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Existence results for elliptic equation involving polyharmonic operator and a critical growth

Asma Benhamida*, Rejeb Hadiji† and Habib Yazidi ‡

Abstract

In this work, we study the two following minimization problems for $r \in \mathbb{N}^*$,

$$S_{0,r}(\varphi) = \inf_{u \in H_0^r(\Omega), \|u+\varphi\|_{L^{2^*r}}=1} \|u\|_r^2 \quad \text{and} \quad S_{\theta,r}(\varphi) = \inf_{u \in H_\theta^r(\Omega), \|u+\varphi\|_{L^{2^*r}}=1} \|u\|_r^2,$$

where $\Omega \subset \mathbb{R}^N$, $N > 2r$, is a smooth bounded domain, $2^*r = \frac{2N}{N-2r}$, $\varphi \in L^{2^*r}(\Omega) \cap C(\Omega)$ and the norm $\|\cdot\|_r = \int_{\Omega} |(-\Delta)^\alpha \cdot|^2 dx$ where $\alpha = \frac{r}{2}$ if r is even and $\|\cdot\|_r = \int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} \cdot|^2 dx$ where $\alpha = \frac{r-1}{2}$ if r is odd. Firstly, we prove that, when $\varphi \neq 0$, the infimum in $S_{0,r}(\varphi)$ and $S_{\theta,r}(\varphi)$ are achieved. Secondly, we show that $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$ for a large class of φ .

Keywords : Polyharmonic operator, minimizing problem, critical Sobolev exponent.

2010 AMS subject classifications: 35J20, 35J25, 35H30, 35J60.

1 Introduction and main results

Let $r \in \mathbb{N}^*$ and Ω be a bounded domain of \mathbb{R}^N with $N \geq 2r + 1$. We define the space $H_0^r(\Omega)$ and $H_\theta^r(\Omega)$ by

$$H_0^r(\Omega) := \{f \in H^r(\Omega) \mid D^k f = 0 \text{ on } \partial\Omega \text{ for } k = 0, 1, \dots, r-1\}$$

and

$$H_\theta^r(\Omega) := \{f \in H^r(\Omega) \mid (-\Delta)^k f = 0 \text{ on } \partial\Omega \text{ for } 0 \leq k \leq [(r+1)/2]\}.$$

where $D^k f$ denote any derivative of order k of the function f and $[(r+1)/2]$ is the integer part of $(r+1)/2$.

Define the following norm

$$\|f\|_r^2 = \begin{cases} \int_{\Omega} |(-\Delta)^{r/2} f|^2 dx & \text{if } r \text{ is even,} \\ \int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} f|^2 dx & \text{if } r \text{ is odd.} \end{cases}$$

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Then, we consider the following minimizing problem

$$S_{0,r}(\varphi) = \inf_{u \in H_0^r(\Omega), \|u+\varphi\|_{L^{2^*r}}=1} \|u\|_r^2 \quad (1)$$

and

$$S_{\theta,r}(\varphi) = \inf_{u \in H_\theta^r(\Omega), \|u+\varphi\|_{L^{2^*r}}=1} \|u\|_r^2, \quad (2)$$

where the function $\varphi \in L^{2^*r}(\Omega) \cap C(\Omega)$ and $2^*r = \frac{2N}{N-2r}$ is the limiting Sobolev exponent in the imbedding $H_0^r(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q \leq 2^*r$.

The problem under consideration in this paper is related to many geometrical equations involving where lack of compactness occurs. The statement of this problem on the bounded domain is associated with problems of the resolution of some minimization problem from geometry and physics, where the goal of our minimization problem is to determine the existence of a non-trivial minimum.

In 1986, Brezis considered in [1] the first formulation of a problem which $r = 1$ and $\varphi = 0$, see also [5], [11] and [15]. In [6], Brezis and Nirenberg provided the first positive answer to this problem for $r = 1$ stated in terms of an existence result from the infimum under the condition $\varphi \not\equiv 0$. Since then, this problem has received many attention and this result has been improved in several ways. This include some results in the case where $r = 2$ and $\varphi = 0$ (see for instance [18] and [19]) that $S_{\theta,2}(0) = S_{0,2}(0) = S$ is the best Sobolev constant and S is not achieved. In the papers [2] and [12] the authors studied the problems (1) and (2) for the biharmonic operator $(-\Delta)^2$, see also [10] for other study of biharmonic operator. Polyharmonic equations have been considered in several works, see for example [9]. This type of problems have many applications, we can cite for example the study of quantitative properties of solutions of semi-linear problems or the Paneitz type operator which appears in Willmore surfaces and in geometry, see [16]. In this paper we are interested in the two minimization problems (1) and (2) where the function φ is given in $L^{2^*r}(\Omega) \cap C(\Omega)$. More precisely, we consider the case $r \geq 2$ and φ is not identically 0 which is a natural generalization of the previous works.

Since we have $H_0^r(\Omega) \subset H_\theta^r(\Omega)$ we always have $S_{\theta,r} \leq S_{0,r}$. A natural question arises is do we still have $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$ or $S_{\theta,r}(\varphi) = S_{0,r}(\varphi)$? In the case where the two infima $S_{\theta,r}$ and $S_{0,r}$ are achieved respectively by u_θ and u_0 we will be interested by some regularity of u_θ , namely is u_θ in $H_0^r(\Omega)$? and also by the sign of the Lagrange multipliers associated to the two problems.

Note that, if φ is smooth, using the change of variable $u + \varphi = v$ in the energy, we are led to problems with nonvanishing boundary datum, see [13].

Our main results can be stated as follows.

Theorem 1.

*Let Ω be a smooth bounded domain in \mathbb{R}^N with $N \geq 2r+1$ and $\varphi \in L^{2^*r} \cap C(\Omega) \setminus \{0\}$. Then $S_{0,r}(\varphi)$ and $S_{\theta,r}(\varphi)$ are achieved.*

Theorem 2.

*Let Ω be a smooth bounded domain in \mathbb{R}^N with $N \geq 2r+1$ and $\varphi \in L^{2^*r} \cap C(\Omega) \setminus \{0\}$. We have*

- (i) If $\|\varphi\|_{L^{2^*r}} < 1$ and φ has a constant sign on Ω , then every minimizer of $S_{\theta,r}(\varphi)$ is not in $H_0^r(\Omega)$ and we have $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$.
- (ii) If $\varphi \in (H_0^r(\Omega))^\perp$, where $(H_0^r(\Omega))^\perp$ is the orthogonal space of $H_0^r(\Omega)$ in $H_\theta^r(\Omega)$, then every minimizer of $S_{\theta,r}(\varphi)$ is not in $H_0^r(\Omega)$ and we have $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$.
- (iii) If $\|\varphi\|_{L^{2^*r}} > 1$ and $\varphi \in H_0^r(\Omega)$ then $S_{\theta,r}(\varphi) = S_{0,r}(\varphi)$.

Remark 1.1.

The proof of cases (i)-(ii) and (iii) are completely different, the last case is treated using the convexity of the problems, for more details see [8].

The rest of paper is organized as follows: In section 2, we prove that the infimum in (1) and (2) are achieved where $\varphi \neq 0$ using some technical steps. In section 3, we present the proof of Theorem 2, more precisely we establish that $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$, for φ satisfying suitable conditions.

2 Existence of minimizers

In this section, we will prove Theorem 1. This result is a natural generalization of the works [6] and [12].

Proof of Theorem 1.

We prove that $S_{\theta,r}$ is achieved. The proof for $S_{0,r}$ is similar. We follow an idea introduced in [6] see also [12].

Let $\{u_j\}$ be a minimizing sequence for $S_{\theta,r}(\varphi)$, that is,

$$\|u_j + \varphi\|_{L^{2^*r}} = 1 \tag{3}$$

and

$$\|u_j\|_r^2 = S_{\theta,r}(\varphi) + o(1). \tag{4}$$

An easy computations give that $\{u_j\}$ is bounded in $H_\theta^r(\Omega)$. Then, there exists a subsequence, still noted, $\{u_j\}$ such that

$$\begin{aligned} u_j &\rightharpoonup u && \text{weakly in } H_\theta^r(\Omega), \\ u_j &\rightarrow u && \text{strongly in } L^t(\Omega) \text{ for any } t < 2^*r, \\ u_j &\rightarrow u && \text{a.e on } \Omega, \\ u_j &\rightharpoonup u && \text{weakly in } L^{2^*r}(\Omega). \end{aligned}$$

Using the lower semi-continuity in (3) and (4), we obtain that

$$\|u + \varphi\|_{L^{2^*r}} \leq 1,$$

and

$$\|u\|_r^2 \leq S_{\theta,r}(\varphi). \tag{5}$$

In order to prove that $S_{\theta,r}(\varphi)$ is achieved by u , we need to establish $\|u + \varphi\|_{L^{2^*r}} = 1$. We proceed by contradiction, then we suppose that

$$\|u + \varphi\|_{L^{2^*r}} < 1. \tag{6}$$

We will prove the contradiction in four steps.

