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Quantization effects for multi-component Ginzburg-Landau vortices

Rejeb Hadiji ^{*}, Jongmin Han[†] and Juhee Sohn[‡]

Abstract

In this paper, we are concerned with n -component Ginzburg-Landau equations on \mathbb{R}^2 . By introducing a diffusion constant for each component, we discuss that the n -component equations are different from n -copies of the single Ginzburg-Landau equations. Then, the results of Brezis-Merle-Riviere for the single Ginzburg-Landau equation can be nontrivially extended to the multi-component case. First, we show that if the solutions have their gradients in L^2 space, they are trivial solutions. Second, we prove that if the potential is square summable, then it has quantized integrals, i.e., there exists one-to-one correspondence between the possible values of the potential energy and \mathbb{N}^n . Third, we show that different diffusion coefficients in the system are important to obtain nontrivial solutions of n -component equations.

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Keywords : n -component Ginzburg-Landau equations, quantization effect

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply connected domain. The classical Ginzburg-Landau(abbreviated by GL) energy on Ω is given by

$$G_\varepsilon^b(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2 dx. \quad (1.1)$$

Here, $u : \Omega \rightarrow \mathbb{R}^2$ is an order parameter and $u = g$ on $\partial\Omega$ for a smooth function $g : \Omega \rightarrow S^1$. For convenience, we often regard $u : \Omega \rightarrow \mathbb{R}^2$ as $u : \Omega \rightarrow \mathbb{C}$. The corresponding Euler-Lagrange equations are

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

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For the last three decades, there have been lots of studies on the solution structures of (1.2), in particular, on the asymptotics as $\varepsilon \rightarrow 0$.

Such studies started from the seminal work of [4] and [5] which deals fairly in detail with the asymptotic behaviors of minimizers u_ε for G_ε^b on

$$H_g^1(\Omega, \mathbb{R}^2) = \{u \in H^1(\Omega, \mathbb{R}^2) : u = g \text{ on } \partial\Omega\}.$$

The main issue in the analysis is the value of the degree of g , say d . If $d = 0$, then $H_g^1(\Omega, S^1) \neq \emptyset$ and u_ε converges to a harmonic map u_0 in $H_g^1(\Omega, \mathbb{R}^2)$. Here, u_0 minimizes

$$\int_{\Omega} |\nabla u|^2 dx \quad \text{on } H_g^1(\Omega, S^1)$$

and is the unique solution of

$$-\Delta u_0 = u_0 |\nabla u_0|^2 \text{ on } \Omega, \quad u_0 = g \text{ on } \partial\Omega, \quad |u_0| = 1 \text{ on } \Omega. \quad (1.3)$$

The main point is that u_0 serves as a test function for G_ε^b . This gives a uniform boundedness of $G_\varepsilon^b(u_\varepsilon)$ and hence H^1 -boundedness of u_ε . However, if $d \neq 0$, then $H_g^1(\Omega, S^1) = \emptyset$ and the energy is not bounded:

$$G_\varepsilon^b(u_\varepsilon) = 2\pi d \ln \frac{1}{\varepsilon} + O(1). \quad (1.4)$$

This phenomena gives rise to the formation of singularities $\{a_1, \dots, a_n\} \subset \Omega$ such that $u_\varepsilon \rightarrow u_* \in H_{loc}^1(\Omega \setminus \{a_1, \dots, a_n\})$ where u_* is a harmonic map that has singularities a_1, \dots, a_n .

On the other hand, if Ω is star-shaped, then the Pohozaev identity implies that the potential is uniformly bounded in the limit $\varepsilon \rightarrow 0$:

$$E_\varepsilon^b(u_\varepsilon) = \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u_\varepsilon|^2)^2 dx \leq C. \quad (1.5)$$

This also implies by (1.4) that

$$\int_{\Omega} |\nabla u_\varepsilon|^2 \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0. \quad (1.6)$$

If we set $\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, then \tilde{u}_ε satisfies

$$-\Delta \tilde{u}_\varepsilon = \tilde{u}_\varepsilon (1 - |\tilde{u}_\varepsilon|^2) \quad \text{in } \Omega_\varepsilon = \frac{1}{\varepsilon} \Omega. \quad (1.7)$$

If we take the limit $\varepsilon \rightarrow 0$, \tilde{u}_ε converges to a function u in $C_{loc}^2(\mathbb{R}^2)$ such that

$$-\Delta u = u(1 - |u|^2) \quad \text{in } \mathbb{R}^2. \quad (1.8)$$

More generally, let us consider for $\lambda > 0$

$$-\lambda \Delta u = u(1 - |u|^2) \quad \text{in } \mathbb{R}^2. \quad (1.9)$$

This can be derived from the limit $\varepsilon \rightarrow 0$ for a variant of (1.2):

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.10)$$

The weight function $a(x) > 0$ appears in (1.10) when we study the equation (1.2) on a domain equipped with a background Riemannian metric. See [1, 2, 3] for the study of asymptotics of solutions to (1.10). If we set $\tilde{a}(x) = a(\varepsilon x)$ and $\tilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ for a solution u_ε of (1.10), then \tilde{u}_ε satisfies

$$-\operatorname{div}(\tilde{a}(x)\nabla \tilde{u}_\varepsilon) = \tilde{u}_\varepsilon (1 - |\tilde{u}_\varepsilon|^2) \quad \text{in } \Omega_\varepsilon. \quad (1.11)$$

By taking the limit $\varepsilon \rightarrow 0$, we are led to (1.9) with $\lambda = a(0)$. Although (1.9) can be obtained directly from (1.8) by a scaling $u(x) \mapsto u(x/\sqrt{\lambda})$, it is helpful to consider (1.10) when we study n -component generalization on \mathbb{R}^2 as we shall see.

Meanwhile, since $E_\varepsilon^b(u_\varepsilon)$ is convergent up to a subsequence in view of (1.5), it is interesting to find the value of

$$E^b(u) = \int_{\mathbb{R}^2} (1 - |u|^2)^2 dx \quad (1.12)$$

for solutions of (1.9). Remarkably, it was verified in [6] that if u is a solution of (1.9), then $E^b(u)$ allows only quantized values. Meanwhile, in the point of (1.6), it was proved that if u is a solution of (1.9) and $\nabla u \in L^2(\mathbb{R}^2)$, then u is a trivial solution. This can be summarized as

Theorem A. [6]

- (i) If u is a solution of (1.9), then $E^b(u) = 2\pi\lambda d^2$ for $d = 0, 1, 2, \dots, \infty$.
- (ii) If u is a solution of (1.9) and $\nabla u \in L^2(\mathbb{R}^2)$, then either $u \equiv 0$ or $u \equiv c_0$ for a constant c_0 with $|c_0| = 1$.

The classical model with the energy (1.1) can be generalized as an n -component model. For a pair of maps $(u_1, \dots, u_n) \in H_{g_1}^1(\Omega, \mathbb{C}) \times \dots \times H_{g_n}^1(\Omega, \mathbb{C})$, we consider an energy

$$G_\varepsilon(u_1, \dots, u_n) = \frac{1}{2} \int_{\Omega} \sum_{j=1}^n |\nabla u_j|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 dx. \quad (1.13)$$

The Euler-Lagrange equations are

$$\begin{cases} -\Delta u_i = \frac{1}{\varepsilon^2} u_i \left(n - \sum_{j=1}^n |u_j|^2 \right) & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \end{cases} \quad (1.14)$$

for $i = 1, \dots, n$. This direct generalization to an n -component model originates from a semilocal gauge field model [10, 11]. The energy (1.13) was used to explain some issues in cosmology related to a particular state of the early universe that possesses both a local and a global gauge invariance. It also describes interactions of multiple order parameters. See [8, 11] for more physical motivations.

