

Continuum mechanics of defective media: an approach using fiber bundles [★]

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Abstract. The kinematics of a micro-structured material is geometrically modeled through the framework of fiber bundle geometry. The material continuum is a fiber bundle $\mathcal{M} \rightarrow \mathbb{B}$ where \mathbb{B} is compact and orientable. It is commonly agreed that connections with curvature and torsion can describe defect densities in micro-structured materials. The aim of this work is to introduce a method to derive these objects from the kinematics in an intrinsic way. The material bundle \mathcal{M} is therefore placed in the Euclidean fiber bundle $\mathcal{E} \equiv T\mathbb{E} \rightarrow \mathbb{E}$ using a placement map $\varphi : \mathcal{M} \rightarrow \mathcal{E}$. A first-order transformation $\mathbf{F} : T\mathcal{M} \rightarrow T\mathcal{E}$ generalizing $T\varphi$ is then introduced. Finally, using \mathbf{F} , a metric on \mathbb{B} , a connection on \mathcal{M} and a solder form on \mathcal{M} are inferred from the Euclidean structure on \mathcal{E} . These new objects are grouped into a single one, called a pseudo-metric, which allows us to describe the current state of matter through, among other things, the curvature (disclinations) and torsion (dislocations) tensors. On one hand, we see that the torsion tensor can be non-zero even in the holonomic $\mathbf{F} = T\varphi$ case. On the other-hand, in order for the material to have a non-zero curvature tensor, we see that one must have a non-holonomic first-order transformation: that is, $\mathbf{F} \neq T\varphi$.

Keywords: Generalized continuum · Fiber bundles · Connection · mechanics

Introduction

We are interested in the study of micro-structured materials. That is, materials in which the microscopic state at a point plays an import role in its macroscopic behavior and therefore have to be included in its geometric description. Mathematically, an n -dimensional micro-structured material with a k -dimensional micro-structure (or a $n \times k$ material for short) is therefore a fiber bundle manifold $\mathcal{M} \xrightarrow{\pi_{\mathcal{M}}} \mathbb{B}$ of rank k over a compact orientable n -dimensional manifold \mathbb{B} ,

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which is the classical macroscopic body.

At a point b in the macroscopic body \mathbb{B} , the micro-structure of a micro-element is represented by the fiber \mathcal{M}_b of \mathcal{M} over b . Those elements are interpreted as first order infinitesimal neighborhoods of geometrical macroscopic points (eventually embedded in a bigger space of higher dimension), as it is commonly considered for crystal modeling. For this reason, \mathcal{M} is often closely related to a tangent space (not necessarily $T\mathbb{B}$). In our case, the material is embedded in the Euclidean space. Therefore, $n \in \{1, 2, 3\}$ and $k = 3$.

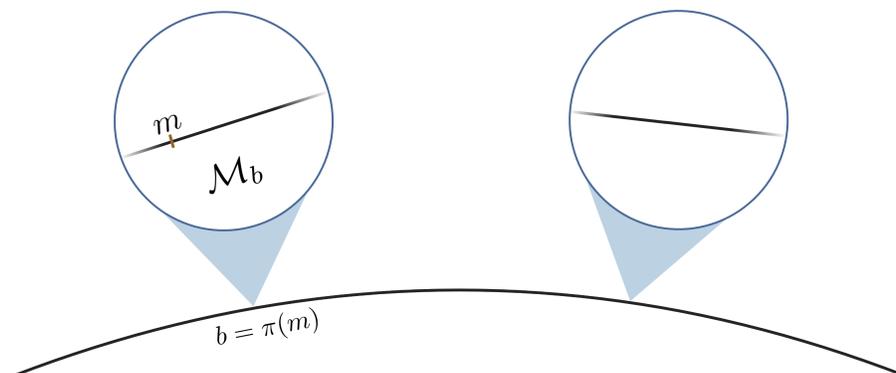


Fig. 1: The standard horizontal representation of a generic material bundle. The macroscopic space is represented horizontally and micro-elements are shown to be inside the macroscopic points. Note that neither \mathbb{B} nor \mathcal{M}_b is assumed to be 1-dimensional, this is only done for illustration purpose.