• **Step 1**

We have

$$S_{\theta,r}(\varphi) - \|u\|_r^2 \geq S_r \left[1 - \int_{\Omega} |u + \varphi|^{2^*r} \right]^{\frac{2}{2^*r}}. \quad (7)$$

Indeed, let $v_j = u_j - u$. We have

$$v_j \rightharpoonup 0 \quad \text{weakly in } H_{\theta}^r(\Omega).$$

$$v_j \rightarrow 0 \quad \text{a.e on } \Omega.$$

Looking in the definition of S_r we write

$$\|v_j\|_r^2 \geq S_r \|v_j\|_{L^{2^*r}}^2. \quad (8)$$

From (3), we see that

$$1 = \|u + \varphi\|_{L^{2^*r}}^{2^*r} + \|v_j\|_{L^{2^*r}}^{2^*r} + o(1),$$

By Brezis-Lieb Lemma [4], we have

$$\|v_j\|_{L^{2^*r}}^2 = \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}} + o(1), \quad (9)$$

Inserting (9) into (8), we get

$$\|v_j\|_r^2 \geq S_r \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}}. \quad (10)$$

On the other hand, from (4), we write

$$\|v_j\|_r^2 = S_{\theta,r}(\varphi) - \|u\|_r^2 + o(1). \quad (11)$$

Inserting (10) into (11) we obtain (7).

• **Step 2**

Let $v \in H_{\theta}^r$ such that $\|v + \varphi\|_{L^{2^*r}} \leq 1$, we have

$$S_{\theta,r}(\varphi) - \|v\|_r^2 \leq S_r \left[1 - \|v + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}}, \quad (12)$$

and thus

$$S_{\theta,r}(\varphi) - \|u\|_r^2 = S_r \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}}. \quad (13)$$

Indeed, let $v \in H_{\theta}^r(\Omega)$ such that $\|v + \varphi\|_{L^{2^*r}} \leq 1$. Suppose that $\|v + \varphi\|_{L^{2^*r}} < 1$, otherwise (12) comes directly from the definition of $S_{\theta,r}$. There exists $c_{\varepsilon} > 0$ such that $\|v + \varphi + c_{\varepsilon} u_{x_0,\varepsilon}\|_{L^{2^*r}} = 1$ where $u_{x_0,\varepsilon}$ is an extremal function associate to the best Sobolev constant S_r defined by

$$u_{x_0,\varepsilon}(x) = \frac{\varepsilon^{\frac{N-2r}{2}} \xi}{(\varepsilon^2 + |x - x_0|^2)^{\frac{N-2r}{2}}}. \quad (14)$$

where $x_0 \in \Omega$ and $\xi \in C_0^\infty(B(x_0, R))$ be a fixed cut-off function satisfying $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $B(x_0, \frac{R}{2})$ with R a positive constant.

We have from [17],

$$(-\Delta)^j u_{x_0, \varepsilon}(t) = \frac{\varepsilon^{\frac{N-2r+4j}{2}} \sum_{i=0}^j G(i, j) t^{2i}}{(\varepsilon^2 + t^2)^{\frac{N-2r+4j}{2}}}, \quad \text{for } j = 1, 2, \dots, r,$$

where

$$G(i, j) = 2^i \binom{j}{i} K_j D(i, j) E(i, j),$$

with

$$K_j = \prod_{h=0}^{j-1} (N - 2r + 2h),$$

$$D(i, j) = \begin{cases} 1 & \text{if } i = 0 \\ \prod_{h=0}^{j-1} (r - h) & \text{if } i = 1, 2, \dots, j \end{cases}$$

and

$$E(i, j) = \begin{cases} \prod_{h=0}^{j-1} (N + 2h) & \text{if } i = 0, 1, \dots, j-1 \\ 1 & \text{if } i = j \\ 0 & \text{if } i \geq j+1. \end{cases}$$

From [7], we have

$$\|u_{x_0, \varepsilon}\|_{L^{2^*r}}^2 = \frac{K}{S_r} + O(\varepsilon^{N-2r}), \quad (15)$$

$$\|u_{x_0, \varepsilon}\|_r^2 = K + O(\varepsilon^{N-2r}), \quad (16)$$

$$u_{x_0, \varepsilon} \rightharpoonup 0 \quad \text{in } H^r(\Omega), \quad (17)$$

where K is a positive constant.

Now, we have $\|v + \varphi + c_\varepsilon u_{x_0, \varepsilon}\|_{L^{2^*r}} = 1$. Using Bresiz-Lieb Lemma, we write

$$c_\varepsilon^{2^*r} \|u_{x_0, \varepsilon}\|_{L^{2^*r}}^{2^*r} = 1 - \|v + \varphi\|_{L^{2^*r}}^{2^*r} + o(1),$$

Therefore

$$c_\varepsilon^2 = \frac{S_r}{K} \left[1 - \|v + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}} + o(1). \quad (18)$$

On the other hand, we have

If r is even

$$\begin{aligned} S_{\theta, r}(\varphi) &\leq \|v + c_\varepsilon u_{x_0, \varepsilon}\|_r^2 \\ &\leq \|v\|_r^2 + c_\varepsilon^2 \|u_{x_0, \varepsilon}\|_r^2 + 2c_\varepsilon \int_\Omega (\Delta)^{\frac{r}{2}} v (\Delta)^{\frac{r}{2}} u_{x_0, \varepsilon} dx \\ &\leq \|v\|_r^2 + c_\varepsilon^2 K + 2\varepsilon^{\frac{N}{2}} c_\varepsilon \int_\Omega (\Delta)^{\frac{r}{2}} v \frac{\sum_{i=0}^{\frac{r}{2}} G(i, \frac{r}{2}) t^{2i}}{(\varepsilon^2 + t^2)^{\frac{N}{2}}} dx + o(\varepsilon^{\frac{N-2r}{2}}). \end{aligned} \quad (19)$$

If r is odd

$$\begin{aligned}
S_{\theta,r}(\varphi) &\leq \|v + c_\varepsilon u_{x_0,\varepsilon}\|_r^2 \\
&\leq \|v\|_r^2 + c_\varepsilon^2 \|u_{x_0,\varepsilon}\|_r^2 + 2c_\varepsilon \int_\Omega |\nabla(-\Delta)^{\frac{r-1}{2}} v| |\nabla(-\Delta)^{\frac{r-1}{2}} u_{x_0,\varepsilon}| dx \\
&\leq \|v\|_r^2 + c_\varepsilon^2 K + 2\varepsilon^{\frac{N-2r+4j}{2}} c_\varepsilon \int_\Omega |\nabla(-\Delta)^{\frac{r-1}{2}} v| \left[\frac{\sum_{i=0}^j G(i,j) t^{2i}}{(\varepsilon^2 + t^2)^{\frac{N-2}{2}}} - \frac{t^2}{2} \frac{(N-2)}{(\varepsilon^2 + t^2)^{N-2}} \right] dx \\
&\quad + o(\varepsilon^{\frac{N-2r}{2}}).
\end{aligned} \tag{20}$$

In the two cases of r , using (16) and (18), the inequality (19) or (20) becomes

$$S_{\theta,r}(\varphi) \leq \|v\|_r^2 + S_r \left[1 - \|v + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}} + o(\varepsilon^{\frac{N-2r}{2}}).$$

Therefore we deduce (12). Also, replace v par u in (12) and using step 1 we get (13).

• **Step 3**

According to assumption (6):

If r is even then

$$\int_\Omega (\Delta)^{r/2} u (\Delta)^{r/2} v dx = S_r \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}-1} \int_\Omega |u + \varphi|^{2^*r-2} (u + \varphi) v dx \tag{21}$$

for every $v \in H_\theta^r(\Omega)$.

If r is odd then

$$\int_\Omega \nabla(-\Delta)^{\frac{r-1}{2}} u \nabla(-\Delta)^{\frac{r-1}{2}} v dx = S_r \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}-1} \int_\Omega |u + \varphi|^{2^*r-2} (u + \varphi) v dx \tag{22}$$

for every $v \in H_\theta^r(\Omega)$.

Indeed, let $v \in H_\theta^r(\Omega)$. Since $\|u + \varphi\|_{L^{2^*r}} < 1$, there exists $t_0 > 0$ such that for all $|t| < t_0$ we have

$$\|u + \varphi + tv\|_{L^{2^*r}} < 1.$$

Therefore, from Step 2, we have

$$S_{\theta,r}(\varphi) - \|u + tv\|_r^2 \leq S_r \left[1 - \|u + tv + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}}.$$

At this stage, we distinguish two cases:

If r is even then

$$S_{\theta,r}(\varphi) - \|u\|_r^2 - 2t \int_\Omega (\Delta)^{\frac{r}{2}} u (\Delta)^{\frac{r}{2}} v dx + o(t) \leq S_r \left[1 - \|u + tv + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}},$$

some computations give

$$\begin{aligned}
&S_{\theta,r}(\varphi) - \|u\|_r^2 - 2t \int_\Omega (\Delta)^{\frac{r}{2}} u (\Delta)^{\frac{r}{2}} v dx + o(t) \leq S_r \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}} \\
&\times \left[1 - 2t(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} \int_\Omega |u + \varphi|^{2^*r-2} (u + \varphi) v dx + o(t) \right].
\end{aligned}$$

Using (13), we obtain

$$\begin{aligned} -2t \int_{\Omega} (\Delta)^{\frac{r}{2}} u (\Delta)^{\frac{r}{2}} v dx + o(t) &\leq -2t S_r (1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{\frac{2}{2^*r}-1} \int_{\Omega} |u + \varphi|^{2^*r-2} (u + \varphi) v dx \\ &+ o(t). \end{aligned}$$

We deduce (21) by letting t goes to 0^{\pm} .

