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THE STOCHASTIC BESSEL OPERATOR AT HIGH TEMPERATURES

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Abstract

We know from Ramírez and Rider [12] that the hard edge of the spectrum of the Beta-Laguerre ensemble converges, in the high-dimensional limit, to the bottom of the spectrum of the stochastic Bessel operator. Using stochastic analysis techniques, we show that, in the high temperatures limit, the rescaled eigenvalues point process of the stochastic Bessel operator converges to a limiting point process characterized with coupled stochastic differential equations.

Keywords: Stochastic Bessel operator; Beta-Laguerre ensemble

1. Context and motivation

Coulomb gases, also known as β -ensembles, are probability measures on sets of points. In statistical physics, these points are elementary particles on the real line confined under a random potential while repelling each other. The Dyson parameter β acts as an inverse temperature and can affect both the shape of the potential and the strength of the repulsive force. In recent years, β -ensembles have attracted significant interest (see for instance [5], [6], [8], [13], [1], [2], [14], [16], [17], [18]).

We introduce the classical β -Laguerre ensemble of size n , whose ordered points (also called eigenvalues) $\lambda_0^{(n)} \leq \lambda_1^{(n)} \dots \leq \lambda_{n-1}^{(n)}$ have the density on the real line, with respect to the Lebesgue measure:

$$\frac{1}{Z_{n,\beta,a}} \prod_{i < j} |\lambda_i^{(n)} - \lambda_j^{(n)}|^\beta \times \prod_{k=0}^{n-1} (\lambda_k^{(n)})^{\frac{\beta}{2}(m-n+1)-1} e^{-\frac{\beta}{2}\lambda_k^{(n)}} \mathbf{1}_{\lambda_k^{(n)} > 0}, \quad (1)$$

where $Z_{n,\beta,a}$ acts as a normalization constant. When $\beta = 1$ (resp. $\beta = 2$, $\beta = 4$), (1) is the density of the eigenvalues of a real (resp. complex, unitary) $n \times n$ Wishart matrix. This connection to random matrix theory was extended to any $\beta > 0$ by Dimitriu and Edelman [5, Theorem 3.4], adapting earlier work from Silverstein [15]. They found a set of bidiagonal random matrices $(L_{n,m}^\beta)_{m \geq n}$ such that the density of the (positive) eigenvalues of $L_{n,m}^\beta (L_{n,m}^\beta)^T$ is (1).

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When m and n tend to infinity with $n/m \rightarrow \gamma \in]0, 1]$, the rescaled empirical measure of the eigenvalues converges weakly a.s. to the Marchenko-Pastur distribution:

$$\mu_\gamma(dx) = \frac{1}{2\pi\gamma x} \sqrt{(\gamma_+ - x)(x - \gamma_-)} \mathbf{1}_{[\gamma_-, \gamma_+]}(x) dx, \quad \gamma_\pm = (1 \pm \sqrt{\gamma})^2.$$

We refer to [7] for a review of the asymptotic global statistics of classical β -ensembles.

From now on, we restrict ourselves to the asymptotic regime $m - n \rightarrow a$, for a fixed $a > 0$, thus $\gamma = 1$. The eigenvalues of the β -Laguerre ensemble can't go below $\gamma_- = 0$ because of the positivity constraint, so this is called a *hard edge*. To study the local statistics of the β -Laguerre ensemble at the hard edge in the high-dimensional limit, Ramírez and Rider [12] introduced a stochastic operator as limit object for the β -Laguerre ensemble.

Definition 1. (*The stochastic Bessel operator.*)

Let B be a standard Brownian motion on \mathbb{R}_+ . For $\beta > 0$, the stochastic Bessel operator (known as SBO) is the random differential operator

$$\mathfrak{G}_a^\beta = -\frac{1}{m_a(x)} \frac{d}{dx} \left(\frac{1}{s_a(x)} \frac{d}{dx} \right),$$

$$m_a(x) = \exp\left(- (a+1)x - \frac{2}{\sqrt{\beta}} B(x)\right), \quad s_a(x) = \exp\left(ax + \frac{2}{\sqrt{\beta}} B(x)\right),$$

defined on a subset of $L^2(\mathbb{R}_+, m_a)$ with Dirichlet and Neumann boundary conditions at 0 and at infinity, respectively.

They showed the connection between the eigenvalues of the β -Laguerre ensemble at the hard edge and the low-lying eigenvalues of the SBO:

Proposition 1. (Limit of the β -Laguerre ensemble at the hard edge [12, Theorem 1].)

With probability one, when restricted to the positive half-line with Dirichlet conditions at the origin, \mathfrak{G}_a^β has a discrete spectrum with single eigenvalues $0 < \Lambda_{\beta,a}(0) < \Lambda_{\beta,a}(1) < \dots \uparrow \infty$. Moreover, with $0 < \lambda_0^{(n)} < \lambda_1^{(n)} < \dots < \lambda_n^{(n)}$ the ordered points of the β -Laguerre ensemble of size n ,

$$\{n\lambda_0^{(n)}, n\lambda_1^{(n)}, \dots, n\lambda_k^{(n)}\} \implies \{\Lambda_{\beta,a}(0) < \Lambda_{\beta,a}(1) < \dots < \Lambda_{\beta,a}(k),\}$$

(jointly in law) for any fixed $k < \infty$ as $n \uparrow \infty$.

In this research paper, we study the convergence of the lowest eigenvalues of the SBO in the high temperatures limit, when the inverse temperature parameter β tends to 0.

2. Our result

Dumaz, Li and Valkó [4, Proposition 7] showed that, if ψ solves the equation $\mathfrak{G}_a^\beta \psi = \lambda \psi$ with deterministic initial conditions $\psi(0) = c_0$, $\psi'(0) = c_1$ then (ψ, ψ') is the unique strong solution of the

stochastic differential equation system:

$$d\psi(x) = \psi'(x)dx, \quad d\psi'(x) = \frac{2}{\sqrt{\beta}}\psi'(x)dB(x) + \left(\left(a + \frac{2}{\beta}\right)\psi'(x) - \lambda e^{-x}\psi(x) \right) dx,$$

with the corresponding initial conditions.