Examples of micro-structured materials include the 1-dimensional beams (1×3 materials), the 2-dimensional plates (2×3 materials) and the 3-dimensional Cosserat continuum (a 3×3 material). Some authors [5, p138-141] choose to use the principal bundle (see section 2.2). It is a fiber bundle over the same base manifold \mathbb{B} whose fibers are not the micro-elements but their possible configurations (a sub-group of $GL(3)$). This is why, at first, those examples may seem to have different micro-structure as, in practice, one often ends up manipulating the principal bundle. Those could for example, depending on assumptions made on the material, be isomorphic to the 1-dimensional circle $SO(2) \simeq S^1$ (for planar inextensible beams), the 2-dimensional space $SO(2) \times \mathbb{R}^*$ (for planar extensible beams) or the 3-dimensional space $SO(3)$ (for some plates and Cosserat continuum).

The standard interpretation

Figure 1 shows the standard representation of a generic material bundle. This interpretation of the mathematical model is central to our problem. It states that micro-elements are inside macroscopic points and are therefore imperceptible to an experimenter. In general, this interpretation states that the projection, to the macroscopic spaces \mathbb{B} and \mathbb{E} , of any kinematic object of our model can be seen as a kinematic object of the classical continuum mechanics. For example, this implies that $\bar{\varphi} : \mathbb{B} \rightarrow \mathbb{E}$ (defined later on) corresponds to an usual macroscopic placement map that can be observed and measured by an experimenter.

That the macroscopic kinematics is classical does not mean that the macroscopic dynamics will be. Macroscopic kinematic object are independent of the micro-structure, but microscopic or mixed objects can depend on the macroscopic states. This means that a coupling between the macroscopic and microscopic variables can append in the dynamics' equations. In other words, in general, $\bar{\varphi}$ will not behave classically.

1 First formalism

1.1 Placement map

By analogy with \mathcal{M} , the ambient space is modeled as the Euclidean fiber bundle $\mathcal{E} \equiv T\mathbb{E} \xrightarrow{\pi_{\mathcal{E}}} \mathbb{E}$ of rank 3 over the usual ambient space \mathbb{E} , the 3-dimensional Euclidean space. We chose a Lagrangian approach. Accordingly, one defines a placement map. This latter is a fiber bundle embedding $\varphi : \mathcal{M} \rightarrow \mathcal{E}$ which maps the micro-element over $b \in \mathbb{B}$ to the micro-place over $\bar{\varphi}(b) \in \mathbb{E}$ where $\bar{\varphi} : \mathbb{B} \rightarrow \mathbb{E}$ is called the shadow of φ (see the commutative diagram bellow). This placement map φ defines a macroscopic material placement map, via its shadow $\bar{\varphi}$ and the differential $T\bar{\varphi}$.

$$\begin{array}{ccccccc}
 T\mathcal{E} & \xrightarrow{T\pi_{\mathcal{E}}} & T\mathbb{E} & \xrightarrow{\pi_{T\mathbb{E}}} & \mathbb{E} & \xleftarrow{\pi_{\mathcal{E}}} & \mathcal{E} & \xleftarrow{\pi_{T\mathcal{E}}} & T\mathcal{E} \\
 \mathbf{F} \uparrow & & T\bar{\varphi} \uparrow & & \bar{\varphi} \uparrow & & \varphi \uparrow & & \mathbf{F} \uparrow \\
 T\mathcal{M} & \xrightarrow{T\pi_{\mathcal{M}}} & T\mathbb{B} & \xrightarrow{\pi_{T\mathbb{B}}} & \mathbb{B} & \xleftarrow{\pi_{\mathcal{M}}} & \mathcal{E} & \xleftarrow{\pi_{T\mathcal{M}}} & T\mathcal{M}
 \end{array}$$

In order to also have a description of the material's defects, one must have a generalized material placement map. To complete the description of a generalized material placement map, one additionally has to introduce a generalized first order placement map $\mathbf{F} : T\mathcal{M} \rightarrow T\mathcal{E}$. The standard interpretation then requires $\mathbf{F}(V\mathcal{M}) \subset V\mathcal{E}$. Physically, this means that a microscopic change in the material causes a microscopic change in the ambient image. In particular, this implies that \mathbf{F} can be seen as a fiber bundle morphism between $T\mathcal{M} \xrightarrow{T\pi_{\mathcal{M}}} T\mathbb{B}$ and $T\mathcal{E} \xrightarrow{T\pi_{\mathcal{E}}} T\mathbb{E}$ (notice the change of base spaces and projection maps, see the commutative diagram above).

