Passivity techniques and Hamiltonian structures in discrete time

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Abstract. The object of this paper is to show the impact of representing discrete-time dynamics as two coupled difference/differential equations in establishing passivity properties and describing port-Hamiltonian structures as well as the related energy-based control strategies.

Keywords: Discrete-time systems; Port-Hamiltonian structures; Passivity based control, Energy-based control

1 Introduction

Energy-based modeling and control are fundamental evergreen concepts that are extensively investigated because of their sustained impact towards the newest methodologies and technologies (see the textbooks or survey-oriented contributions [58, 1, 51, 45, 49, 52, 3]). The underlying idea consists in deducing a system representation explicitly catching the energy features of the dynamics through dissipative or conservative components. Then, energy based control strategies can be designed. In this framework, the class of port-Hamiltonian systems is paradigmatic [50, 47, 9]. From a theoretical point of view, most of the literature is devoted to the continuous-time setting in spite of a pervasive interest in computer-oriented applications and thus in digital structures at large. The general obstacles to perform an equivalent analysis in a discrete-time framework are well-known and related to the difficulty to describe the geometric structure behind the state and output discrete evolutions [32]. In this sense, the definition of dissipation itself is an open question [5, 4, 44, 33]. As a consequence, a shared definition of discrete-time port-Hamiltonian structures has been missing in spite of a variety of approaches often inspired by discretization schemes aimed to preserve the energy properties and/or power-balance exchanges (to cite a few see [13, 11, 55, 16, 17, 57, 56, 56, 53, 54, 7, 60, 15, 2, 8]).

The present work presents recent advances in the definition of dissipative notions for discrete-time systems when adopting a Differential Difference Representation (DDR) and their impact for characterizing port-Hamiltonian structures. The idea behind such a representation consists in separating discrete-time dynamics into two components: a difference equation that describes the control-free evolution and a differential equation that models the variation of the dynamics with respect to the control variable. This hybrid structure is adapted to cope with the intrinsic nonlinearity (that becomes more and more complex through iterative composition along successive time steps) in both the state and input variables when representing discrete-time dynamics in the map form. Splitting the free evolution from the controlled part leads naturally to a description of a discrete-time dynamics through two coupled difference and differential equations. An exponential representation of the discrete-time flow results through the integration of the differential controlled part. This is useful for further composing the state evolutions along successive time steps; the composition of exponential flows replace the composition of nonlinear functions. Analogously, when considering an output function, one gets an exponential form representation of the Volterra series expansions characterizing the input-to-output evolutions. Recalling, if necessary, that typical numerical burdens have to be faced when considering both purely discrete systems or systems issued from sampling due to their intrinsic nonlinearities, these exponential forms provide efficient computational tools. Accordingly, the combinatorial properties of the series expansions that describe the solutions can be exploited to qualify suitable polynomial approximations of increasing order. Section 2 recalls these developments as useful prolegomena to the paper while details can be found in [27, 28, 30, 35].

In Section 3, the aforementioned representation is used to characterize passivity. This notion, and more generally the one of dissipativity, relies upon energy exchanges and the way the system interacts with the environment. Roughly speaking, a system is passive when the internally stored energy does not exceed the one externally supplied [58, 59]. This property is caught by characterizing the variation of a particular function along the system's trajectories [49, 52, 45]. Such a function, referred to as the storage function, generally represents the energy and is strictly linked to Lyapunov and/or Hamiltonian functions. With this in mind and for a fixed storage quantity, the DDR form we propose immediately allows us to characterize average passivity [33]; a novel notion of passivity in discrete time. This is done by isolating the control dependent part and thus defining in a very natural way the corresponding average passive output that is, in turn, a conjugate quantity whose product with the control variable is a power unit. The notion of average dissipation can be then exploited for control purposes by extending the usual concept of negative output damping to cope with stabilization at the origin. This approach can be further generalized to describe Passivity-Based Control (PBC) at large for nonlinear discrete-time systems towards the so-called second generation of passivity-based control, aimed at managing the system energy to satisfy control specifications. Those techniques endow Energy Balancing (EB) and Interconnection and Damping Assignment

(IDA) to deal with more complex systems including networked or cascade dynamics. The foundations of average passivity-based techniques are recalled in Section 3, while more detailed studies are in [36, 34].

As already mentioned, port-Hamiltonian dynamics are of pervasive interest due to their mathematical structure and their foundations in physics (to cite a few [18, 19, 20, 46, 50, 9, 48]). Hereinafter, a novel state-space representation of discrete-time port-Hamiltonian structures that represents a breakthrough in the literature is naturally deduced from the DDR form. The so defined forms are endowed with average passivity properties that validate the proposed choice of conjugate output. Further on, the fundamental characteristics of port-Hamiltonian structures as the qualifying closeness property under power-preserving interconnection [40, 38], are validated. As a consequence, these forms are efficient for the design of average passivity based control strategies for complex and networked discrete-time dynamics as illustrated in [41, 22, 23, 38]. Port-Hamiltonian structures are discussed in Section 4 while dedicated studies are in [39, 41, 38].

1.1 Notations

Throughout the paper all the functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. The sets $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of real, natural numbers including 0 and integers respectively. For any vector $v \in \mathbb{R}^n$, $|v|$ and $v^\top$ define the norm and transpose of v respectively. I_d denotes the identity function on the definition space while I denotes the identity operator and the identity matrix when related to a linear operator. The symbols " > 0" and " < 0" denote positive and negative definite functions (or matrices), respectively. The symbol $\circ$ denotes the composition of two functions or operators, depending on the context. Given a real-valued function $V : \mathbb{R}^n \to \mathbb{R}$ assumed differentiable, setting $\nabla = col\{\frac{\partial}{\partial x_i}\}_{i=1,n}$, $\nabla V(x)$ represents the gradient column-vector. Given a vector-valued function $F(x) = col(F_1(x), \ldots, F_n(x))$, the operator

$$J_x[F](x) = \frac{\partial F(x)}{\partial x} = \left\{ \frac{\partial F_i}{\partial x_j}(x) \right\}_{i,j=1,n}$$

denotes the Jacobian of the function F evaluated at x. Given a smooth vector field over $\mathbb{R}^n$, e^f) denotes the exponential Lie operator $e^f := I + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$ with $\mathrm{L}_f = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$ and indicating by L_f^i the operator at power i with respect to the usual composition of vector fields (for a linear vector field, the exponential Lie operator recovers the exponential of the matrix representing the operator). For any smooth function $h : \mathbb{R}^n \to \mathbb{R}$, one verifies $e^f h(x) = h(e^f(x)) = e^f h|_x$ where $|_x$ denotes the evaluation of the function at x. Given two vector fields f, g, their Lie bracket is again a vector field denoted by $ad_f g = [f, g] = (\mathrm{L}_f \circ \mathrm{L}_g I_d - \mathrm{L}_g \circ \mathrm{L}_f) I_d$ and, iteratively for $i \geq 1$, $ad_f^i g = ad_f^{i-1} \circ ad_f g$ with $ad_f^1 g = ad_f g$. A function $R(x, u)$ is said in $O(u^p)$ for $p \geq 1$ if, whenever it is defined, it can be written as $R(x, u) = u^{p-1}\tilde{R}(x, u)$ and there exist a function $\theta \in \mathcal{K}_\infty$ and $u^\star > 0$ such that $\forall u \leq u^\star$, $|\tilde{R}(x, u)| \leq \theta(u)$.