Consider the family of coupled diffusions $(p_\lambda^\beta, \lambda \in \mathbb{R}_+^*)$ with initial condition $p_\lambda^\beta(0) = +\infty$:

$$dp_\lambda^\beta(t) = \frac{2}{\sqrt{\beta}}p_\lambda^\beta(t)dB(t) + \left(\left(a + \frac{2}{\beta}\right)p_\lambda^\beta(t) - p_\lambda^\beta(t)^2 - \lambda e^{-t} \right) dt, \quad (2)$$

where the diffusion p_λ^β may explode to $-\infty$, in which case it immediately restarts from $+\infty$.

Using a Riccati transform (introduced by Halperin in [9]), Ramírez and Rider [12] showed a fundamental connection between the family $(p_\lambda^\beta, \lambda \in \mathbb{R}_+^*)$ and the eigenvalues of \mathfrak{S}_a^β :

$$\forall k \in \mathbb{N}, \{p_\lambda^\beta \text{ explodes at most } k \text{ times on } (0, \infty)\} = \{\Lambda_{\beta,a}(k) > \lambda\}. \quad (3)$$

It is crucial to note that the *same* Brownian motion drives the whole family of SDEs from (2). It implies important properties such as the monotonicity of the number of explosions of p_λ^β (which turns out to be finite). In fact, the number of explosions of p_λ^β over \mathbb{R}_+ corresponds to N_λ^β , the counting function of the eigenvalues of the SBO.

When β tends to 0, the smallest eigenvalues get close to the hard edge at 0 at an exponential rate. In order to get a non trivial limit, we therefore consider the rescaled eigenvalues $\mu_{\beta,a}(i) := \beta \ln(1/\Lambda_{\beta,a}(i))$ for $i \geq 0$ (note that this reverses the ordering of the eigenvalues).

Our main result is the following theorem:

Theorem 1. (Convergence of the low-lying eigenvalues of the SBO.) *When β tends to 0, the rescaled eigenvalue point process of the SBO $(\mu_{\beta,a}(i), i \geq 0)$ converges in law towards a random simple point process on \mathbb{R}_+ which can be described using coupled SDEs.*

The convergence holds for a well chosen topology of Radon measures on \mathbb{R}_+ , corresponding to a left-vague/right-weak topology (see below for more details). We will also give a characterization (similar to the one for the SBO eigenvalues) of the limiting point process through coupled diffusions. This characterization enables one to compute various statistics on the limiting point process.

Usually, one expects that, when the temperature is high, the limiting point process is no longer repulsive and corresponds to a Poisson point process as the noise becomes dominant. Here, we get a different result. It comes from the competition between the strong repulsive interaction and attraction at the hard edge for small β (see (1)). Because of this interaction, the repulsive factor does not disappear in the limit.

3. Strategy of proof and limiting point process

3.1. Rescaled diffusions

We study the small beta limit of the family of diffusions (p_λ^β) from (2) when λ is properly rescaled with β , i.e. when λ is such that $\beta \ln(1/\lambda)$ is of order 1. The equality (3) tells us that the number of explosions of p_λ^β on \mathbb{R}_+ is the number of eigenvalues of \mathfrak{G}_a^β below λ .

Notice that, when p_λ^β reaches 0, the random term vanishes and the diffusion drift is negative. It implies that p_λ^β never reaches 0 from below. It is easy to check that the hitting times of 0 form a discrete point process.

Let us fix $\mu > 0$ and set $\Lambda^\beta := \exp(-\mu/\beta)$. Using the property above, we define the diffusion $q_\mu^\beta(t)$, which equals

$$\begin{aligned} q_\mu^+(t) &:= \beta \ln(p_{\Lambda^\beta}^\beta(t/(4\beta))) \text{ when } p_{\Lambda^\beta}^\beta(t/(4\beta)) > 0, \\ q_\mu^-(t) &:= -\beta \ln(-p_{\Lambda^\beta}^\beta(t/(4\beta))) - \mu - t/4 \text{ when } p_{\Lambda^\beta}^\beta(t/(4\beta)) < 0, \end{aligned}$$

The diffusions $q_\mu^+(t)$ and $q_\mu^-(t)$ follow the SDEs:

$$dq_\mu^+ = dW(t) + \frac{1}{4} \left(a - \exp(q_\mu^+(t)/\beta) - \exp(-(q_\mu^+(t) + t/4 + \mu)/\beta) \right) dt, \quad (4)$$

$$dq_\mu^- = dW(t) + \frac{1}{4} \left(-(a+1) - \exp(q_\mu^-(t)/\beta) - \exp(-(q_\mu^-(t) + t/4 + \mu)/\beta) \right) dt, \quad (5)$$

where W is a Brownian motion corresponding to (different) rescaling of the initial Brownian motion B . The diffusions q_μ^\pm may explode to $-\infty$ in a finite time. By definition, the diffusion q_μ^β alternates between q_μ^+ and q_μ^- : it starts to follow q_μ^+ and each time $q_\mu^\beta = q_\mu^+$ (resp. q_μ^-) reaches $-\infty$, q_μ^β immediately restarts from $+\infty$ and follows q_μ^- (resp. q_μ^+).

Let us define the critical line:

$$c_\mu(t) := -\mu - t/4. \quad (6)$$

Figure 1 shows a sample path of the diffusion q_μ^β . On this event, the diffusion q_μ^β explodes one time as q_μ^+ (blue), then one time as q_μ^- (red), and then stays above the critical line $c_\mu(t)$ as q_μ^+ (blue) and does not explode anymore.