It is very natural to ask that the properties of the classical model (1.1) are still valid for the generalized model (1.13). Regarding this question, a detailed analysis was firstly performed in [8]. A remarkable thing is that the convergence of minimizers of (1.13) is not so seriously affected by the degrees

$$\deg(g_j, \partial\Omega) = d_j \in \mathbb{N} \cup \{0\} \quad \text{for } j = 1, \dots, n. \quad (1.15)$$

Indeed, a natural generalization of $H_g^1(\Omega, S^1)$ is the space

$$\begin{aligned} & \mathcal{X}(g_1, \dots, g_n; \Omega) \\ &= \left\{ (u_1, \dots, u_n) \in H_{g_1}^1(\Omega; \mathbb{R}^2) \times \dots \times H_{g_n}^1(\Omega; \mathbb{R}^2) : \sum_{j=1}^n |u_j|^2 = n \text{ a.e. on } \Omega \right\}. \end{aligned}$$

Then, it is known by [8, Theorem 1.1] that $\mathcal{X}(g_1, \dots, g_n; \Omega) \neq \emptyset$ regardless of the values of d_j 's. For $n \geq 2$, the condition $\sum_{j=1}^n |u_j|^2 = n$ excludes the

possibility of singular behavior of each u_j near its zero. So, if (u_1^*, \dots, u_n^*) is a minimizer of

$$\sum_{j=1}^n \int_{\Omega} |\nabla u_j|^2 dx \quad \text{on } \mathcal{X}(g_1, \dots, g_n; \Omega),$$

then for a sequence of minimizers $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ for (1.13), we obtain

$$\frac{1}{2} \int_{\Omega} \sum_{j=1}^n |\nabla u_{j,\varepsilon}|^2 dx + \frac{1}{4\varepsilon^2} \int_{\Omega} \left(n - \sum_{j=1}^n |u_{j,\varepsilon}|^2 \right)^2 dx \leq G_{\varepsilon}(u_1^*, \dots, u_n^*). \quad (1.16)$$

So, the asymptotics of minimizers can be analyzed as in the degree zero case for the classical model (1.1). We note that (u_1^*, \dots, u_n^*) satisfies

$$\begin{cases} -\Delta u_i = \frac{1}{n} u_i \sum_{j=1}^n |\nabla u_j|^2 & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega, \quad \sum_{j=1}^n |u_j|^2 = n & \text{in } \Omega. \end{cases} \quad (1.17)$$

It is an open problem whether (1.17) allows a unique solution.

As in the equation (1.10), let us introduce a positive weight function $a_i(x)$ for u_i for each i . So, one may consider the following system: for $i = 1, \dots, n$

$$\begin{cases} -\operatorname{div}(a_i(x) \nabla u_i) = \frac{1}{\varepsilon^2} u_i \left(n - \sum_{j=1}^n |u_j|^2 \right) & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega. \end{cases} \quad (1.18)$$

Given a solution $(u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ of (1.18), if we set $\tilde{u}_{i,\varepsilon}(x) = u_{i,\varepsilon}(\varepsilon x)$ and $\tilde{a}_i(x) = a_i(\varepsilon x)$, then $\tilde{u}_{i,\varepsilon}$'s satisfy

$$-\operatorname{div}(\tilde{a}_i(x) \nabla \tilde{u}_i) = \tilde{u}_{i,\varepsilon} \left(n - \sum_{j=1}^n |\tilde{u}_{j,\varepsilon}|^2 \right) \quad \text{in } \Omega_{\varepsilon}. \quad (1.19)$$

By taking the limit $\varepsilon \rightarrow 0$, we are led to

$$-\lambda_i \Delta u_i = u_i \left(n - \sum_{j=1}^n |u_j|^2 \right) \quad \text{in } \mathbb{R}^2, \quad (1.20)$$

where $\lambda_i = a_i(0)$. We recall that (1.9) can be reduced to (1.8), i.e. the case $\lambda = 1$, by the change of variables $u(x) \mapsto u(\sqrt{\lambda}x)$. However, if $n > 1$, such scale argument does not work anymore for (1.19). A role of the constants $\lambda_1, \dots, \lambda_n$ will be addressed briefly in Theorem 1.2.

A natural question regarding (1.20) is whether the conclusion of Theorem A for the single GL system (1.9) is still valid for the solutions of the n -component GL system (1.20) or not. There is a big difference in this analysis between the single case (1.9) and the multi-component case (1.20). If a solution u of (1.9) satisfies (1.5), then one can show two important things

$$|u|^2 \leq 1 \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad |u(x)|^2 \rightarrow 1 \quad \text{as } |x| \rightarrow \infty. \quad (1.21)$$

These properties play central roles in the proof of Theorem A. Meanwhile, for a solution pair (u_1, \dots, u_n) of (1.20) satisfying

$$E(u_1, \dots, u_n) = \int_{\mathbb{R}^2} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 dx < \infty, \quad (1.22)$$

we can show by Lemma 3.1 below that

$$\sum_{j=1}^n |u_j|^2 \leq n \text{ in } \mathbb{R}^2 \quad \text{and} \quad \sum_{j=1}^n |u_j|^2 \rightarrow n \text{ as } |x| \rightarrow \infty. \quad (1.23)$$

Unlike the single case (1.21), the properties (1.23) do not provide any individual behaviors of u_j . This makes the analysis for solutions of (1.20) sophisticated and motivates a new research problem. The purpose of this paper is to verify how such obstruction can be overcome for a conclusion similar to Theorem A. Especially, we focus on the quantization of the potential $E(u_1, \dots, u_n)$. Our idea in this paper can be applicable for other types of n -component Ginzburg-Landau equations. We will address this topic elsewhere.

We start with a problem analogous to Theorem A (ii). In other words, we consider the question whether the equation (1.20) allows only trivial solutions whenever the gradients of u_j are square summable. Regarding this problem, we obtain the following result.

Theorem 1.1. *Let (u_1, \dots, u_n) be a solution of (1.20) satisfying*

$$\sum_{j=1}^n \int_{\mathbb{R}^2} |\nabla u_j|^2 dx < \infty. \quad (1.24)$$

Then, either $(u_1, \dots, u_n) \equiv (0, \dots, 0)$ or $(u_1, \dots, u_n) = (c_1, \dots, c_n)$ for some constants $c_1, \dots, c_n \in \mathbb{C}$ with $|c_1|^2 + \dots + |c_n|^2 = n$.

By Theorem 1.1, if (u_1, \dots, u_n) is a solution of (1.20) satisfying (1.24), then $E(u_1, \dots, u_n)$ is either 0 or ∞ . Hence, if a solution (u_1, \dots, u_n) takes a finite nonzero value of the potential $E(u_1, \dots, u_n)$, then $\nabla u_j \notin L^2(\mathbb{R}^2)$ for some j . A simple example of such solutions can be constructed from solutions of (1.9). Indeed, let $U(x)$ be a solution of (1.9) such that $|U(x)| \rightarrow 1$. If we assume $\lambda_1 = \dots = \lambda_n = \lambda$ and set $u_j(x) = U(\sqrt{n}x)$ for each j , then (u_1, \dots, u_n) is a solution of (1.20) with $\nabla u_j \notin L^2(\mathbb{R}^2)$.