If r is odd then

Using again Step 2, we have

$$S_{\theta,r}(\varphi) - \|u\|_r^2 - 2t \int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u \nabla(-\Delta)^{\frac{r-1}{2}} v dx + o(t) \leq S_r \left[1 - \|u + tv + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}},$$

some computations give

$$\begin{aligned} S_{\theta,r}(\varphi) - \|u\|_r^2 - 2t \int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u \nabla(-\Delta)^{\frac{r-1}{2}} v dx + o(t) &\leq S_r \left[1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{2}{2^*r}} \\ \times \left[1 - 2t(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} \int_{\Omega} |u + \varphi|^{2^*r-2} (u + \varphi) v dx + o(t) \right]. \end{aligned}$$

Using (13), we obtain

$$\begin{aligned} &-2t \int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u \nabla(-\Delta)^{\frac{r-1}{2}} v dx + o(t) \\ &\leq -2t S_r (1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{\frac{2}{2^*r}-1} \int_{\Omega} |u + \varphi|^{2^*r-2} (u + \varphi) v dx + o(t). \end{aligned}$$

We get (22) by letting t goes to 0^{\pm} .

Now, we will show that the hypothesis (6) is not true and leads to a contradiction with (13).

• **Step 4**

The assumption (6) implies that

$$S_{\theta,r}(\varphi) - \|u\|_r^2 < S_r \left[1 - \int_{\Omega} |u + \varphi|^{2^*r} \right]^{\frac{2}{2^*r}}. \quad (23)$$

Indeed, we have that $u + \varphi \not\equiv 0$, otherwise, from (21) we obtain $\|u\|_r = 0$ therefore $u = 0$ and $\varphi = 0$ which is false. Since we may replace u by $-u$ and φ by $-\varphi$, we may assume, without loss of generality, that $u + \varphi > 0$ in a set Σ of a positive measure in a ball $B(x_0, \frac{R}{2}) \subset \Omega$ with R a positive constant. Then, let $x_0 \in \Sigma$ such that $(u + \varphi)(x_0) > 0$.

As in the proof of Step 2 there exists $c_{\varepsilon} > 0$ such that $\|u + \varphi + c_{\varepsilon} u_{x_0, \varepsilon}\|_{L^{2^*r}} = 1$, where c_{ε} is defined in (18).

We use $u + c_{\varepsilon} u_{x_0, \varepsilon}$ as a testing function of (??) gives that

If r is even

$$S_{\theta,r}(\varphi) \leq \|u\|_r^2 + c_{\varepsilon}^2 \int_{\Omega} ((\Delta)^{\frac{r}{2}} u_{x_0, \varepsilon})^2 dx + 2c_{\varepsilon} \int_{\Omega} (\Delta)^{\frac{r}{2}} u_{x_0, \varepsilon} (\Delta)^{\frac{r}{2}} u dx. \quad (24)$$

Let δ_{ε} and c_0 be given by

$$c_{\varepsilon} = c_0(1 - \delta_{\varepsilon}), \quad c_0^2 = \frac{S_r}{K} [1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r}]^{\frac{2}{2^*r}}. \quad (25)$$

Therefore

$$[1 - \|u + \varphi\|_{L^{2^{*r}}}^{2^{*r}}]^{-1} = c_0^{-2^{*r}} \left(\frac{S_r}{K}\right)^{\frac{2^{*r}}{2}}. \quad (26)$$

Using (16) and applying Step 3 with $v = u_{x_0, \varepsilon}$, (24)

$$\begin{aligned} S_{\theta, r}(\varphi) - \|u\|_r^2 &\leq c_0^2(1 - \delta_\varepsilon)^2(K + O(\varepsilon^{N-2r})) \\ &+ 2c_\varepsilon S_r [1 - \|u + \varphi\|_{L^{2^{*r}}}^{2^{*r}}]^{\frac{2}{2^{*r}}-1} \int_{\Omega} |u + \varphi|^{2^{*r}-2}(u + \varphi)u_{x_0, \varepsilon} dx. \end{aligned}$$

Using (18) we obtain

$$\begin{aligned} S_{\theta, r}(\varphi) - \|u\|_r^2 &\leq c_0^2(1 - \delta_\varepsilon)^2(K + O(\varepsilon^{N-2r})) \\ &+ 2c_\varepsilon c_0^2 K [1 - \|u + \varphi\|_{L^{2^{*r}}}^{2^{*r}}]^{-1} \int_{\Omega} |u + \varphi|^{2^{*r}-2}(u + \varphi)u_{x_0, \varepsilon} dx. \end{aligned} \quad (27)$$

Using (25) we write

$$\begin{aligned} S_{\theta, r}(\varphi) - \|u\|_r^2 &\leq S_r [1 - \|u + \varphi\|_{L^{2^{*r}}}^{2^{*r}}]^{\frac{2}{2^{*r}}} (1 - \delta_\varepsilon)^2 (1 + O(\varepsilon^{N-2r})) \\ &+ 2c_\varepsilon c_0^2 K [1 - \|u + \varphi\|_{L^{2^{*r}}}^{2^{*r}}]^{-1} \int_{\Omega} |u + \varphi|^{2^{*r}-2}(u + \varphi)u_{x_0, \varepsilon} dx. \end{aligned} \quad (28)$$

We distinguish two cases:

If $2^{*r} \geq 3$ we apply the following inequality

$$(x + y)^p - x^p - y^p - px^{p-1}y - pxy^{p-1} \geq 0, \quad x, y \geq 0, \quad p \geq 3.$$

For $x = u + \varphi$ and $y = c_\varepsilon u_{x_0, \varepsilon}$, using (15) and (25) we write

$$\begin{aligned} c_\varepsilon \int_{\Omega} |u + \varphi|^{2^{*r}-2}(u + \varphi)u_{x_0, \varepsilon} dx &\leq \frac{1}{2^{*r}} [1 - \|u + \varphi\|_{L^{2^{*r}}}^{2^{*r}} - c_\varepsilon^{2^{*r}} \|u_{x_0, \varepsilon}\|_{L^{2^{*r}}}^{2^{*r}}] \\ &- c_\varepsilon^{2^{*r}-1} \int_{\Omega} |u_{x_0, \varepsilon}|^{2^{*r}-1}(u + \varphi) dx \\ &\leq \frac{1}{2^{*r}} c_0^{2^{*r}} \left(\frac{K}{S_r}\right)^{\frac{2^{*r}}{2}} \\ &- \frac{1}{2^{*r}} c_0^{2^{*r}} (1 - \delta_\varepsilon)^{2^{*r}} \left(\left(\frac{K}{S_r}\right)^{\frac{2^{*r}}{2}} + O(\varepsilon^N)\right) \\ &- c_\varepsilon^{2^{*r}-1} \int_{\Omega} |u_{x_0, \varepsilon}|^{2^{*r}-1}(u + \varphi) dx \end{aligned}$$

On the other hand, a easy computation gives

$$\int_{\Omega} |u_{x_0, \varepsilon}|^{2^{*r}-1}(u + \varphi) dx = D\varepsilon^{\frac{N-2r}{2}}(u + \varphi)(x_0) + o(\varepsilon^{\frac{N-2r}{2}}) \quad (29)$$

where $D = \int_{\mathbb{R}^N} \frac{1}{(1 + |y|^2)^{\frac{N+2r}{2}}}$.

Then

$$\begin{aligned} c_\varepsilon \int_{\Omega} |u + \varphi|^{2^{*r}-2}(u + \varphi)u_{x_0, \varepsilon} dx &\leq c_0^{2^{*r}} \left(\frac{K}{S_r}\right)^{\frac{2^{*r}}{2}} (\delta_\varepsilon - \frac{1}{2}(2^{*r} - 1)\delta_\varepsilon^2 + o(\delta_\varepsilon^2) + o(\varepsilon^N)) \\ &- \frac{1}{2^{*r}} c_\varepsilon^{2^{*r}-1} D\varepsilon^{\frac{N-2r}{2}}(u + \varphi)(x_0) + o(\varepsilon^{\frac{N-2r}{2}}). \end{aligned} \quad (30)$$

Inserting (30) into (28), we get

$$\begin{aligned}
S_{\theta,r}(\varphi) - \|u\|_r^2 &\leq c_0^2 K(1 - 2\delta_\varepsilon + \delta_\varepsilon^2) + O(\varepsilon^{N-2r}) \\
&+ 2c_0^2 \left(\frac{K}{S_r}\right)^{\frac{2^*r}{2}} K(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} (\delta_\varepsilon - \frac{1}{2}(2^*r - 1)\delta_\varepsilon^2 + o(\delta_\varepsilon^2)) + o(\varepsilon^N) \\
&- 2c_\varepsilon^{2^*r-1} c_0^2 K(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} D\varepsilon^{\frac{N-2r}{2}} (u + \varphi)(x_0) + o(\varepsilon^{\frac{N-2r}{2}}).
\end{aligned}$$

