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## Output feedback control of discrete-time Lur'e systems through FIR Zames-Falb multipliers<sup>★</sup>

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**Abstract:** This paper is concerned with output feedback control of discrete-time Lur'e systems with slope bounded nonlinearities that can be non-odd. Using FIR (Finite Impulse Response) Zames-Falb multipliers of any given order, stability conditions based on positive realness are proposed in terms of linear matrix inequalities and linear constraints that assure the  $\ell_1$  norm bound for the multiplier. Importantly, the proposed approach does not require line searches. Applying a relaxation on the stability and exploiting the fact that the matrices of the system appear affinely in the conditions, an iterative method is constructed to search for the FIR Zames-Falb multiplier certifying stability. An extension to cope with the design of a control law is also presented, dealing with state or output feedback gains that may include a term proportional to the output of the nonlinear block. Numerical examples illustrate the advantages of the proposed approach, that can be numerically more efficient than other techniques from the literature.

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**Keywords:** Discrete-time Lur'e systems, FIR Zames-Falb multipliers, Absolute stability, Output-feedback control, Linear matrix inequalities.

### 1. INTRODUCTION

Lur'e systems are feedback interconnections between a linear time-invariant (LTI) plant with a static nonlinear function (Khalil, 2002, Chapter 7), as presented in Figure 1. Under assumptions on the nonlinearity, as sector and slope bounds, conditions for absolute stability can be formulated in time domain, where the stability certificates are Lyapunov functions (Gonzaga et al., 2012; Ahmad et al., 2015; Park, 1997, 2002; Bertolin et al., 2022a; Park et al., 2019), or in the frequency domain, where the stability certificates are LTI systems called multipliers (Khalil, 2002; Turner et al., 2009; Turner and Drummond, 2021; Ahmad et al., 2013; Ahmad and Carrasco, 2016; Haddad and Bernstein, 1994, 1993; Carrasco et al., 2012a, 2020, 2016; Safonov and Wyetzner, 1987; Carrasco et al., 2012b, 2014).

For slope-bounded nonlinearities, the Zames-Falb (ZF) multipliers (Zames and Falb, 1968; O'Shea, 1966, 1967; Turner, 2021) are the most effective method to assess stability. A positive realness condition must be verified for the overall system and the  $\ell_1$  norm of the impulse response of the multiplier must be bounded. For discrete-time Lur'e systems, the requirements on the bound of the  $\ell_1$  norm (called star norm) for infinite impulse response ZF multipliers (Ahmad et al., 2013) and positive realness conditions are expressed in terms of linear ma-

trix inequalities (LMIs). These formulations consider a state-space realization of the multiplier. Although being effective to some extent, the  $\ell_1$  bound constraint can be conservative, and requires a scalar search. Moreover, the conditions apply only to odd nonlinearities. The *finite impulse response* (FIR) ZF multipliers (Wang et al., 2014; Carrasco et al., 2020) offer a simpler solution to bound the  $\ell_1$  norm of the impulse response and the positive realness condition is still expressed with an LMI constraint. Besides, FIR multipliers handle odd or non-odd nonlinearities, and can be causal, anti-causal or noncausal.

Regarding control design for discrete-time Lur'e systems, most of the existing works are based on Lur'e-Lyapunov functions (Bertolin et al., 2022a; Kim and Braatz, 2017; Louis et al., 2015b,a), with the exception of (Bertolin et al., 2022b) that used star-norm bounds for the  $\ell_1$  constraint and positive realness to derive synthesis conditions. In this paper, the aim is to propose a method for the output feedback control of discrete-time Lur'e systems based on FIR ZF multipliers. An iterative algorithm, where an LMI optimization problem is solved at each iteration, is first presented for absolute stability analysis and its extension to cope with control design is immediate. A control law using state or output feedback, that may also include the feedback of the output of the nonlinearity, can be designed jointly with the FIR ZF multiplier. Odd or non-odd nonlinearities, causal or anti-causal FIR multipliers of any given order can be considered. Numerical examples from the literature illustrate the performance of the proposed method.

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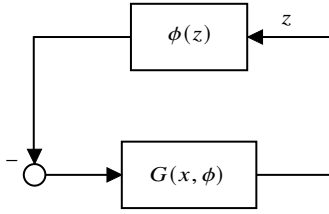


Fig. 1. Block diagram of a Lur'e system.

**Notation:** For a symmetric matrix,  $A > 0$  ( $A < 0$ ) means that  $A$  is positive (negative) definite. For matrices or vectors ( $^T$ ) indicates the transpose,  $\text{He}(A) = A + A^T$  and  $\text{diag}(A_1, \dots, A_n)$  represents a block diagonal matrix formed by the square matrices (or elements)  $A_1, \dots, A_n$ . The symbol  $\star$  represents a term induced by symmetry in a square matrix,  $\sim$  indicates a state-space realization of a transfer function and  $\text{Re}\{M(j\omega)\}$  means the real part of  $M(j\omega)$ . The identity and the zero matrices are denoted, respectively, by  $I$  and  $0$ .  $\mathcal{B}_\perp$  represents a basis for the null space of  $\mathcal{B}$ , i.e.,  $\mathcal{B}\mathcal{B}_\perp = 0$ . Throughout the text the dimensions of the matrices may be omitted for simplicity (being inferred from the context).