The standard interpretation then also requires that the shadows of \mathbf{F} with respect to the usual structures and these new structures are, respectively, $\varphi : \mathcal{M} \rightarrow \mathcal{E}$ and $T\bar{\varphi} : T\mathbb{B} \rightarrow T\mathbb{E}$. Those two requirements reduces the degrees of freedom of \mathbf{F} to $k \cdot (k+n) \in \{12, 15, 18\}$ real coefficients, which are scalar fields on \mathcal{M} (i.e. \mathbf{F} lives in a $(k+n) \times (k \cdot (k+n))$ vectorial fiber bundle). In the classical case \mathbf{F} ought to be the differential of φ – i.e. $\mathbf{F} = T\varphi$ – but as it turns out, this assumption is too restrictive and corresponds to the absence of disclinations in the micro-structure (see [7]).

1.2 Metric, connection and solder form

The existence of a first order placement map allows one to superimpose the tangent of the body over the tangent of the ambient space. Doing so, first order functions of the ambient space such as tensors, can be seen in the material space. Mathematically, this correspond to a pull-back by the first order placement map.

First, there exists a metric \mathbf{g} on \mathbb{E} : the Euclidean metric. This allows the measure of angles and lengths. Using the first order macroscopic placement $T\bar{\varphi}$, one can pull it back as a metric \mathbf{G} on \mathcal{M} , which gives the Cauchy-Green tensor (see¹ the equation and diagram below). In the case of generalized continua, the macroscopic metric is accompanied by two micro-structured linear applications: the connection and the solder form.

$$\forall b \in \mathbb{B} \quad \mathbf{G}_b := T_{\bar{\varphi}(b)}^* \bar{\varphi} \cdot \mathbf{g}_{\bar{\varphi}(b)} \cdot T_b \bar{\varphi}$$

$$\begin{array}{ccc} T^*\mathbb{B} & \xleftarrow{T^*\bar{\varphi}} & T^*\mathbb{E} \\ \mathbf{G} \uparrow & & \uparrow \mathbf{g} \\ T\mathbb{B} & \xrightarrow{T\bar{\varphi}} & T\mathbb{E} \end{array}$$

Figure 2 shows the vertical representation of a generic connection on a material bundle. A connection on \mathcal{E} can be defined as a right-inverse $\Gamma : T\mathbb{B} \rightarrow T\mathcal{M}$ of $T\pi_{\mathcal{M}} : T\mathcal{M} \rightarrow T\mathbb{B}$ (i.e. $T\pi_{\mathcal{M}} \cdot \Gamma = \text{Id}_{T\mathbb{B}}$ – please note that other equivalent definitions also exist [1,6,5]). While it does not induce a notion of length or angle, it still induces a notion of local geodesics and, as shown on the figure, parallel transport of micro-elements along the horizontal lift of a macroscopic path. This later, allows to connect close microscopic spaces by following the flow generated by Γ . If the macroscopic path is a closed curve in \mathbb{B} , one may witness some lack of closure of the horizontal lift in \mathcal{M} . This lack of closure is measured by the curvature and torsion in a way similar to the way Frank and Burgers vectors are introduced and used in crystallography (see [3]).