2 Basic on discrete-time dynamics

2.1 Differential/Difference Representation

A single-input nonlinear discrete-time dynamics over $\mathbb{R}^n$ is usually represented by a function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, smooth in both the state and input variables

$$x_{k+1} = F(x_k, u_k) = x_k + F_0(x_k) + g(x_k, u_k)u_k. \tag{1}$$

For any pair of state and input variables (x_k, u_k) fixed at time instant $k \in \mathbb{N}$, x_{k+1} denotes the state reached at time $k+1$ from x_k, under the action of the control u_k. For convenience that will be clear later on and without loss of generality, the free evolution is decomposed as $F(x, 0) = x + F_0(x)$ while $g(x, u)u$ (with $g(x, 0) \neq 0$) represents the control dependent part of the dynamics.
In [30], we proposed to represent discrete dynamics as two coupled difference and differential equations (DDR). More in detail, for all time step k and under mild conditions (e.g., submersivity of $F(x, u)$), (1) can be represented as

$$x^+(0) = x + F_0(x) \tag{2a}$$

$$\frac{dx^+(u)}{du} = G(x^+(u), u) \tag{2b}$$

when denoting $(x, u) = (x_k, u_k) \in \mathbb{R}^n \times \mathbb{R}$, any pair of state and input variables at generic time instant k, $x^+(u) = x^+(u_k) = x_{k+1}$, the state reached at time $k+1$ starting from x_k under the action of u_k and $x^+(0)$, the state reached along free evolution $u_k = 0$. In doing so, we underline that $x^+(u)$ is viewed as a curve in $\mathbb{R}^n$, parameterized by the control variable $u \in \mathbb{R}$. $G(\cdot, u)$ is a properly defined vector field on $\mathbb{R}^n$, parameterized by u, satisfying $G(F(x, u), u) = \frac{\partial F}{\partial u}(x, u)$. With this in mind, the difference equation (2a) describes the jump of the state evolving in free evolution while the differential equation (2b) models the rate of change of the state dynamics with respect to control variation.

Remark 1. Provided $F(x, 0)$ be invertible, the vector field $G(x, u)$ can be defined for u sufficiently small as

$$G(x, u) := \left.\frac{\partial F(x, u)}{\partial u}\right|_{x=F^{-1}(x,u)}, \tag{3}$$

and expanded, around $u = 0$, as

$$G(x, u) = G_1(x) + \sum_{i \geq 1} \frac{u^i}{i!} G_{i+1}(x); \quad G_1(x) = G(x, 0). \tag{4}$$

The vector fields G_i in the power expansion (4) define a family of canonical vector fields associated to the dynamics (2). The $(G_i)'s$ have been proved useful in the study of analysis and design problems linked to the geometry of the evolutions in the state space (e.g.[27, 28, 30, 6, 31]).

Remark 2. The DDR form can be generalized to multi-input dynamics by modeling the rate of change of the dynamics under the action of each control. Setting $\boldsymbol{u} = (u^1, \ldots, u^m)^\top$, one replaces (2b) by the set of partial derivative equations

$$\frac{\partial x^+(\boldsymbol{u})}{\partial u^j} = G^j(x^+(\boldsymbol{u}), \boldsymbol{u}) \quad \text{for} \quad j = (1, \ldots, m) \quad \text{with} \quad x^+(0) = F(x, 0) \tag{5}$$

and $G^j(x, \boldsymbol{u})$ satisfying the condition $G^j(F(x, \boldsymbol{u}), \boldsymbol{u}) := \frac{\partial F(x, \boldsymbol{u})}{\partial u^j}$.

2.2 Some useful manipulations

To better grasp the DDR forms, some useful concepts are specified below. Particular classes can be discussed besides the linear one that corresponds to set in (2), $F_0(x) = Ax$ with constant control vector $G(x, u) = B$ and matrices (A, B) of suitable dimensions. A peculiar class is represented by the input-affine one.

Proposition 1. - *Input-affine DDR - Assuming $G(x, u) = G_1(x)$ in (2), the dynamics in map form (1) preserves a nonlinear dependence on u which admits the exponential representation*

$$F(x, u) = e^{uG_1}(x)\Big|_{x+F_0(x)} \tag{6}$$

where e^{uG_1} denotes the flow associated with the solution to the differential equation (2b) when $G(x, u) = G_1(x)$.

This result follows from the fact that integrating (2b) from 0 and u with initial condition $x^+(0) = x + F_0(x)$, one recovers a dynamics in map form (1); i.e.

$$x^+(u) = x + F_0(x) + \int_0^u G(x^+(s), s)\mathrm{d}s = x + F_0(x) + g(x, u)u. \tag{7}$$

When assuming $G(x, u) = G_1(x)$, this integral form admits the simple exponential representation described in Proposition 1 that is generalized in the sequel to a control dependent vector field $G(\cdot, u)$.

Remark 3. We note for completeness that given an input-affine dynamics in map form, $g(x, u) = g(x)$ in (1), the associated control vector field $G(x, u)$ in (4) is no more linear in u and satisfies the algebraic constraint $G(F(x, u), u) = g(x)$.

Easy manipulations show how DDR forms are transformed under coordinates change and feedback.

Lemma 1. *Let the coordinates change $z = T(x)$ defined by the diffeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$, then the DDR dynamics (2) is transformed into*

$$\begin{array}{l} z^+(0) \ = z + \bar{F}_0(z) \\ \frac{dz^+(u)}{du} = \bar{G}(z^+(u), u) \end{array} \tag{8}$$

with

$$z + \bar{F}_0(z) = T(x + F_0(x))\Big|_{x=T^{-1}(z)}$$
$$\bar{G}(z,u) = Ad_T G(z,u)$$

where $Ad_T G(\cdot,u)$ *indicates the transport of the vector field* $G(\cdot,u)$ *along* T*; i.e.*

$$Ad_T G(z,u) = \left[\tfrac{\partial T}{\partial x} G(\cdot,u)\right]_{x=T^{-1}(z)} = \mathrm{L}_{G(\cdot,u)} T(x)\big|_{x=T^{-1}(z)}. \tag{9}$$

Lemma 2. *Let the state feedback* $u(x,v) = \alpha(x) + v$ *with* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *smooth and external control* $v \in \mathbb{R}$*, then the DDR dynamics (2) is transformed into*

$$\begin{array}{l} x_\alpha^+(0) \ = x + F_\alpha(x) \\ \frac{dx_\alpha^+(v)}{dv} = G_\alpha(x_\alpha^+(v), v) \end{array} \tag{10}$$

where

$$\begin{aligned} x_\alpha^+(v) &= F(x, \alpha(x) + v) = x^+(\alpha(x) + v) \\ x_\alpha^+(0) &= F(x, \alpha(x)) = x + F_\alpha(x) = x^+(\alpha(x)) \\ G_\alpha(x,v) &= G(x, \alpha(x) + v). \end{aligned}$$

The proof works out showing that

$$\frac{dx_\alpha^+(v)}{dv} = \frac{\partial F(x,u)}{\partial u}\Big|_{u=\alpha(x)+v} \frac{\partial(\alpha(x)+v)}{\partial v} = G(x_\alpha^+(v), \alpha(x)+v) = G_\alpha(x_\alpha^+(v), v).$$

2.3 Input-to-state and input-to-output trajectories

In our context, a discrete-time system $\Sigma_d(h)$ is given by a state dynamics in the DDR form (2) and a smooth output function $h : \mathbb{R}^n \to \mathbb{R}$. The generalization of Proposition 1 to a non input-affine form is reported and further extended to describe input-output evolutions over one or several time-steps.