## 2. PROBLEM DEFINITION

Consider the discrete-time nonlinear Lur'e system given by

$$\begin{cases} x(k+1) = Ax(k) + B_\phi \phi(z(k)) + B_u u(k) \\ z(k) = C_z x(k) + D_{z\phi} \phi(z(k)) \\ y(k) = C_y x(k) \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control input,  $z \in \mathbb{R}$ ,  $\phi: \mathbb{R} \rightarrow \mathbb{R}$ , and  $y \in \mathbb{R}^{n_y}$  is the measured output. Matrices  $A$ ,  $B_\phi$ ,  $B_u$ ,  $C_z$ ,  $C_y$ , and  $D_{z\phi}$  are real and have appropriate dimensions. The nonlinearity  $\phi$  is a monotonic function (odd or non-odd), that verifies the slope bound

$$0 \leq \frac{\phi(\hat{z}) - \phi(z)}{(\hat{z} - z)} \leq \Lambda \quad (2)$$

for all  $z, \hat{z} \in \mathbb{R}$ ,  $z \neq \hat{z}$ , and  $\phi(\hat{z}), \phi(z) \in [0, \Lambda]$  where  $\Lambda$  is a given positive scalar.

System (1) is assumed to be well-posed, that is, the algebraic loop involving  $z$  and  $\phi(z)$  has a unique solution. A sufficient condition for well-posedness is given by

$$1 + \Lambda D_{z\phi} > 0. \quad (3)$$

The main goal of this paper is to design a stabilizing static output-feedback controller for system (1). Considering the control law  $u = Ky + K_\phi \phi(z)$ , the state-space representation for the closed-loop system is given by

$$\begin{cases} x(k+1) = A_{cl}x(k) + B_{cl}\phi(z(k)) \\ z(k) = C_zx(k) + D_{z\phi}\phi(z(k)) \end{cases} \quad (4)$$

with  $A_{cl} = A + B_u K C_y$  and  $B_{cl} = B_\phi + B_u K_\phi$ .

As an important result that is used to derive the proposed conditions, the Finsler's Lemma (de Oliveira and Skelton, 2001) is reproduced below.

**Lemma 1.** (Finsler's lemma). Consider matrices  $Q \in \mathbb{R}^{\ell \times \ell}$  and  $\mathcal{B} \in \mathbb{R}^{m \times \ell}$ , with  $\text{rank}(\mathcal{B}) < \ell$  and  $\mathcal{B}\mathcal{B}_\perp = 0$ . Then, the following conditions are equivalent:

- i)  $\mathcal{B}_\perp^T Q \mathcal{B}_\perp < 0$ ;
- ii)  $\exists X \in \mathbb{R}^{\ell \times m}$  such that  $Q + X\mathcal{B} + \mathcal{B}^T X^T < 0$ .

### 2.1 Stability conditions

By performing a loop transformation, stability analysis of the closed-loop system (4) can be investigated through the existence of a ZF multiplier  $M(z)$  (Zames and Falb, 1968), such that

$$\text{Re}\{M(e^{j\omega})\tilde{G}_{cl}(e^{j\omega})\} > 0, \forall \omega \in [0, 2\pi] \quad (5)$$

is verified, with  $\tilde{G}_{cl}(e^{j\omega}) = (1 + \Lambda G_{cl}(e^{j\omega}))$ . Consider, without loss of generality,  $M(z) \sim (A_m, B_m, C_m, 1)$ , with state space realization of  $M(z)\tilde{G}_{cl}(z)$  given by

$$A_I = \begin{bmatrix} A_p & 0 \\ B_m C_p & A_m \end{bmatrix}, \quad B_I = \begin{bmatrix} B_p \\ B_m D_p \end{bmatrix} \\ C_I = [C_p \quad C_m], \quad D_I = D_p,$$

where  $A_p = A_{cl}$ ,  $B_p = B_{cl}$ ,  $C_p = \Lambda C_z$  and  $D_p = (1 + \Lambda D_{z\phi})$ . Then condition (5) can be verified through the following lemma (Ahmad et al., 2013).

**Lemma 2.** System  $M(z)\tilde{G}_{cl}(z) \sim (A_I, B_I, C_I, D_I)$  is stable and positive real if there exists a positive definite matrix  $P = P^T$ , such that

$$\begin{bmatrix} A_I^T P A_I - P & A_I^T P B_I - C_I^T \\ \star & B_I^T P B_I - D_I^T - D_I \end{bmatrix} < 0. \quad (6)$$

### 2.2 FIR multipliers

As presented in the previous subsection, besides the positive realness condition,  $M(z)$  must belong to the class of ZF multipliers to certify the absolute stability of the closed-loop Lur'e system (4). In this work, the following causal FIR multiplier is adopted

$$M(z) = \sum_{i=0}^{n_M} m_i z^{-i} \quad (7)$$

where  $n_M \geq 1$  is the multiplier order and  $m_0 = D_m = 1$ .

This particular structure for  $M(z)$  simplifies the constraint to bound the  $\ell_1$  norm  $M(z)$ . As a matter of fact, such condition can be established in terms of linear programming involving the coefficients  $m_i$ , as presented in the next lemma for both odd and non-odd nonlinearities (Carrasco et al., 2020).

