inequality relies on the linearization of the function  $q(x) = x^k$  at  $x_0 = \mathbb{E}(\bar{X}_m) = \frac{1}{r+1}$ . Specifically, the following inequality holds for all  $x \in [0, 1]$

$$x^k = q(x) \geq q(x_0) + q'(x_0)(x - x_0) = \frac{1}{(r+1)^k} + \frac{k}{(r+1)^{k-1}} \left( x - \frac{1}{r+1} \right). \quad (6)$$

Moreover one may choose  $c \geq 0$  for which the following inequality is true for all  $x \in [0, 1]$  (see Lemma 5.2)

$$x^k \leq q(x_0) + q'(x_0)(x - x_0) + c(x - x_0)^2. \quad (7)$$

Proposition 3.1 below is a consequence of inequalities (6) and (7).

**Proposition 3.1.** *Let  $r, m \in \mathbb{N}_{>0}$  and  $k \in \mathbb{N}$ . The following inequality holds*

$$\frac{m^k}{(r+1)^k} \leq \mathcal{M}_r(k, m) \leq \frac{m^k}{(r+1)^k} + m^{k-1} \frac{(r+1)^k - 1 - kr}{(r+1)^k r(r+2)}. \quad (8)$$

The leading term when  $m$  is large in both the lower and upper bounds is  $m^k / (r+1)^k$ . Therefore, lower and upper bounds are asymptotically equivalent when  $k \in \mathbb{N}_{\geq 0}$  and  $r \in \mathbb{N}_{>0}$  are fixed and  $m$  tends to  $+\infty$ . These bounds are accurate when  $m$  is large since  $\bar{X}_m$  converges to  $\mathbb{E}(\bar{X}_m)$  and both inequalities (6) and (7) are accurate on the neighbourhood of  $\mathbb{E}(\bar{X}_m)$ .

### 3.2 Sharp upper and lower bounds when $k$ is large

Asymptotic behaviour of moments, when  $k$  is large, depends on the density of  $X_1 + \dots + X_m$  on the tail, *i.e.* on the neighbourhood of  $m$ . This motivates us to introduce the following inequality proved in Corollary 5.4:

$$g_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1}, \text{ for all } x \in [0, m] \quad (9)$$

where  $g_r$  is given by (2) and  $g_r^{*m}$  is  $m$ -th convolution of  $g_r$ . Moreover, this inequality is an equality for  $x \in [m-1, m]$ . We derive from this fact a lower and upper bounds for  $\mathcal{M}_r(k, m)$  given in Proposition 3.2 below.

**Proposition 3.2.** *For any  $r \in \mathbb{N}_{>0}$ , for any  $k \in \mathbb{N}$  for any  $m \in \mathbb{N}_{>0}$  the following inequalities hold.*

$$\begin{aligned} \mathcal{M}_r(k, m) &\leq \frac{(r!)^m}{(mr-1)!} \int_0^m x^k (m-x)^{mr-1} dx = \frac{k!(r!)^m m^{k+rm}}{(k+mr)!}. \\ \mathcal{M}_r(k, m) &\geq \frac{(r!)^m}{(mr-1)!} \int_{m-1}^m x^k (m-x)^{mr-1} dx \\ &\geq \frac{k!(r!)^m m^{k+mr}}{(k+mr)!} \left( 1 - \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i \right). \end{aligned}$$

These bounds are accurate when  $k$  is large since

$$\lim_{k \rightarrow +\infty} \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i = 0.$$

As a consequence of Proposition 3.2, we observe that  $S_r(p, m) \leq m^p/m!$ . Moreover, lower and upper bounds are asymptotically equivalent when  $k$  tends to  $+\infty$  therefore  $S_r(p, m) \sim m^p/m!$  when  $p$  is large. This approximation, well known when  $r = 1$  (see [2]), remains true when  $r > 1$ .

### 3.3 Sharp upper bound when $r$ is large

Proposition 3.3 proves that the moment  $\mathcal{M}_r(k, m)$  is bounded, up to an explicit expression, by the moment of a sum of independent random variables having the same standard exponential distribution.