Theorem 1. *Given* $\Sigma_d(h)$ *and a pair of state and input variables* (x_k, u_k) *at generic time instant* k*, one gets at time* $k+1$

$$x_{k+1} = x^+(u_k) = e^{u_k \mathcal{G}(\cdot,u_k)} I_d|_{x_k + F_0(x_k)} = F(x_k, u_k)$$
$$h(x_{k+1}) = h(x^+(u_k)) = e^{u_k \mathcal{G}(\cdot,u_k)} h|_{x_k + F_0(x_k)} = h(F(x_k, u_k))$$

where the series exponent is a Lie series $\mathcal{G}(\cdot,u) \in Lie\{G_1, \ldots, G_p, \ldots\}$ *that can be described through its expansion in powers of* u*. For the first term one reports*

$$\mathcal{G}(\cdot,u) = G_1 + \frac{u}{2} G_2 + \frac{u^2}{3!}(G_3 + [G_1, G_2]) + O(u^3).$$

The proof, developed in [27, 35], follows from the Lie properties endowed by the flow characterizing the solutions to nonlinear u-dependent ordinary differential equations of the form (2b), when expressed through a chronological exponential.

A major property of these exponential representations of discrete flows is to be easily composed along successive time steps. For, it is instrumental to define, according to (9), the transport of any vector field G_i along the free evolution $(Id + F_0)$ so getting for $j \geq 0$, the family of transported vector fields $(G_i^j)_s^{'}$ with

$$G_i^{j+1}(x) = Ad_{I_d+F_0} G_i^j(x) = \mathrm{L}_{G_i^j}(x + F_0(x))\Big|_{(I_d+F_0)^{-1}(x)}$$

and iteratively $G_i^{j+1}(x) = Ad_{I_d+F_0}^{j+1} G_i(x)$ with $G_i^0 = G_i$.

The following result holds when denoting by $\mathcal{G}^j(\cdot, u); j \geq 0$, the series exponent

$$\mathcal{G}^j(\cdot, u) \in Lie\{G_1^j, \ldots, G_p^j, \ldots\}$$

defined as in Theorem 1 with respect to the transported vector fields $(G_i^j)_s^{'}$.

Theorem 2. *Given $\Sigma_d(h)$, initial state value $x_0 \in \mathbb{R}^n$ and input sequences $\{u_0, \ldots, u_k\}$, $k \in \mathbb{N}$, then at generic time instant $k+1 > 0$, one gets*

$$\begin{aligned} x_{k+1} &= x^+(u_k, \ldots, u_0) = e^{u_0 \mathcal{G}^k(\cdot, u_0)} \circ \cdots \circ e^{u_k \mathcal{G}^0(\cdot, u_k)}|_{(I_d+F_0)^{k+1}(x_0)} \\ y_{k+1} &= y^+(u_k, \ldots, u_0) = e^{u_0 \mathcal{G}^k(\cdot, u_0)} \circ \cdots \circ e^{u_k \mathcal{G}^0(\cdot, u_k)} h|_{(I_d+F_0)^{k+1}(x_0)}. \end{aligned}$$

According to these results it becomes clear that the vector fields $(G_i)_s^{'}$ and their transport along the drift term enter in a differential geometric characterization of the structure of the accessible states. In fact, accessibility can be reported to the properties of the orbit of the associated Lie groups (see [10, 14] for further details). Further on, invariance can be characterized in terms of the Lie algebra $Lie\{G_1^j, \ldots, G_p^j, \ldots\}$ as discussed in [25]. The same holds regarding the properties of controlled invariance, feeback linearization [31] or decoupling [6], up to characterize the corresponding control solutions.

2.4 Concluding comments

In this section, an alternative description to the usual map form is proposed for discrete-time dynamics. The free evolution that generates a purely discrete evolution defines the initial condition of the differential equation modeling the control action. This suggests a modeling approach that would separately identify the free evolution as a map F_0 and the variation of the dynamics with respect to u as a vector field $G(\cdot, u)$, that may depend on u. In doing so, a family of control vector fields, the $(G_i)_s^{'}$, that enters in the characterization of the structural and control properties of the dynamics, is defined. Accordingly, it becomes possible to combine a more visual geometric approach with a more computational algebraic one to provide intriguing relationships between the continuous-time and

discrete-time settings that should be further investigated towards a unified vision. Further on, being the solution to an ordinary differential equation, the state dynamics can be rewritten in terms of the exponential operator characterizing the associated flow. It follows that the whole differential geometry apparatus behind this operator form can be used to describe the input-to-state and input-to-output behaviours along successive time steps. In particular, the composition of nonlinear maps is replaced with the composition of exponential forms that is more tractable in practice. This is well illustrated with the characterization of the Volterra series and the computation of its kernels in terms of the vector fields $(G_i^j)_s'$ and their Lie brackets. We refer to [26, 29] for further discussion along these lines in relation with realization problems.

3 Passivity techniques

The notion of average passivity has been introduced in [34] to weaken the necessary requirement of direct throughput when adopting the standard notion in a discrete-time framework. This definition is in fact directly inspired by the splitting of the state dynamics into free and control parts.

3.1 Average passivity

Denoting by $\Sigma_d(h)$ a discrete-time system described by the dynamics (1) (equivalently (2)) with output $h : \mathbb{R}^n \to \mathbb{R}$, the definition below is recalled from [34].

Definition 1 (Average passivity). *$\Sigma_d(h)$ is said average passive if there exists a positive semi-definite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (the storage function) such that for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ the following inequality holds*

$$S(x^+(u)) - S(x) \leq \int_0^u h(x^+(s))\mathrm{d}s := uh^{av}(x, u) \tag{11}$$

with the average output defined as

$$h^{av}(x, u) := \frac{1}{u}\int_0^u h(x^+(s))\mathrm{d}s. \tag{12}$$

This definition is directly inspired by the DDR form of the dynamics that yields to rewrite the rate of change of the storage function S, between two successive states, in an integral form as

$$S(x^+(u)) - S(x) = S(x^+(0)) - S(x) + \int_0^u L_{G(\cdot,s)}S(x^+(s))\mathrm{d}s.$$

The so-defined average map $h^{av}(\cdot, u)$ in (12), introduces a direct input-output link in such a way that average passivity with respect to h is in fact equivalent to usual passivity with respect to $h^{av}(\cdot, u)$.

As an immediate consequence of average passivity, setting $u = 0$ in (11), one verifies $S(x^+(0)) - S(x) \leq 0$, so concluding stability of any equilibrium $x_e \in \mathbb{R}^n$ when S qualifies as a Lyapunov function at x_e ($S(x_e) = 0$ and $S(x) > 0$ for $x \neq x_e$) and asymptotic stability provided $S(x^+(0)) - S(x) < 0$.

