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Decomposition of State Spaces into Subobjects in Quantum Field Theory

Pierre Gosselin*

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Abstract

This paper introduces a comprehensive formalism for decomposing the state space of a quantum field into several entangled subobjects, i.e., fields generating a subspace of states. Projecting some of the subobjects onto degenerate background states reduces the system to an effective field theory depending on parameters representing the degeneracies. Notably, these parameters are not exogenous. The entanglement among subobjects in the initial system manifests as an interrelation between parameters and non-projected subobjects. Untangling this dependency necessitates imposing linear first-order equations on the effective field. The geometric characteristics of the parameter spaces depend on both the effective field and the background of the projected subobjects. The system, governed by arbitrary variables, has no dynamics, but the projection of some subobjects can be interpreted as slicing the original state space according to the lowest eigenvalues of a parameter-dependent family of operators. The slices can be endowed with amplitudes similar to some transitions between each other, contingent upon these eigenvalues. Averaging over all possible transitions shows that the amplitudes are higher for maps with increased eigenvalue than for maps with decreasing eigenvalue.

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1 Introduction

Composite objects in quantum field theory are typically treated as combinations of various states or fields through perturbative computations or non-perturbative methods such as operator product expansion. This work takes a different approach, starting with an arbitrary field theory and considering states that decompose into constrained substates, built from subfields referred to as subobjects throughout this paper. These subobjects define a field theory themselves, but due to the constraints arising from the decomposition, these fields become entangled with each other. Thus, the decomposition describes interacting tensor products of fields defined by subobjects.

For each decomposition, we consider the projection of the states of one or several subobjects onto some background or some operator eigenspace. Doing so, the projected subobject keeps track of some degrees of freedom of the field we started with and of the interactions with the remaining subobjects. The projected subobjects constitute a background on which the remaining subobjects

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are defined. The degeneracy of the background state induces the emergence of effective parameter-dependent fields, reducing the initial field theory to an effective field theory depending on these parameters. These parameters are exogenous in the first approximation, and this effective field theory may be considered to be defined on an a priori parameter space.

However, this parameter space keeps track of the characteristics of the initial system and the projected subobject as well as their interactions with remaining subobjects. Due to the entanglement of the subobjects, the remaining field's degrees of freedom globally condition the geometry of the parameters. Imposing independence to the first order between the effective field variation and parameters leads to some field equations. Both the remaining degrees of freedom and the projected background determine the metric of the parameter space. This dependency in the projected background is the trace of the states from which the parameter space has emerged.

Alternatively, this approach can be described in terms of operators and states formalism. The intertwined objects can be projected onto some operator's lowest eigenspace with degeneracy. This projection operator acts on the state space of one or several subobjects, so that the projected states are similar to the subobject background previously described.

Due to the subobjects' entanglement, the projection operators along with their eigenvalues, on which we project onto, depend on the remaining subobject degrees of freedom. The resulting state space is decomposed into "slices," each consisting of states of the effective field theory such that the lowest eigenvalue depending on these states has a given value. These slices are not orthogonal with each other, as the state-dependency of the eigenvalue implies that two different states correspond to different projection operators: the eigenvalues are not the eigenvalues of a single operator, but all of them are the minimum eigenvalues of a state-dependent set of operators.

These projections result in describing the effective system in terms of states and operators depending on some degeneracy parameters, one of these parameters being distinguished from others. The field equations resulting from the independence between field variations and the parameter space are recovered. This approach allows computing amplitudes between states with different distinguished eigenvalues: for each eigenvalue, we consider the corresponding slice. Then, we can compute transitions between slices. Indeed, the states for a given eigenvalue generate a state subspace, and we can define a map between these subspaces. They are defined by summing infinitesimal transitions maps between close slices. The form of the transitions is similar to some path integrals between slices. These maps and the corresponding amplitudes depend on the states and on the background constituted by the projected subobjects. The geometry underlying the transition is local, since the projection defining the effective states varies with the states themselves. It implies that the transition between states includes both the projected subobjects and the non projected subobject. That is, the apparent geometry underlying transitions depends on the states but also on some apparently inert background.

The field theory presented here is defined on an abstract parameter space, and no dynamics occur. However, after projection on the background, the set of lowest eigenvalues defining the slices in the state space along with their amplitudes, allows defining an analog of such dynamics. Considering the averaged transitions between slices, we can show that under some conditions about the number of states in each slice, the amplitudes are weighted for transitions corresponding to an increase in eigenvalues. This asymmetry results from the characteristics of the slices. The number of transition maps between these spaces increases with the eigenvalue, which induces a bias of transition amplitude towards an increase in this variable.