On \mathcal{E} , one has the Levi-Civita connexion, which is defined as the unique connexion with no torsion whose geodesics are those of the Euclidean metric. Coordinate-wise, it corresponds to the inclusion $T\mathbb{E} \simeq \mathcal{E} \hookrightarrow \mathcal{E} \times \{0\} \subset \mathcal{E} \times \mathbb{R}^6 \simeq T\mathcal{E}$. In a similar way to what has been done for the metric, the connection on \mathcal{M} ,

¹ Here $T^*\bar{\varphi} : T^*\mathbb{E} \rightarrow T^*\mathbb{B}$ is the transpose of the differential $T\bar{\varphi} : T\mathbb{B} \rightarrow T\mathbb{E}$ of $\bar{\varphi} : \mathbb{B} \rightarrow \mathbb{E}$.

which is not a tensor, is then obtained by pulling back this Euclidean Levi-Civita connection γ of \mathcal{E} into a connection Γ on \mathcal{M} . This involves the use of the fiber bundle morphism $\mathbf{F} : \mathbb{T}\mathcal{M} \rightarrow \mathbb{T}\mathcal{E}$ to superimpose $\mathbb{T}\mathcal{M}$ over $\mathbb{T}\mathcal{E}$. The formula and associated diagram are as follows:

$$\forall b \in \mathbb{B}, \forall m \in \mathcal{M}_b, \quad \mathbf{\Gamma}_m := \mathbf{F}_b^{-1} \cdot \gamma_{\varphi(m)} \cdot \mathbb{T}_b \bar{\varphi}$$

$$\begin{array}{ccc} \mathbb{T}\mathcal{M} & \xrightarrow{\mathbf{F}} & \mathbb{T}\mathcal{E} \\ \mathbf{\Gamma} \uparrow \downarrow \mathbb{T}\pi_{\mathcal{M}} & & \mathbb{T}\pi_{\mathcal{E}} \uparrow \downarrow \gamma \\ \mathbb{T}\mathbb{B} & \xrightarrow{\mathbb{T}\bar{\varphi}} & \mathbb{T}\mathbb{E} \end{array}$$

A solder form on \mathcal{E} is, in an analog fashion, an injective² map $\vartheta : \mathbb{T}_x \mathbb{E} \hookrightarrow \mathbb{V}_p \mathcal{E}$ for any $x \in \mathbb{E}, p \in \mathcal{E}_x$. While a connection glues microscopic grains together, a solder form glues a grain to the macroscopic space. From a vector at a macroscopic point, it generates a vector field inside the microscopic grain at that macroscopic point. As an ambient solder form, we choose the canonical Euclidean solder form ϑ which comes from the identification $\mathcal{E} \simeq \mathbb{T}\mathbb{E}$. Just like for γ , ϑ can be pulled back onto \mathcal{M} as $\mathbf{\Theta} : \mathbb{T}\mathbb{B} \hookrightarrow \mathbb{V}\mathcal{M}$. The formula and associated diagram are as follows:

$$\forall b \in \mathbb{B}, \forall m \in \mathcal{M}_b, \quad \mathbf{\Theta}_m := \mathbf{F}_b^{-1} \cdot \vartheta_{\varphi(m)} \cdot \mathbb{T}_b \bar{\varphi}$$

$$\begin{array}{ccc} \mathbb{V}\mathcal{M} & \xrightarrow{\mathbf{F}} & \mathbb{V}\mathcal{E} \\ \mathbf{\Theta} \uparrow \downarrow \mathbb{T}\bar{\varphi} & & \vartheta^{-1} \uparrow \downarrow \vartheta \\ \mathbb{T}\mathbb{B} & \xrightarrow{\mathbb{T}\bar{\varphi}} & \mathbb{T}\mathbb{E} \end{array}$$

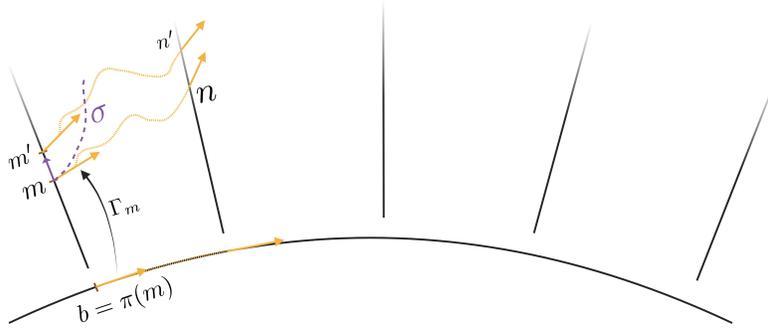


Fig. 2: The vertical representation of a generic material bundle. The macroscopic space is represented horizontally and micro-elements are unfolded vertically above it. Note that neither \mathbb{B} nor \mathcal{M}_b is assumed to be 1-dimensional, this is only done for illustration purpose.