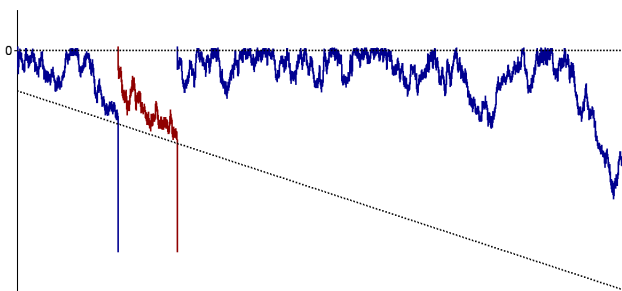


FIGURE 1: Sample trajectory of the diffusion q_μ^β .

Roughly, the diffusion q_μ^+ behaves as follows after each explosion time. First, it quickly goes down to values around 0. Then, it spends some time between the line $t \mapsto c_\mu(t)$ and 0, where it behaves as a reflected (downwards) Brownian motion with drift $a/4$. If it reaches the line $t \mapsto c_\mu(t)$ in a finite time then it quickly explodes to $-\infty$ after this hitting time.

The behaviour of the diffusion q_μ^- is similar except that in the interval $[c_\mu(t), 0]$, it behaves as a reflected (downwards) Brownian motion with drift $-(a+1)/4$. Therefore, it almost surely hits $c_\mu(t)$ when $a > 0$.

There are two types of explosions for q_μ^β : either q_μ^β explodes at a time ξ^+ such that $q_\mu^\beta(\xi^+) = q_\mu^+(\xi^+)$, which corresponds to the (rescaled) hitting times of 0 by the initial diffusion $p_{\Lambda^\beta}^\beta$, or q_μ^β explodes at time ξ^- such that $q_\mu^\beta(\xi^-) = q_\mu^-(\xi^-)$, in which case we get the (rescaled) explosion times of the initial diffusion $p_{\Lambda^\beta}^\beta$.

In the following, we denote by $\xi_\beta^+(0) < \xi_\beta^-(0) < \xi_\beta^+(1) < \xi_\beta^-(1) < \dots$ the explosion times of the diffusion q_μ^β and by

$$\nu_\mu^\beta := \sum_{i \geq 0} \delta_{\xi_\beta^-(i)}. \quad (7)$$

the measure corresponding to the (rescaled) explosions of $p_{\Lambda^\beta}^\beta$.

We will prove that, for a well-chosen topology, the trajectory of the diffusion q_μ^β converges in law, when β tends to 0, towards a non-trivial limit, that we describe in the following paragraph.

3.2. Description of the limiting point process

Let us define now the limiting diffusion r_μ , which will characterize the limiting point process. Its definition involves Brownian motions with drift reflected downwards at 0.

By definition, a Brownian motion with drift q reflected downwards at 0 is a diffusion Markov process

with infinitesimal operator $G : f \in \mathcal{D} \mapsto \frac{1}{2}f'' + qf'$ acting on the domain

$$\mathcal{D} := \left\{ f \in C_b[0, +\infty[, Gf \in C_b([0, +\infty[), \lim_{x \downarrow 0} f'(x) = 0 \right\},$$

where $C_b([0, +\infty[)$ denotes the continuous and bounded functions on $[0, +\infty[$. Using the Skorohod problem, we can write this diffusion as

$$(W(t) + qt) - \sup_{s \leq t} (W(s) + qs) \vee 0,$$

where W is a Brownian motion starting at 0 or any negative point.

The limit diffusion r_μ starts at 0 at time 0, i.e. $r_\mu(0) = 0$. It then follows a Brownian motion with drift $a/4$ reflected downwards at 0, that we denote r^+ , until its first hitting time of the critical line $c_\mu(t)$ from (6). If it reaches $t \mapsto c_\mu(t)$ in a finite time, then it immediately restarts at 0 at this time then follows a reflected downwards at 0 Brownian motion with another drift $-(a+1)/4$, denoted r^- , and so on, alternating between r^+ and r^- each time it hits $t \mapsto c_\mu(t)$ and restarts at 0. Note that the probability that r^+ reaches the critical line decreases with time. On the other hand, since $a > 0$, r^- almost surely hits the critical line in a finite time.

Let us denote by $\xi_0^+(0) < \xi_0^-(0) < \xi_0^+(1) < \xi_0^-(1) < \dots$ the hitting times of the critical line by the diffusion r_μ , and define the random measure associated to the point process $(\xi_0^-(k), k \geq 0)$:

$$\nu_\mu^0 := \sum_{i \geq 0} \delta_{\xi_0^-(i)}.$$

We then use the coupled random measures $(\nu_\mu^0, \mu > 0)$ to define a discrete point process on \mathbb{R}_+ . Since $\mu \in \mathbb{R}_+ \mapsto \nu_\mu(\mathbb{R}_+)$ decreases from $+\infty$ to 0 almost surely, it is easy to prove the following proposition:

Proposition 2. (Limiting point process.) *There is a unique random variable M_0 valued on the Borel sets of $]0, +\infty[$, such that, for all fixed $\mu_1 < \dots < \mu_k$,*

$$M_0[\mu_i, +\infty[= \nu_{\mu_i}^0(\mathbb{R}_+).$$

Almost surely, the measure M_0 is discrete, bounded from above and has an accumulation point at 0.

3.3. Convergence towards the limiting measures

We can now state the desired convergence result:

Proposition 3. (Convergence of the explosion times of $p_{\Lambda^\beta}^\beta$.)

When β tends to 0, the measure ν_μ^β converges in law to the measure ν_μ^0 for the topology of weak convergence.

It is immediate to extend this proposition for the joint law of $\nu_{\mu_i}^\beta$ when μ_1, \dots, μ_k are fixed positive numbers. It directly implies the following result on the finite dimensional laws of the point process $\{\mu_{\beta,a}(i), i \geq 0\}$. Let us denote by M_β the measure associated to this point process, i.e. $M_\beta := \sum_{i \geq 0} \delta_{\mu_{\beta,a}(i)}$.

Proposition 4. (Convergence of the finite-marginals of the eigenvalue process.)