**Lemma 3.**  $M(z)$  given in (7) is a ZF multiplier if

i)

$$\sigma = \sum_{i=0}^{n_M} m_i \geq 0, \quad (8)$$

with  $m_i \leq 0$ , for  $i = 1, \dots, n_M$  and  $m_0 = 1$  for non-odd nonlinearities.

ii)

$$\sigma = \sum_{i=0}^{n_M} m_i^+ + \sum_{i=0}^{n_M} m_i^- \leq 2, \quad (9)$$

where  $m_i = m_i^+ - m_i^-$ , and  $m_i^+, m_i^- \geq 0$ , for  $i = 1, \dots, n_M$ ,  $m_0^+ = 1$  and  $m_0^- = 0$  for odd nonlinearities.

In this paper, the search for  $M(z) \sim (A_m, B_m, C_m, 1)$  is simplified by adopting the following controllable canonical form

$$A_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_m = \begin{bmatrix} m_{n_M} \\ m_{(n_M-1)} \\ \vdots \\ m_1 \end{bmatrix}^T. \quad (10)$$

The search for anti-causal FIR ZF multipliers can be performed by transforming  $A_p$ ,  $B_p$ ,  $C_p$  and  $D_p$  as proposed in Carrasco et al. (2020). Although in Carrasco et al. (2020) a non-causal multiplier can be computed by solving a convex optimization problem, the resulting conditions are not easily extended for control design, which is the main purpose of this paper.

### 3. MAIN RESULTS

As the main contribution of this work, the present section deals with the design of the static output-feedback gains  $K$  and  $K_\phi$  associated to the closed-loop system (4).

**Theorem 1.** Let the positive integer  $n_M$  and the matrices  $\bar{X}_i \in \mathbb{R}^{(n_x+n_M) \times (n_x+n_M)}$ ,  $i = 1, 3, 4$ ,  $\bar{X}_2 \in \mathbb{R}^{(n_x+n_M) \times 1}$ , with  $\bar{X}_4$  of full rank, be given. If there exist  $X_i \in \mathbb{R}^{(n_x+n_M) \times (n_x+n_M)}$ ,  $i = 1, 3, 4$ ,  $X_2 \in \mathbb{R}^{1 \times (n_x+n_M)}$ ,  $0 < P^T = P \in \mathbb{R}^{(n_x+n_M) \times (n_x+n_M)}$ ,  $K \in \mathbb{R}^{n_u \times n_y}$ ,  $K_\phi \in \mathbb{R}^{n_u \times 1}$  and  $C_m \in \mathbb{R}^{1 \times n_M}$  such that

$$Q + \text{He} \left( \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}}_X \underbrace{\begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 & \bar{X}_4 \end{bmatrix}}_{\bar{X}} \right) < 0, \quad (11)$$

where

$$Q = \begin{bmatrix} -P & -C_I^T & 0 & A_I^T \\ \star & -2D_I & 0 & B_I^T \\ \star & \star & P & -I \\ \star & \star & \star & 0 \end{bmatrix},$$

and the conditions of Lemma 3 ( $\phi$  odd or non-odd) are verified, then the gains  $K$  and  $K_\phi$  stabilize the closed-loop system (4) and  $M(z)$  given in (7) is a FIR ZF multiplier.

**Proof.** Consider

$$\mathcal{B}_\perp = \left[ \frac{I}{-\bar{X}_4^{-1}\bar{X}_1 \quad -\bar{X}_4^{-1}\bar{X}_2 \quad -\bar{X}_4^{-1}\bar{X}_3} \right]$$

as basis for the null space of  $\bar{X}$ , that is,  $\bar{X}\mathcal{B}_\perp = 0$ . Invoking Lemma 1, condition (11) is equivalent to  $\mathcal{B}_\perp^T Q \mathcal{B}_\perp < 0$ , that can be rewritten as

$$\begin{bmatrix} -P & -C_I^T & 0 \\ \star & -2D_I & 0 \\ \star & \star & P \end{bmatrix} + \text{He} \left( \begin{bmatrix} (\bar{X}_4^{-1}\bar{X}_1)^T \\ (\bar{X}_4^{-1}\bar{X}_2)^T \\ (\bar{X}_4^{-1}\bar{X}_3)^T \end{bmatrix} \begin{bmatrix} A_I & B_I & -I \end{bmatrix} \right) < 0. \quad (12)$$

The above inequality is, again, in the form *ii*) of Lemma 1. Then, computing a basis for the null space of  $\begin{bmatrix} A_I & B_I & -I \end{bmatrix}$ , the following condition, equivalent to (6), can be obtained

$$\begin{bmatrix} A_I^T P A_I - P & A_I^T P B_I - C_I^T \\ B_I^T P A_I - C_I & B_I^T P B_I - 2D_I \end{bmatrix} < 0, \quad (13)$$

proving the positive realness of the interconnected system. The feasibility of (8) (non-odd) or (9) (odd) assures that  $M(z)$  is a ZF multiplier. As a consequence, the closed-loop system is stable.

The main advantage of Theorem 1 is that the variables  $P$ ,  $C_m$  and the matrices of the system appear affinely in the conditions. Thus, the stabilizing gains can be directly computed, without resorting to change of variables or any other algebraic techniques, which in general are difficult to be applied for static output-feedback control, even for linear time-invariant systems. Moreover, the multiplier  $M(z)$  can be of any given order  $n_M \geq 1$ . However, note that inequality (11) is convex

because  $\bar{X}$  is a fixed matrix. Consequently, due to the lack of a general approach for choosing suitable matrices  $\bar{X}_i$ ,  $i = 1, \dots, 4$ , the results obtained through this sufficient condition may be conservative. A strategy to overcome this difficulty is the application of relaxations, as shown in the next theorem.