² A quick comparison of dimensions gives $\dim(\mathbb{T}_x \mathbb{E}) = 2 \dim(\mathbb{E}) = 6 = 2 \text{rank}(\mathcal{E}) = \dim(\mathbb{V}_p \mathcal{E})$. Hence, in the Euclidean case, ϑ is a bijection.

1.3 Pseudo-metric

Another way to interpret the connection and solder form is as embeddings. The connection can be seen as an application which takes a macroscopic vector $\mathbf{v} \in \mathbb{T}\mathbb{E}$ and embeds it in the total tangent space as $\gamma(\mathbf{v}) \in \mathbb{T}\mathcal{E}$ which can be interpreted as the corresponding vector with the same macroscopic part ($\mathbb{T}\pi_{\mathcal{E}} \cdot \gamma \cdot \mathbf{v} = \mathbf{v}$) and no vertical³ part ($v_{\gamma} \cdot \gamma \cdot \mathbf{v} = 0$).

Similarly, $\vartheta(\mathbf{v}) \in \mathbb{V}\mathcal{E} \subset \mathbb{T}\mathcal{E}$ can be interpreted as the corresponding vector with the same vertical part ($\vartheta^{-1} \cdot v_{\gamma} \cdot \vartheta \cdot \mathbf{v} = \mathbf{v}$) and no horizontal part ($h_{\gamma} \cdot \vartheta \cdot \mathbf{v} = 0$). In this interpretation, $\vartheta^{-1} \cdot v_{\gamma} : \mathbb{T}\mathcal{E} \rightarrow \mathbb{T}\mathbb{E}$ can be seen as a "microscopic" projection. In particular,

$$\mathbb{T}\varpi = \mathbb{T}\pi + \vartheta^{-1} \cdot v_{\gamma} : \mathbb{T}\mathcal{E} \rightarrow \mathbb{T}\mathbb{E}$$

is also a projection which will be called **the projection of interpretation** as it gives the sum of the macroscopic part and the microscopic part, both seen in the macroscopic space $\mathbb{T}\mathbb{E}$, which is how one interprets \mathbf{v} .

This projection of interpretation gives an alternative approach for the generalization of the metric. Indeed, one can simply pull-back the Euclidean metric $\mathbf{g} : \mathbb{T}\mathbb{E} \rightarrow \mathbb{T}^*\mathbb{E}$ onto $\mathbb{T}\mathcal{E}$ using $\mathbb{T}\varpi$. One then obtains

$$\begin{aligned} \mathbf{g} : \mathbb{T}\mathcal{E} &\longrightarrow \mathbb{T}^*\mathcal{E} \\ &:= \mathbb{T}^*\varpi \cdot \mathbf{g} \cdot \mathbb{T}\varpi \end{aligned}$$

Its kernel is the kernel of $\mathbb{T}\varpi$. Those are the vectors whose macroscopic part is the opposite of their microscopic part. In particular their microscopic and macroscopic part are of the same magnitude, which mean these are not physically acceptable vectors. The kernel of \mathbf{g} , as it turns out, is the horizontal space $H_{\gamma-\vartheta} = \text{Im}(\gamma - \vartheta)$ of the connection $\gamma - \vartheta$. The solder form ϑ is, by construction, isometric with respect to $\mathfrak{g}|_{\mathbb{V}\mathcal{E}} : \mathbb{V}\mathcal{E} \rightarrow \mathbb{V}^*\mathcal{E}$ and \mathbf{g} . This means that, from \mathbf{g} , one can extract $\gamma - \vartheta$ and ϑ up to a "rotation" in $\mathcal{O}(\mathbf{g}, \mathfrak{g}|_{\mathbb{V}\mathcal{E}})$.