Fix $\mu_1 < \dots < \mu_k$. When $\beta \rightarrow 0$, the random vector $(M_\beta[\mu_1, +\infty[, \dots, M_\beta[\mu_k, +\infty[)$ converges in law to the random vector $(M_0[\mu_1, +\infty[, \dots, M_0[\mu_k, +\infty[)$.

Consider the space of measures on $]0, \infty[$ with the topology that makes continuous the maps $m \mapsto \langle f, m \rangle$ for any continuous and bounded function f with support bounded to the left. In other words, this is the vague topology towards 0 and the weak topology towards $+\infty$.

The previous proposition shows that the family $(M_\beta)_{\beta > 0}$ is tight: indeed the above convergence provides the required control on the mass at $+\infty$.

As the finite-marginals of any limiting point is identified, we deduce the convergence of the left-vague/right-weak topology of the eigenvalues point process stated in Theorem 1.

The rest of the paper is devoted to the proof of Proposition 3. In Section 4, we control the first explosion time of the diffusion q_μ^β and deduce the weak convergence of its k first explosions times. Then, in Section 5, we show the tightness of the family $(\nu_\mu^\beta)_{\beta > 0}$. Unless specified otherwise, the limits and o pertain to the asymptotics $\beta \rightarrow 0$. For lighter notations, we omit the dependency on β of our variables.

3.4. Useful results

We will use the following estimates for any Brownian motion W :

$$\forall x \geq 0, \mathbb{P}\left(\sup_{s \in [0,1]} W(s) > x\right) \leq \mathbb{P}\left(\sup_{s \in [0,1]} |W(s)| > x\right) \leq 4e^{-x^2/2}, \quad (8)$$

$$\forall x \geq 0, \mathbb{P}\left(\sup_{s \in [0,1]} W(s) > x\right) = \mathbb{P}\left(|W(1)| > x\right) \geq 1 - \sqrt{\frac{2}{\pi}}x. \quad (9)$$

Consider a diffusion y started from 0 and the same diffusion \bar{y} reflected downwards at the origin:

$$\bar{y}(t) = y(t) - \sup_{s \leq t} y(s).$$

For all $\delta > 0$, $\{\bar{y}(t) < -\delta\} = \{\exists s < t, y(t) - y(s) < -\delta\}$, therefore:

$$\sup_{s \in [0,t]} |y(s)| < \delta/2 \Rightarrow \inf_{s \in [0,t]} \bar{y}(s) > -\delta. \quad (10)$$

4. Control of the explosion times

In this section, we fix $\mu > 0$. Recall the definition of the critical line $c = c_\mu$ in (6). Consider the first two explosion times $\xi^+ := \xi_\beta^+(0)$ and $\xi^- := \xi_\beta^-(0)$ of the diffusion q_μ^β . Until the first explosion time ξ^+ , by definition $q_\mu^\beta(0) = +\infty$ and $q_\mu^\beta(t) = q^+(t)$ follows the SDE (4).

Set $\delta = \beta^{1/8}$ and introduce the first hitting times by the diffusion q^+ :

$$\tau_0 := \inf \{t \geq 0, q^+(t) \leq 0\} \text{ and } \tau_c := \inf \{t \geq 0, q^+(t) \leq c(t) + \delta\}.$$

We decompose the trajectory of q^+ into three parts. First, it reaches the axis $x = 0$ in a short time (descent from $+\infty$). Then, it spends a time of order $O(1)$ in the region $[c(t) + \delta, 0]$ and behaves like a reflected Brownian motion with a constant drift $a/4$. Finally, if it approaches the critical line $t \mapsto c(t)$ closer than δ , then it explodes with high probability within a short time (explosion to $-\infty$).

Recall the first hitting times $\xi_0^+(0) < \xi_0^-(0)$ of the critical line c by the diffusion r_μ .

Proposition 5. (Limit behavior of the diffusions q^+ and q^- .)

Set $T > 0$, independent of β . There exist a deterministic $\eta \rightarrow 0$ and an event \mathcal{E}_0 , $\mathbb{P}(\mathcal{E}_0) \rightarrow 1$, on which, for β small enough:

- (a) $\tau_0 < \beta$,
- (b) $\sup_{[\tau_0, \tau_c \wedge T]} |q^+(t) - r^+(t)| < \delta$,
- (c) $\tau_c < T \Rightarrow |\xi^+ - \tau_c| < \eta$ and $|\xi_0^+(0) - \tau_c| < \eta$.

Properties (a), (b) and (c) also hold for the diffusions q^- from (5) and r^- , with their corresponding hitting and explosion times. As a consequence, on the event \mathcal{E}_0 , for β small enough,

$$\xi^- < T \Rightarrow |\xi^- - \xi_0^-(0)| < 2\eta. \quad (11)$$

The control (11) extends to any $\xi_\beta^-(k)$ and $\xi_0^-(k)$ for $k \in \mathbb{N}$ and ensures that, for any $T > 0$, $\mathbb{P}(\xi_\beta^-(k) \leq T) \rightarrow \mathbb{P}(\xi_0^-(k) \leq T)$, thus identifying the measure ν_μ^0 as the unique possible limit for ν_μ^β .

The rest of this section is dedicated to the proof of Proposition 5. Recall that the diffusion q^- differs from its counterpart q^+ only by its constant drift component $-(a+1)/4$ (instead of $+a/4$ for q^+), which makes q^- decrease faster than q^+ . We prove the results for the diffusion q^+ , they extend to the diffusion q^- with the same arguments.

We introduce the stationary diffusion \bar{q} on \mathbb{R}_+ , which we use to approximate q^+ in the region where the drift component $\exp[-\frac{1}{\beta}(c(t) + q^+(t))]$ becomes negligible as β tends to 0:

$$d\bar{q}(t) = dW(t) + \frac{1}{4}(a - e^{\bar{q}(t)/\beta})dt. \quad (12)$$

4.1. Descent from $+\infty$: proof of (a)

It suffices to prove property (a) for the diffusion \bar{q} , which bounds the diffusion q^+ from above. Set the level $l_1 := \beta^{3/4}$, so that $\beta = o(l_1)$. As β tends to 0, when the diffusion \bar{q} is above the level l_1 , the term of leading order in the right-hand side of (12) is $e^{\bar{q}(t)/\beta}$.