Then

$$\begin{aligned}
S_{\theta,r}(\varphi) - \|u\|_r^2 &\leq c_0^2 K - 2c_0^2 K\delta_\varepsilon + c_0^2 K\delta_\varepsilon^2 + o(\varepsilon^{\frac{N-2r}{2}}) \\
&+ 2c_0^{2^*r+2} \left(\frac{K}{S_r}\right)^{\frac{2^*r}{2}} K(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} \delta_\varepsilon \\
&- (2^*r - 1)c_0^{2^*r+2} \left(\frac{K}{S_r}\right)^{\frac{2^*r}{2}} K(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} \delta_\varepsilon^2 + o(\delta_\varepsilon^2) \\
&- 2c_\varepsilon^{2^*r-1} c_0^2 K(1 - \|u + \varphi\|_{L^{2^*r}}^{2^*r})^{-1} D\varepsilon^{\frac{N-2r}{2}} (u + \varphi)(x_0) + o(\varepsilon^{\frac{N-2r}{2}}).
\end{aligned}$$

Using (26), we get

$$\begin{aligned}
S_{\theta,r}(\varphi) - \|u\|_r^2 &\leq c_0^2 K - c_0^2 K(2^{2^*r} - 2)\delta_\varepsilon^2 + o(\delta_\varepsilon^2) \\
&- 2c_\varepsilon K \left(\frac{S_r}{K}\right)^{\frac{2^*r}{2}} D\varepsilon^{\frac{N-2r}{2}} (u + \varphi)(x_0) + o(\varepsilon^{\frac{N-2r}{2}}).
\end{aligned}$$

Consequently

$$S_{\theta,r}(\varphi) - \|u\|_r^2 < c_0^2 K = S_r \left[1 - \| \|u + \varphi\|_{L^{2^*r}}^{L^{2^*r}} \right]^{\frac{2}{2^*r}}.$$

If $2^*r \leq 3$:

We use the following inequality see [[6], Lemma 4] and [12],

$$\left| |x + y|^p - |x|^p - |y|^p - pxy(|x|^{p-2} + |y|^{p-2}) \right| \leq \begin{cases} C|x|^{p-1}|y| & \text{if } |x| \leq |y|, \\ C|x||y|^{p-1} & \text{if } |x| \geq |y|, \end{cases} \quad (31)$$

for $x, y \in \mathbb{R}$, where $C = C(p)$ a positive a constant.

Define

$$\begin{aligned}
A_\varepsilon &:= 1 - \int_\Omega |u + \varphi|^{2^*r} dx - c_\varepsilon^{2^*r} \int_\Omega |u_{x_0,\varepsilon}|^{2^*r} dx - 2^*r c_\varepsilon^{2^*r-1} \int_\Omega |u_{x_0,\varepsilon}|^{2^*r-1} (u + \varphi) dx \\
&- 2^*r c_\varepsilon \int_\Omega |u + \varphi|^{2^*r-2} (u + \varphi) u_{x_0,\varepsilon} dx.
\end{aligned} \quad (32)$$

Applying the inequality (31) with $x = u + \varphi$, $y = c_\varepsilon u_{x_0,\varepsilon}$ and we suppose that $x_0 = 0$ for simplicity, we write

$$\begin{aligned}
|A_\varepsilon| &\leq C \left\{ c_\varepsilon \int_{\{|u+\varphi| \leq c_\varepsilon u_{x_0,\varepsilon}\}} |u + \varphi|^{2^*r-1} u_{x_0,\varepsilon} dx + c_\varepsilon^{2^*r-1} \int_{\{|u+\varepsilon| \geq c_\varepsilon u_{x_0,\varepsilon}\}} |u + \varphi|^{2^*r-1} dx \right\} \\
|A_\varepsilon| &\leq A_\varepsilon^1 + A_\varepsilon^2.
\end{aligned}$$

On one hand, we have

$$A_\varepsilon^1 \leq C_1 \varepsilon^{\frac{N-2r}{2}} \int_0^{c_1 \varepsilon^{\frac{1}{2}}} \frac{z^{N-1}}{(\varepsilon^2 + z^2)^{\frac{N-2r}{2}}} dz \quad \text{since } \{|u + \varphi| \leq c_\varepsilon u_{x_0,\varepsilon}\} \subset \{|x| \leq c_1 \varepsilon^{\frac{1}{2}}\},$$

and

$$A_\varepsilon^2 \leq C_2 \varepsilon^{\frac{N+2r}{2}} \int_{c_2 \varepsilon^{\frac{1}{2}}}^{c_3} \frac{z^{N-1}}{(\varepsilon^2 + z^2)^{\frac{N+2r}{2}}} dz \quad \text{since } \{|u + \varphi| \geq c_\varepsilon u_{x_0, \varepsilon}\} \subset \{c_2 \varepsilon^{\frac{1}{2}} \leq |x| \leq c_3\},$$

where c_1, c_2 and c_3 are some positive constants.

On the other hand, some computations give

$$\int_0^{c_1 \varepsilon^{\frac{1}{2}}} \frac{z^{N-1}}{(\varepsilon^2 + z^2)^{\frac{N+2r}{2}}} dz \leq \int_0^{c_1 \varepsilon^{\frac{1}{2}}} z^{2r-1} dz = \frac{1}{2r} c_1 \varepsilon^r = O(\varepsilon^r),$$

and

$$\int_{c_2 \varepsilon^{\frac{1}{2}}}^{c_3} \frac{z^{N-1}}{(\varepsilon^2 + z^2)^{\frac{N+2r}{2}}} dz \leq \int_{c_2 \varepsilon^{\frac{1}{2}}}^{c_3} z^{-2r-1} dz = -\frac{1}{2r} [c_3^{-2r} - (c_2)^{-2r} \varepsilon^{-r}] = \varepsilon^{-r} (K_2 - K_3 \varepsilon^r) = O(\varepsilon^{-r}).$$

Therefore $A_\varepsilon^1 = O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2r}{2}})$ and $A_\varepsilon^2 = O(\varepsilon^{\frac{N}{2}}) = o(\varepsilon^{\frac{N-2r}{2}})$. Thus

$$A_\varepsilon = o(\varepsilon^{\frac{N-2r}{2}}). \quad (33)$$

Combining (15), (32) and (33) we get

$$\begin{aligned} c_\varepsilon \int_\Omega |u + \varphi|^{2^{**}-2} (u + \varphi) u_{x_0, \varepsilon} dx &= \frac{1}{2^{**}} \left[1 - \|u + \varphi\|_{L^{2^{**}}}^{2^{**}} - c_0^{2^{**}} (1 - 2^{**} \delta_\varepsilon) \left(\frac{K}{S_r} \right)^{\frac{2^{**}}{2}} + o(\varepsilon^{N-2r}) \right] \\ &\quad - c_\varepsilon^{2^{**}-1} \int_\Omega |u_{x_0, \varepsilon}|^{2^{**}-1} (u + \varphi) dx + o(\delta_\varepsilon) + o(\varepsilon^{\frac{N-2r}{2}}). \end{aligned}$$

Using (26), an easy computation gives

$$c_\varepsilon \int_\Omega |u + \varphi|^{2^{**}-2} (u + \varphi) u_{x_0, \varepsilon} dx = \delta_\varepsilon c_0^{2^{**}} \left(\frac{K}{S_r} \right)^{\frac{2^{**}}{2}} - c_0^{2^{**}-1} \int_\Omega |u_{x_0, \varepsilon}|^{2^{**}-1} (u + \varphi) dx + o(\delta_\varepsilon) + o(\varepsilon^{\frac{N-2r}{2}}). \quad (34)$$

On the other way, we have

$$c_\varepsilon \int_\Omega |u + \varphi|^{2^{**}-2} (u + \varphi) u_{x_0, \varepsilon} dx = c_\varepsilon \varepsilon^{\frac{N-2r}{2}} \int_\Omega \frac{(u + \varphi)^{2^{**}-1}(x) dx}{|x|^{N-2r}} + o(\varepsilon^{\frac{N-2r}{2}}) = O(\varepsilon^{\frac{N-2r}{2}}) \quad (35)$$

Putting (29) and (35) into (34) we deduce

$$\delta_\varepsilon = O(\varepsilon^{\frac{N-2r}{2}}). \quad (36)$$

Now, returning to (27) and using (34), we write

$$\begin{aligned} S_{\theta, r}(\varphi) - \|u\|_r^2 &\leq c_0^2 K - 2\delta_\varepsilon c_0^2 K + 2c_0^2 K (1 - \|u + \varphi\|_{2^{**}}^{2^{**}})^{-1} \\ &\quad \times \left[\delta_\varepsilon c_0^{2^{**}} \left(\frac{K}{S_r} \right)^{\frac{2^{**}}{2}} - c_0^{2^{**}-1} \int_\Omega |u_{x_0, \varepsilon}|^{2^{**}-1} (u + \varphi) dx \right] + o(\delta_\varepsilon) + o(\varepsilon^{\frac{N-2r}{2}}). \end{aligned}$$

From (36), we get

$$S_{\theta, r}(\varphi) - \|u\|_r^2 \leq c_0^2 K - 2c_0 K \left(\frac{S_r}{K} \right)^{\frac{2^{**}}{2}} \int_\Omega |u_{x_0, \varepsilon}|^{2^{**}-1} (u + \varphi) + o(\varepsilon^{\frac{N-2r}{2}}). \quad (37)$$

Using (29) we write

$$S_{\theta,r}(\varphi) - \|u\|_r^2 \leq c_0^2 K - 2c_0 K \left(\frac{S_r}{K}\right)^{\frac{2^*r}{2}} D(u + \varphi)(x_0) \varepsilon^{\frac{N-2r}{2}} + o(\varepsilon^{\frac{N-2r}{2}}),$$