**Proposition 3.3.** *Let  $k \in \mathbb{N}$ ,  $m \in \mathbb{N}_{>0}$ ,  $r \in \mathbb{N}_{>1}$  and  $\mathcal{E}_1, \dots, \mathcal{E}_m$  be *i.i.d* random variables having standard exponential distribution with density  $\exp(-x)$ .*

*i) The following inequality holds*

$$r^k \mathcal{M}_r(k, m) \leq \left( \frac{r}{r-1} \right)^{2k} \mathbb{E}(\mathcal{E}_1 + \dots + \mathcal{E}_m)^k = \left( \frac{r}{r-1} \right)^{2k} \frac{(m-1+k)!}{(m-1)!}. \quad (10)$$

*ii) The upper bound given in i) is sharp since the following limit holds*

$$\lim_{r \rightarrow +\infty} r^k \mathcal{M}_r(k, m) = \mathbb{E}(\mathcal{E}_1 + \dots + \mathcal{E}_m)^k = \frac{(m-1+k)!}{(m-1)!}. \quad (11)$$

It seems difficult for authors to find lower bounds which are sharp when  $r$  tends to  $+\infty$ . Finally, we recap hereafter lower and upper bounds for  $r$ -associated Stirling number of the second kind:

- Proposition 3.1 provides the following lower and upper bounds

$$\begin{cases} S_r(p, m) \geq \frac{p! m^{p-rm}}{m!(r!)^m (p-rm)!(r+1)^{p-rm}} \\ S_r(p, m) \leq \frac{p! m^{p-rm}}{m!(r!)^m (p-rm)!(r+1)^{p-rm}} \left( 1 + \frac{(r+1)^{p-rm} - 1 - r(p-rm)}{mr(r+2)} \right). \end{cases} \quad (12)$$

These bound are equivalent when  $p - rm$ ,  $r$  are fixed and when  $m$  tends to  $+\infty$ .

- Proposition 3.2 provides the following lower and upper bounds

$$\begin{cases} S_r(p, m) \geq \frac{m^p}{m!} - \frac{(m-1)^{p-rm}}{m!} \sum_{i=1}^{mr} \binom{p}{p-rm+i} (m-1)^i \\ S_r(p, m) \leq \frac{m^p}{m!}. \end{cases} \quad (13)$$

These bound are equivalent when  $m, r$  are fixed and when  $p$  tends to  $+\infty$ .

- Proposition 3.3 provides the following upper bound when  $r \geq 2$

$$S_r(p, m) \leq \frac{p!r^{2(p-rm)}(m-1+p-rm)!}{m!(r!)^m(p-rm)!(r-1)^{2(p-rm)}(m-1)!}. \quad (14)$$

This upper bound is equivalent to  $S_r(p, m)$  when  $p-rm, m$  are fixed and when  $r$  tends to  $+\infty$ .

## 4 Numerical experiments

### 4.1 Upper and lower bounds of Stirling numbers of the second kind

According to Propositions 3.1 and 3.2, for all  $m \in \mathbb{N}$  and all  $p \in \mathbb{N}_{>0}$ , Stirling numbers of the second kind do satisfy the following inequalities:

$$\begin{aligned} S_1(p, m) &\leq \underbrace{\min \left\{ \frac{m^p}{m!}, \binom{p}{m} \left( \frac{m}{2} \right)^{p-m} \left( 1 + \frac{2^{p-m} + m - p - 1}{3m} \right) \right\}}_{U(p, m)} \\ S_1(p, m) &\geq \underbrace{\max \left\{ \frac{m^p}{m!} - \frac{(m-1)^{p-m}}{m!} \sum_{i=1}^m \binom{p}{m-i} (m-1)^i, \binom{p}{m} \left( \frac{m}{2} \right)^{p-m} \right\}}_{L(p, m)}. \end{aligned}$$

First of all we are going to compare these bounds  $U(m, p)$  and  $L(p, m)$  to bounds given in Rennie and Dobson [10] reported below:

$$\underbrace{\frac{1}{2}(m^2 + m + 2)m^{p-m-1} - 1}_{L_{rd}(p, m)} \leq S_1(p, m) \leq \underbrace{\frac{1}{2} \binom{p}{m} m^{p-m}}_{U_{rd}(p, m)}. \quad (15)$$