The tensor \mathbf{g} is called **the ambient pseudo-metric** as it is symmetric and positive but its kernel is non-trivial (otherwise it would be a metric). As \mathbf{g} with $\mathbb{T}\bar{\varphi}$, \mathbf{g} can be pulled-back on $\mathbb{T}\mathcal{M}$ using \mathbf{F} . One obtains the material pseudo-metric

$$\begin{aligned} \mathfrak{G} : \mathbb{T}\mathcal{M} &\longrightarrow \mathbb{T}^*\mathcal{M} \\ &:= \mathbf{F}^* \cdot \mathbf{g} \cdot \mathbf{F} \end{aligned}$$

Similarly to the ambient case, one has that $\ker(\mathfrak{G}) = H_{\Gamma-\Theta}$ and that Θ is an isometry for \mathbf{G} and $\mathfrak{G}|_{\mathbb{V}\mathcal{M}}$. This means that, if \mathbf{G} is known, one can extract the connection $\Gamma - \Theta$ and the orbit of Θ under $\mathcal{O}(\mathbf{G}, \mathfrak{G}|_{\mathbb{V}\mathcal{M}})$ from \mathfrak{G} . Nevertheless, $\mathbf{G} : \mathbb{T}\mathbb{B} \rightarrow \mathbb{T}^*\mathbb{B}$ cannot be extracted from $\mathfrak{G} : \mathbb{T}\mathcal{M} \rightarrow \mathbb{T}^*\mathcal{M}$ directly, unless

³ We use here the standard notation $h_{\gamma} = \gamma \cdot \mathbb{T}\pi_{\mathcal{E}}$ and $v_{\gamma} = \text{Id}_{\mathbb{T}\mathcal{E}} - h_{\gamma}$ such that $\mathbb{T}\pi_{\mathcal{E}} \cdot h_{\gamma} = \mathbb{T}\pi_{\mathcal{E}}$ and $\mathbb{T}\pi_{\mathcal{E}} \cdot v_{\gamma} = \mathbf{0}$.

$\mathbf{F} : \mathbb{T}\mathcal{M} \rightarrow \mathbb{T}\mathcal{E}$ is entirely known (in which case \mathbf{G} is also known). The pseudo-metric \mathfrak{G} is a canonical extension of the right Cauchy-Green tensor \mathbf{G} . As \mathbf{G} plays a central role in the classical case, one expects \mathfrak{G} to also play a key role in the generalized case. These observations therefore justify the use of \mathbf{G} , $\mathbf{\Gamma}$ and \mathfrak{G} as our primary tools.

2 Constitutive equations and structure group

2.1 Stress and constitutive equations

By formulating continuum mechanics using differential geometry (as in [4,1,2]) one obtains a geometric formulation of static equilibrium that can be generalized to the micro-structured case. For example, the macroscopic Cauchy stress $\bar{\mathbf{S}}$ and the macroscopic second Piola tensor $\bar{\mathbf{\Sigma}}$ are:

$$\bar{\mathbf{S}} : \mathbb{T}^*\mathbb{E} \longrightarrow \mathbb{T}\mathbb{E} \qquad \bar{\mathbf{\Sigma}} : \mathbb{T}^*\mathbb{B} \longrightarrow \mathbb{T}\mathbb{B}$$

In the generalized version, tensors on \mathbb{E} (resp. \mathbb{B}) naturally become tensors on \mathcal{E} (resp. \mathcal{M}). This gives

$$\mathbf{S} : \mathbb{T}^*\mathcal{E} \longrightarrow \mathbb{T}\mathcal{E} \qquad \mathbf{\Sigma} : \mathbb{T}^*\mathcal{M} \longrightarrow \mathbb{T}\mathcal{M}$$