Another example is a radially symmetric vortex solution. If we put $u_i(x) = e^{d_i \theta} f_i(r)$ where $x = re^{i\theta}$ and d_i is a nonnegative integer, then we can rewrite (1.20) as

$$\begin{cases} f_i'' + \frac{1}{r} f_i' - \frac{d_i^2}{r^2} f_i + \frac{1}{\lambda_i} f_i \left(n - \sum_{j=1}^n |f_j|^2 \right) = 0 & \text{for } r > 0, \\ f_i(r) = b_i r^{d_i} + o(r^{d_i}) & \text{for some } b_i > 0 \text{ as } r \searrow 0. \end{cases} \quad (1.25)$$

For the case $n = 1$, by letting $u(r) = e^{d\theta} f(r)$, we can transform (1.9) into

$$\begin{cases} f'' + \frac{1}{r} f' - \frac{d^2}{r^2} f + \frac{1}{\lambda} f(1 - f) = 0 & \text{for } r > 0, \\ f(r) = br^d + o(r^d) & \text{for some } b > 0 \text{ as } r \searrow 0. \end{cases} \quad (1.26)$$

The existence and properties of solutions to (1.26) are well-known in [7, 9]. In particular, there exists a unique solution f such that

$$f(r) \nearrow 1 \text{ as } r \rightarrow \infty, \quad (1.27)$$

and u satisfies the quantization property of Theorem A. When $n > 1$ and (f_1, \dots, f_n) is a solution of (1.25), then the condition (1.27) can be generalized by

$$f_i(r) \nearrow \alpha_i \text{ for some } \alpha_i \in (0, n) \text{ with } \alpha_1^2 + \dots + \alpha_n^2 = n. \quad (1.28)$$

Since one may consider infinitely many choices of $(\alpha_1, \dots, \alpha_n)$ whereas $\alpha_1 = 1$ is the unique choice for $n = 1$, it is expected that the solution structures of (1.25) is richer than those of (1.26). We will address this topic elsewhere. Instead, we concentrate on the quantization of potential $E(u_1, \dots, u_n)$ in this paper.

Before proceeding to the quantization problem, it is worthwhile to see that the constants $\lambda_1, \dots, \lambda_n$ play an important role to get nontrivial solutions of (1.25). We have seen that if f is the unique solution of (1.26), then (f_1, \dots, f_n) with $f_j(r) = f(\sqrt{n}r)$ for each j is a solution of (1.25) when $d_1 = \dots = d_n = d$ and $\lambda_1 = \dots = \lambda_n = \lambda$. The next theorem tells us that if $\lambda_1 = \dots = \lambda_n$, then all d_j 's must be equal. So, given a solution (f_1, \dots, f_n) of (1.25), there is a possibility that each f_j is a scale of a solution of (1.26). Therefore, it is necessary to assume that all λ_j 's are not equal in order to get a nontrivial solution of (1.25) in the sense that it doesn't come from n -copies of solutions of (1.26).

Theorem 1.2. *Let (f_1, \dots, f_n) be a solution of (1.25) that satisfies (1.28). If $\lambda_1 = \dots = \lambda_n$, then $d_1 = \dots = d_n$.*

To state the quantization effect, we need a notation. For a nonnegative real number α and a smooth function $u : \mathbb{R}^2 \rightarrow \mathbb{C}$, we set

$$I_\alpha(u) = \int_{\mathbb{R}^2} (\alpha^2 - |u|^2)^2 dx. \quad (1.29)$$

The third result of this paper is the following.

Theorem 1.3. *Let (u_1, \dots, u_n) be a solution of (1.20) such that*

$$E(u_1, \dots, u_n) < \infty. \quad (1.30)$$

Suppose that there are nonnegative real numbers $\alpha_1, \dots, \alpha_n$ such that

$$\alpha_1^2 + \dots + \alpha_n^2 = n \quad \text{and} \quad I_{\alpha_j}(u_j) < \infty. \quad (1.31)$$

Then, there exist nonnegative integers $d_j \in \{0, 1, 2, \dots\}$ for $1 \leq j \leq n$ such that

$$E(u_1, \dots, u_n) = 2\pi \sum_{j=1}^n \lambda_j \alpha_j^2 d_j^2. \quad (1.32)$$

The basic strategy for the proof of Theorem 1.1 and Theorem 1.3 is based on the proof of Theorem A given in [6]. However, there arise many nontrivial situations in the proof of Theorem 1.3 that do not appear in the analysis of solutions of (1.9). For instance, any nontrivial solution u of (1.9) with (1.5) satisfies (1.21). Whereas if (u_1, \dots, u_n) is a solution of (1.20), then $|u_1|^2 + \dots + |u_n|^2 \rightarrow n$ but there is no information on the behavior of individual u_j . Moreover, the maximum principle only gives the inequality $|u_1|^2 + \dots + |u_n|^2 \leq n$ but do not provide any pointwise estimate of individual u_j . These properties make our problem difficult and require some new ideas. Throughout a detailed analysis, we will deduce why the condition (1.31) is reasonable and necessary in the proof.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 and Theorem 1.2. In Section 3, we prove Theorem 1.3. The main idea for the proofs of Theorem 1.1 and Theorem 1.3 is to derive Pohozaev identities since we will use finiteness conditions of functional-like quantities such as (1.24) and (1.30). Meanwhile, Theorem 1.2 is proved by a technique of integration by parts for the ODE system (1.25).

We close this section by introducing some notations. We denote \bar{a} by the complex conjugate of $a \in \mathbb{C}$. We write $B_R = B(0, R)$ and $S_R = \partial B_R$. We set $\xi_k(x) = \xi(x/k)$ where $\xi \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ for $|x| \leq 1$, and $\xi = 0$ for $|x| \geq 2$. By the symmetry property of (1.20), we may assume that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \quad (1.33)$$

2 Proof of Theorem 1.1 and Theorem 1.2

This section is devoted to the proof of Theorem 1.1 and Theorem 1.2. Throughout this section, we put

$$f = \sum_{j=1}^n |u_j|^2 \quad \text{and} \quad h = \sum_{j=1}^n \frac{1}{\lambda_j} |u_j|^2. \quad (2.1)$$

We have $\lambda_n^{-1} f \leq h \leq \lambda_1^{-1} f$ by (1.33). We often use the following identities:

$$\begin{cases} -\lambda_i \Delta u_i = u_i(n - f), \\ -\Delta f = 2h(n - f) - 2 \sum_{j=1}^n |\nabla u_j|^2. \end{cases} \quad (2.2)$$

Lemma 2.1. *If (u_1, \dots, u_n) is a solution of (1.20) satisfying (1.24), then $f = |u_1|^2 + \cdots + |u_n|^2 \leq n$ in \mathbb{R}^2 . Moreover, $\nabla u_j \in L^\infty(\mathbb{R}^2)$ for each j .*

Proof. Let $\varphi = \sqrt{f} - \sqrt{n}$. By using (2.2), we derive

$$\begin{aligned} \Delta \varphi &= \varphi h f^{-\frac{1}{2}} (\sqrt{f} + \sqrt{n}) - \frac{1}{4} f^{-\frac{3}{2}} \left(|\nabla f|^2 - 4f \sum_{j=1}^n |\nabla u_j|^2 \right) \\ &\geq \varphi h f^{-\frac{1}{2}} (\sqrt{f} + \sqrt{n}), \end{aligned}$$

where the inequality comes from the Cauchy-Schwartz inequality:

$$|\nabla f|^2 = 4 \left(\sum_{j=1}^n \operatorname{Re}[\bar{u}_j \nabla u_j] \right)^2 \leq 4f \sum_{j=1}^n |\nabla u_j|^2.$$