**Theorem 2.** Let  $\bar{X} = \begin{bmatrix} 0 & 0 & -\nu I & I \end{bmatrix}$ , where  $0 < \nu < 2(1 + \Lambda D_{z\phi})^{-1}$  and the change of variables  $\bar{M}(z) \sim (A_m, B_m, \bar{C}_m, 1)$ , for

$$\begin{aligned} \bar{C}_m &= [\bar{m}_{n_M} \quad \bar{m}_{(n_M-1)} \quad \dots \quad m_1] \\ &= [(\rho^{(n_M-1)} m_{n_M}) \quad (\rho^{(n_M-2)} m_{(n_M-1)}) \quad \dots \quad m_1], \end{aligned} \quad (14)$$

and

$$\begin{aligned} \bar{A}_p &= \bar{A}_{cl} = \rho A + B_u \hat{K} C_y \\ \bar{B}_p &= \bar{B}_{cl} = \rho B_\phi + B_u \hat{K}_\phi \\ \bar{C}_p &= \rho C_p \end{aligned} \quad (15)$$

with  $\rho > 0$ ,  $\hat{K} = \rho K$  and  $\hat{K}_\phi = \rho K_\phi$ . Then, the conditions of Theorem 1 always have a feasible solution with a sufficiently small value of  $\rho$ .

**Proof.** The ZF multiplier  $\bar{M}(z) \sim (A_m, B_m, \bar{C}_m, 1)$  can be equivalently rewritten as,

$$\begin{aligned} \bar{M}(z) &= \bar{C}_m(zI - A_m)^{-1} B_m + 1 \\ &= C_m(zI - \rho A_m)^{-1} B_m + 1. \end{aligned} \quad (16)$$

Then, by (16) and from the choices made in (15), one has

$$\begin{aligned} \bar{A}_I &= \begin{bmatrix} \rho A + B_u \hat{K} C_y & 0 \\ \rho B_m C_p & \rho A_m \end{bmatrix}, \quad \bar{B}_I = \begin{bmatrix} \rho B_\phi + B_u \hat{K}_\phi \\ B_m(1 - \Lambda D_\phi) \end{bmatrix}, \\ \bar{C}_I &= [\rho C_p \quad C_m]. \end{aligned}$$

With  $\bar{A}_I$ ,  $\bar{B}_I$ ,  $\bar{C}_I$  in inequality (11), and fixing  $X_1 = X_2 = 0$  and  $X_3 = -\nu X_4 = 0.5I$ , one has

$$\begin{bmatrix} -P & -\bar{C}_I^T & 0 & \bar{A}_I^T \\ \star & -2D_I & 0 & \bar{B}_I^T \\ \star & \star & P - \nu I & 0 \\ \star & \star & \star & -\frac{1}{\nu} I \end{bmatrix} < 0.$$

With a sufficiently small value of  $\rho$ ,  $K = K_\phi = 0$  and  $C_m = 0$  (note that the conditions of Lemma 3 remain feasible), and applying a Schur complement, one has

$$\begin{bmatrix} P & 0 & 0 \\ \star & 2D_I - \nu \bar{B}_I^T \bar{B}_I & 0 \\ \star & \star & \nu I - P \end{bmatrix} > 0.$$

Blocks (1,1) and (3,3) of the above inequality are feasible with  $0 < P < \nu I$ . Finally, block (2,2) is feasible with the choice  $0 < \nu < 2(1 + \Lambda D_{z\phi})^{-1}$ , because

$$\begin{aligned} 2D_I - \nu \bar{B}_I^T \bar{B}_I &= 2(1 + \Lambda D_{z\phi}) - \nu(1 + \Lambda D_{z\phi})^2 B_m^T B_m \\ &= 2 - \nu(1 + \Lambda D_{z\phi}) > 0, \end{aligned}$$

since  $(1 + \Lambda D_{z\phi}) > 0$  by assumption (3).

Note that, adopting the initialization proposed by Theorem 2, Theorem 1 always produces a feasible solution thanks to the parameter  $\rho$ , which is a relaxation factor in matrices  $A_p$ ,  $B_p$ ,  $C_p$  of the closed-loop system. With the proposed change of variables,  $\rho$  appears affinely in condition (11). This way,  $\rho$  can be maximized as an objective function to find the stabilizing gains  $K$  and  $K_\phi$  and also a valid multiplier  $\bar{M}(z)$  which certifies the stability of the original Lur'e system ( $\rho = 1$ ). While  $\rho < 1$  the resulting matrices  $X_i^T$ ,  $i = 1, \dots, 4$  can be used as new values of  $\bar{X}_i$ ,  $i = 1, \dots, 4$ , thus generating a new feasible solution. This

is possible because (11) is structured in the form *ii*) of Finsler's Lemma, i.e.,  $Q + \text{He}(X\mathcal{B}) < 0$ . Then, as  $\text{He}(X\mathcal{B}) = \text{He}(\mathcal{B}^T X^T)$ , the choice  $\mathcal{B} = X^T$  guarantees a feasible solution in a new test, ensuring that the value of  $\rho$  cannot decrease.