Consider the ordinary differential equation on \mathbb{R}_+ :

$$dy(t) := \frac{1}{4}\left(a - \frac{1}{2}e^{y/\beta}\right)dt, \quad y(0) = +\infty,$$

which has for solution $y(t) = -\beta \ln\left(\frac{1}{2a}(1 - e^{-at/4\beta})\right)$. The time t_1 at which y reaches the level $l_1/2$ has the asymptotics

$$t_1 = 8\beta e^{-l_1/(2\beta)} + o(\beta e^{-l_1/\beta}).$$

Introduce the diffusion $\bar{q}_1(t) := \bar{q}(t) - W(t)$. Its evolution writes:

$$d\bar{q}_1(t) = \frac{1}{4}\left(a - e^{(\bar{q}_1(t)+W(t))/\beta}\right)dt.$$

Let $\mathcal{E}'_1 := \left\{\sup_{[0,t_1]} |W(t)| \leq \beta^2\right\}$. By the Brownian tail bound (8), $\mathbb{P}(\mathcal{E}'_1) \rightarrow 1$.

On the event \mathcal{E}'_1 , $e^{(\bar{q}_1(t)+W(t))/\beta} \geq e^{-1/\beta}e^{\bar{q}_1(t)/\beta}$, so the diffusion \bar{q}_1 is bounded from above by y for β small enough and hits the level $l_1/2$ before time t_1 . Since $|\bar{q}_1(t) - \bar{q}(t)| \leq \beta^2$ before time t_1 , for β small enough, the diffusion \bar{q} hits the level l_1 before time t_1 .

After the level l_1 is reached, we use the Brownian motion $W(t_1 + \cdot) - W(t_1)$ to reach $x = 0$ in a short additional time. Set the event

$$\mathcal{E}''_1 := \left\{\inf_{[0,\beta/2]} \left\{W(t_1+t) - W(t_1) + \frac{a}{4}t\right\} < -l_1\right\},$$

on which $\tau_0 < t_1 + \beta/2$. Since $\mathbb{P}(\mathcal{E}''_1) \geq \mathbb{P}\left(\inf_{[0,\beta/2]} W(t_1+t) - W(t) < -l_1 - \frac{a}{8}\beta\right)$, the lower bound (9) and the asymptotics $\beta = o(l_1)$ and $l_1 = o(\sqrt{\beta})$ imply that $\mathbb{P}(\mathcal{E}''_1) \rightarrow 1$, thus proving the property (a) on the event $\mathcal{E}_1 = \mathcal{E}'_1 \cap \mathcal{E}''_1$.

4.2. Convergence to r^+ : proof of (b)

The bound (8) shows that the probability of the following event tends to 1 as β tends to 0:

$$\mathcal{E}'_2 = \left\{\sup_{t \in [0,\beta]} |W(t)| \leq \beta^{1/4}\right\}.$$

Note that $\beta^{1/4} = o(\delta)$ and that the diffusion $r^+(\tau_0 + t) - r^+(\tau_0)$ is equal in law to the diffusion r^+ , by the strong Markov property. Thus, to prove property (b), it suffices to show that, with overwhelming probability as β tends to 0, for β small enough,

$$\sup_{[0,\tau'_c \wedge T]} |q_0^+(t) - r^+(t)| < \delta/2, \quad (13)$$

where q_0^+ denotes the diffusion q^+ started from $x = 0$ at time $t = 0$ and τ'_c its first hitting time of $c(t) + \delta$. We write \bar{q}_0 the stationary diffusion \bar{q} from (12) started from $x = 0$ at time $t = 0$. For $t \in [0, \tau'_c \wedge T]$, $q_0^+(t) \geq c(t) + \delta$, so we have the bounds:

$$\forall t \in [0, \tau'_c \wedge T], \bar{q}_0(t) - e^{-\delta/\beta} T \leq q_0^+(t) \leq \bar{q}_0(t).$$

Since $e^{-\delta/\beta} = o(\delta)$, to prove property (13), it is enough to show that, on an event \mathcal{E}'_2 of probability going to 1 as β tends to 0, for β small enough,

$$\sup_{[0, \tau'_c \wedge T]} |\bar{q}_0(t) - r^+(t)| < \delta/4. \quad (14)$$

To that end, we bound the diffusion $\bar{q}_0(t)$ from below and above by two reflected diffusions r_1^+ and r_2^+ that converge to r^+ as β tends to 0.

We set the level $l_2 := \beta^{1/6}$, so that $l_2 = o(\delta)$ and $\delta = o(\sqrt{l_2})$.

4.2.1. *Lower bound* For any stochastic process Z , we denote by $L_t^x(Z)$ its local time at position x and time t . Since the element of drift $-e^{\bar{q}_0(t)/\beta}$ is decreasing on $]-\infty, -l_2]$, we have the lower bound:

$$\forall t \in [0, \tau'_c \wedge T], \bar{q}_0(t) \geq r_1^+(t), \quad (15)$$

where $r_1^+(t)$ is the following diffusion (reflected downwards at the barrier $-l_2$):

$$r_1^+(t) := -l_2 + W(t) + at/4 - e^{-\frac{\delta}{\beta}} T/4 - L_t^0 \left(W(t) + at/4 - e^{-\frac{\delta}{\beta}} T/4 \right),$$

It is straightforward that

$$\forall t \in [0, \tau'_c \wedge T], r^+(t) - e^{-\frac{\delta}{\beta}} T/4 - l_2 \leq r_1^+(t) \leq r^+(t) - l_2.$$

Since $e^{-\frac{\delta}{\beta}} = o(l_2)$, for β small enough,

$$\sup_{[0, \tau'_c \wedge T]} |r^+(t) - r_1^+(t)| < 2l_2.$$

4.2.2. *Upper bound* We wish to bound the diffusion $\bar{q}_0(t)$ from above by the diffusion $r^+(t) + l_2$. To prove that this upper bound holds with high probability as β tends to 0, we use the following result, that shows how unlikely it becomes for the diffusion $\bar{q}_0(t)$ to hit the level l_2 before any negative level.