Hence, by Kato's inequality and (1.33)

$$\Delta \varphi^+ \geq \chi_{\{\varphi > 0\}} \Delta \varphi \geq \frac{1}{\lambda_n} \varphi^+ \sqrt{f} (\sqrt{f} + \sqrt{n}) \geq \frac{2n}{\lambda_n} \varphi^+. \quad (2.3)$$

Again by the Cauchy-Schwartz inequality and (1.24),

$$|\nabla \varphi^+| = \frac{1}{\sqrt{f}} \left| \sum_{j=1}^n \operatorname{Re}[\bar{u}_j \nabla u_j] \right| \leq \left(\sum_{j=1}^n |\nabla u_j|^2 \right)^{\frac{1}{2}} \in L^2(\mathbb{R}^2).$$

So, multiplying (2.3) by ξ_k , we can see that

$$\begin{aligned} \frac{2n}{\lambda_n} \int_{\mathbb{R}^2} \xi_k \varphi^+ &\leq - \int_{\mathbb{R}^2} \nabla \varphi^+ \cdot \nabla \xi_k \leq \frac{C}{k} \int_{\{k < |x| < 2k\}} |\nabla \varphi^+| \\ &\leq C \left(\int_{\{k < |x| < 2k\}} |\nabla \varphi^+|^2 \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Hence, $\varphi^+ \equiv 0$.

Now, we note that $|\Delta u_j| \leq \lambda_j^{-1} |u_j| (n - f) \leq \lambda_j^{-1} n \sqrt{n}$. Thus, for each $x \in \mathbb{R}^2$, the function $v_{j,x}(y) = u_j(x + y)$ for $y \in B_1(0)$ is uniformly bounded

in $W^{2,p}(B_1)$ for all $p > 1$. Hence, $\nabla v_{j,x}$ is uniformly bounded in $C^1(B_{1/2})$ and thus $\nabla u_j \in L^\infty(\mathbb{R}^2)$. \square

Proof of Theorem 1.1. By multiplying the second equation of (2.2) by ξ_k , we obtain

$$2 \int_{\mathbb{R}^2} \xi_k h(n-f) = \int_{\mathbb{R}^2} \nabla \xi_k \cdot \nabla f + 2 \sum_{j=1}^n \int_{\mathbb{R}^2} \xi_k |\nabla u_j|^2. \quad (2.4)$$

Since $u_j \in L^\infty(\mathbb{R}^2)$ and $\nabla u_j \in L^\infty(\mathbb{R}^2)$, it follows from (1.24) that

$$\left| \int_{\mathbb{R}^2} \nabla \xi_k \cdot \nabla f \right| \leq C \sum_{j=1}^n \left(\int_{k \leq |x| \leq 2k} |\nabla u_j|^2 \right)^{\frac{1}{2}} \rightarrow 0$$

as $k \rightarrow \infty$. Thus, letting $k \rightarrow \infty$, we are led to

$$\frac{1}{\lambda_n} \int_{\mathbb{R}^2} f(n-f) \leq \int_{\mathbb{R}^2} h(n-f) = \sum_{j=1}^n \int_{\mathbb{R}^2} |\nabla u_j|^2. \quad (2.5)$$

Since $\nabla u_j \in L^\infty(\mathbb{R}^2)$, this implies that the set $\{x : 1 \leq f(x) \leq 3/2 < n\}$ is bounded. Thus, we can conclude from (2.5) that either $f \in L^1(\mathbb{R}^2)$ or $(n-f) \in L^1(\mathbb{R}^2)$.

Meanwhile, by multiplying (1.20) by $\xi_k x \cdot \nabla u_i$, we obtain

$$\begin{aligned} (LHS) &= - \sum_{i=1}^n \int_{\mathbb{R}^2} \lambda_i \xi_k \Delta u_i (x \cdot \nabla u_i) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^2} \xi_k u_i (f-n) (x \cdot \nabla u_i) = (RHS). \end{aligned}$$

We have

$$(LHS) = \sum_{i=1}^n \lambda_j \int_{\mathbb{R}^2} (\nabla \xi_k \cdot \nabla u_i) (x \cdot \nabla u_i) - \sum_{i=1}^n \frac{\lambda_i}{2} \int_{\mathbb{R}^2} |\nabla u_i|^2 (x \cdot \nabla \xi_k).$$

So,

$$|(LHS)| \leq C \sum_{i=1}^n \int_{\{k < |x| < 2k\}} |\nabla u_i|^2 \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Now, suppose that $f \in L^1(\mathbb{R}^2)$. Then, since $f \leq n$ and $f \in L^1(\mathbb{R}^2)$,

$$\begin{aligned} (RHS) &= \frac{1}{2} \int_{\mathbb{R}^2} \xi_k (x \cdot \nabla f) (f-n) \\ &= \int_{\mathbb{R}^2} \xi_k f \left(n - \frac{f}{2} \right) + \frac{1}{2} \int_{\mathbb{R}^2} f \left(n - \frac{f}{2} \right) (x \cdot \nabla \xi_k) \\ &\geq \frac{n}{2} \int_{\mathbb{R}^2} \xi_k f + o(1) \end{aligned}$$

as $k \rightarrow \infty$. This implies that $f \equiv 0$ and thus $u_i \equiv 0$ for each i . On the other

hand, if $(n - f) \in L^1(\mathbb{R}^2)$, then

$$\begin{aligned} (RHS) &= \frac{1}{4} \int_{\mathbb{R}^2} \xi_k x \cdot \nabla (f - n)^2 \\ &= -\frac{1}{2} \int_{\mathbb{R}^2} \xi_k (f - n)^2 - \frac{1}{4} \int_{\mathbb{R}^2} (x \cdot \nabla \xi_k) (f - n)^2 \\ &= -\frac{1}{2} \int_{\mathbb{R}^2} \xi_k (f - n)^2 + o(1) \end{aligned}$$

as $k \rightarrow \infty$. So, we have $f \equiv n$, which implies by (2.4) that $\nabla u_j \equiv 0$ for each j . This gives us the desired conclusion of Theorem 1.1. \square

Proof of Theorem 1.2. Let us assume that $\lambda_1 = \dots = \lambda_n = \lambda$. For instance, suppose that $d_1 > d_2$. Then, we have from (1.25)

$$\begin{aligned} f_1'' + \frac{1}{r} f_1' &= \frac{d_1^2}{r^2} f_1 + \frac{1}{\lambda} f_1 \left(\sum_{j=1}^n f_j^2 - n \right), \\ f_2'' + \frac{1}{r} f_2' &= \frac{d_2^2}{r^2} f_2 + \frac{1}{\lambda} f_2 \left(\sum_{j=1}^n f_j^2 - n \right). \end{aligned}$$

Choose $r_0 > 0$ such that $f_1 \geq \alpha_1/2$ and $f_2 \geq \alpha_2/2$ for $r \geq r_0$. So, for all $r \geq r_0$

$$\frac{1}{r} [r(f_1' f_2 - f_1 f_2')] = \frac{d_1^2 - d_2^2}{r^2} f_1 f_2 \geq \frac{\alpha_1 \alpha_2 (d_1^2 - d_2^2)}{4} \cdot \frac{1}{r^2} \equiv \frac{c_0}{r^2}.$$