More in general, average dissipativity can be defined making reference to the average output and a supply rate function $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, so requiring that the dissipation inequality below be verified for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$

$$S(x^+(u)) - S(x) \leq s(u, h^{\mathrm{av}}(x, u)).$$

The notion of average passivity can be extended to average passivity from some nominal non-zero constant value $\bar{u} \neq 0$, [21, 40, 38]. It is instrumental when discussing the action of a feedback law over passivity.

Definition 2 (Average passivity from $\bar{u}$). *$\Sigma_d(h)$ is said* average passive from a given $\bar{u}$ *with $\bar{u} \in U \subseteq \mathbb{R}$, if there exists a positive semi-definite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (the storage function) such that, for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$*

$$S(x^+_{\bar{u}}(u)) - S(x) \leq u h^{av}_{\bar{u}}(x, u) \tag{13}$$

with u-average output from $\bar{u}$ defined as

$$h^{av}_{\bar{u}}(x, u) := \frac{1}{u} \int_0^u h(x^+(\bar{u} + s))\mathrm{d}s.$$

When $\bar{u} = 0$, one recovers average passivity since $h^{av}_0(x, u) = h^{av}(x, u)$.

3.2 Feedback stabilization and interconnection

On these bases, Passivity Based Control (PBC) techniques can be developed as discussed below. Two basic ingredients of control strategies exploiting passivity are revisited hereafter making reference to the average notion. Firstly, stabilization through negative output feedback and additional damping is specified. Secondly, closeness under power preserving input-output interconnection is discussed so enlarging the control design to interconnected dynamics. The next definition is instrumental.

Definition 3 (Zero-state detectability). *Given $\Sigma_d(h)$ let $x_e \in \mathbb{R}^n$ be an equilibrium and $\mathcal{Z} \in \mathbb{R}^n$ be the largest invariant set contained in the set $\{x \in \mathbb{R}^n$ s.t. $h(x^+(0), 0) = 0\}$. $\Sigma_d(h)$ is said zero-state detectable (ZSD) if $x = x_e$ is an asymptotically stable equilibrium conditionally to $\mathcal{Z}$.*

The following theorem characterizes the negative output damping feedback.

Theorem 3 (Negative average output feedback). *Given $\Sigma_d(h)$ with equilibrium $x_e \in \mathbb{R}^n$, ($F_0(x_e) = 0$), assumed average passive with storage function $S > 0$, then the feedback $u = \alpha(x)$ solving the algebraic equality*

$$u = -\kappa h^{av}(x, u) \quad \textit{with} \quad \kappa > 0 \tag{14}$$

ensures asymptotic stability provided that $\Sigma_d(h)$ is Zero State Detectability (ZSD). Moreover, setting $u(x,v) = \alpha(x) + v$ with external control $v \in \mathbb{R}$, then the closed loop dynamics is average passive again with respect to h

$$S(x_\alpha^+(v)) - S(x) \leq v h_\alpha^{av}(x,v). \tag{15}$$

To conclude average passivity of the closed-loop dynamics it is sufficient to note that under the feedback $u(x,v) = \alpha(x) + v$, the average output associated with h along the closed-loop dynamics described in Lemma 2, recovers the average output from $\bar{u} = \alpha(x)$ defined in Definition 2. In fact, one has

$$h_\alpha^{av}(x,v) := \frac{1}{v}\int_0^v h(x_\alpha^+(s))\mathrm{d}s.$$

Then, by definition of $\alpha(x)$, one gets

$$\begin{aligned}
\int_0^{\alpha(x)+v} h(x^+(s))\mathrm{d}s &= \int_0^{\alpha(x)} h(x^+(s))\mathrm{d}s + \int_{\alpha(x)}^{\alpha(x)+v} h(x^+(s))\mathrm{d}s \\
&= \int_0^{\alpha(x)} h(x^+(s))\mathrm{d}s + \int_0^{v} h(x^+(\alpha(x)+s))\mathrm{d}s \\
&= -\kappa(h^{av}(x,\alpha(x)))^2 + \int_0^v h(x_\alpha^+(s))\mathrm{d}s \\
&\leq v h_\alpha^{av}(x,v)
\end{aligned}$$

so concluding that average passivity from $\alpha(x)$ coincides with average passivity under preliminary feedback $\alpha(x)$ with respect to the same output map h.

The negative average output feedback solving (14) is the first step towards stabilizing strategy through additional damping or average PBC feedback.

Remark 4. Computational aspects are beyond the scope of the paper. However, we underline that the control solution being expressed as the solution to the algebraic equality of the form (14), its computation may be a difficult. In practice, it can be performed according to suitable approximation methods as discussed in [21, 22].

The second fundamental property to verify regards the interconnection of two average passive systems through their respective input and output variables when setting $u = \phi(h^{av}(x,u))$. Given two average passive systems $\Sigma_d^i(h^i)$ with storage function S^i for $i = 1, 2$, a power-preserving input-output interconnection making reference to the average outputs can be naturally defined as in [40, 43].

Definition 4 (Power preserving interconnection). *The input-output interconnection between $\Sigma_d(h^1)$ and $\Sigma_d(h^2)$ is said* power preserving *if it satisfies the integral equality*

$$\int_0^{u^1} h^1({x^1}^+(s))\mathrm{d}s + \int_0^{u^2} h^2({x^2}^+(s))\mathrm{d}w = 0 \tag{16}$$

equivalently rewritten as

$$u^1 h^{1^{av}}(x^1, u^1) + u^2 h^{2^{av}}(x^2, u^2) = 0. \tag{17}$$

We easily note that the simplest way to solve (17) is to set

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \phi \begin{pmatrix} h^{1^{av}}(x^1, u^1) \\ h^{2^{av}}(x^2, u^2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h^{1^{av}}(x^1, u^1) \\ h^{2^{av}}(x^2, u^2) \end{pmatrix} \tag{18}$$

so recovering the classical power preserving interconnection expressed with respect to the average outputs. The solution to the implicit equality (17) defines a preliminary power preserving state-feedback that we denote $\alpha(x) = (\alpha^1(x), \alpha^2(x))^\top$ with $x = (x^1, x^2)^\top$. The following Theorem can be stated.

Theorem 4 (Average passivity under power preserving interconnection). *Let, for $i = 1, 2$, the systems $\Sigma_d(h^i)$ be average passive with respective storage functions S^i. Let $\alpha(x)$ be the power-preserving interconnection satisfying (18) and set $u = \alpha(x) + v$ with external control $v = (v^1, v^2)^\top$. Then, the interconnected system*

$$x^{1+}_{\alpha^1}(v^1) = F^1_{\alpha^1}(x, v^1) \tag{19a}$$

$$x^{2+}_{\alpha^2}(v^2) = F^2_{\alpha^2}(x, v^2) \tag{19b}$$

with output $h = (h^1, h^2)^\top$ is average passive with storage function $S(x) := S^1(x^1) + S^2(x^2)$. Namely, the dissipation inequality holds; i.e.

$$S^1(x^{1+}_{\alpha^1}(v^1)) - S^1(x^1) + S^2(x^{2+}_{\alpha^2}(v^2)) - S^2(x^2) \leq v^\top h^{av}_\alpha(x, v) \tag{20}$$

with the average output of the closed-loop dynamics defined as

$$h^{av}_\alpha(x, v) = \begin{pmatrix} \frac{1}{v^1} \int_0^{v^1} h^1(x^{1+}_{\alpha^1}(s)) \mathrm{d}s \\ \frac{1}{v^2} \int_0^{v^2} h^2(x^{2+}_{\alpha^2}(s)) \mathrm{d}s \end{pmatrix}. \tag{21}$$

The proof, detailed in [40], works out in two steps. First, one concludes that under power-preserving input-output interconnection, the interconnected system is average passive from $\alpha(x)$ according to the Definition 2. Then, because of the feedback structure, average passivity from α recovers average passivity of the dynamics under preliminary feedback $\alpha(x)$, so concluding the claim.