Numerical comparison between  $U(p, m)$  and  $U_{rd}(p, m)$  is not relevant since

$$\frac{1}{2} \binom{p}{m} m^{p-m} \geq \binom{p}{m} \left( \frac{m}{2} \right)^{p-m} \left( 1 + \frac{2^{p-m} + m - p - 1}{3m} \right)$$

holds for  $m < p$ . Contrarily to the upper bound, the lower bound  $L(p, m)$  is not uniformly larger than the one given by Rennie and Dobson; for instance,  $31 = L_{rd}(6, 2) > L(6, 2) = 28.5$ . Numerical experiments in Figure 1 illustrate that for most integers  $p, m$ ,  $L(p, m)$  is a better approximation of  $S_1(p, m)$  than  $L_{rd}(p, m)$ .

Figure 2 provides a comparison between  $L(p, m), U(p, m)$  and  $S_1(p, m)$ .

### 4.2 Upper bounds of Bell numbers

The Bell number  $B(p)$ , where  $p \in \mathbb{N}_{>0}$ , represents the number of partition of  $\{1, \dots, p\}$ . Since a Bell number is a sum of Stirling numbers of the second kind  $B(p) = \sum_{m=1}^p S_1(p, m)$  then, the following inequality occurs:

$$B(p) \leq \underbrace{\sum_{m=0}^p \min \left\{ \frac{m^p}{m!}, \binom{p}{m} \left( \frac{m}{2} \right)^{p-m} \left( 1 + \frac{2^{p-m} + m - p - 1}{3m} \right) \right\}}_{=U(p)} \quad (16)$$

for all  $p \in \mathbb{N}_{>0}$ . In Figure 3 we compare  $U(p)$  with the upper bound:  $B(p) \leq U_{bt}(p) = \left( \frac{0.792p}{\ln(p+1)} \right)^p$  given in Berend and Tassa [1].

Note that  $U(p) \leq \sum_{m=0}^{+\infty} m^p/m! = eB(p)$  (the last equality is due to the Dobiński formula). In Figure 4 we show that  $U(p)/B(p)$  is very close to  $e$  when  $p$  is large.

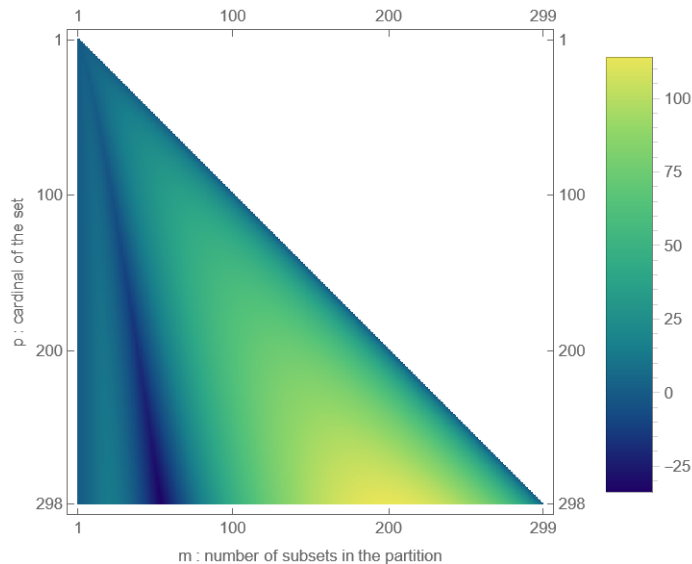


Figure 1: This figure report  $\ln(L(p, m)) - \ln(L_{rd}(p, m))$  as a function of  $m$  (on the  $x$ -axis) and  $p$  (on the  $y$ -axis). One may observe that for most integers the lower bound  $L(p, m)$  is a better approximation of  $S_1(p, m)$  than  $L_{rd}(p, m)$  (as  $\ln(L(p, m)) - \ln(L_{rd}(p, m)) > 0$ ).