The iterative procedure is presented in Algorithm 1 (the index  $(\cdot)_{it}$  represents the value of the variable at iteration  $it$ ). The input parameters are  $it_{max}$ , which defines a stopping criterion,  $n_M$ , multiplier order, and  $\bar{X}$ , the initial matrix required by Theorem 1. If  $\rho \geq 1$  at some iteration, then the stability of the closed-loop Lur'e system is certified. Unfortunately, although the sequence of values for  $\rho$  is non-decreasing, the convergence to a value greater than 1 cannot be assured.

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**Algorithm 1** Iterative procedure for stabilization
 

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Input parameters:  $it_{max}$ ,  $n_M$ ,  $\bar{X}$ ;  
 Make the changes of variables as in (14) and (15);  
 $it \leftarrow 1$ ;  
**while**  $it \leq it_{max}$  **do**  
   **maximize**  $\rho_{it}$  **subject to:** (8) and (11) ( $\phi$  non-odd) or (9) and (11) ( $\phi$  odd);  
   **if**  $\rho_{it} \geq 1$  **then**  
     **return**  $\rho_{it}$ ,  $\bar{C}_{m,it}$  and  $X$   
   **end**  
    $\bar{X} = X^T$ ;  
    $it = it + 1$ ;  
**end**

---

Next theorem states that, if  $\rho > 1$  is obtained, then the controller gains and the FIR ZF multiplier  $M(z)$  for the original system ( $\rho = 1$ ) can be computed as a convex combination of the solutions of two consecutive iterations.

**Theorem 3.** Let  $\bar{A}_p$ ,  $\bar{B}_p$ ,  $\bar{C}_p$  and  $\bar{C}_m$  be as given in Theorem 2 and consider the existence of two consecutive solutions of conditions of Lemma 3 and (11) provided by Algorithm 1, respectively at iterations  $k$  and  $k+1$ , such that

$$Q_k + \text{He}(X_k X_{k-1}^T) < 0, \quad \rho_k < 1, \quad (17)$$

$$Q_{k+1} + \text{He}(X_{k+1} X_k^T) < 0, \quad \rho_{k+1} > 1. \quad (18)$$

Then, the controller gains  $K$  and  $K_\phi$  and the ZF multiplier  $M(z)$  for the original system (i.e., for  $\rho = 1$ ) are obtained from a convex combination of the solutions at iterations  $k$  and  $k+1$ .

**Proof.** Consider the nonnegative scalars  $\zeta_k$  and  $\zeta_{k+1}$  such that,

$$\zeta_k + \zeta_{k+1} = 1, \quad \zeta_k = \frac{\rho_{k+1} - 1}{\rho_{k+1} - \rho_k}, \quad \zeta_{k+1} = \frac{1 - \rho_k}{\rho_{k+1} - \rho_k}.$$

Multiplying (17) by  $\zeta_k$  and (18) by  $\zeta_{k+1}$ , and summing up, one has

$$\begin{aligned} & \zeta_k (Q_k + \text{He}(X_k X_{k-1}^T)) + \zeta_{k+1} (Q_{k+1} + \text{He}(X_{k+1} X_k^T)) \\ &= \zeta_k Q_k + \zeta_{k+1} Q_{k+1} + \text{He}(\zeta_k X_k X_{k-1}^T + \zeta_{k+1} X_{k+1} X_k^T) < 0. \end{aligned} \quad (19)$$

Regarding the term  $\zeta_k Q_k + \zeta_{k+1} Q_{k+1}$ , one has for the block  $A_I$

$$\begin{aligned} & \zeta_k \begin{bmatrix} \rho_k A_p + B_u \hat{K}_k C_y & 0 \\ \rho_k B_m C_p & A_m \end{bmatrix} + \zeta_{k+1} \begin{bmatrix} \rho_{k+1} A_p + B_u \hat{K}_{k+1} C_y & 0 \\ \rho_{k+1} B_m C_p & A_m \end{bmatrix} \\ &= \begin{bmatrix} A_p + B_u (\zeta_k \hat{K}_k + \zeta_{k+1} \hat{K}_{k+1}) C_y & 0 \\ B_m C_p & A_m \end{bmatrix} = \begin{bmatrix} A_p + B_u \bar{K} C_y & 0 \\ B_m C_p & A_m \end{bmatrix} \end{aligned}$$

and similarly for  $B_I$  and  $C_I$

$$\begin{aligned} & \zeta_k \begin{bmatrix} \rho_k B_\phi + B_u \hat{K}_{\phi,k} \\ B_m D_p \end{bmatrix} + \zeta_{k+1} \begin{bmatrix} \rho_{k+1} B_\phi + B_u \hat{K}_{\phi,k+1} \\ B_m D_p \end{bmatrix} \\ &= \begin{bmatrix} B_\phi + B_u (\zeta_k \hat{K}_{\phi,k} + \zeta_{k+1} \hat{K}_{\phi,k+1}) \\ B_m D_p \end{bmatrix} = \begin{bmatrix} B_\phi + B_u \bar{K}_\phi \\ B_m D_p \end{bmatrix}, \\ & \zeta_k [\rho_k C_p \bar{C}_{m,k}] + \zeta_{k+1} [\rho_{k+1} C_p \bar{C}_{m,k+1}] \\ &= [C_p (\zeta_k \bar{C}_{m,k} + \zeta_{k+1} \bar{C}_{m,k+1})] = [C_p \bar{C}_m]. \end{aligned}$$

Since (19) holds and the convex combination also preserves the positive realness condition, the ZF multiplier associated with  $\rho = 1$  given by  $M(z) \sim (A_m, B_m, \bar{C}_m, 1)$  assures closed-loop stability with the gains  $K = \bar{K}$  and  $K_\phi = \bar{K}_\phi$ .