Which, for consistency, are required to be morphisms for both structures $\mathbb{T}\mathcal{E} \xrightarrow{\pi_{\mathbb{T}\mathcal{E}}} \mathcal{E}$ and $\mathbb{T}\mathcal{E} \xrightarrow{\mathbb{T}\pi_{\mathcal{E}}} \mathbb{T}\mathbb{E}$ (resp. $\mathbb{T}\mathcal{M} \xrightarrow{\pi_{\mathbb{T}\mathcal{M}}} \mathcal{M}$ and $\mathbb{T}\mathcal{M} \xrightarrow{\mathbb{T}\pi_{\mathcal{M}}} \mathbb{T}\mathbb{B}$). This is a manifestation of the standard interpretation, just like it was for \mathbf{F} , which also requires that the shadows be $\text{Id}_{\mathcal{E}}$ and $\bar{\mathbf{S}}$ (resp. $\text{Id}_{\mathcal{M}}$ and $\bar{\mathbf{\Sigma}}$).

The notion of elasticity and hyper-elasticity can be generalized to the micro-structured case. The elasticity simply states that \mathbf{S} (resp. $\mathbf{\Sigma}$) is a linear function $\hat{\mathbf{S}}$ (resp. $\hat{\mathbf{\Sigma}}$) of \mathbf{F} , or a linear function of \mathfrak{G} in the isotropic case. The hyper-elasticity on the other side states that this function $\hat{\mathbf{S}}$ is obtainable as the differential $\mathbb{T}\mathbf{W}$ of a functional \mathbf{W} of \mathbf{F} (resp. \mathfrak{G} in the isotropic case) called the energy density. Such an energy, in the isotropic case, can depend on invariants of \mathbf{G} , $\mathbf{\Gamma}$ and \mathfrak{G} . Already existing formulae for the macroscopic case can be extended by replacing \mathbf{G} with \mathfrak{G} . For example, identifying $\mathbb{T}\mathcal{M}$ and $\mathbb{T}^*\mathcal{M}$ for simplicity (this is roughly equivalent to having a reference pseudo-metric), the Saint Venant-Kirchhoff constitutive law gives, under the substitution $\mathbf{G} \rightarrow \mathfrak{G}$:

$$\begin{aligned} \mathbf{W}(\mathfrak{G}) &:= \frac{a}{4} \text{tr}(\mathfrak{G} - \text{Id}) + \frac{b}{4} \text{tr}((\mathfrak{G} - \text{Id})^2) \\ \mathbf{\Sigma} &:= a \text{tr}(\mathfrak{G} - \text{Id}) \text{Id} + b(\mathfrak{G} - \text{Id}) \end{aligned}$$

Under a specific frame induced by $\mathbf{\Gamma}$, \mathfrak{G} splits into blocks as

$$\begin{bmatrix} \mathbb{T}\pi_{\mathcal{M}}^* \cdot \mathbf{G} \cdot \mathbb{T}\pi_{\mathcal{M}} & 0 \\ 0 & (\text{Id} - \mathbf{\Gamma} \cdot \mathbb{T}\pi_{\mathcal{M}})^* \cdot \mathfrak{G} \cdot (\text{Id} - \mathbf{\Gamma} \cdot \mathbb{T}\pi_{\mathcal{M}}) \end{bmatrix}$$

Hence, we see that even with a naive extension of the Saint Venant-Kirchhoff formula, there is a micro/macro coupling occurring through the term

$$\text{tr}(\mathfrak{G} - \text{Id}) = \text{tr}(\mathbf{G}) - k + \text{tr}((\text{Id} - \mathbf{\Gamma} \cdot \text{T}\pi_{\mathcal{M}})^* \cdot \mathfrak{G} \cdot (\text{Id} - \mathbf{\Gamma} \cdot \text{T}\pi_{\mathcal{M}})) - 3$$

If we place ourselves in the holonomic case $\varphi \equiv \text{T}\bar{\varphi}$ and $\mathbf{F} = \text{T}\varphi$. That is, if the microscopic kinematics is a copy of the macroscopic one. Further computations show that one has $\text{tr}(\mathfrak{G}) = 2\text{tr}(\mathbf{G})$ and, up to a constant rescaling of a and b , $\mathbf{\Sigma}$ is just two (diagonal) copies of $\bar{\mathbf{\Sigma}}$. That is, the microscopic part of the stress is just an independent copy of its macroscopic part and therefore the dynamics is classical. In general, however, it is not the case.