3.3 Passivating output map

Theorem 3 shows how stabilization under feedback can be achieved by exploiting average passivity. However, stabilization to some target equilibria that are local extrema of suitably shaped energy functions can be requested. For, the following proposition is instrumental, it specifies a dummy output function that preserves average passivity and is zero at the local minima of the storage function. It is reminiscent of the continuous-time context (see [34, 38] for details).

Proposition 2 (Average passivating output). *Let $\Sigma_d(h)$ be average passive with storage function S, then it is also average passive with respect to the dummy output function*

$$Y(x,u) = \mathrm{L}_{G(\cdot,u)}S(x). \tag{22}$$

$Y(\cdot,u)$, computed as the Lie derivative of S along $G(\cdot,u)$, is referred to as an average passivating output because it satisfies the Energy Balance Equality (EBE)

$$\underbrace{S(x^+(u)) - S(x)}_{\text{stored energy}} = \underbrace{S(x^+(0)) - S(x)}_{\text{dissipated energy}} + \underbrace{S(x^+(u)) - S(x^+(0))}_{\text{supplied energy}}.$$

Average passivity with respect to the output (22) holds since by definition

$$S(x^+(u)) - S(x^+(0)) = u\Big(\mathrm{L}_{G(\cdot,u)}S\Big)^{av}(x,u) = uY^{av}(x,u)$$

with $S(x^+(0)) - S(x) \leq 0$ from the average passivity assumption of $\Sigma_d(h)$. Moreover, $Y(x,u)$ is zero at local extrema of S since $Y(x,u) = \frac{\partial S}{\partial x}G(x,u)$.

Remark 5. Proposition 2 generalizes to assuming $\Sigma_d(h)$ average passive from a given $\bar{u}$. In that case, average passivity from $\bar{u}$ with respect to $Y(\cdot,u)$ follows

$$S(x^+(\bar{u}+u)) - S(x) \leq S(x^+(\bar{u}+u)) - S(x^+(\bar{u})) = uY^{av}_{\bar{u}}(x,u)$$

because by assumption $S(x^+(\bar{u})) - S(x) \leq 0$ and by definition

$$\Big(\mathrm{L}_{G_{\bar{u}}(\cdot,u)}S\Big)^{av}(x,u) = \frac{1}{u}\int_0^u \mathrm{L}_{G(\cdot,\bar{u}+s)}S(x^+(\bar{u}+s))\mathrm{d}s = Y^{av}_{\bar{u}}(x,u)$$

with $G_{\bar{u}}(\cdot,s) = G(\cdot,\bar{u}+s)$.

Remark 6. Specifying the result in Theorem 3 on such output function (22), the stabilizing feedback $u = \alpha(x)$ results to be the solution to the algebraic equality

$$u = -\kappa\Big(\mathrm{L}_{G(\cdot,u)}S\Big)^{av}(x); \qquad \kappa > 0$$

that can be solved in first approximation around $u = 0$ so getting

$$\alpha_{ap}(x) = -\kappa\lambda(x)\big(\frac{\partial S(x)}{\partial x}G_1(x)\big)\Big|_{x=x+F_0(x)}$$

with a suitable gain $\lambda(x) > 0$ as discussed in [21]. Setting $u = \alpha(x)+v$, the closed loop dynamics with output $\mathrm{L}_{G_\alpha(\cdot,v)}S(x) = Y_\alpha(x,v)$ remains average passive.

In the present paper, oriented to characterize Hamiltonian dynamics in discrete time, it is instrumental to relate the passivating output defined in (22) to a certain discrete gradient function. The definition below is recalled from [24, 38].

Definition 5 (Discrete gradient function). *Given a smooth real-valued function $S : \mathbb{R}^n \to \mathbb{R}$, its* discrete gradient *is a function of two variables $\bar{\nabla} S|_x^z : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying for all $x, z \in \mathbb{R}^n$*

$$(z-x)^\top \bar{\nabla} S|_x^z = S(z) - S(x) \quad \textit{with} \quad \lim_{z \to x} \bar{\nabla} S|_x^z = \nabla S(x). \tag{23}$$

Definition 5 properly states that the discrete gradient function satisfying (23) describes the rate of change of this function between two states. It is not uniquely defined and different methods to solve the equality can be worked out [11, 24, 12]. Through component-wise integration, one gets the computable expression below

$$\bar{\nabla} S|_x^z = \left[\bar{\nabla}_1 S|_{x_1}^{z_1} \ \dots \ \bar{\nabla}_n S|_{x_n}^{z_n}\right]^\top$$

with

$$\bar{\nabla}_i S|_{x_i}^{z_i} = \frac{1}{z_i - x_i} \int_{x_i}^{z_i} \frac{\partial S(x_1, ..., x_{i-1}, s, z_{i+1}, ..., z_n)}{\partial s} \, \mathrm{d}s.$$

Remark 7. When $S(x) = \frac{1}{2} x^\top P x$ with $P \in \mathbb{R}^{n \times n}$, the discrete gradient is uniquely expressed as

$$\bar{\nabla} S|_x^z = \frac{1}{2} P(x + z). \tag{24}$$

By definition of the discrete-gradient function, the EBE in Proposition 2 rewrites

$$\begin{aligned} S(x^+(u)) - S(x) &= \big(S(x^+(0)) - S(x)\big) + \big(S(x^+(u)) - S(x^+(0))\big) \\ &= \underbrace{S(x^+(0)) - S(x)}_{= F_0^\top(x) \bar{\nabla} S|_x^{x^+(0)}} + \underbrace{\int_0^u \mathrm{L}_{G(\cdot,s)} S(x^+(s)) \mathrm{d}s}_{= u g^\top(x,u) \bar{\nabla} S|_{x^+(0)}^{x^+(u)}}. \end{aligned}$$

It is instrumental to describe the passivating output defined in (22) in terms of the discrete gradient function. One gets.