Therefore

$$S_{\theta,r}(\varphi) - \|u\|_r^2 < S_r \left[1 - \|u + \varphi\|_{2^*r}^{2^*r}\right]^{\frac{2}{2^*r}}.$$

If r is odd

We use again $u + c_\varepsilon u_{x_0,\varepsilon}$ as a testing function of (2), we have

$$S_{\theta,r}(\varphi) \leq \|u\|_r^2 + c_\varepsilon^2 \int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} u_{x_0,\varepsilon}|^2 dx + 2c_\varepsilon \int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} u_{x_0,\varepsilon}| |\nabla(-\Delta)^{\frac{r-1}{2}} u| dx.$$

Using (22) and applying the same technics used in the case where r is even, we obtain

$$S_{\theta,r}(\varphi) - \|u\|_r^2 < S_r \left[1 - \|u + \varphi\|_{2^*r}^{2^*r}\right]^{\frac{2}{2^*r}}.$$

Note that if instead of $(u + \varphi)(x_0) > 0$ we had $(u + \varphi)(x_0) < 0$, then we would choose $c_\varepsilon > 0$ such that $\|u + \varphi - c_\varepsilon u_{x_0,\varepsilon}\|_{L^{2^*}} = 1$. Which completes the proof of Step 4 and then the proof of Theorem 1 is done. \square

Remark 2.1. Let us note that any minimizers $u_\theta \in H_\theta^r(\Omega)$ of $S_{\theta,r}$, respectively $u_0 \in H_0^r(\Omega)$ of $S_{0,r}$, satisfy the following Euler-Lagrange equations:

$$\begin{cases} (-\Delta)^r u_\theta = \Lambda_\theta |u_\theta + \varphi|^{2^*r-2} (u_\theta + \varphi) & \text{in } \Omega, \\ \Delta^{r-1} u_\theta = \dots = \Delta u_\theta = u_\theta = 0 & \text{on } \partial\Omega, \end{cases} \quad (38)$$

and

$$\begin{cases} (-\Delta)^r u_0 = \Lambda_0 |u_0 + \varphi|^{2^*r-2} (u_0 + \varphi) & \text{in } \Omega, \\ \frac{\partial^{r-1} u_0}{(\partial\nu)^{r-1}} = \dots = \frac{\partial u_0}{\partial\nu} = u_0 = 0 & \text{on } \partial\Omega, \end{cases} \quad (39)$$

where Λ_θ is the Lagrange multiplier associated to u_θ and Λ_0 is the Lagrange multiplier associated to u_0 .

By analogy of the case $r = 2$ in [14], we can find the sign of the Lagrange multipliers which depends on $\|\varphi\|_{L^{2^*r}}$ and we have

Proposition 2.1.

(a) If $\|\varphi\|_{L^{2^*r}} < 1$ then $\Lambda_\theta > 0$. and $\Lambda_0 > 0$.

(b) If $\|\varphi\|_{L^{2^*r}} > 1$ then $\Lambda_\theta < 0$. and $\Lambda_0 < 0$.

Proof. We shall prove the results for Λ_θ , the proof of results for Λ_0 are similar.

We begin by noticing that Λ_θ can be written as:

$$S_{\theta,r} = \Lambda_\theta \left[1 - \int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) \varphi \right]. \quad (40)$$

Indeed, we have (see Remark 2.1)

$$S_{\theta,r} = \Lambda_\theta \int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) u_\theta,$$

and

$$\int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) (u_\theta + \varphi) = \int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) u_\theta + \int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) \varphi,$$

and since,

$$\int_{\Omega} |u_\theta + \varphi|^{2^{**r}} = 1.$$

Therefore we deduce (40).

Then, if we suppose $\|\varphi\|_{L^{2^{**r}}} < 1$, and by the Hölder inequality we have

$$\int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) \varphi \leq \left[\int_{\Omega} (|u_\theta + \varphi|^{2^{**r}-1})^{\frac{2^{**r}}{2^{**r}-1}} dx \right]^{\frac{2^{**r}-1}{2^{**r}}} \left[\int_{\Omega} |\varphi|^{2^{**r}} dx \right]^{\frac{1}{2^{**r}}}.$$

Since $u \not\equiv 0$ except for $\|\varphi\|_{L^{2^{**r}}} = 1$ which is an obvious case.

Thus,

$$\int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) \varphi \leq \left[\int_{\Omega} |\varphi|^{2^{**r}} dx \right]^{\frac{1}{2^{**r}}} < 1$$

and then $\Lambda_\theta > 0$. Now we assume that $\|\varphi\|_{L^{2^{**r}}} > 1$ and set, as in [14],

$$h(t) = \int_{\Omega} |tu_\theta + \varphi|^{2^{**r}}.$$

This function admits a derivative given by the formula

$$h'(t) = 2^{**r} \int_{\Omega} |tu_\theta + \varphi|^{2^{**r}-2} (tu_\theta + \varphi) u_\theta.$$

Now, the function u_θ satisfies

$$(-\Delta)^r u_\theta = \Lambda_\theta |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi).$$

Then, multiplying by u_θ and integrating by parts, we get

If r is even

$$\int_{\Omega} |(-\Delta)^{\frac{r}{2}} u_\theta|^2 dx = \Lambda_\theta \int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) u_\theta = \frac{\Lambda_\theta}{2^{**r}} h'(1).$$

If r is odd

$$\int_{\Omega} |\nabla (-\Delta)^{\frac{r-1}{2}} u_\theta|^2 dx = \Lambda_\theta \int_{\Omega} |u_\theta + \varphi|^{2^{**r}-2} (u_\theta + \varphi) u_\theta = \frac{\Lambda_\theta}{2^{**r}} h'(1).$$

So item (b) is verified because $h(1) = 1$, and we see that $h(t) \geq 1$ for all $t \in [0, 1]$. So, we conclude that $h'(1) \leq 0$. Otherwise, since h is continuous and $h(0) > 1$, there exists $0 < s < 1$ such that

$h(s) = 1$.

Therefore,

$$\int_{\Omega} |su_{\theta} + \varphi|^{2^*r} = 1.$$

When su_{θ} as a testing function in(2), we have

If r is even

$$S_{\theta,r} = \int_{\Omega} |(-\Delta)^{r/2} su_{\theta}|^2 dx \leq s^r \int_{\Omega} |(-\Delta)^{r/2} u_{\theta}|^2 dx.$$

If r is odd

$$S_{\theta,r} = \int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} su_{\theta}|^2 dx \leq s^{r-1} \int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} u_{\theta}|^2 dx.$$

We get a contradiction and the proof is completed. \square

Remark 2.2. In [9], the author considered the following semi-linear polyharmonic problem:

$$\begin{cases} (-\Delta)^r u = |u|^{2^*r-2} u + f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ (-\Delta)^{r-1} u = \dots = (-\Delta)u = u = 0 & \text{on } \partial\Omega. \end{cases} \quad (41)$$

This problem is equivalent to (38) when $\Lambda_{\theta} > 0$ is fixed. The author prove the existence of positive solutions under the sufficient conditions on f and the domain Ω .

3 Proof of Theorem 2

By definitions of (1) and (2) we have $S_{\theta,r} \leq S_{0,r}$. In this section we present a gap phenomenon between $S_{\theta,r}$ and $S_{0,r}$ under suitable hypothesis on φ .

Proof of (i).

Let φ be a positive function not identically zero. We adapt the argument of Van der Vorst [19] to the present situation. Let u_{θ} a the minimizer of (2). We give reason by contradiction. We assume that u_{θ} is in $H_0^r(\Omega)$.

If r is even

Let v be the solution of the following problem

$$\begin{cases} (-\Delta)^{\frac{r}{2}} v = |(-\Delta)^{\frac{r}{2}} u_{\theta}| & \text{in } \Omega, \\ (-\Delta)^{\frac{r}{2}-1} v = \dots = -\Delta v = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (42)$$

We get

$$\begin{cases} (-\Delta)(-\Delta)^{\frac{r}{2}-1}(v - u_{\theta}) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\frac{r}{2}-1}(v - u_{\theta}) = 0 & \text{on } \partial\Omega, \end{cases} \quad (43)$$

and

$$\begin{cases} (-\Delta)(-\Delta)^{\frac{r}{2}-1}(v + u_{\theta}) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\frac{r}{2}-1}(v + u_{\theta}) = 0 & \text{on } \partial\Omega. \end{cases} \quad (44)$$

In equations (43) and (44), using successively the maximum principle we obtain $v > |u_{\theta}|$ or $v = -u_{\theta}$ or $v = u_{\theta}$.

By taking the equation (42) with $v = u_{\theta}$ and $v = -u_{\theta}$, we find the function $(-\Delta)^{\frac{r}{2}} u_{\theta}$ has a

constant sign. These two cases $v = u_\theta$ or $-u_\theta$ when $u_\theta = \frac{\partial u_\theta}{\partial \nu} = \dots = \frac{\partial^{\frac{r}{2}-1} u_\theta}{(\partial \nu)^{\frac{r}{2}-1}} = 0$ on $\partial\Omega$ are false if we use the maximum principle when we consider $u_\theta = 0$ in Ω . So we have $v > |u_\theta|$ in Ω .