#### 4. EXAMPLES

In this section, numerical examples are presented to illustrate the performance of Algorithm 1 (named A1) in stability analysis (fix  $K = K_\phi = 0$ ), and in control design with state, namely with  $C_y = I$ , and output feedback. Both FIR multipliers for odd and non-odd nonlinearities are investigated. The initial choices proposed for A1 are  $it_{max} = 250$  and  $v = (1 + \Lambda D_{z\phi})^{-1}$ . Although the proposed method could deal with any given order for the ZF multipliers,  $n_M = 2$  has been fixed in all tests for simplicity.

In the experiments, A1 is compared with the method from Bertolin et al. (2022b) (B22), that performs a line search of parameter  $\alpha \in (0, 1)$  (related with a bound to the  $\ell_1$  norm of the ZF multiplier) as follows: *i*) A bisection is used to find the maximum value of  $\Lambda$ , considering 10 equally spaced points for  $\alpha \in (0, 1)$ ; *ii*) For the value of  $\alpha$  that provided the best  $\Lambda$  in the previous step (defined by  $\alpha^*$ ), two new bisections are performed, first considering 10 equally spaced points in the interval  $\alpha \in [0.75\alpha^*, 1.25\alpha^*]$ , obtaining a new  $\alpha^*$  and then using 50 equally spaced points for  $\alpha \in [0.9\alpha^*, 1.1\alpha^*]$  (70 searches in total). The implementation was made in Matlab-2022 using the Yalmip parser (Löfberg (2004)) and the Mosek solver (Andersen and Andersen (2000)), on a computer with Windows 10 operating system and Core i7 processor.

##### Experiment 1 – Stability analysis

Seven Lur'e systems from the literature are investigated. Systems 1 to 6 are from Carrasco et al. (2020) and system 7 is from Bertolin et al. (2022b). The aim of the experiment is to search for the maximum value of  $\Lambda$  such that each system remains stable. For this purpose, the search for the maximum value is performed by a bisection procedure in  $\Lambda$  with precision given by  $10^{-3}$ . Using a transformation on the system matrices, A1 can provide ZF multipliers  $M(z)$  that are causal (*c*) or anti-causal (*ac*). Table 1 shows the results obtained using algorithm A1, and also (for comparison) the results from Bertolin et al. (2022b) (B22) also considering second order ZF multipliers, and from Carrasco et al. (2020) (C20) — noncausal FIR ZF multipliers with  $n_f = n_b$ , i.e., same number of causal and anti-causal terms, presented in parenthesis.

As can be noted in Table 1, C20 provides the best results for stability analysis, both in terms of larger values for  $\Lambda$  and smaller computational effort. Neither A1 nor B22 compete with C20 in terms of stability analysis but, differently from C20, both can be extended to cope with the problem of control design for discrete-time Lur'e systems, as presented next. In a

Table 1. Maximum values of  $\Lambda$  in stability analysis obtained by A1, B22 for causal ( $c$ ) and anti-causal ( $ac$ ) multipliers, and C20 (noncausal  $nc$ ), considering  $\phi$  odd and non-odd (shaded). The numerical complexity can be inferred from the time ( $t$ ) in seconds required to test each system. The value of  $n_f = n_b$  of C20 is given in parenthesis.

System		1		2		3		4		5		6		7	
		$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$
A1	$c$	12.978	2.8	0.428	9.5	0.241	8.1	3.288	8.2	2.404	2.2	0.965	1.2	14.207	3.8
		12.876	1.9	0.420	9.5	0.136	8.6	2.582	5.3	1.023	1.8	0.910	1.1	12.876	2.8
	$ac$	1.132	3.1	1.083	9.9	0.269	11.0	3.152	4.3	2.447	1.8	1.086	2.1	14.849	2.4
		0.787	2.3	0.194	8.6	0.200	10.3	1.528	4.5	2.447	1.4	0.650	0.9	5.570	2.0
B22	$c$	12.984	404.1	0.768	790.1	0.238	663.4	3.361	822.2	2.447	197.3	0.960	241.9	14.893	349.5
	$ac$	2.494	531.9	0.547	902.3	0.299	740.4	3.246	1090.4	2.447	80.8	1.086	54.9	14.894	140.2
C20	$nc$	13.511 (17)	0.4	1.105 (2)	0.2	0.312 (3)	0.5	3.824 (2)	0.5	2.447 (1)	0.7	1.086 (1)	0.4	14.895 (1)	0.8
		13.028 (7)	0.9	0.802 (15)	0.7	0.312 (14)	1.2	3.824 (5)	0.6	2.447 (1)	0.6	0.911 (2)	0.6	14.897 (6)	1.1

comparison between A1 and B22, one can note that A1 presents superior values of  $\Lambda$  on almost all systems for both causal and anti-causal ZF multipliers. Moreover, the computational time spent by A1 is substantially smaller (order of hundred times) than the one required by B22, due to the scalar search performed on  $\alpha \in (0, 1)$  by the method from B22.