Lemma 3 (Average passivating output in discrete gradient form). *Given the dynamics* (1) *and a real-valued smooth function map $S : \mathbb{R}^n \to \mathbb{R}$, the following equalities hold*

$$\begin{aligned} S(x^+(0)) - S(x) &= S(x + F_0(x)) - S(x) = F_0^\top(x) \bar{\nabla} S|_x^{x^+(0)} \\ S(x^+(u)) - S(x^+(0)) &= S(F(x,u)) - S(x + F_0(x)) = u g^\top(x,u) \bar{\nabla} S|_{x^+(0)}^{x^+(u)}. \end{aligned}$$

with the relation

$$u g^\top(x,u) \bar{\nabla} S|_{x^+(0)}^{x^+(u)} = \int_0^u \mathrm{L}_{G(\cdot,s)} S(x^+(s)) \mathrm{d}s = u \Big(\mathrm{L}_{G(\cdot,u)} S\Big)^{av}(x,u). \tag{25}$$

Example 1. The discrete integrator

$$x^+(u) = x + u; \quad y = h(x) = x$$

is the simplest storage element. Setting $S(x) = \frac{1}{2}x^2$, as storage function, the system is average passive as

$$S(x^+(u)) - S(x) = xu + \frac{1}{2}u^2 = \int_0^u x^+(v)dv = \int_0^u (x+s)ds = uh^{av}(x,u)$$

with $h^{av}(x,u) = x + \frac{1}{2}u$. Accordingly, one conclude passivity of the input-state-output system

$$x^+(u) = x + u; \quad h^{av}(x,u) = x + \frac{1}{2}u.$$

The associated negative average output feedback satisfies the algebraic equality $u = -\kappa(x + \frac{1}{2}u)$ so computing $u = -\frac{\kappa}{1+\frac{\kappa}{2}}x; \kappa > 0$ that recovers negative output feedback with suitably shaped gain.

3.4 Concluding comments

In this section, the notion of average passivity is introduced and is shown to be qualifying for the design of discrete-time average passivity based stabilizing strategies through damping (Theorem 3) or interconnection (Theorem 4). A passivating output and its related negative average output feedback are described in Proposition 2, so enlarging the control objectives to stabilization to target equilibria. This method of passivation can be further exploited to describe cascade stabilizing procedures for triangular state dynamics through backstepping or feedforward strategies in a discrete-time context, as in [21]. Further on, the second generation of average passivity based control including an energy-shaping component to shape the energy of the system and fulfil required control specifications can be developed. Such extensions include Interconnection and Damping Assignment (IDA) techniques that modify the internal port-Hamiltonian structure to assign a new equilibrium, or Control by Interconnection (CbI) techniques that manage energy exchanges through an interconnection pattern. Preliminary works in this direction are in [39, 40, 41, 22]. Finally, Lemma 3, that rewrites the average passivating output in its discrete gradient form, directly inspires the novel port-control Hamiltonian structure we propose in the next section.

4 Port-Hamiltonian structures in discrete time

Port-Hamiltonian structures have a pervasive impact in numerous applied domains enlarging the more traditional mechanical one. These structures are more essentially described in the continuous-time domain while in discrete time, a consensus on a specific structure is not reached in spite of a rich literature. In this section, a novel description of port-Hamiltonian structures is proposed exploiting the DDR form and the average passivating output map introduced in Proposition 2.

4.1 Control-free port-Hamiltonian dynamics

From [39, 38, 43], we first recall the definition of a control-free port-Hamiltonian structure. A unified definition that mimics the continuous-time structure exists in the discrete-time framework, just replacing the gradient function with the discrete gradient function. Let $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, be a smooth real-valued function that denotes the Hamiltonian function.

Definition 6. *A control-free discrete-time port-Hamiltonian dynamics over $\mathbb{R}^n$ can be described by the first-order difference equation*

$$x^+ - x = (J(x) - R(x))\bar{\nabla} H|_x^{x^+} \tag{26}$$

where $J(x) = -J^\top(x)$, $R(x) = R^\top(x) \succeq 0$, are matrices of functions representing the interconnection and resistive parts respectively.

By construction, one immediately verifies that:

• any local extremum of $H(x)$ ($\bar{\nabla} H|_{x_e}^{x_e} = \nabla H(x_e) = 0$), is an equilibrium;
• the rate of change of the Hamiltonian along the dynamics satisfies the equality

$$H(x^+) - H(x) = \underbrace{-\bar{\nabla} H^\top|_x^{x^+} J(x) \bar{\nabla} H|_x^{x^+}}_{=0} \underbrace{-\bar{\nabla} H^\top|_x^{x^+} R(x) \bar{\nabla} H|_x^{x^+}}_{\leq 0}$$

by skew symmetry of $J(x)$ and semi-positiveness of $R(x)$. Taking the sum of these increments, energy dissipation from time 0 to time k is described by the equality

$$H(x_k) - H(x_0) = \underbrace{-\sum_{i=0}^{k-1} \bar{\nabla} H^\top|_{x_i}^{x_i^+} R(x_i) \bar{\nabla} H|_{x_i}^{x_i^+}}_{\text{dissipated energy} \leq 0}.$$

• When $J(x) = 0$, the dynamics is dissipative. The simplest example is the gradient dynamics defined with $R(x) = I$ and $J(x) = 0$ that is

$$x^+ - x = -\bar{\nabla} H|_x^{x^+}$$

satisfying

$$H(x^+) - H(x) = -\bar{\nabla} H^\top|_x^{x^+} \bar{\nabla} H|_x^{x^+} = -||\bar{\nabla} H|_x^{x^+}||_2 \leq 0.$$

• When $R(x) = 0$, the dynamics is conservative

$$H(x^+) = H(x)$$

so concluding that the Hamiltonian is a constant of motion for (26).

4.2 Canonical discrete Hamiltonian dynamics

Let us illustrate the proposed definition on a peculiar class. Canonical Hamiltonian dynamics are defined over $\mathbb{R}^{2n}$, when setting as skew symmetric interconnection matrix

$$J_c = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

Setting $x = (x^1, x^2)^\top$, $x^i \in \mathbb{R}^n$ for $i = 1, 2$ and $\bar{\nabla} H|_x^{x^+} = col(\bar{\nabla}_1 H|_{x^1}^{x^{1+}}, \bar{\nabla}_2 H|_{x^2}^{x^{2+}})$, the canonical discrete Hamiltonian vector field associated to a given H, denoted by $\bar{X}_H$, satisfies

$$\begin{pmatrix} x^{1+} - x^1 \\ x^{2+} - x^2 \end{pmatrix} = J_c \bar{\nabla} H|_x^{x^+} = \begin{pmatrix} \bar{\nabla}_2 H|_{x^2}^{x^{2+}} \\ -\bar{\nabla}_1 H|_{x^1}^{x^{1+}} \end{pmatrix} = \bar{X}_H \tag{27}$$

For completeness, we note that for a given Hamiltonian function over R^{2n}, the canonical Hamiltonian dynamics is solution for all v of the condition

$$\Omega(x^+ - x, v) = \left\langle \bar{\nabla} H|_x^{x^+}, v \right\rangle$$

where $\Omega(u, v) =< u, J_c v >$ denotes the usual symplectic form. In fact, easy computations show that this equality rewritten as

$$< (x^+ - x), J_c v >= \left\langle \bar{\nabla} H(x), v \right\rangle$$

is solved by $x^+ - x = \bar{X}_H$ defined in (27).

Further on, for any given real-valued smooth function $C : \mathbb{R}^{2n} \to \mathbb{R}$, its rate of change along the Hamiltonian dynamics $\bar{X}_H$ is given by

$$C(x^+) - C(x) = \{C, H\}_D$$

where $\{C, H\}_D$ indicates the discrete Poisson bracket interestingly defined as the usual Poisson bracket but with respect to the discrete gradient; i.e.

$$\{C, H\}_D := \sum_{i=1}^{n} \bar{\nabla}_{1i} C|_{x^{1i}}^{x^{1i+}} \bar{\nabla}_{2i} H|_{x^{2i}}^{x^{2i+}} - \bar{\nabla}_{2i} C|_{x^{2i}}^{x^{2i+}} \bar{\nabla}_{1i} H|_{x^{1i}}^{x^{1i+}}.$$

Canonical discrete Hamilton's equations can be alternately written as

$$C(x^+) - C(x) = \{C, H\}_D, \quad \forall C : \mathbb{R}^{2n} \to \mathbb{R}.$$

In our formalism, any function C satisfying $\{C, H\}_D = 0$ is referred to as a discrete integral or constant of the motion with respect to the discrete dynamics generated by $\bar{X}_H$.