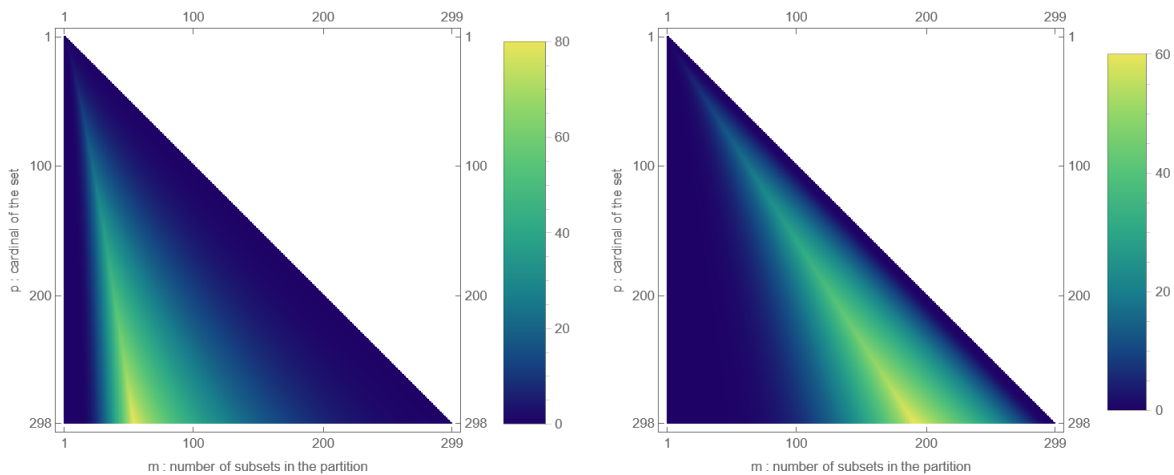


Figure 2: This figure report  $\ln(S_1(p, m)) - \ln(L(p, m))$  (on the left) and  $\ln(U(p, m)) - \ln(S_1(p, m))$  (on the right) as a function of  $m$  and  $p$ . These numerical experiments comply with Propositions 3.1 and 3.2 since both lower and upper bounds accurately approximate  $S_1(p, m)$  when  $p$  is large and  $m$  is small or when  $m$  is large and  $p - m$  is small.

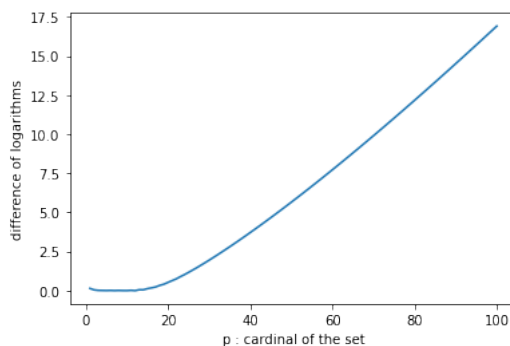


Figure 3: This figure report  $\ln(U_{bt}(p)) - \ln(U(p))$  as a function of  $p$ . One may observe that when  $p \geq 13$ ,  $U(p)$  is more accurate upper bound for  $S_1(p, m)$  than  $U_{bt}(p)$  (as  $\ln(U_{bt}(p)) - \ln(U(p)) > 0$  for  $p \geq 13$ ).

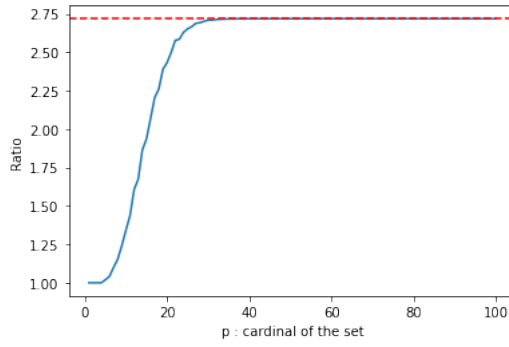


Figure 4: This figure report  $U(p)/B(p)$  as a function of  $p$ . One may observe that  $U(p)/B(p)$  is approximately equal to  $e$  when  $p$  is large.

## 5 Proofs

### 5.1 Proof of Theorem 2.1

The identity given in Lemma 5.1 below, combined with the multinomial formula, allows us to complete the proof of Theorem 2.1.