Integrating this inequality on (r_0, r) , we are led to

$$f_1' f_2 - f_1 f_2' \geq \frac{c_0}{r} \ln r + \frac{c_1}{r}$$

for some $c_1 \in \mathbb{R}$. Set $h = f_1/f_2$. Then, $h(r) \rightarrow \alpha_1/\alpha_2$ as $r \rightarrow \infty$ and

$$h' = \frac{f_1' f_2 - f_1 f_2'}{f_2^2} \geq \frac{1}{\alpha_2^2} (f_1' f_2 - f_1 f_2') \geq \frac{c_0}{\alpha_2^2 r} \ln r + \frac{c_1}{\alpha_2^2 r} \quad \text{for } r \geq r_0.$$

This leads us to a contradiction: as $r \rightarrow \infty$,

$$h(r) \geq \frac{c_0}{2\alpha_2^2} [(\ln r)^2 - (\ln r_0)^2] + \frac{c_1}{\alpha_2^2} \ln \frac{r}{r_0} + h(r_0) \rightarrow \infty.$$

Consequently, $d_1 = d_2$. \square

3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The key part is to classify the values of potentials $I_{\alpha_i}(u_i)$ into four cases. Throughout this section, let (u_1, \dots, u_n) be a solution of (1.20) satisfying

$$E = E(u_1, \dots, u_n) = \int_{\mathbb{R}^2} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 < \infty. \quad (3.1)$$

We begin with the following lemma.

Lemma 3.1. *Let (u_1, \dots, u_n) be any solution of (1.20) satisfying (3.1). Then, the followings hold.*

(i) We have

$$\sum_{j=1}^n |u_j|^2 \leq n \quad \text{and} \quad \sum_{j=1}^n |u_j|^2 \rightarrow n \quad \text{as} \quad |x| \rightarrow \infty. \quad (3.2)$$

(ii) We have

$$\nabla u_j \in L^\infty(\mathbb{R}^2) \quad \text{for each } j. \quad (3.3)$$

(iii) There exists a constant C independent of R such that

$$\sum_{j=1}^n \int_{B_R} |\nabla u_j|^2 dx \leq CR. \quad (3.4)$$

Proof. (i) Let $f = |u_1|^2 + \cdots + |u_n|^2$ and $\varphi = f - n$. By (2.2),

$$\Delta \varphi \geq 2h\varphi.$$

Multiplying this equation by $\xi_k \varphi^+$, we are led to

$$\int_{\mathbb{R}^2} (|\nabla \varphi^+|^2 + 2h|\varphi^+|^2) \xi_k dx \leq \frac{1}{2} \int_{\mathbb{R}^2} \Delta \xi_k |\varphi^+|^2 dx \leq \frac{C}{k^2} \int_{\mathbb{R}^2} |\varphi^+|^2 dx.$$

Letting $k \rightarrow \infty$, we see by (3.1) that $\varphi^+ \equiv 0$. This implies the first part of (3.2).

Next, to show the second part of (3.2), we argue by a contradiction. Assume that there exists a sequence $|x_k| \rightarrow \infty$ such that $f(x_k) \leq n - 2\delta$ for some $0 < \delta < 2/5$. Let $r = \min \{1, \delta/(2Mn\sqrt{n} + nM^2)\}$, where we denote $M = \max_j \|\nabla u_j\|_{L^\infty}$ in the rest of this paper. For $x \in B(x_k, r)$, we have by the Mean Value Theorem

$$\begin{aligned} f(x) &\leq \sum_{j=1}^n (|u_j(x_k)| + |u_j(x) - u_j(x_k)|)^2 \\ &\leq \sum_{j=1}^n |u_j(x_k)|^2 + 2M|x_k - x| \sum_{j=1}^n |u_j(x_k)| + nM^2|x_k - x|^2 \\ &< n - 2\delta + (2Mn\sqrt{n} + nM^2)r \leq n - \delta. \end{aligned}$$

Then, it follows that

$$\int_{B(x_k, r)} (f - n)^2 dx \geq \pi r^2 \delta^2.$$

This is a contradiction since $f - n \in L^2(\mathbb{R}^2)$ by (3.1).

(ii) This follows from Lemma 2.1.

(iii) By multiplying (1.20) by u_i for each i and integrating them over B_R , we obtain

$$\begin{aligned} \sum_{i=1}^n \int_{B_R} \lambda_i |\nabla u_i|^2 dx &= \sum_{i=1}^n \int_{\partial B_R} \lambda_i u_i \frac{\partial u_i}{\partial \nu} dS + \int_{B_R} f(n - f) dx \\ &\leq n\sqrt{n}\lambda_n M |\partial B_R| + nE^{\frac{1}{2}} |B_R|^{\frac{1}{2}} \leq CR. \end{aligned}$$

Here ν denotes the outward normal to B_R . \square

We note that although the statement $|u_1|^2 + \cdots + |u_n|^2 \leq n$ in Lemma 2.1 and in (3.2) is the same, it is proved under different conditions. The following lemma helps us classify the values of $I_{\alpha_j}(u_j)$.

Lemma 3.2. Let $n \geq 3$ and (u_1, \dots, u_n) be a solution of (1.20) satisfying (3.1). For $1 \leq j \leq n$, let $\alpha_j \in [0, \sqrt{n}]$ be given such that $\alpha_1^2 + \dots + \alpha_n^2 = n$. If $k \in \{0, 1, \dots, n\}$ is the number of α_j such that $I_{\alpha_j}(u_j) = \infty$, then $k \neq 1$. Moreover, if $I_{\alpha_j}(u_j) < \infty$, then

$$|u_j| \rightarrow \alpha_j \quad \text{uniformly as } |x| \rightarrow \infty. \quad (3.5)$$

Proof. Suppose that $I_{\alpha_j}(u_j) < \infty$ for $1 \leq j \leq n-1$. Let

$$X_R = \left[\int_{B_R} \left(\sum_{j=1}^{n-1} \alpha_j^2 - \sum_{j=1}^{n-1} |u_j|^2 \right)^2 dx \right]^{\frac{1}{2}},$$

$$Y_R = \left[\int_{B_R} (\alpha_n^2 - |u_n|^2)^2 dx \right]^{\frac{1}{2}}.$$

Since $I_{\alpha_j}(u_j) < \infty$ for $1 \leq j \leq n-1$, it follows that $X_R \leq C$ where C is independent of R . By (3.1) and the Cauchy-Schwartz inequality,

$$\begin{aligned} \infty > E &= \int_{\mathbb{R}^2} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 dx \\ &\geq X_R^2 + Y_R^2 + 2 \int_{B_R} \left(\sum_{j=1}^{n-1} \alpha_j^2 - \sum_{j=1}^{n-1} |u_j|^2 \right) (\alpha_n^2 - |u_n|^2) dx \\ &\geq (X_R - Y_R)^2. \end{aligned}$$

As a consequence, Y_R is also uniformly bounded as $R \rightarrow \infty$ and thus $I_{\alpha_n}(u_n) < \infty$. This implies that either $k = 0$ or $2 \leq k \leq n$. The behavior (3.5) follows from the same argument for the proof of (i) in Lemma 3.1. \square

By Lemma 3.2, if (u_1, \dots, u_n) is a solution of (1.20) satisfying (3.1), then one of the following four cases holds:

(P1) there exist $\alpha_j \in [0, \sqrt{n}]$ for $1 \leq j \leq n$ such that

$$\alpha_j > 0 \quad \forall j, \quad \sum_{j=1}^n \alpha_j^2 = n, \quad \text{and} \quad I_{\alpha_j}(u_j) < \infty \quad \forall j;$$

(P2) there exist $l \in \{1, \dots, n-1\}$ and $\alpha_j \in [0, \sqrt{n}]$ for $1 \leq j \leq n$ such that

$$\begin{cases} \alpha_j > 0 \text{ for } 1 \leq j \leq l, & \alpha_j = 0 \text{ for } l+1 \leq j \leq n, \\ \sum_{j=1}^n \alpha_j^2 = n, & \text{and } I_{\alpha_j}(u_j) < \infty \quad \forall j; \end{cases}$$

(P3) there exist $l \in \{1, \dots, n-2\}$ and $\alpha_j \in [0, \sqrt{n}]$ for $1 \leq j \leq n$ such that

$$\begin{cases} I_{\alpha_j}(u_j) < \infty \text{ for } 1 \leq j \leq l, & I_{\alpha_j}(u_j) = \infty \text{ for } l+1 \leq j \leq n, \\ \sum_{j=1}^n \alpha_j^2 = n; \end{cases}$$

(P4) $I_{\alpha_j}(u_j) = \infty$ for any choice of $\alpha_j \in [0, \sqrt{n}]$ satisfying $\alpha_1^2 + \dots + \alpha_n^2 = n$.

Remark 3.3.

(a) In view of Lemma 3.2, $l \neq n - 1$ in the case **(P3)**. In particular, **(P3)** does not happen for $n = 2$.

(b) Let us explain the above classification for the case $n = 3$. Suppose that $\alpha_1, \alpha_2, \alpha_3 \in [0, \sqrt{3}]$ and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 3$. Assume that $I_{\alpha_1}(u_1) < \infty$. By the same argument of Lemma 3.2, it is obvious that

$$I_{\alpha_1}(u_1) < \infty \quad \text{if and only if} \quad J = \int_{\mathbb{R}^2} (\alpha_2^2 + \alpha_3^2 - |u_2|^2 - |u_3|^2)^2 < \infty.$$

When $J < \infty$, we have that $I_{\alpha_2}(u_2) < \infty$ if and only if $I_{\alpha_3}(u_3) < \infty$. So, if $I_{\alpha_1}(u_1) < \infty$, then either

$$(i) \quad I_{\alpha_1}(u_1) < \infty, \quad I_{\alpha_2}(u_2) < \infty, \quad I_{\alpha_3}(u_3) < \infty,$$

or

$$(ii) \quad I_{\alpha_1}(u_1) < \infty, \quad I_{\alpha_2}(u_2) = \infty, \quad I_{\alpha_3}(u_3) = \infty.$$

If (i) is true and $\alpha_j > 0$ for all $j = 1, 2, 3$, then it results in the case **(P1)**. If (i) is true and either $\alpha_1, \alpha_2 > 0$ with $\alpha_3 = 0$ or $\alpha_1 = \sqrt{3}$ with $\alpha_2 = \alpha_3 = 0$, then we are led to the case **(P2)**. If (ii) is true, then we have **(P3)**. Finally, if $I_{\alpha_1}(u_1) = I_{\alpha_2}(u_2) = I_{\alpha_3}(u_3) = \infty$ for any choice of α_j with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 3$, then we get the case **(P4)**.

(c) We believe that no solutions of (1.20) satisfy both $E(u_1, \dots, u_n) < \infty$ and **(P3)** (or **(P4)**). Of course, one can find an example of (u_1, \dots, u_n) that violates one of three conditions: (i) (u_1, \dots, u_n) is a solution, (ii) $E(u_1, \dots, u_n) < \infty$, and (iii) **(P3)** or **(P4)** is true.

For instance, if $n \geq 2$ and we set

$$\begin{aligned} u_1(x, y) &= \sqrt{n} \sin(x^2 + y^2), \\ u_j(x, y) &= \sqrt{n} \cos^{j-1}(x^2 + y^2) \sin(x^2 + y^2) \quad \text{for } 2 \leq j \leq n-1, \\ u_n(x, y) &= \sqrt{n} \cos^{n-1}(x^2 + y^2), \end{aligned}$$

then (u_1, \dots, u_n) satisfies $E(u_1, \dots, u_n) < \infty$ and **(P4)**. However, (u_1, \dots, u_n) is not a solution of (1.20).

Another example is $u_j(x, y) = A_j e^{i\omega x}$ where A_j and ω are positive real numbers for $1 \leq j \leq n$ such that $A_1^2 + \dots + A_n^2 + \omega^2 = n$. Then, one may check that (u_1, \dots, u_n) is a solution of (1.20). Moreover, if $\alpha_j = A_j$ for $1 \leq j \leq n-1$ and $\alpha_n = \sqrt{A_n^2 + \omega^2}$, then $I_{\alpha_j}(u_j) = 0$ for $1 \leq j \leq n-1$ and $I_{\alpha_n}(u_n) = \infty$. Thus, (u_1, \dots, u_n) satisfies **(P3)** with $l = n-1$ but we obtain $E(u_1, \dots, u_n) = \infty$. \square

Based on the above classification and Remark 3.3, we will focus on the quantization problem only for the cases **(P1)** and **(P2)**. Thus, the condition (1.31) in Theorem 1.3 is quite a reasonable assumption. Now, Theorem 1.3 is a consequence of the following proposition.

Proposition 3.4. *Let (u_1, \dots, u_n) be a solution pair of (1.20) satisfying (3.1). Suppose **(P1)** or **(P2)** is true. That is, there exist $l \in \{1, \dots, n\}$ and $\alpha_i \in$*

$[0, \sqrt{n}]$ for $1 \leq i \leq n$ such that

$$\begin{cases} \alpha_i > 0 & \text{for } 1 \leq i \leq l, \\ \alpha_i = 0 & \text{for } l+1 \leq i \leq n \quad \text{if } l < n, \\ \alpha_1^2 + \cdots + \alpha_n^2 = n, \\ I_{\alpha_i}(u_i) < \infty \quad \forall i. \end{cases}$$

Then, there exist l nonnegative integers d_1, \dots, d_l such that

$$\int_{\mathbb{R}^2} \left(\sum_{j=1}^n |u_j|^2 - n \right)^2 dx = 2\pi \sum_{i=1}^l \lambda_i \alpha_i^2 d_i^2. \quad (3.6)$$

Proof. We follow the argument of [6] that are based on the Pohozaev identities. Let us choose small $\delta \in (0, 1)$ and a number $\mu > 1$ such that

$$\begin{aligned} 0 < \delta &< \min \left\{ \frac{\alpha_i^2}{2}, \frac{1}{4} \mid 1 \leq i \leq l \right\}, \\ \max \left\{ \frac{\alpha_i^2}{\alpha_i^2 - \delta} \mid 1 \leq i \leq l \right\} &< 2(1 - \delta), \\ \max \left\{ \frac{\alpha_i^2}{\alpha_i^2 - \delta} \mid 1 \leq i \leq l \right\} &< \mu < 2(1 - \delta). \end{aligned}$$

By (3.5), there exists $R_0 > 0$ such that for all $|x| \geq R_0$ and $1 \leq i \leq l$,

$$\alpha_i^2 - \delta < |u_i(x)|^2 < \mu(\alpha_i^2 - \delta). \quad (3.7)$$

For $R > R_0$, the degrees

$$d_i = \deg(u_i, S_R) \quad \text{for } 1 \leq i \leq l$$

are well-defined. We notice that given a solution (u_1, \dots, u_n) of (1.20), if we replace some u_j by \bar{u}_j , then it gives also a solution of (1.20). For instance, if $n = 2$ and (u_1, u_2) is a solution, then (\bar{u}_1, u_2) , (u_1, \bar{u}_2) , and (\bar{u}_1, \bar{u}_2) are also solutions. So, without loss of generality, we may assume that $d_j \geq 0$ for $1 \leq j \leq l$. In addition, there exist smooth real valued functions $\psi_j(x)$ on $\mathbb{R}^2 \setminus B_{R_0}$ such that

$$u_j(x) = |u_j(x)| e^{i(d_j \theta + \psi_j)} \equiv \rho_j(x) e^{i\varphi_j} \quad \text{on } \mathbb{R}^2 \setminus B_{R_0} \quad \text{for } 1 \leq j \leq l. \quad (3.8)$$

We note that ψ_j is a globally defined smooth function whereas φ_j is locally defined.