Lemma 1. (Levels hitting times for the diffusion $\bar{q}_0(t)$.)

For any $\gamma < 0$,

$$\mathbb{P} \left(\inf \{t \geq 0, \bar{q}_0(t) = l_2\} < \inf \{t \geq 0, \bar{q}_0(t) = \gamma\} \right) \longrightarrow 0.$$

Lemma 1 is proved in the Appendix using standard tools of diffusion analysis.

The choice of level $\gamma = -\mu(T)$ in Lemma 1 provides the existence of an event \mathcal{E}'_2 of probability going to 1 as β tends to 0 on which the diffusion $\bar{q}_0(t)$ hits the barrier $c(t)$ before the level l_2 , and thus:

$$\forall t \in [0, \tau'_c \wedge T], \bar{q}_0(t) \leq r^+(t) + l_2. \quad (16)$$

4.2.3. *Conclusion* Gathering (15) and (16), we get that, on \mathcal{E}'_2 , for β small enough,

$$\forall t \in [0, \tau'_c \wedge T], r_1^+(t) \leq \bar{q}_0(t) \leq r^+(t) + l_2,$$

which implies $\sup_{[0, \tau'_c \wedge T]} |\bar{q}_0(t) - r^+(t)| < 2l_2$. This in turn implies (14) and thus proves property (b).

4.3. Explosion to $-\infty$: proof of (c)

We denote by q_δ^+ (resp. $q_{-\delta}^+$) the diffusion q^+ started at time $t = 0$ from position $-\mu + \delta$ (resp. $-\mu - \delta$). We introduce the first hitting time τ_δ of the level $c(t) - \delta$ by the diffusion $q_{+\delta}^+$, and the explosion time $\tau_{-\infty}$ of the diffusion $q_{-\delta}^+$ to $-\infty$.

Recall that $l_2 = \beta^{1/6}$. To prove property (c), we choose $\eta := 2l_2$ and show that there exists an event of probability going to 1 as β tends to 0 on which, for β small enough, $\tau_\delta < \eta/2$ and $\tau_{-\infty} < \eta/2$.

4.3.1. *Control of τ_δ* Recall that $\delta = \beta^{1/8}$, so that $l_2 = o(\delta)$ and $\delta = o(\sqrt{l_2})$.

We use the variations of the Brownian motion W to cross the critical line $c(t)$. The upper bound $q_\delta^+(t) \leq -\mu + \delta + W(t) + \frac{a}{4}t$ implies that $\tau_\delta < l_2$ on the event

$$\mathcal{E}_3 := \left\{ \inf_{[0, l_2]} \left\{ W(s) + \frac{a}{4}s + \frac{1}{4}s + 2\delta \right\} < 0 \right\},$$

and the Brownian tail bound from (9) shows that $\mathbb{P}(\mathcal{E}_3) \rightarrow 1$.

Note that the inclusion of events $\mathcal{E}'_3 \subset \left\{ \inf \{t \geq 0, r^+(t) + t/4 \leq -2\delta\} < l_2 \right\}$ ensures that, on a subevent of \mathcal{E}_2 (where $|q^+(\tau_c) - r^+(\tau_c)| < \delta$ if $\tau_c < +\infty$) of probability going to $\mathbb{P}(\mathcal{E}_2)$ as β tends to 0, the diffusion r^+ hits the critical line $t \mapsto c(t)$ while the diffusion q^+ crosses this line, between times τ_c and $\tau_c + \eta$.

4.3.2. *Control of $\tau_{-\infty}$* On each Brownian trajectory, the diffusion $q_{-\delta}^+$ is bounded from above by the diffusion z , with

$$dz(t) := dW(t) + \frac{1}{4}(a - e^{-\frac{1}{\beta}(t/4 + \mu_\beta + z(t))})dt, \quad z(0) = -\mu - \delta.$$

Define the diffusion $z_1(t) = z(t) + \mu - W(t) - at/4$, with evolution

$$dz_1(t) := -\frac{1}{4}e^{-\frac{1}{\beta}(z_1(t) + t/4 + W(t) + at/4)}dt, \quad z_1(0) = -\delta.$$

Consider the event $\mathcal{E}'_3 := \left\{ \sup_{[0, \beta]} |W(t)| \leq \delta/4 \right\}$, with $\mathbb{P}(\mathcal{E}'_3) \rightarrow 1$.

On \mathcal{E}'_3 , for β small enough:

$$\forall t \leq \beta, \quad W(t) + \frac{a+1}{4}t \leq \delta/2,$$

so the diffusion z_1 is bounded from above by the diffusion z_2 with evolution

$$dz_2(t) := -\frac{1}{4}e^{-\frac{1}{\beta}(z_2(t) + \delta/2)}dt, \quad z_2(0) := -\delta.$$

The diffusion z_2 has the solution

$$z_2(t) = -\delta/2 + \beta \ln \left(e^{-\delta/(2\beta)} - \frac{t}{4\beta} \right),$$

which explodes to $-\infty$ in a time $4\beta e^{-\delta/(2\beta)}$, smaller than β for β small enough. This remains true for z_1 and thus for z , since $|z_1 - z| \leq \mu + a\beta/4 + \beta$ while $t \leq \beta$. Since $\beta = o(\eta)$, this proves the desired control on $\tau_{-\infty}$.