**Lemma 5.1.** *Let  $m, r \in \mathbb{N}_{>0}$  and  $p \in \mathbb{N}$  where  $p \geq rm$ . The  $r$ -associated Stirling numbers of the second kind satisfy the following equality*

$$S_r(p, m) = \frac{p!}{m!} \sum_{i_1 + \dots + i_m = p - rm} \frac{1}{(r + i_1)! \times \dots \times (r + i_m)!}$$

where the sum is computed over all the integers  $i_1, \dots, i_m \in \{0, \dots, p - rm\}$  satisfying  $i_1 + \dots + i_m = p - rm$ .

*Proof.* Given  $i_1, \dots, i_m \in \mathbb{N}$  such that  $i_1 + \dots + i_m = p - rm$ , let us count the number of ordered partitions of  $\{1, \dots, p\}$  in  $m$  parts where the first part has  $r + i_1$  elements, the second part has  $r + i_2$  elements and so on.

There are  $\binom{p}{r+i_1}$  possibilities for the first part. There are  $\binom{p-r-i_1}{r+i_2}$  possibilities for the second part and so on. Therefore the number of ordered partitions where the first part has  $r + i_1$  elements, the second part has  $r + i_2$  elements and so on is

$$\frac{p!}{(r + i_1)! \times \dots \times (r + i_m)!}.$$

Consequently, the number of ordered partitions of  $\{1, \dots, p\}$  in  $m$  parts having at least  $r$  elements is

$$\sum_{i_1 + \dots + i_m = p - rm} \frac{p!}{(r + i_1)! \times \dots \times (r + i_m)!}.$$

Finally, when the order is not taken into account, by dividing by  $m!$ , one may deduce that

$$S_r(p, m) = \frac{p!}{m!} \sum_{i_1 + \dots + i_m = p - rm} \frac{1}{(r + i_1)! \times \dots \times (r + i_m)!}.$$

□

*Proof.* Theorem 2.1 Let us recall the multinomial formula. Given  $x_1, \dots, x_m \in \mathbb{R}$  and  $k \in \mathbb{N}$ , we have

$$(x_1 + \dots + x_m)^k = \sum_{i_1 + \dots + i_m = k} \frac{k!}{i_1! \times \dots \times i_m!} x_1^{i_1} \times \dots \times x_m^{i_m}.$$

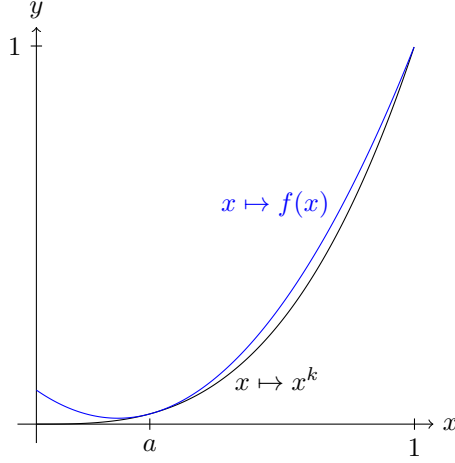


Figure 5: Illustration of the inequality given in Lemma 5.2.

Let  $k = p - rm$ . Since  $\mathbb{E}(X_1^s) = \frac{s!r!}{(s+r)!}$ , the multinomial formula and Lemma 5.1 give

$$\begin{aligned}
\mathbb{E} \left[ (X_1 + \dots + X_m)^k \right] &= \sum_{i_1 + \dots + i_m = k} \frac{k!}{i_1! \times \dots \times i_m!} \mathbb{E}(X_1^{i_1}) \times \dots \times \mathbb{E}(X_m^{i_m}) \\
&= (r!)^m k! \sum_{i_1 + \dots + i_m = k} \frac{1}{(r + i_1)! \times \dots \times (r + i_m)!} \\
&= \frac{m!(r!)^m k!}{(k + rm)!} S_r(k + rm, m) \\
&= \frac{m!(r!)^m (p - rm)!}{p!} S_r(p, m)
\end{aligned}$$

which finishes the proof. □

## 5.2 Proof of Proposition 3.1

Proposition 3.1 is a consequence of the following lemma.