### Experiment 2 – Control design

Since the main objective of this work is the synthesis of output feedback control laws for discrete-time Lur'e systems, this experiment evaluates the performance of A1 for the control design of state ( $sf$ ) and output ( $of$ ) feedback gains. For that, matrices  $B_u$  are given by (as in Bertolin et al. (2022b))

$$(1, 6, 7) : B_u^T = [0 \ 1], \quad (2, 3) : B_u^T = [0 \ 1 \ 0 \ 0]$$

$$(4) : B_u^T = [0 \ 1 \ 0 \ 0 \ 0], \quad (5) : B_u^T = [0 \ 1 \ 0],$$

and  $C_y = C_z$ . As in the analysis experiment, the maximum value of  $\Lambda$  for each system is computed through a bisection procedure. In all the cases, A1 is first used with the conditions that ensure stability for non-odd nonlinearities. After convergence, the value of  $X^T$  obtained at the last iteration is used as  $\bar{X}$  to initialize the algorithm in the search for the largest  $\Lambda$  associated with odd nonlinearities. This strategy can always be used since the feasibility of (8) implies that (9) always admits a solution. Moreover, with this initialization, the values of  $\Lambda$  cannot be smaller than the ones previously obtained for non-odd nonlinearities, resulting in less computational effort for A1 when dealing with the odd case.

The results (maximum values of  $\Lambda$  and time demanded  $t$ , in seconds) are shown in Table 2. As can be noted, when compared with B22, A1 demands considerably lower computational times for all the cases, being also superior in terms of the achievable values of  $\Lambda$  for almost all the systems investigated. Moreover, A1 can deal with both odd and non-odd nonlinearities while B22 is restricted to the case of odd nonlinear functions. It is worth to mention that other values of  $\Lambda$  (possibly greater) could be obtained by increasing the order  $n_M$  of the ZF multiplier.

## 5. CONCLUSIONS

An LMI-based iterative method has been proposed for the stability analysis of discrete-time Lur'e systems through FIR ZF multipliers of any given order, with immediate extension to cope with static output and state feedback control. Both odd and non-odd nonlinearities are considered. Anti-causal multipliers

can also be obtained with a transformation on the system matrices. Future work will investigate noncausal FIR ZF multipliers with arbitrary number of causal and anti-causal terms for both stability analysis and control design using integral quadratic constraints (IQC) based strategies.

## REFERENCES

- Ahmad, N.S. and Carrasco, J. (2016). Convex LMI approach for stability of critically stable systems with slope-restricted nonlinearities. In *Proceedings of the 55th IEEE Conference on Decision and Control*, 3228–3233. Las Vegas, NV, USA.
- Ahmad, N.S., Carrasco, J., and Heath, W.P. (2013). LMI searches for discrete-time Zames-Falb multipliers. In *Proceedings of the 52nd IEEE Conference on Decision and Control*, 5258–5263. Florence, Italy.
- Ahmad, N.S., Carrasco, J., and Heath, W.P. (2015). A less conservative LMI condition for stability of discrete-time systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control*, 60(6), 1692–1697.
- Andersen, E.D. and Andersen, K.D. (2000). The MOSEK interior point optimizer for linear programming: An implementation of the homogeneous algorithm. In H. Frenk, K. Roos, T. Terlaky, and S. Zhang (eds.), *High Performance Optimization*, volume 33 of *Applied Optimization*, 197–232. Springer US. <http://www.mosek.com>.
- Bertolin, A.L.J., Oliveira, R.C.L.F., Valmorbidia, G., and Peres, P.L.D. (2022a). Control design of uncertain discrete-time Lur'e systems with sector and slope bounded nonlinearities. *International Journal of Robust and Nonlinear Control*, 32(12), 7001–7015.
- Bertolin, A.L.J., Oliveira, R.C.L.F., Valmorbidia, G., and Peres, P.L.D. (2022b). An LMI approach for stability analysis and output-feedback stabilization of discrete-time Lur'e systems using Zames-Falb multipliers. *IEEE Control Systems Letters*, 6, 710–715.
- Carrasco, J., Heath, W.P., Li, G., and Lanzon, A. (2012a). Comments on “On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities”. *IEEE Transactions on Automatic Control*, 57(9), 2422–2428.
- Carrasco, J., Heath, W.P., Zhang, J., Ahmad, N.S., and Wang, S. (2020). Convex searches for discrete-time Zames-Falb multipliers. *IEEE Transactions on Automatic Control*, 65(11), 4538–4553.
- Carrasco, J., Maya-Gonzalez, M., Lanzon, A., and Heath, W.P. (2012b). LMI search for rational anticausal Zames-Falb multipliers. In *Proceedings of the 51st IEEE Conference on Decision and Control*, 7770–7775. Maui, HI, USA.

Table 2. Maximum values of  $\Lambda$  with  $\phi$  odd and non-odd (shaded) in control design obtained by A1 and B22 for causal ( $c$ ) and anti-causal ( $ac$ ) multipliers, considering state ( $sf$ ) and output ( $of$ ) feedback. The numerical complexity can be inferred from the time ( $t$ ) in seconds required to test each system. The notation +100 means that values of  $\Lambda$  larger than 100 have been obtained.