5. Tightness of the explosion times measures

In this section, we fix $\mu > 0$. Recall the measure of the explosion times ν_μ^β from (7). We prove in this section that there are $\beta_0 > 0$ and $\alpha > 0$, such that, for all $\epsilon > 0$, there exist a finite time T_ϵ and a finite number of explosions N_ϵ so that:

$$\inf_{\beta < \beta_0} \mathbb{P} \left(\left\{ \nu_\mu^\beta([0, \alpha T_\epsilon]) \leq N_\epsilon \right\} \cap \left\{ \nu_\mu^\beta([\alpha T_\epsilon, +\infty[) = 0 \right\} \right) > 1 - \epsilon. \quad (17)$$

Introduce \mathcal{L}_μ^β , the law of the random measure ν_μ^β . The bound (17) gives us the tightness condition:

$$\exists \beta_0, \forall \epsilon > 0, \exists K_\epsilon \text{ compact, } \sup_{\beta < \beta_0} \mathcal{L}_\beta(K_\epsilon) \geq 1 - \epsilon.$$

Indeed, the closure K_ϵ of $\{\mu \in \mathcal{P}, \mu([0, \alpha T_\epsilon]) \leq N_\epsilon \text{ and } \mu([\alpha T_\epsilon, +\infty[) = 0\}$, where \mathcal{P} is the space of locally finite measures on \mathbb{R}_+ , satisfies the Kallenberg criterion for weak relative compactness (see [10]):

$$\begin{aligned} \sup_{\mu \in K_\epsilon} \mu(\mathbb{R}_+) &< +\infty, \\ \inf_{t > 0} \sup_{\mu \in K_\epsilon} \mu([t, +\infty[) &= 0. \end{aligned}$$

The rest of this section is dedicated to the proof of (17), which uses Prokhorov theorem to prove the relative compactness of the family $(\mathcal{L}_\beta)_{\beta < \beta_0}$ in $\mathcal{M}^1(\mathcal{P})$, the set of probability measures on \mathcal{P} , and thus concludes the proof of Proposition 3.

We show a preliminary result that will be helpful to control the number of explosions. Recall that $\xi^+ = \xi_\beta^+(0)$ is the first explosion time of the diffusion q^+ from (4), started from $+\infty$ at time 0.

Lemma 2. (Lower bound on the explosion time of q^+ .)

For β small enough,

$$\forall t > 0, \mathbb{P}(\xi^+ > t) \geq 1 - 4e^{-\mu^2/(32t)}.$$

Proof. We fix a deterministic δ_0 such that $0 < \delta_0 < \mu/4$. Recall the definition of the critical line c_μ in (6). When the diffusion q^+ is in the region between $-c_\mu(t) + \delta_0$ and $-\delta_0$, we have the lower bound, for β small enough:

$$a - e^{q^+(t)/\beta} - e^{-\frac{1}{\beta}(\mu(t) + q^+(t))} \geq a - 2e^{-\delta_0/\beta} \geq a/2. \quad (18)$$

Introduce the diffusion \widehat{q} on \mathbb{R}_+ , started from $-\delta_0$ at time 0 and reflected downwards at $-\delta_0$:

$$\widehat{q}(t) := -\epsilon + W(t) + at/8 - L_t^0(W(t) + at/8).$$

The bound (18) shows that, for β small enough, the diffusion q^+ is bounded from below by the diffusion \widehat{q} on each Brownian trajectory, up until the first hitting time of $c_\mu(t) + \delta_0$ by the diffusion q^+ .

Set $t > 0$ and introduce the event $\mathcal{E} := \left\{ \sup_{s \in [0, t]} |W(s)| < \mu/4 \right\}$. By the Brownian tail bound (8), we have $\mathbb{P}(\mathcal{E}) \geq 1 - 4e^{-\mu^2/(32t)}$. Besides, on the event \mathcal{E} , using (10),

$$\forall s \in [0, t], \widehat{q}(s) > -\delta_0 - \mu/2.$$

This means that the diffusion \widehat{q} stays above $-\mu$ until time t . Thus, for β small enough so that (18) holds, on the event \mathcal{E} , we have $\xi^+ > t$. \square

We now turn to the proof of (17). Fix $\epsilon > 0$. We control the diffusion q_μ^β with two diffusions. The first diffusion Q_1 starts at time T_ϵ at position -1 and is reflected below the horizontal line -1 with drift $a/8$. The second diffusion Q_2 starts at time $2T_\epsilon$ at position $c_\mu(T_\epsilon)$, has a drift $a/8$ as well and is also reflected below -1 .

We can choose T_ϵ high enough such that the diffusions Q_1 and Q_2 do not hit $c_\mu(t) + 1$ with probability greater than $1 - \epsilon/10$. Indeed, the sublinearity of the Brownian motion $W(T_\epsilon + t) - W(T_\epsilon)$ is such that

$$\exists T_0, \forall t \geq T_0, |W(T_\epsilon + t) - W(T_\epsilon)| < t/16 \text{ with probability greater than } 1 - \epsilon/40. \quad (19)$$

On the event where (19) holds, the diffusion Q_1 stays above $-1 - t/8$ after time T_0 , and thus above the critical line $c_\mu(t)$. We now choose T_ϵ high enough so that $|W(T_\epsilon + t) - W(t)| < c_\mu(T)/4$ with probability greater than $1 - \epsilon/40$ until time T_0 , thus Q_1 stays above $c_\mu(t)$ before time T_0 as well. Therefore, on an event of probability greater than $1 - \epsilon/20$, the diffusion Q_1 never hits $c_\mu(t)$. Similar arguments can be used for the second diffusion Q_2 .

The term $-(a+1)/4$ in the drift of the diffusion q^- implies the existence of $\alpha > 0$ such that, almost surely, when started before time $2T_\epsilon + 1$, the diffusion q^- explodes before time αT_ϵ .

If, at time T_ϵ , the diffusion q_μ^β evolves as the diffusion q^- , then it almost surely explodes before time αT_ϵ , after which it evolves as q^+ and stays above Q_1 (for β small enough such that $2e^{-1/\beta} < a/8$) and does not explode anymore.

If, at time T_ϵ , the diffusion q_μ^β evolves as the diffusion q^+ , then we distinguish between three cases: First, if the diffusion q_μ^β hits -1 between times T_ϵ and $2T_\epsilon$, then q_β stays above Q_1 and therefore does not explode.