**Lemma 5.2.** *Let  $k \geq 2$ ,  $a \in (0, 1)$  and  $f: x \in [0, 1] \rightarrow a^k + ka^{k-1}(x-a) + c(x-a)^2$  where  $c \geq 0$  is such that  $f(1) = 1$  (namely  $c = \frac{a^{k-1}(ak-a-k)+1}{(1-a)^2}$ ). Then  $f(x) \geq x^k$  for all  $x \in [0, 1]$ .*

*Proof.* First note that for all  $x \in [0, 1]$  the condition  $f(x) \geq x^k$  is equivalent to

$$ka^{k-1}(x-a) + c(x-a)^2 \geq x^k - a^k = (x-a)(x^{k-1} + ax^{k-2} + \dots + a^{k-1}).$$

Note that this inequality holds if and only if

$$\begin{aligned}
ka^{k-1} + c(x-a) &\geq x^{k-1} + ax^{k-2} + \dots + a^{k-1} \text{ for all } x \in [a, 1] \text{ and} \\
ka^{k-1} + c(x-a) &\leq x^{k-1} + ax^{k-2} + \dots + a^{k-1} \text{ for all } x \in [0, a].
\end{aligned}$$

Let  $d(x) = ka^{k-1} + c(x-a)$  and  $p(x) = x^{k-1} + ax^{k-2} + \dots + a^{k-1}$ . Thus  $p(a) = d(a)$  and  $p(1) = d(1)$ , by construction of  $c$ . Because  $p$  is convex and  $d$  is affine, one may deduce that  $d(x) \leq p(x)$  once  $x \in [0, a]$  and  $d(x) \geq p(x)$  once  $x \in [a, 1]$ , which completes the proof. □

*Proof.* Proposition 3.1 By Lemma 5.2, for  $a = \mathbb{E}(\bar{X}_m) = \frac{1}{r+1}$ ,  $r > 0$ , for all  $x \in [0, 1]$  we get that

$$\begin{aligned}
x^k &\leq \mathbb{E}(\bar{X}_m)^k + k\mathbb{E}(\bar{X}_m)^{k-1} (x - \mathbb{E}(\bar{X}_m)) + c (x - \mathbb{E}(\bar{X}_m))^2 \\
&\leq \frac{1}{(r+1)^k} + \frac{k}{(r+1)^{k-1}} \left( x - \frac{1}{r+1} \right) + c \left( x - \frac{1}{r+1} \right)^2
\end{aligned}$$



where  $c = (1 + \frac{1}{r})^2 \left(1 - \frac{kr+1}{(r+1)^k}\right)$ . This inequality implies that

$$\begin{aligned} \mathcal{M}_r(k, m) = m^k \mathbb{E}(\overline{X}_m^k) &\leq m^k (\mathbb{E}(\overline{X}_m)^k + \text{cvar}(\overline{X}_m)) \\ &\leq \frac{m^k}{(1+r)^k} + c \frac{rm^{k-1}}{(1+r)^2(2+r)} \\ &\leq \frac{m^k}{(r+1)^k} + \frac{(r+1)^k - 1 - kr}{(r+1)^k r (r+2)} m^{k-1}. \end{aligned}$$

□

### 5.3 Proof of Proposition 3.2

We use in the proof a well-known beta integral (see, for example, [5]): let  $a, b \in \mathbb{N}$  and  $x \in \mathbb{R}_{>0}$  then we have

$$\int_0^x (x-t)^a t^b dt = \frac{a!b!}{(a+b+1)!} x^{a+b+1}. \quad (17)$$

To compute explicitly the density of  $X_1 + \dots + X_m$  on the tail  $[m-1, m]$ , we use the following technical lemma. Let  $h_r$  be the following density

$$h_r(x) = \begin{cases} rx^{r-1} & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}. \quad (18)$$

Actually,  $h_r$  is the density of  $1 - X$  where the density  $X$  is  $g_r$  (2). Convolution computations are slightly easier to handle with  $h_r$  than  $g_r$ .