4.3 Port-controlled Hamiltonian structures

A novel description of port-controlled Hamiltonian structures exploiting the DDR form and the passivating average output map defined in Proposition 2 can now be proposed. This form is further validated by the Energy Balance Equation that it satisfies and its relation with feedback strategies and power-preserving interconnection. The definition below is recalled from [39, 38].

Definition 7 (Port-controlled Hamiltonian system (pH)). *Given a smooth real-valued function $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a discrete-time port-Hamiltonian system Σ_d^H over $\mathbb{R}^n$ can be described according to the input-state-output form below*

$$x^+(0) = x + (J(x) - R(x))\bar{\nabla} H|_x^{x^+(0)} \tag{28a}$$

$$\frac{dx^+(u)}{du} = G(x^+(u), u) \tag{28b}$$

$$Y(x, u) = L_{G(\cdot,u)} H(x) \tag{28c}$$

$J(x) = -J^\top(x)$, $R(x) = R^\top(x) \succeq 0$, are matrices of functions representing the interconnection and the resistive parts.

Accordingly, the following result holds.

Theorem 5. *Given a discrete-time port-Hamiltonian system of the form (28), then the following holds:*
- *any local extremum of $H(x)$ is an equilibrium;*
- *the rate of change of the Hamiltonian along the dynamics satisfies*

$$H(x^+(u)) - H(x) = \underbrace{-(\bar{\nabla} H|_x^{x^+(0)})^\top R(x) \bar{\nabla} H|_x^{x^+(0)}}_{\leq 0} + \underbrace{\int_0^u \mathrm{L}_{G(\cdot,s)} H(x^+(s))\, \mathrm{d}s}_{= u Y^{av}(x,u)} \tag{29}$$

so concluding average passivity with respect to the output map (28c).

Some comments are in order.

We note that the so defined output (28c), that corresponds to the passivating output introduced in Proposition 2 when substituting the storage function S with the Hamiltonian function H, qualifies as conjugate output: the product $uY^{av}(x, u)$ describes the energy brought to the system through the external input and output variables between two successive time steps.

Taking the sum of each increment (29) from time 0 to k, one gets the Energy Balance Equality in a form that perfectly splits in the total stored energy, the internally dissipated energy from the one supplied by the input/output variables:

$$\underbrace{H(x_k) - H(x_0)}_{\text{stored energy}} = \underbrace{-\sum_{i=0}^{k-1} \bar{\nabla} H^\top|_{x_i}^{x_i^+(0)} R(x_i) \bar{\nabla} H|_{x_i}^{x_i^+(0)}}_{\text{dissipated energy}} + \underbrace{\sum_{i=0}^{k-1} u_i Y^{av}(x_i, u_i)}_{\text{supplied energy}}.$$

As an alternative to the DDR form of Σ_d^H in (28), integration with respect to u transforms the port-Hamiltonian system into its map form. Adopting the discrete gradient form representation of the average output described in Lemma 3, one gets equivalently to (28) the port-Hamiltonian structure in map form. The following proposition specifies the equivalence between these two forms.

Proposition 3 (Port-Hamiltonian systems in map form). *The pH structure* (28) *can be equivalently rewritten in map form as*

$$x^+(u) = x + (J(x) - R(x))\bar{\nabla} H|_x^{x^+(0)} + ug(x,u) \tag{30a}$$

$$y(x,u) = g^\top(x,u)\bar{\nabla} H|_{x^+(0)}^{x^+(u)} \tag{30b}$$

where by definition and from Lemma 3

$$ug(x,u) := \int_0^u G(x^+(s),s)\mathrm{d}s; \qquad y(x,u) = Y^{av}(x,u) = \frac{H(x^+(u)) - H(x^+(0))}{u}.$$

Remark 8. The average representation of the conjugate output is instrumental to describe its series expansion in power of u that gives for the first terms

$$y(\cdot,u) = Y^{av}(x,u) = \mathrm{L}_{G_1} H|_{x^+} + \frac{u}{2}\left(\mathrm{L}_{G_1}^2 + \mathrm{L}_{G_2}\right) H|_{x^+} + O(u^2) \tag{31}$$

where $O(u^2)$ contains all the remaining terms of higher order in the control variable u. Further details regarding the complete series expansion and its iterative computation are in [34, 38].

It is interesting to highlight a matrix representation of the port-Hamiltonian structure we propose as a preamble to describe the associated Dirac structure. Easy computations show that the equations (28) satisfy

Lemma 4. *Easy computations show that the equations (28) satisfy*

$$\begin{pmatrix} x^+(0) - x \\ dx^+(u) \\ -Y(x^+(u),u) \end{pmatrix} = \begin{pmatrix} J(x) - R(x) & 0 & 0 \\ 0 & 0 & G(x^+(u),u) \\ 0 & -G^\top(x^+(u),u) & 0 \end{pmatrix} \begin{pmatrix} \bar{\nabla} H|_x^{x^+(0)} \\ \nabla H(x^+(u)) \\ du \end{pmatrix}$$

that underlines the hybrid representation that couples a one step ahead difference regarding the free evolution with a differential form with respect to u-variation.

4.4 Average PBC strategies for port-Hamiltonian systems

Port-controlled Hamiltonian systems represent a common class of average passive systems over which PBC strategies can be applied. Specifying the result in Theorem 3, one gets.

Theorem 6. *Let the port-Hamiltonian system Σ_d^H described in* (28) *(equivalently in (30)) be ZSD with H having a minimum in $x_\star$. Then, the feedback $\alpha(x)$, solution of the implicit damping equality*

$$\alpha(x) = -\kappa Y^{av}(x, \alpha(x)) = -\frac{\kappa}{\alpha(x)} \int_0^{\alpha(x)} \mathrm{L}_{G(\cdot,s)} H(x^+(s)) \mathrm{d}s \tag{32}$$

equivalently rewritten in terms of the discrete gradient function as

$$\alpha(x) = -\kappa g_\alpha^\top(x) \bar{\nabla} H|_{x^+}^{x^+(\alpha(x))}. \tag{33}$$

with $g_\alpha(x) := g(x, \alpha(x))$ achieves asymptotic stabilization of the equilibrium $x_\star$.