Considering this inequality and the fact that $\varphi \geq 0$, we get $u_\theta + \varphi < v + \varphi$ in Ω and $-u_\theta - \varphi < v + \varphi$ in Ω ; therefore $|u_\theta + \varphi| < |v + \varphi|$ in Ω and as result we have

$$\int_{\Omega} |v + \varphi|^{2^*r} dx > 1.$$

Currently, take the function $f(t) = \int_{\Omega} |tv + \varphi|^{2^*r} dx$ for $t \in [0, 1]$. Since f is continuous, $f(0) < 1$ and $f(1) > 1$, there exists $s \in]0, 1[$ such that $f(s) = 1$.

But we have

$$\int_{\Omega} |(-\Delta)^{\frac{r}{2}} sv|^2 dx \leq s^2 \int_{\Omega} |(-\Delta)^{\frac{r}{2}} v|^2 dx,$$

this gives a contradiction with the definition of $S_{\theta,r}(\varphi)$.

If r is odd

Let v be the solution of the next problem:

$$\begin{cases} (-\Delta)^{\frac{r-1}{2}} v = |(-\Delta)^{\frac{r-1}{2}} u_\theta| & \text{in } \Omega, \\ (-\Delta)^{\frac{r-1}{2}-1} v = \dots = -\Delta v = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (45)$$

We obtain

$$\begin{cases} (-\Delta)(-\Delta)^{\frac{r-1}{2}-1}(v - u_\theta) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\frac{r-1}{2}-1}(v - u_\theta) = 0 & \text{on } \partial\Omega, \end{cases} \quad (46)$$

and

$$\begin{cases} (-\Delta)(-\Delta)^{\frac{r-1}{2}-1}(v + u_\theta) \geq 0 & \text{in } \Omega, \\ (-\Delta)^{\frac{r-1}{2}-1}(v + u_\theta) = 0 & \text{on } \partial\Omega. \end{cases} \quad (47)$$

In (46) and (47), using successively the maximum principle we obtain $v > |u_\theta|$ or $v = -u_\theta$ or $v = u_\theta$.

By taking the equation (45) with $v = u_\theta$ and $v = -u_\theta$, we find the function $(-\Delta)^{\frac{r-1}{2}} u_\theta$ has a constant sign. These two cases $v = u_\theta$ or $-u_\theta$ when $u_\theta = \frac{\partial u_\theta}{\partial \nu} = \dots = \frac{\partial^{\frac{r-1}{2}-1} u_\theta}{(\partial \nu)^{\frac{r-1}{2}-1}} = 0$ on $\partial\Omega$ are false if we use the maximum principle when we consider $u_\theta = 0$ in Ω . Thus, we have $v > |u_\theta|$ in Ω . Using this inequality and the fact that $\varphi \geq 0$, we get $u_\theta + \varphi < v + \varphi$ in Ω and $-u_\theta - \varphi < v + \varphi$ in Ω ; therefore $|u_\theta + \varphi| < |v + \varphi|$ in Ω and as result we have

$$\int_{\Omega} |v + \varphi|^{2^*r} dx > 1.$$

Currently, let us consider the function $f(t) = \int_{\Omega} |tv + \varphi|^{2^*r} dx$ for $t \in [0, 1]$. Since f is continuous, $f(0) < 1$ and $f(1) > 1$, there exists $s \in]0, 1[$ such that $f(s) = 1$.

But we have

$$\int_{\Omega} |\nabla((-\Delta)^{\frac{r-1}{2}} sv)|^2 dx \leq s^2 \int_{\Omega} |(-\Delta)^{\frac{r-1}{2}} v|^2 dx,$$

that contradiction the definition of $S_{\theta,r}(\varphi)$.

This finish the proof of (i).

Proof of (ii). We will prove it into two cases.

Case 1: Assume that φ is in $(H_0^r(\Omega))^\perp$ and $\|\varphi\|_{L^{2^*r}} > 1$. Let u_θ a solution of (2). Multiplying (38) by $u_\theta + \varphi$ and integrating by parts, we get

If r is even

$$\int_{\Omega} |(-\Delta)^{\frac{r}{2}} u_\theta|^2 + \int_{\Omega} (-\Delta)^{\frac{r}{2}} u_\theta \cdot (-\Delta)^{\frac{r}{2}} \varphi dx = \Lambda_\theta.$$

If r is odd

We have

$$\int_{\Omega} |\nabla(-\Delta)^{\frac{r-1}{2}} u_\theta|^2 dx + \int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u_\theta \cdot \nabla(-\Delta)^{\frac{r-1}{2}} \varphi dx = \Lambda_\theta.$$

As $\|\varphi\|_{L^{2^*r}} > 1$, we have $\Lambda_\theta < 0$, therefore $\int_{\Omega} (-\Delta)^{\frac{r}{2}} u_\theta \cdot (-\Delta)^{\frac{r}{2}} \varphi dx < 0$, if r is even and $\int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u_\theta \cdot \nabla(-\Delta)^{\frac{r-1}{2}} \varphi dx < 0$ if r is odd. Which improve that u_θ is not in $H_0^r(\Omega)$; in conclusion we have $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$.

Case 2: Assume that φ is in $(H_0^r(\Omega))^\perp$ and $\|\varphi\|_{L^{2^*r}} < 1$. Let $x_0 \in \Omega$ and $u_{x_0,\varepsilon}$ defined in (14). From [7], we have $\int_{\Omega} |u_{x_0,\varepsilon}|^{2^*r} = \frac{K}{S_r} + o(1)$, and $\int_{\Omega} |\Delta u_{x_0,\varepsilon}|^2 = K + o(1)$, where K is a positive constant.

Since $\|\varphi\|_{L^{2^*r}} < 1$, there exists $c_\varepsilon > 0$ such that $\|\varphi + c_\varepsilon u_{x_0,\varepsilon}\|_{L^{2^*r}} = 1$. By Brezis-Lieb identity (see [4]) we have

$$c_\varepsilon^{2^*r} = \left(\frac{S_r}{K}\right)^{\frac{2^*r}{2}} \left[1 - \|\varphi\|_{L^{2^*r}}^{2^*r}\right] + o(1),$$

then

$$c_\varepsilon^2 K = S_r \left[1 - \|\varphi\|_{L^{2^*r}}^{2^*r}\right]^{\frac{N-2r}{N}} + o(1).$$

when $o(1)$ tends to 0. At limit, we have

$$c_\varepsilon^2 K = S_r \left[1 - \|\varphi\|_{L^{2^*r}}^{2^*r}\right]^{\frac{N-2r}{N}}. \quad (48)$$

Afterwards, since $\|\varphi + c_\varepsilon u_{x_0,\varepsilon}\|_{L^{2^*r}}^2 = 1$ we write

$$S_{\theta,r}(\varphi) \leq c_\varepsilon^2 \|u_{a,\varepsilon}\|_r^2 = S_r \left[1 - \|\varphi\|_{L^{2^*r}}^{2^*r}\right]^{\frac{N-2r}{N}} + o(1).$$

Using (48), direct computations show that

$$S_{\theta,r}(\varphi) \leq S_r \left(1 - \|\varphi\|_{L^{2^*r}}^{2^*r}\right)^{\frac{N-2r}{N}}. \quad (49)$$

On the other way, multiplying (38) by u_θ and integrating, we get

$$S_{\theta,r}(\varphi) = \Lambda_\theta \int_{\Omega} |u_\theta + \varphi|^{2^*r-2} (u_\theta + \varphi) u_\theta dx.$$

Thus, the Hölder inequality gives

$$S_{\theta,r}(\varphi) \leq \Lambda_\theta \left[\int_{\Omega} (|u_\theta + \varphi|^{2^*r-1})^{\frac{2^*r}{2^*r-1}} dx \right]^{\frac{2^*r-1}{2^*r}} \left[\int_{\Omega} |u_\theta|^{2^*r} dx \right]^{\frac{1}{2^*r}}$$

$$S_{\theta,r}(\varphi) \leq \Lambda_\theta \|u_\theta\|_{L^{2^*r}}. \quad (50)$$

Applying the Sobolev inequality we obtain that

$$S_{\theta,r}(\varphi) \leq \Lambda_\theta \frac{1}{S_r^{\frac{1}{2}}} \|u\|_r. \quad (51)$$

Or

$$\|u_\theta\|_r^2 = S_{\theta,r}(\varphi). \quad (52)$$

Combining (52) and (51) we find

$$S_{\theta,r}(\varphi) \leq \Lambda_\theta \left[1 - \|\varphi\|_{L^{2^*r}}^{2^*r} \right]^{\frac{1}{2^*r}}. \quad (53)$$

Now, multiplying (38) by $(u_\theta + \varphi)$ and integrating, using (53) we acquire

If r is even

$$\int_{\Omega} (-\Delta)^{\frac{r}{2}} u_\theta \cdot (-\Delta)^{\frac{r}{2}} \varphi dx = \Lambda_\theta - S_{\theta,r}(\varphi). \quad (54)$$

Combining (53) and (54) we are lead to

$$\int_{\Omega} (-\Delta)^{\frac{r}{2}} u_\theta \cdot (-\Delta)^{\frac{r}{2}} \varphi dx \geq \Lambda_\theta \left[1 - \left(1 - \int_{\Omega} |\varphi|^{2^*r} \right)^{\frac{N-2r}{2N}} \right] > 0,$$

If r is odd

$$\int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u_\theta \cdot \nabla(-\Delta)^{\frac{r-1}{2}} \varphi dx = \Lambda_\theta - S_{\theta,r}(\varphi). \quad (55)$$

Combining (53) and (55) we have

$$\int_{\Omega} \nabla(-\Delta)^{\frac{r-1}{2}} u_\theta \cdot \nabla(-\Delta)^{\frac{r-1}{2}} \varphi dx \geq \Lambda_\theta \left[1 - \left(1 - \int_{\Omega} |\varphi|^{2^*r} \right)^{\frac{N-2r}{2N}} \right] > 0.$$

This means that u_θ is not in $H_0^r(\Omega)$ and we have $S_{\theta,r}(\varphi) < S_{0,r}(\varphi)$.