**Lemma 5.3.** *Let  $m \in \mathbb{N}_{>0}$  then the following equality and inequality hold*

$$h_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} x^{mr-1}, \text{ for all } x \in [0, 1]. \quad (19)$$

$$h_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} x^{mr-1}, \text{ for all } x \in \mathbb{R}_{\geq 0}. \quad (20)$$

*Proof.* Let us prove (19) by induction. When  $m = 1$ , one may notice that whatever  $x \in [0, 1]$  we have  $h_r^{*m}(x) = h_r(x)$ . Let  $m \in \mathbb{N}_{>0}$  such that

$$h_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \forall x \in [0, 1].$$

Therefore for  $x \in [0, 1]$  we have that

$$\begin{aligned} h_r^{*(m+1)}(x) &= \int_{\mathbb{R}} h_r^{*m}(x-t) h_r(t) dt = \int_0^x h_r^{*m}(x-t) r t^{r-1} dt \\ &= \frac{r(r!)^m}{(mr-1)!} \int_0^x (x-t)^{mr-1} t^{r-1} dt \\ &= \frac{r(r!)^m}{(mr-1)!} \frac{(mr-1)!(r-1)!}{((m+1)r-1)!} x^{(m+1)r-1} \\ &= \frac{(r!)^{m+1}}{((m+1)r-1)!} x^{(m+1)r-1}. \end{aligned}$$

The proof of (20) by induction is quite similar than the proof of (19). When  $m = 1$ , the result is straightforward. Let  $m \in \mathbb{N}_{>0}$  such that

$$h_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} x^{mr-1} \quad \forall x \in \mathbb{R}_{\geq 0}.$$

Therefore for  $x \in \mathbb{R}_{\geq 0}$  we have

$$\begin{aligned}
h_r^{*m+1}(x) &= \int_{\mathbb{R}} h_r^{*m}(x-t)h_r(t)dt = \int_0^x h_r^{*m}(x-t)h_r(t)dt \\
&\leq \frac{r(r!)^m}{(mr-1)!} \int_0^x (x-t)^{mr-1}t^{r-1}dt \\
&\leq \frac{r(r!)^m}{(mr-1)!} \frac{(mr-1)!(r-1)!}{((m+1)r-1)!} x^{(m+1)r-1} \\
&\leq \frac{(r!)^{m+1}}{((m+1)r-1)!} x^{(m+1)r-1}.
\end{aligned}$$

□

Note that the only difference between the proof of (19) and (20) is the majorization of  $h_r(t)$  by  $rt^{r-1}$ . We are now ready to prove the explicit formula for the  $m$ -th convolution  $g_r^{*m}$  on  $[m-1, m]$  and an upper bound on  $[0, m]$ .

**Corollary 5.4.** *For all  $x \in [0, m]$  we have that*

$$g_r^{*m}(x) \leq \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1}.$$

Moreover if  $x \in [m-1, m]$  then

$$g_r^{*m}(x) = \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1}.$$

*Proof.* By Lemma 5.3, it suffices to prove that  $g_r^{*m}(x) = h_r^{*m}(m-x)$  for all  $x \in [0, m]$ . Let  $x \in [0, m]$  and set  $z = m-x$ , because the density of  $(1-X_1) + \dots + (1-X_m)$  is  $h_r^{*m}$  then the following equalities occur

$$\begin{aligned}
\frac{\partial}{\partial z} \mathbb{P}((1-X_1) + \dots + (1-X_m) \leq z) &= h_r^{*m}(z) \\
\frac{\partial}{\partial z} \mathbb{P}(X_1 + \dots + X_m \geq m-z) &= h_r^{*m}(z) \\
g_r^{*m}(m-z) &= h_r^{*m}(z) \\
g_r^{*m}(x) &= h_r^{*m}(m-x).
\end{aligned}$$

□

One may notice that if  $r = 1$  and  $x \in [m-1, m]$ , then  $g_r^{*m}(x) = (m-x)^{m-1}/(m-1)!$  which is the density in the tail of the Irvin-Hall distribution (see [7, 8]).

Upper and lower bounds given in Proposition 3.2 are straightforward consequences of Corollary 5.4. Indeed, the upper bound is just the beta integral of

$$\int_0^m \frac{(r!)^m}{(mr-1)!} (m-x)^{mr-1} x^k dx.$$

*Proof.* Proposition 3.2 The lower bound is derived from the following computations

$$\begin{aligned}
&\frac{(r!)^m}{(mr-1)!} \int_0^{m-1} x^k (m-x)^{mr-1} dx \\
&= \frac{(r!)^m}{(mr-1)!} \int_0^{m-1} x^k (1+(m-1)-x)^{mr-1} dx \\
&= \frac{(r!)^m}{(mr-1)!} \int_0^{m-1} x^k \sum_{i=0}^{mr-1} \binom{mr-1}{i} (m-1-x)^i dx \\
&= \frac{(r!)^m}{(mr-1)!} \sum_{i=0}^{mr-1} \binom{mr-1}{i} \frac{k!i!(m-1)^{k+i+1}}{(k+i+1)!} \\
&= \frac{(r!)^m k!}{(k+mr)!} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^{k+i}.
\end{aligned}$$