We claim that

$$\sum_{i=1}^l \int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \psi_i|^2 dx < \infty, \quad (3.9)$$

$$\sum_{i=1}^l \int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \rho_i|^2 dx < \infty, \quad (3.10)$$

$$\sum_{i=l+1}^n \int_{\mathbb{R}^2} |\nabla u_i|^2 < \infty. \quad (3.11)$$

We postpone the proof of these estimates to the end of this proof.

By the Pohozaev identity for (1.20), i.e., multiplying $x \cdot \nabla u_i$ on (1.20), we

deduce that for $r > 0$,

$$\begin{aligned} & \sum_{i=1}^n \int_{S_r} \lambda_i \left| \frac{\partial u_i}{\partial \nu} \right|^2 + \frac{1}{r} \int_{B_r} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2 \\ &= \sum_{i=1}^n \int_{S_r} \lambda_i \left| \frac{\partial u_i}{\partial \tau} \right|^2 + \frac{1}{2} \int_{S_r} \left(n - \sum_{j=1}^n |u_j|^2 \right)^2. \end{aligned} \quad (3.12)$$

Set

$$E(r) = \int_{B_r} \left(\sum_{j=1}^n |u_j|^2 - n \right)^2.$$

Then, $E(r) \rightarrow E$ as $r \rightarrow \infty$. By integrating (3.12) for $r \in (0, R)$, we see that

$$\sum_{i=1}^n \int_{B_R} \lambda_i \left| \frac{\partial u_i}{\partial \nu} \right|^2 + \int_0^R \frac{E(r)}{r} dr = \sum_{i=1}^n \int_{B_R} \lambda_i \left| \frac{\partial u_i}{\partial \tau} \right|^2 + \frac{1}{2} E(R). \quad (3.13)$$

By (3.9) and (3.10), it holds that

$$\sum_{i=1}^l \int_{B_R} \lambda_i \left| \frac{\partial u_i}{\partial \nu} \right|^2 \leq \sum_{i=1}^l \int_{B_R} (\lambda_i |\nabla \rho_i|^2 + \lambda_i \rho_i^2 |\nabla \varphi_i|^2) \leq C.$$

Moreover, it follows from (3.11) that

$$\sum_{i=l+1}^n \left(\int_{B_R} \lambda_i \left| \frac{\partial u_i}{\partial \nu} \right|^2 + \int_{B_R} \lambda_i \left| \frac{\partial u_i}{\partial \tau} \right|^2 \right) \leq C.$$

Hence, by dividing (3.13) by $\log R$ and letting $R \rightarrow \infty$, we are led to

$$E = \sum_{i=1}^l \lim_{R \rightarrow \infty} \frac{1}{\log R} \int_{B_R} \lambda_i \left| \frac{\partial u_i}{\partial \tau} \right|^2. \quad (3.14)$$

For $R > R_0$, by letting $A_R = B_R \setminus B_{R_0}$, we have

$$\begin{aligned} & \int_{A_R} \left| \frac{\partial u_i}{\partial \tau} \right|^2 = \int_{A_R} \left| \frac{\partial \rho_i}{\partial \tau} \right|^2 + \int_{A_R} \rho_i^2 \left| \frac{d_i}{r} + \frac{\partial \psi_i}{\partial \tau} \right|^2 \\ &= \int_{A_R} \left\{ \left| \frac{\partial \rho_i}{\partial \tau} \right|^2 + \frac{d_i^2}{r^2} (\rho_i^2 - \alpha_i^2) + \rho_i^2 \left| \frac{\partial \psi_i}{\partial \tau} \right|^2 \right\} + \int_{A_R} \frac{2d_i \rho_i^2}{r} \frac{\partial \psi_i}{\partial \tau} + \int_{A_R} \frac{\alpha_i^2 d_i^2}{r^2}. \end{aligned}$$

The first integral is uniformly bounded with respect to R by (3.9), (3.10) and the conditions $I_{\alpha_i}(u_i) < \infty$. Moreover,

$$\int_{A_R} \frac{\rho_i^2}{r} \frac{\partial \psi_i}{\partial \tau} = \int_{A_R} \frac{\rho_i^2 - \alpha_i^2}{r} \frac{\partial \psi_i}{\partial \tau} \leq \frac{1}{R_0} I_{\alpha_i}(u_i)^{\frac{1}{2}} \left(\int_{B_{R_0}^c} |\nabla \psi_i|^2 \right)^{\frac{1}{2}} \leq C.$$

Consequently, we have

$$\int_{A_R} \left| \frac{\partial u_i}{\partial \tau} \right|^2 = 2\pi \alpha_i^2 d_i^2 \log R + O(1) \quad \text{as } R \rightarrow \infty. \quad (3.15)$$

Then, we obtain the desired result (3.6) from (3.14) and (3.15). It remains to prove (3.9), (3.10), and (3.11).

Proof of (3.9): Inserting (3.8) in (1.20) and taking real and imaginary

parts, we obtain that for each i and $|x| \geq R_0$,

$$\nabla \cdot (\rho_i^2 \nabla \varphi_i) = \rho_i (\rho_i \Delta \varphi_i + 2 \nabla \rho_i \cdot \nabla \varphi_i) = 0, \quad (3.16)$$

$$-\lambda_i \Delta \rho_i + \lambda_i \rho_i |\nabla \varphi_i|^2 = \rho_i \left(n - \sum_{j=1}^n \rho_j^2 \right). \quad (3.17)$$

Letting $x^\perp = (-x_2, x_1)$ for $x = (x_1, x_2)$, we can rewrite (3.16) as

$$\nabla \cdot \left(\rho_i^2 \left(\frac{d_i}{r^2} x^\perp + \nabla \psi_i \right) \right) = 0 \quad \text{for } |x| \geq R_0. \quad (3.18)$$

Integrating (3.18) on $\mathbb{R}^2 \setminus B_R$ for $R > R_0$, we are led to

$$0 = \int_{S_R} \rho_i^2 \left(\frac{d_i}{r^2} x^\perp + \nabla \psi_i \right) \cdot \nu = \int_{S_R} \rho_i^2 \frac{\partial \psi_i}{\partial \nu}. \quad (3.19)$$

Here, ν is the outward unit normal vector to ∂B_R . Let

$$\begin{aligned} \psi_i^R &= \frac{1}{2\pi R} \int_{S_R} \psi_i dx \quad \text{for } i = 1, \dots, l \\ \eta_0(R) &= \sum_{i=1}^l \int_{A_R} (\alpha_i^2 - \delta) |\nabla \psi_i|^2 dx. \end{aligned}$$