It should be noted that, simply extending classical formulae this way does not provide all possible energies, as other invariants exist. For example, even $\text{tr}(\mathbf{G})$ will not be accessible as it will be replaced with $\text{tr}(\mathfrak{G})$ in the generalized case. Furthermore, this would not necessarily lead to geometrically exact laws as the micro-structure is ignored. Therefore, invariants of the pseudo-metric \mathfrak{G} need to be used instead. That is, invariants of the connection $\mathbf{\Gamma}$ and the torsion $\mathbf{\Theta}$ on \mathcal{M} must be used, in addition to those of the metric \mathbf{G} on \mathbb{B} . In theory, this could allow the formulation of geometrically exact constitutive equations making use of the micro-structure. Equations which could use, for instance, the torsion and curvature tensors of the connection to quantify these generalized strain and stress.

2.2 Structure group and principal formalism

Although we took care to omit this detail until now, fiber bundles are not merely spaces locally isomorphic to Cartesian products. They are actually equipped with a group, called the structure group [1], acting on its typical fiber. This group can be physically interpreted as the group of admissible change of coordinates or the set of microscopic configurations. Objects of the theory are therefore required to interact properly with it. The structure group is usually simpler than the typical fiber of \mathcal{M} . For this reason, using the principal bundle associated to \mathcal{M} is sometimes easier. The principal bundle associated to a fiber bundle $\mathcal{M} \rightarrow \mathbb{B}$ is a fiber bundle $\mathcal{P}^{\mathcal{M}}$ whose base space, structure group and (in some sens) transition maps are the same but whose typical fiber is the structure group $\mathcal{G}_{\mathcal{M}}$ itself. For example, in the case of planar and inextensible Timoshenko–Ehrenfest beams, the structure group is $\text{SO}(2)$ which is isomorphic to the circle S^1 . Therefore, the principal material bundle is $\mathcal{P}^{\mathcal{M}} \rightarrow \mathbb{B}$ whose typical fiber is $\mathcal{G}_{\mathcal{M}} \simeq S^1$.

Objects defined on \mathcal{M} and \mathcal{E} can canonically be defined (or transferred) on $\mathcal{P}^{\mathcal{M}}$ and $\mathcal{P}^{\mathcal{E}}$ (see [1,5]). A placement map $\varphi : \mathcal{M} \rightarrow \mathcal{E}$ can for example be decomposed as a macroscopic placement map $\bar{\varphi} : \mathbb{B} \rightarrow \mathbb{E}$ and a material section (i.e. a microscopic configuration) $\mu : \mathbb{B} \rightarrow \mathcal{G}_{\mathcal{M}}$. In the case of the planar and inextensible Timoshenko–Ehrenfest beam, we see that a material placement map is therefore described by a classical placement map $\bar{\varphi} : \mathbb{B} \rightarrow \mathbb{E}$ and an angle $\theta_b \in S^1 \simeq [0, 2\pi[$ at each $b \in \mathbb{B}$ (see figure 3).

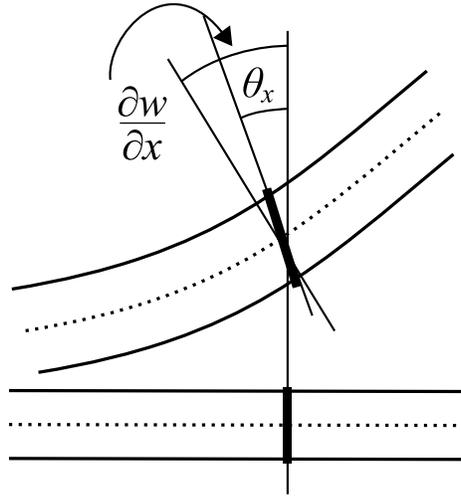


Fig. 3: Deformation of a Timoshenko beam. The normal section rotates by an amount θ_x with $\frac{\partial w}{\partial x} \neq \sin(\theta_x)$ ($\frac{\partial w}{\partial x}$ is the z -aligned part of $T\bar{\varphi}$). Source: Wikipedia

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