Because  $\mathcal{M}_r(k, m) \geq \frac{(r!)^m}{(mr-1)!} \int_{m-1}^m x^k (m-x)^{mr-1} dx$  one may deduce the following inequality

$$\mathcal{M}_r(k, m) \geq \frac{k!(r!)^m m^{k+mr}}{(k+mr)!} \left( 1 - \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i \right).$$

Let us assume that  $k > mr$ , then  $\binom{k+mr}{k+i} \leq \binom{k+mr}{k}$ , for  $1 \leq i \leq mr$ , hence

$$\begin{aligned} & \frac{(m-1)^k}{m^{k+mr}} \sum_{i=1}^{mr} \binom{k+mr}{k+i} (m-1)^i \\ & \leq \frac{1}{m^{mr}} \left( 1 - \frac{1}{m} \right)^k \sum_{i=1}^{mr} \binom{k+mr}{k} (m-1)^i \\ & \leq \left( 1 - \frac{1}{m} \right)^k \binom{k+mr}{k} \frac{((m-1)^{mr+1} - (m-1))}{m-2} \\ & \leq \left( 1 - \frac{1}{m} \right)^k (k+mr)^{mr} \frac{(m-1)^{2mr}}{(mr)!} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

□

## 5.4 Proof of Proposition 3.3

*Proof.* Proposition 3.3 i) Note that the density of the random variable  $Y_i = (r-1)X_i$  on  $[0, r-1]$  is

$$f(x) = \frac{r}{r-1} \left( 1 - \frac{x}{r-1} \right)^{r-1}.$$

Due to the following inequality for all  $x \in [0, r-1]$

$$\frac{r}{r-1} \left( 1 - \frac{x}{r-1} \right)^{r-1} \leq \frac{r}{r-1} \exp(-x)$$

one may deduce that

$$\begin{aligned} (r-1)^k \mathbb{E} \left( \sum_{i=1}^m X_i \right)^k &= \mathbb{E} \left( \sum_{i=1}^m Y_i \right)^k \leq \left( \frac{r}{r-1} \right)^k \mathbb{E} \left( \sum_{i=1}^m \mathcal{E}_i \right)^k \\ &\leq \left( \frac{r}{r-1} \right)^k \frac{(m-1+k)!}{(m-1)!} \end{aligned}$$

since  $\sum_{i=1}^m \mathcal{E}_i$  has an Erlang distribution (the density of  $\sum_{i=1}^m \mathcal{E}_i$  is  $h(x) = \frac{x^{m-1} \exp(-x)}{(m-1)!}$ ) whose moment of order  $k$  is  $\frac{(m-1+k)!}{(m-1)!}$ . This inequality completes the proof of i).

ii) The density of the random variable  $(r-1)X_1$  converges pointwise to the density of  $\mathcal{E}_1$  namely  $\lim_{r \rightarrow +\infty} f(x) = \exp(-x)$ , for all  $x \in [0, +\infty)$ . Therefore, the following limit holds

$$\lim_{r \rightarrow +\infty} x^k f^{*m}(x) = \frac{x^{m+k-1} \exp(-x)}{(m-1)!}.$$

Finally, the Dominated Convergence Theorem gives that

$$\begin{aligned} \lim_{r \rightarrow +\infty} (r-1)^k \mathbb{E} \left( \sum_{i=1}^m X_i \right)^k &= \lim_{r \rightarrow +\infty} \int_0^{+\infty} \frac{x^{m+k-1} \exp(-x)}{(m-1)!} dx \\ &= \mathbb{E} \left( \sum_{i=1}^m \mathcal{E}_i \right)^k = \frac{(m-1+k)!}{(m-1)!}. \end{aligned}$$

□

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