Accordingly the closed loop dynamics rewrites

$$x^+(\alpha(x)) = x + (J(x) - R(x)) \bar{\nabla} H|_x^{x^+} - \kappa g_\alpha(x) g_\alpha^\top(x) \bar{\nabla} H|_{x^+}^{x^+(\alpha(x))}. \tag{34}$$

Setting now $u(x, v) = \alpha(x) + v$ and $x_\alpha^+(v) = x^+(\alpha(x) + v)$, the closed loop port-Hamiltonian structure can be represented in matrix form over $\mathbb{R}^{3n+1}$ as

$$\begin{pmatrix} x^+ - x \\ x^+(\alpha(x)) - x^+ \\ \mathrm{d}x_\alpha^+(v) \\ -Y_\alpha(x_\alpha^+(v), v) \end{pmatrix}$$

$$= \begin{pmatrix} J(x) - R(x) & 0 & 0 & 0 \\ 0 & -\kappa g_\alpha(x) g_\alpha^\top(x) & 0 & 0 \\ 0 & 0 & 0 & G_\alpha(x_\alpha^+(v), v) \\ 0 & 0 & -G_\alpha^\top(x_\alpha^+(v), v) & 0 \end{pmatrix} \begin{pmatrix} \bar{\nabla} H|_x^{x^+(0)} \\ \bar{\nabla} H|_{x^+(0)}^{x^+(\alpha(x))} \\ \nabla H(x_\alpha^+(v)) \\ \mathrm{d}v \end{pmatrix}$$

with symmetric matrix

$$\begin{pmatrix} R(x) & 0 \\ 0 & \kappa g_\alpha(x) g_\alpha^\top(x) \end{pmatrix} \succeq 0$$

expressing the modified closed-loop dissipation matrix. The conjugate output $Y_\alpha^{av}(x, v)$, computed as the average from $\alpha(x)$ of the output $Y(x, u) = \mathrm{L}_{G(\cdot,u)} H(x)$ can be rewritten in discrete gradient form as

$$Y_\alpha^{av}(x, v) = \frac{1}{v} \int_0^v \mathrm{L}_{G_\alpha(\cdot,s)} H(x_\alpha^+(s)) \mathrm{d}s = \tilde{g}^\top(x, v) \bar{\nabla} H|_{x_\alpha^+}^{x_\alpha^+(v)}.$$

with

$$\begin{aligned} v\tilde{g}(x, v) &= \int_0^v G(x^+(\alpha(x) + s), \alpha(x) + s) ds = \int_0^v G_\alpha(x_\alpha^+(s), s) \mathrm{d}s \\ &= \alpha(x) g(x, \alpha(x) + v) - \alpha(x) g(x, \alpha(x)) + v g(x, \alpha(x) + v). \end{aligned}$$

Theorem 6 is the first step towards a variety of stabilizing techniques. Specifying the result in Theorem 4 to port-Hamiltonian structures, one gets preservation of the port-Hamiltonian structure under power-preserving interconnection; a qualifying property to discuss energy management based control schemes. Examples in this direction are developed in [41, 43, 42, 22, 23, 38].

Remark 9. For completeness, we report the port-controlled Hamiltonian structure usually proposed in the literature [16, 2, 60]. One sets

$$x^+(u) = x + (J(x) - R(x))\bar{\nabla}H|_x^{x^+(u)} + ug_{lit}(x,u) \tag{35a}$$

$$h_{lit}(x,u) = g_{lit}^\top(x,u)\bar{\nabla}H|_x^{x^+(u)} \tag{35b}$$

where, with respect to the form we propose, the discrete gradient of H from x to $x^+(0)$ is substituted with the discrete gradient of H from x to $x^+(u)$ plus an additive controlled part $ug_{lit}(x,u)$. It results that the rate of change of H between two successive time instants rewrites as

$$H(x^+(u)) - H(x) = \underbrace{-(\bar{\nabla}H|_x^{x^+(u)})^\top R(x)\bar{\nabla}H|_x^{x^+(u)}}_{\leq 0} + ug_{lit}^\top(x,u)\bar{\nabla}H|_x^{x^+(u)} \tag{36}$$

so naturally concluding passivity with respect to the output map $h_{lit}(x,u)$. However, the fact that the resistive part depends on the input variable in an unpredictable way through the term $\bar{\nabla}H|_x^{x^+(u)}$ may represent an obstacle to managing damping or energy exchanges under feedback. As a consequence in (36), the inner product $ug_{lit}^\top(x,u)\bar{\nabla}H|_x^{x^+(u)}$, does not contain the total power supplied to the system because the resistive part has a control dependent element too.

Example 2. Specifying the state equations (28) (equivalently (30)) to a linear dynamics with quadratic Hamiltonian function $H(x) = \frac{1}{2}x^\top Px$ and symmetric positive matrix P, a linear Port Hamiltonian structure can be be described as

$$x^+(0) = x + (J-R)\frac{P}{2}(x + x^+(0); \quad \frac{dx^+(u)}{du} = B; \quad Y(x) = B^\top Px \tag{37}$$

or equivalently in map form as

$$x^+(u) = x + (J-R)\frac{P}{2}(x + x^+(0)) + Bu; \quad Y^{av}(x,u) = B^\top\frac{P}{2}(x^+(0) + x^+(u)) \tag{38}$$

with matrices of appropriate dimensions and constant elements. Because the discrete gradient function can be explicitly expressed as a function of x and $x^+(0)$, the state equations (37) (equivalently (38)) can be rewritten in their explicit form so getting a linear dynamics with an output map admitting a feedthrough term as it is required to encompass a passivity property. One gets

$$x^+(u) = A_H x + Bu; \quad y(x,u) = B^\top PA_H x + \frac{B^\top PB}{2}u \tag{39}$$

with

$$A_H = \left(I - \tfrac{(J-R)P}{2}\right)^{-1}\left(I + \tfrac{(J-R)P}{2}\right).$$

Easy computations show that the output $y(x,u)$ is exactly the average output associated with $Y(x) = B^\top Px$ since by definition

$$\begin{aligned}\left(B^\top Px\right)^{av}(x,u) &= \frac{1}{u}\int_0^u B^\top Px^+(s)ds \\ &= \frac{1}{u}\int_0^u B^\top P(A_Hx + Bs)ds = B^\top PA_Hx + \frac{B^\top PB}{2}u.\end{aligned}$$

It is also possible to raise the question: when and how a passive system satisfies a pH structure? In the linear case, the question can be answered.

Proposition 4. *Consider the average passive linear system*

$$x^+(u) = Ax + Bu; \qquad Y(x) = B^\top Px$$

with positive definite storage $\frac{1}{2}x^\top Px$, then it can be rewritten in the port-Hamiltonian form (37) (equivalently (38)) according to the decomposition in skew-symmetric and symmetric part as follows

$$2(A-I)(I+A)^{-1}P^{-1} = J - R; \quad J = -J^\top; \quad R = R^\top \geq 0.$$

5 Concluding comments

In the proposed differential algebraic framework, discrete-time Port-controlled Hamiltonian structures that validate the usually required energy balance properties are described in the proposed differential algebraic framework. Accordingly, the basic stabilizing techniques behind energy-based control strategies are revisited to confirm the open perspectives regarding control design through energy management along the lines developed in [41, 22, 23, 38]. All the material discussed in this paper regards a purely discrete time setting but it can be specified to the sampled-data context. How port-Hamiltonian structures are transformed under sampling when assuming both the measurements and control variables available at discrete time instants is thus a natural and challenging question addressed in [30, 35, 37]. In that digital framework, it comes out that the sampled-data dynamics are necessarily parameterized by the sampling period as well as the control solutions that are described around the continuous-time ones by infinite series expansions. As a result, the solutions can be computed through an iterative procedure and approximated at any order so rendering the approach constructive in a digital environment. Finally, we stress that in our opinion the proposed approach sets an unifying framework to investigate controlled finite-dimensional dynamical systems in discrete-time as well as under sampling.

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