System			1		2		3		4		5		6		7	
			$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$	$\Lambda$	$t$
A1	c	sf	29.608	7.0	7.605	0.5	28.284	24.9	+100	0.1	+100	0.2	61.891	44.4	+100	0.0
			29.299	10.4	4.727	38.2	14.182	72.0	+100	38.7	+100	20.1	31.290	31.9	+100	1.8
		of	28.966	2.054	2.303	1.399	0.323	0.6	7.283	3.9	2.540	0.6	1.156	4.5	+100	0.01
			28.576	5.1	2.234	30.4	0.322	13.1	5.279	8.3	2.533	1.3	1.155	6.1	+100	5.8
	ac	sf	29.408	7.7	12.761	0.6	19.637	6.9	+100	0.3	+100	0.1	63.714	35.6	+100	0.1
			19.858	2.4	8.209	46.7	18.767	78.5	+100	24.4	+100	16.2	48.152	32.7	+100	0.4
		of	21.196	5.2	2.571	1.6	0.324	0.9	8.650	0.9	2.541	0.7	1.167	2.3	+100	0.1
			19.927	2.5	2.371	26.7	0.323	15.6	8.647	22.0	2.541	0.9	1.154	17.0	+100	1.2
B22	c	sf	24.093	662.3	7.010	1636.9	20.626	2069.8	+100	2335.7	+100	493.8	29.220	1170.0	+100	180.3
			27.500	519.6	2.272	1636.4	0.324	1741.7	8.080	1828.5	2.541	667.0	1.155	593.5	+100	517.9
	ac	sf	24.063	806.7	5.586	1599.8	13.857	1907.3	+100	1710.5	+100	828.7	85.938	850.8	+100	43.7
			22.131	654.2	2.579	1741.8	0.325	1782.7	8.567	1839.1	2.541	844.3	1.165	553.2	+100	219.9

- Carrasco, J., Maya-Gonzalez, M., Lanzon, A., and Heath, W.P. (2014). LMI searches for anticausal and noncausal rational Zames–Falb multipliers. *Systems & Control Letters*, 70, 17–22.
- Carrasco, J., Turner, M.C., and Heath, W.P. (2016). Zames–Falb multipliers for absolute stability: From O’Shea’s contribution to convex searches. *European Journal of Control*, 28, 1–19.
- de Oliveira, M.C. and Skelton, R.E. (2001). Stability tests for constrained linear systems. In S.O. Reza Moheimani (ed.), *Perspectives in Robust Control*, volume 268 of *Lecture Notes in Control and Information Science*, 241–257. Springer-Verlag, New York, NY.
- Gonzaga, C.A.C., Jungers, M., and Daafouz, J. (2012). Stability analysis of discrete-time Lur’e systems. *Automatica*, 48(9), 2277–2283.
- Haddad, W.M. and Bernstein, D.S. (1993). Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. Part I: Continuous-time theory. *International Journal of Robust and Nonlinear Control*, 3(4), 313–339.
- Haddad, W.M. and Bernstein, D.S. (1994). Parameter-dependent Lyapunov functions and the discrete-time Popov criterion for robust analysis. *Automatica*, 30(6), 1015–1021.
- Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 3rd edition.
- Kim, K.K. and Braatz, R.D. (2017). Robust static and fixed-order dynamic output feedback control of discrete-time parametric uncertain Lur’e systems: Sequential SDP relaxation approaches. *Optimal Control Applications and Methods*, 38(1), 36–58.
- Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control Systems Design*, 284–289. Taipei, Taiwan. <http://yalmip.github.io>.
- Louis, J., Jungers, M., and Daafouz, J. (2015a). Sufficient LMI stability conditions for Lur’e type systems governed by a control law designed on their Euler approximate model. *International Journal of Control*, 88(9), 1841–1850.
- Louis, J., Jungers, M., and Daafouz, J. (2015b). Switching control consistency of switched Lur’e systems with application to digital control design with non uniform sampling. In *Proceedings of the 2015 European Control Conference*, 1754–1759. Linz, Austria.
- O’Shea, R. (1966). A combined frequency-time domain stability criterion for autonomous continuous systems. *IEEE Transactions on Automatic Control*, 11(3), 477–484.
- O’Shea, R. (1967). An improved frequency time domain stability criterion for autonomous continuous systems. *IEEE Transactions on Automatic Control*, 12(6), 725–731.
- Park, J., Lee, S.Y., and Park, P. (2019). A less conservative stability criterion for discrete-time Lur’e systems with sector and slope restrictions. *IEEE Transactions on Automatic Control*, 64(10), 4391–4395.
- Park, P. (1997). A revisited Popov criterion for nonlinear Lur’e systems with sector-restrictions. *International Journal of Control*, 68(3), 461–469.
- Park, P. (2002). Stability criteria of sector- and slope-restricted Lur’e systems. *IEEE Transactions on Automatic Control*, 47(2), 308–313.
- Safonov, M. and Wyetzner, G. (1987). Computer-aided stability analysis renders Popov criterion obsolete. *IEEE Transactions on Automatic Control*, 32(12), 1128–1131.
- Turner, M.C. (2021). Zames–Falb multipliers: don’t panic. <https://doi.org/10.48550/arXiv.2106.15913>.
- Turner, M.C. and Drummond, R. (2021). Discrete-time systems with slope restricted nonlinearities: Zames–Falb multiplier analysis using external positivity. *International Journal of Robust and Nonlinear Control*, 31(6), 2255–2273.
- Turner, M.C., Kerr, M., and Postlethwaite, I. (2009). On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control*, 54(11), 2697–2702.
- Wang, S., Heath, W.P., and Carrasco, J. (2014). A complete and convex search for discrete-time noncausal FIR Zames–Falb multipliers. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, 3918–3923. Los Angeles, CA, USA.
- Zames, G. and Falb, P.L. (1968). Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal on Control*, 6(1), 89–108.