Multiplying (3.18) by $(\psi_i - \psi_i^R)$ and integrating it over A_R for each i , we have by (3.7) and (3.19)

$$\begin{aligned} \eta_0(R) &\leq \sum_{i=1}^l \int_{A_R} \rho_i^2 |\nabla \psi_i|^2 + \sum_{i=1}^l \left\{ - \int_{S_{R_0}} \rho_i^2 \frac{\partial \psi_i}{\partial \nu} \psi_i + \right. \\ &\quad \left. \int_{S_R} \rho_i^2 \frac{\partial \psi_i}{\partial \nu} (\psi_i - \psi_i^R) + \int_{A_R} \left[\frac{d_i}{r} (\alpha_i^2 - \rho_i^2) \frac{\partial \psi_i}{\partial \tau} \right] \right\} \\ &= (I) + (II) + (III). \end{aligned} \quad (3.20)$$

By the Poincaré inequality

$$\int_{S_R} |\psi_i - \psi_i^R|^2 \leq R^2 \int_{S_R} \left| \frac{\partial \psi_i}{\partial \tau} \right|^2,$$

we obtain

$$\begin{aligned} \int_{S_R} \left| \frac{\partial \psi_i}{\partial \nu} \right| \cdot |\psi_i - \psi_i^R| &\leq \left(\int_{S_R} \left| \frac{\partial \psi_i}{\partial \nu} \right|^2 \right)^{\frac{1}{2}} \left(R^2 \int_{S_R} \left| \frac{\partial \psi_i}{\partial \tau} \right|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{R}{2} \int_{S_R} \left(\left| \frac{\partial \psi_i}{\partial \nu} \right|^2 + \left| \frac{\partial \psi_i}{\partial \tau} \right|^2 \right) = \frac{R}{2} \int_{S_R} |\nabla \psi_i|^2. \end{aligned}$$

Hence, by the choice of μ

$$\begin{aligned} (II) &\leq \mu \sum_{i=1}^l \int_{S_R} (\alpha_i^2 - \delta) \left| \frac{\partial \psi_i}{\partial \nu} \right| |\psi_i - \psi_i^R| \\ &\leq \frac{1}{2} \mu R \sum_{i=1}^l \int_{S_R} (\alpha_i^2 - \delta) |\nabla \psi_i|^2 = \frac{1}{2} \mu R \eta_0'(R). \end{aligned}$$

Furthermore, since $I_{\alpha_i}(u_i) < \infty$, it comes from Young's inequality that

$$(III) \leq \sum_{i=1}^l \frac{d_i}{R_0} I_{\alpha_i}(u_i)^{\frac{1}{2}} \left(\int_{A_R} |\nabla \psi_i|^2 \right)^{\frac{1}{2}} \leq C_0 + \delta \eta_0(R).$$

As a consequence, we deduce from (3.20) that

$$\eta_0(R) \leq \frac{\mu R}{2(1-\delta)} \eta'_0(R) + \frac{C_0}{1-\delta} \equiv \frac{R}{\nu} \eta'_0(R) + C_1, \quad (3.21)$$

where C_0 and C_1 are independent of R . We claim that $\eta_0(R) \leq C_1$ for all $R > R_0$. Otherwise, there exists $R_1 > R_0$ such that $\eta_0(R_1) > C_1$. Set $\eta_1(R) = \eta_0(R) - C_1$. Then, we get $[R^{-\nu} \eta_1(R)]' > 0$, which implies that

$$\eta_1(R) \geq \left(\frac{R}{R_1}\right)^\nu \eta_1(R_1) \quad \text{for } R > R_1.$$

However, we see from (3.4) that $\eta_1(R) \leq CR$ for some $C > 0$ and for all large $R > R_1$. Since $\nu = 2(1-\delta)/\mu > 1$ by the choice of δ and μ , this yields a contradiction.

Proof of (3.10): Multiplying (3.17) by $(\alpha_i - \rho_i)\xi_R$ and integrating it on $\mathbb{R}^2 \setminus B_{R_0}$, we have

$$\begin{aligned} & \lambda_i \int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \rho_i|^2 \xi_R \\ & \leq \lambda_i \int_{S_{R_0}} \left| \frac{\partial \rho_i}{\partial \nu} \right| \cdot |\alpha_i - \rho_i| + \frac{\lambda_i}{2} \left| \int_{\mathbb{R}^2 \setminus B_{R_0}} \nabla \xi_R \cdot \nabla (\alpha_i - \rho_i)^2 \right| \\ & \quad + \lambda_i \int_{\mathbb{R}^2 \setminus B_{R_0}} |\rho_i(\alpha_i - \rho_i)| \cdot |\nabla \varphi_i|^2 + \int_{\mathbb{R}^2 \setminus B_{R_0}} \left| \rho_i(\alpha_i - \rho_i) \left(n - \sum_{j=1}^n \rho_j^2 \right) \right| \xi_R \\ & \leq C \left\{ 1 + \frac{1}{R^2} I_{\alpha_i}(u_i) + \|\nabla \varphi_i\|_{L^2(\mathbb{R}^2 \setminus B_{R_0})} + I_{\alpha_i}(u_i)^{\frac{1}{2}} E^{\frac{1}{2}} \right\} < \infty, \end{aligned}$$

where the last inequality comes from $E < \infty$, $I_{\alpha_i}(u_i) < \infty$, and (3.9).

Proof of (3.11):

Let $i \in \{l+1, \dots, n\}$ be fixed. We remind that

$$I_{\alpha_i} = \int_{\mathbb{R}^2} |u_i|^2 = \int_0^\infty \int_{\partial B_r} |u_i|^2 dS dr < \infty. \quad (3.22)$$

So, there exists $r_n \rightarrow \infty$ such that

$$\int_{\partial B_{r_n}} |u_i|^2 dS < \frac{1}{r_n}.$$

We multiply (1.20) by $\xi_{r_n}^2 u_i$ and integrate on B_{r_n} , and then we have

$$\begin{aligned} & \lambda_i \int_{B_{r_n}} \xi_{r_n}^2 |\nabla u_i|^2 + \lambda_i \int_{B_{r_n}} 2\xi_{r_n} u_i \nabla u_i \cdot \nabla \xi_{r_n} \\ & = \lambda_i \int_{\partial B_{r_n}} \frac{\partial u_i}{\partial \nu} \xi_{r_n}^2 u_i + \int_{B_{r_n}} \xi_{r_n}^2 u_i^2 (n-f). \end{aligned}$$

The right side of the above is uniformly bounded by (3.3) and (3.22). Hence,

we are led to

$$\begin{aligned}
\lambda_i \int_{\mathbb{R}^2} \xi_{r_n}^2 |\nabla u_i|^2 &\leq C \int_{\{r_n < |x| < 2r_n\}} |\xi_{r_n} \nabla u_i| |\nabla \xi_{r_n}| \\
&\leq C \left(\int_{\{r_n < |x| < 2r_n\}} \xi_{r_n}^2 |\nabla u_i|^2 \right)^{\frac{1}{2}} \left(\int_{\{r_n < |x| < 2r_n\}} |\nabla \xi_{r_n}|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\mathbb{R}^2} \xi_{r_n}^2 |\nabla u_i|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Letting $r_n \rightarrow \infty$, we obtain the desired result (3.11). \square

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