Indeed, let us note that any minimizer $u_\theta \in H_\theta^r(\Omega)$ of $S_{\theta,r}(\varphi)$ is not in $H_0^r(\Omega)$.

Arguing by contradiction, suppose that $S_{\theta,r}(\varphi) = S_{0,r}(\varphi)$, thus $S_{0,r}(\varphi) = S_{\theta,r}(\varphi) = \|u\|_r^2$ and $\|u + \varphi\|_{L^{2^*r}} = 1$. Therefore u_θ be a minimizer of $S_{0,r}(\varphi)$, as a result $u_\theta \in H_0^r(\Omega)$, which gives a contradiction.

Proof of (iii). Suppose φ be in $H_0^r(\Omega)$. We admit first that for $\|\varphi\|_{L^{2^*r}} > 1$,

If r is even

we have,

$$S_{\theta,r}(\varphi) = \inf_{\substack{u \in H_\theta^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_{\Omega} |(-\Delta)^{\frac{r}{2}} u|^2 dx. \quad (56)$$

We find a convex problem. In this case, based on [8], we will use a duality's method.

For all $p \in L^2(\Omega)$,

Define

$$\beta_\theta = \sup_{\substack{u \in H_\theta^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_{\Omega} p \left((-\Delta)^{\frac{r}{2}} u \right) \text{ and } \beta_0 = \sup_{\substack{u \in H_0^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_{\Omega} p \left((-\Delta)^{\frac{r}{2}} u \right).$$

We get

$$\beta_\theta = \sup_{\substack{v \in H_\theta^r(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega p((-\Delta)^{\frac{r}{2}}v) - \int_\Omega p((-\Delta)^{\frac{r}{2}}\varphi) \quad \text{and} \quad \beta_0 = \sup_{\substack{v \in H_0^r(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega p((-\Delta)^{\frac{r}{2}}v) - \int_\Omega p((-\Delta)^{\frac{r}{2}}\varphi).$$

We will prove that we have

$$\beta_\theta = \beta_0, \quad \text{for all } p \in L^2(\Omega) \quad (57)$$

Initially, we observe that β_θ and β_0 are finite, due to the Hölder inequality

$$\left| \int_\Omega p((-\Delta)^{\frac{r}{2}}v) \right| \leq \|p\|_{L^2} \|(-\Delta)^{\frac{r}{2}}v\|_{L^2}.$$

We include that the linear operator

$$\begin{aligned} L : H_\theta^r(\Omega) &\rightarrow \mathbb{R} \\ v &\rightarrow \int_\Omega p((-\Delta)^{\frac{r}{2}}v) \end{aligned}$$

is continuous for the L^{2^*r} topology. So, there exists $\tilde{p} \in L^{\frac{2N}{N+2r}}$ even that for all $v \in H_\theta^r(\Omega)$ we find $L(v) = \int_\Omega \tilde{p}v$.

We consider that

$$\beta_\theta = \sup_{\substack{v \in H_\theta^r(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega \tilde{p}v - \int_\Omega p((-\Delta)^{\frac{r}{2}}\varphi); \quad \beta_0 = \sup_{\substack{v \in H_0^r(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega \tilde{p}v - \int_\Omega p((-\Delta)^{\frac{r}{2}}\varphi). \quad (58)$$

On the other side, for all $\tilde{p} \in L^{\frac{2N}{N+2r}}$, we have

$$\sup_{\substack{v \in H_\theta^r(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega \tilde{p}v = \sup_{\substack{v \in L^{2^*r}(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega \tilde{p}v = \sup_{\substack{v \in H_0^r(\Omega) \\ \|v\|_{L^{2^*r}} \leq 1}} \int_\Omega \tilde{p}v = \|\tilde{p}\|_{L^{\frac{2N}{N+2r}}}. \quad (59)$$

Actually, in the case where $\|\varphi\|_{L^{2^*r}} > 1$, we have to prove that

$$\frac{1}{2}S_{\theta,r}(\varphi) = \sup_{p \in L^2(\Omega)} \left\{ -\frac{1}{2} \int_\Omega |p|^2 - \sup_{\substack{u \in H_\theta^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_\Omega p((-\Delta)^{\frac{r}{2}}u) \right\} \quad (60)$$

and

$$\frac{1}{2}S_{0,r}(\varphi) = \sup_{p \in L^2(\Omega)} \left\{ -\frac{1}{2} \int_\Omega |p|^2 - \sup_{\substack{u \in H_0^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_\Omega p((-\Delta)^{\frac{r}{2}}u) \right\}. \quad (61)$$

Let us write, for $p \in L^2(\Omega)$ and for $u \in H_\theta^r(\Omega)$,

$$L(u, p) = -\frac{1}{2} \int_\Omega |p|^2 - \int_\Omega ((-\Delta)^{\frac{r}{2}}u) p.$$

We show that

$$\sup_{p \in L^2(\Omega)} L(u, p) = \frac{1}{2} \int_\Omega |(-\Delta)^{\frac{r}{2}}u|^2. \quad (62)$$

Indeed,

Firstly, we have

$$L(u, (-\Delta)^{\frac{r}{2}}u) = -\frac{1}{2} \int_{\Omega} |(-\Delta)^{\frac{r}{2}}u|^2 + \int_{\Omega} ((-\Delta)^{\frac{r}{2}}u)^2,$$

thus,

$$L(u, (-\Delta)^{\frac{r}{2}}u) = \frac{1}{2} \int_{\Omega} |(-\Delta)^{\frac{r}{2}}u|^2.$$

Therefore,

$$\sup_{p \in L^2(\Omega)} L(u, p) \geq L(u, (-\Delta)^{\frac{r}{2}}u). \quad (63)$$

On the other hand, we have,

$$\frac{1}{2} \|(-\Delta)^{\frac{r}{2}}u + p\|_{L^2}^2 = \frac{1}{2} \|(-\Delta)^{\frac{r}{2}}u\|_{L^2}^2 + \frac{1}{2} \|p\|_{L^2}^2 + \int_{\Omega} ((-\Delta)^{\frac{r}{2}}u)p,$$

then,

$$L(u, p) = -\frac{1}{2} \int_{\Omega} |p|^2 - \int_{\Omega} ((-\Delta)^{\frac{r}{2}}u)p \leq -\frac{1}{2} \int_{\Omega} |p|^2 + \frac{1}{2} \int_{\Omega} |(-\Delta)^{\frac{r}{2}}u + p|^2 dx - \int_{\Omega} ((-\Delta)^{\frac{r}{2}}u)p.$$

Thus

$$\sup_{p \in L^2(\Omega)} L(u, p) \leq \frac{1}{2} \int_{\Omega} |(-\Delta)^{\frac{r}{2}}u|^2. \quad (64)$$

Combining (63) and (64), we get (62).

Now, by (56) and (62) we have

$$\frac{1}{2} S_{\theta, r}(\varphi) = \inf_{\substack{u \in H_{\theta}^r(\Omega) \\ \|u + \varphi\|_{L^{2^*r}} \leq 1}} \sup_{p \in L^2(\Omega)} L(u, p). \quad (Y)$$

Let us justify that (Y) = (Y*) where (Y*) is the dual problem of (Y), defined by

$$\sup_{p \in L^2(\Omega)} \inf_{\substack{u \in H_{\theta}^r(\Omega) \\ \|u + \varphi\|_{L^{2^*r}} \leq 1}} L(u, p). \quad (Y^*)$$

Let us define

$$A = \{u \in H_{\theta}^r(\Omega); \|u + \varphi\|_{L^{2^*r}} \leq 1\}.$$

Let u_{θ} be a minimizer that realizes $S_{\theta, r}(\varphi)$ and let $p_{\theta} = -(-\Delta)^{\frac{r}{2}}u_{\theta}$. From (62), we get

$$L(u_{\theta}, p) \leq \sup_{p \in L^2(\Omega)} L(u_{\theta}, p) = \frac{1}{2} S_{\theta, r}(\varphi) \quad \text{for all } p \in L^2(\Omega). \quad (65)$$

Actually, let $u \in A$ we have

$$L(u, p_{\theta}) \geq -\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 - \sup_{u \in A} \int_{\Omega} (-\Delta u)^{\frac{r}{2}} p_{\theta}.$$

Indeed, by definition we write

$$\sup_{u \in A} \int_{\Omega} ((-\Delta)^{\frac{r}{2}}u)p_{\theta} \geq \int_{\Omega} ((-\Delta)^{\frac{r}{2}}u_{\theta})p_{\theta}. \quad (66)$$