Else, following the proof of property (c) from Proposition 5 in Section 4, we can choose a deterministic level $\delta_1 > 0$ so that, if q_μ^β reaches $t \mapsto c_\mu(t) + \delta_1$ between time T_ϵ and $2T_\epsilon$, then it explodes before time $2T_\epsilon + 1$ with probability greater than $1 - \epsilon/10$. After that, q_μ^β behaves as q^- and almost surely explodes one last time before time αT_ϵ , as previously.

Finally, if the diffusion q_μ^β starts above $c_\mu(T_\epsilon) + \delta_1$ at time T_ϵ and stays in the interval $[c_\mu(t) + \delta_1, -1]$ for all $t \in [T_\epsilon, 2T_\epsilon]$, then it is bounded from below by a Brownian motion with a positive drift $a/8$, and therefore it will be above $c_\mu(T)$ at time $2T_\epsilon$ with probability greater than $1 - \epsilon/10$. In this event, q_μ^β stays above the diffusion Q_2 after time $2T_\epsilon$ and thus does not explode.

Gathering the different cases, we thus obtain the existence of an event of probability greater than $1 - \epsilon/2$ on which q_μ^β explodes at most once after time T_ϵ and does not explode after time αT_ϵ .

To conclude the proof of the tightness criterion (17), we apply Lemma 2 to get the existence of a finite N_ϵ and of an event of probability greater than $1 - \epsilon/2$ on which the diffusion q_μ^β explodes at most $N_\epsilon - 1$ times before time T_ϵ .

6. Conclusion

We proved that, in the high temperatures limit $\beta \rightarrow 0$, the properly rescaled point process of the low-lying eigenvalues of the Stochastic Bessel operator converges towards a simple point process on \mathbb{R}_+ , described using coupled SDEs. This limit point process keeps a repulsive factor and therefore differs from the Poisson point process found by Dumaz and Labbé [3] as the high temperatures limit of the Stochastic Airy Operator. Our result opens research perspectives to understand the properties of this new point process.

Appendix: Proof of Lemma 1

Recall that $l_1 = \beta^{3/4}$ and $l_2 = \beta^{1/6}$. In this proof we denote by q the diffusion \bar{q}_0^+ . Set $\gamma < 0$, $\tau := \inf \{t \geq 0, q_0^+(t) = l_2\}$ and $\tau' := \inf \{t \geq 0, q_0^+(t) = \gamma\}$. Let $f_\beta(x) := \frac{1}{4}(a - \exp(x/\beta))$. To compute the hitting times of q , introduce the scale functions \mathfrak{s}_β and \mathfrak{s} :

$$\begin{aligned}\mathfrak{s}_\beta(x) &:= \int_{-1}^x \exp\left(-2 \int_0^u f_\beta(v) dv\right) du, \\ \mathfrak{s}(x) &:= \int_{-1}^x \exp\left(-2 \int_0^u \frac{a}{4} dv\right) du = \frac{2}{a}(e^{ax/2} - e^{ax/2}).\end{aligned}$$

The following lemma explicits the asymptotic behavior of \mathfrak{s}_β .

Lemma 3. (Convergence of the scale functions.)

For any $x_0 < 0$,

$$\mathfrak{s}'_\beta \longrightarrow \mathfrak{s}' \text{ and } \mathfrak{s}_\beta \longrightarrow \mathfrak{s} \text{ uniformly on } [x_0, 0].$$

Furthermore, $\mathfrak{s}_\beta(l_2) \longrightarrow +\infty$.

Since $\mathfrak{s}_\beta(q(\cdot))$ is a local martingale, $\mathfrak{s}_\beta(q(\cdot \wedge \tau(q)) \wedge \tau'(q))$ is a martingale. By the stopping theorem, we get:

$$\mathbb{P}(\tau < \tau') = \frac{\mathfrak{s}_\beta(\gamma) - \mathfrak{s}_\beta(0)}{\mathfrak{s}_\beta(\gamma) - \mathfrak{s}_\beta(l_2)}.$$

Lemma 3 readily implies that this probability tends to 0 as β tends to 0.

Proof of Lemma 3. We have, for all $x \leq 0$,

$$\left| \int_0^x f_\beta(v) - \frac{a}{4} dv \right| \leq \int_x^0 e^{v/\beta} dv \leq \beta \longrightarrow 0,$$

which means that $\ln \mathfrak{s}'_\beta$ converges uniformly to $\ln \mathfrak{s}'$ on $] -\infty, 0]$. Besides, the functions $\ln \mathfrak{s}'_\beta$ and $\ln \mathfrak{s}'$ are bounded on $[x_0, 0]$, and $x \mapsto e^x$ is uniformly continuous on $[x_0, 0]$ so \mathfrak{s}'_β converges uniformly to \mathfrak{s}' on $[x_0, 0]$. Therefore, \mathfrak{s}_β converges uniformly to \mathfrak{s} on $[x_0, 0]$.

Now turning to $\mathfrak{s}_\beta(l_2)$. We can compute explicitly:

$$\ln \mathfrak{s}'_\beta(x) = -2 \int_0^x f_\beta(v) dv = -2ax + 2 \int_0^x e^{v/\beta} dv = -2ax + 2\beta(e^{x/\beta} - 1).$$

A study of the variations of $\ln \mathfrak{s}'_\beta$ shows that \mathfrak{s}'_β decreases before $\beta \ln(2a)$ and increases afterwards. For β small enough, $\ln \mathfrak{s}'_\beta(l_1) \geq \beta e^{l_1/\beta}$, which tends to $+\infty$ as β tends to 0, and \mathfrak{s}'_β increases on $[l_1, l_2]$, so that:

$$\mathfrak{s}_\beta(l_2) \geq \int_{l_1}^{l_2} \mathfrak{s}'_\beta(v) dv \geq (l_2 - l_1) \mathfrak{s}'_\beta(l_1),$$

therefore $\mathfrak{s}_\beta(l_2) \longrightarrow +\infty$. □

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