Multiplying (66) by -1 , we get

$$-\sup_{u \in A} \int_{\Omega} ((-\Delta)^{\frac{r}{2}} u) p_{\theta} \leq -\int_{\Omega} ((-\Delta)^{\frac{r}{2}} u) p_{\theta}, \quad (67)$$

we add $-\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 dx$ in (67), we obtain

$$-\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 - \sup_{u \in A} \int_{\Omega} ((-\Delta)^{\frac{r}{2}} u) p_{\theta} \leq -\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 - \int_{\Omega} ((-\Delta)^{\frac{r}{2}} u) p_{\theta}.$$

Therefore,

$$-\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 dx - \sup_{u \in A} \int_{\Omega} ((-\Delta)^{\frac{r}{2}} u) p_{\theta} dx \leq L(u, p_{\theta}).$$

Now, integrating by part and using the fact that $(-\Delta)^{\frac{r}{2}} u_{\theta} = 0$ on $\partial\Omega$, we have

$$L(u, p_{\theta}) \geq -\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 - \sup_{u \in A} \int_{\Omega} u ((-\Delta)^{\frac{r}{2}} p_{\theta}).$$

On the other hand, using (59), we see that

$$\begin{aligned} -\sup_{u \in A} \int_{\Omega} u ((-\Delta)^{\frac{r}{2}} p_{\theta}) dx &= -\sup_{u \in A} \int_{\Omega} (u + \varphi) (-\Delta)^{\frac{r}{2}} p_{\theta} dx + \int_{\Omega} \varphi (-\Delta)^{\frac{r}{2}} p_{\theta} dx \\ &= -\|(-\Delta)^{\frac{r}{2}} p_{\theta}\|_{L^{\frac{2N}{N+2r}}} + \int_{\Omega} \varphi (-\Delta)^{\frac{r}{2}} p_{\theta} dx \\ &= -\|(-\Delta)^{\frac{r}{2}} p_{\theta}\|_{L^{\frac{2N}{N+2r}}} + \int_{\Omega} ((-\Delta)^{\frac{r}{2}} \varphi) p_{\theta} - \int_{\partial\Omega} \varphi (-\Delta)^{\frac{r}{2}} p_{\theta} dx. \end{aligned}$$

Using $(-\Delta)^{\frac{r}{2}} p_{\theta} = 0$ on $\partial\Omega$ we obtain,

$$-\sup_{u \in A} \int_{\Omega} u ((-\Delta)^{\frac{r}{2}} p_{\theta}) dx = -\|(-\Delta)^{\frac{r}{2}} p_{\theta}\|_{L^{\frac{2N}{N+2r}}} + \int_{\Omega} ((-\Delta)^{\frac{r}{2}} \varphi) p_{\theta},$$

and therefore

$$L(u, p_{\theta}) \geq -\frac{1}{2} \int_{\Omega} |p_{\theta}|^2 - \|(-\Delta)^{\frac{r}{2}} p_{\theta}\|_{L^{\frac{2N}{N+2r}}} + \int_{\Omega} ((-\Delta)^{\frac{r}{2}} \varphi) p_{\theta}. \quad (68)$$

However, the Euler equation (38) for u_{θ} gives

$$\|(-\Delta)^r u_{\theta}\|_{L^{\frac{2N}{N+2r}}} = |\Lambda_{\theta}|. \quad (69)$$

Contrastingly, multiplying the Euler equation (38) by $(u_{\theta} + \varphi)$, we obtain

$$\Lambda_{\theta} = \int_{\Omega} |(-\Delta)^{\frac{r}{2}} u_{\theta}|^2 + \int_{\Omega} (-\Delta)^{\frac{r}{2}} u_{\theta} \cdot (-\Delta)^{\frac{r}{2}} \varphi dx. \quad (70)$$

From Remark 2.1 we know that $\Lambda_{\theta} < 0$ since $\|\varphi\|_{L^{2^*r}} < 1$, therefore (69) and (70) give

$$-\|(-\Delta)^{\frac{r}{2}} p_{\theta}\|_{L^{\frac{2N}{N+2r}}} + \int_{\Omega} p_{\theta} \cdot (-\Delta)^{\frac{r}{2}} \varphi dx = S_{\theta,r}(\varphi). \quad (71)$$

We replace (68) into (71), we find

$$L(u, p_{\theta}) \geq \frac{1}{2} S_{\theta,r}(\varphi) \quad \text{for all } u \in A. \quad (72)$$

Regarding to [8] that (72) and 65 conclude that $(Y) = (Y^*)$. This proof is valid for $\frac{1}{2}S_{\theta,r}(\varphi)$ instead of $\frac{1}{2}S_{0,r}(\varphi)$, then we have proved (60) and (61). Consequently, we have (57) and then we conclude that $S_{\theta,r}(\varphi) = S_{0,r}(\varphi)$.

If r is odd

In the case when r is odd we find the same results just we have

$$S_{\theta,r}(\varphi) = \inf_{\substack{u \in H_{\theta}^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_{\Omega} |\nabla((- \Delta)^{\frac{r-1}{2}} u)|^2$$

instead of $S_{\theta,r}(\varphi) = \inf_{\substack{u \in H_{\theta}^r(\Omega) \\ \|u+\varphi\|_{L^{2^*r}} \leq 1}} \int_{\Omega} |(- \Delta)^{\frac{r}{2}} u|^2$. And then we use the same steps to conclude at the end that $S_{\theta,r}(\varphi) = S_{0,r}(\varphi)$. This ends the proof of the Theorem 2. \square

Remark 3.1.

If $\|\varphi\|_{L^{2^*r}} = 1$ then $S_{\theta,r}(\varphi) = S_{0,r}(\varphi) = 0$ and the infimum are achieved by 0.

Indeed, let $\|\varphi\|_{L^{2^*r}} = 1$,

By the definitions in (1) and (2) and according to Brezis-Lieb Lemma, for all $u \in H_0^r(\Omega)$ we obtain

$$\|u + \varphi\|_{L^{2^*r}} = \|u\|_{L^{2^*r}} + \|\varphi\|_{L^{2^*r}} + o(1),$$

that gives, using $\|\varphi\|_{L^{2^*r}} = 1$,

$$\|u\|_{L^{2^*r}} = 0$$

As a result, we find that $S_{\theta,r}(\varphi) = S_{0,r}(\varphi) = 0$ and the infimum are achieved by 0.

We declare the data availability statement.

References

- [1] H. Brezis, Some variational problems with lack of compactness. Proceedings of Symposia in Pure Mathematics Volume 45 (1986)
- [2] A. Beaulieu, R. Hadiji, Remarks on solutions of a fourth-order problem. Applied Mathematics Letters 19, 661-666 (2006)
- [3] H. Brezis and T. Kato, Remarks on the Schrodinger operator with singular complex potentials. J. Mathématiques Pure et Appliquées 58 , 137-151 (1979)
- [4] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals. Proceedings of the American Mathematical Society 88, 3, 486-490 (1983)
- [5] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Communications on pure and applied mathematics, 36(4), 437-477 (1983)
- [6] H. Brezis, L. Nirenberg, A minimization problem with critical exponent and non-zero data. Symmetry in nature, Scuola Normale Superiore Pisa 1, 129-140 (1989)

- [7] E. Edmunds, D. Fortunato and E. Jannelli, Critical exponents, critical dimensions and the biharmonic operator. *Archive for Rational Mechanics and Analysis* volume 112, 269-289 (1990)
- [8] S. Eklund, B. Temam, *Analyse convexe et problèmes variationnels*. Dunod (1974).
- [9] Y. Ge. Positive solutions in semilinear critical problems for polyharmonic operators, *Journal de Mathématiques Pures et Appliquées* 84, 2,199-245 (2005)
- [10] F. Gazzola, H. Ch. Grunau and M. Squassina, Existence and non existence results for critical growth biharmonic elliptic equation. *Calculus of Variations and Partial Differential Equations* volume 18, 117-143 (2003)
- [11] M. F. Furtado and B. N. Souza, Positive and nodal solutions for an elliptic equation with critical growth. *Communications in Contemporary Mathematics* 18(2), 15-21 (2016)
- [12] M. Guedda, R. Hadiji and C. Picard, A biharmonic problems with constraint involving critical Sobolev exponent. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* ,131, 5 1113-1132 (2001)
- [13] R. Hadiji, A nonlinear problem with a weight and a nonvanishing boundary datum. *Pure and Applied Functional Analysis* 5, f 4, 965-980 (2020),
- [14] R. Hadiji, R. Lewandowski, The sign of Lagrange multiplier for some minimization problem. *Differential Integral Equations* 4(3), 491-492 (1991)
- [15] R. Hadiji and H. Yazidi, *Problem with critical Sobolev exponent and with weight*, *Chinese Annals of Mathematics, Series B*, 327-352 (2007)
- [16] S. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-riemannian manifolds, *Integrability and geometry. Symmetry, Integrability and Geometry: Methods and Applications SIGMA* 4, 036, 1-3(2008)
- [17] C.A. Swanson, The best Sobolev constant. *Applicable Analysis*, 47(4):227-239 (1992)
- [18] R. Van der Vorst, Fourth order elliptic equations with critical growth *C. R. Acad. Sci. Paris Sér. I*, 320, 295-299 (1995)
- [19] V. der Vorst, Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{\frac{2N}{N-4}}(\Omega)$, *Differential and Integral Equations* 6(2), 259-276 (1993)