This work is organized into four parts. The first part presents the formalism for the particular case of a state space defined by functionals of one field. Section 2 presents the initial field theory, the full space of states, and some notations. The states are functionals of a field defined on some

parameter spaces, including some constraints on these parameters. We consider independently the tensor products of the field arising in the states. The states are thus functional of the field tensor power with some constraints on the parameters.

In section 3, the state space is decomposed into two particular subobjects, i.e., entangled fields together with their state spaces. We present the resulting decomposition of the initial field theory. Section 4 derives the projection on the background of one of the subobjects. It results in an effective field theory for the non projected subobject. The degeneracy of the background translates into an effective field depending on parameters describing the symmetries of the background. We describe the effective projected field theory. The remaining subobject absorbs these parameters and becomes a field theory defined on this parameter space. However, due to the constraints between initial subobjects, these parameters are not globally independent from the field.

Then, section 5 studies the consequences of this decomposition. Dependency of parameters in the field translates into joint variations of these two variables. We show that these variables can be considered as independent if the field satisfies some first-order equations. These equations involve the field over the entire parameter space. Averaging over this space yields equations for one value of the parameters, similar to some local equations.

Section 6 focuses on the constraints of the parameter space and the geometry of this space. The mutual dependency between parameters and field translates into a description similar to some set of metric spaces depending functionally on the field and the projected background, which implies that the field dependence of metric depends on some inert quantity.

In section 7, we develop an equivalent approach. Rather than considering solutions to saddle point equations for one of the subobjects, we rather project the space of states onto the minimal eigenstates of an operator acting on this subobject. Due to the constraint between the two subobjects, the eigenvalues and the projected state space depend on the states for the remaining object. This approach allows defining subspaces of effective states, depending on the eigenvalues of the operator. For each eigenvalue, we consider the subspace and parameter values corresponding to this eigenvalue. In section 8, we define the transitions between such subspaces, or slices. Integrating over infinitesimal variation of slices, corresponding to a variation in eigenvalues, yields a path integral formulation for transitions between subspaces of states. These transitions are also defined for operators.

Section 9 builds on this approach to define average transition between slices in the state space. We show that for a given slice, the average amplitudes of transitions towards other slices arise with an increase in eigenvalues.

The second part develops a general formalism for decomposing the state space of a field into several subobjects. Section 10 presents the setup and the main elements. Defining formally a subobject independently from any decomposition, as a composite of states, we consider the sequences of inclusion between subobjects, that is, composed subobjects. In section 11, we introduce all possible decompositions of the initial field and state space. We include the possibility that in a given decomposition, any subobject itself decomposes in subobjects of the subobjects defining the decomposition. The formalism results in describing the system as sets of maps between various constrained parameter spaces. These maps describe sequences of inclusions between subobjects.

Sections 12, 13, 14 and 15 focus on the case of a projection of all subobjects except one. We recover the description of the first part: the effective field, the field equations corresponding to independent parameters, and the geometric aspects for the constraints. In section 16, we also study the transformation properties of the field under reparametrization. In section 17, we consider the general case, where an arbitrary number of subobjects remain unprojected. This allows presenting the emergence of several fields and parameter spaces, arising from an arbitrary initial formalism. Section 18 presents the alternative method of projecting over some eigenspace of an operator and

derives the transitions in the general formalism.

The third part presents an alternative and less general formalism that is closer to the usual formalism of field theory. We also consider decomposition of states into several types of objects, but the formalism starts directly from states in section 19 and with operators in section 20. We recover the main characteristics of the formalism developed in the two first parts. Transitions are considered in section 21. Due to our choice to start with states, these transitions are first derived in a context similar to a first quantized system and then reconsidered in a field second quantization context.

The fourth part concludes this work by outlining several potential developments. Section 22 interpret the preceding formalism as a field theory on a singular fibred space, with the dimension of the fiber contingent upon the points in the basis space. In section 23, we focus on the constraints that define the system. We examine their continuous variations as the state defining the system undergoes changes. These variations in the constraints manifest as differential equation in the state space, akin to some sort of dynamical equations. Section 24 considers discrete, i.e. non continuous, modifications of the constraints. Imposing some consistency conditions between different modifications imply some commutation relations for degeneracy operators.

Part I Decomposition of state space in two subobjects and resulting parameter space.

We present the decomposition of the functional space for a field into two subobjects. The realizations of the field decompose into sums of products of realizations, with each product constrained by relations between the parameters defining the subobjects. Projecting onto a degenerate background state for one of the subobjects provides an effective description of the resulting system. The degeneracy of the background state implies that the system can be described as an effective field theory for a field depending on certain parameters. These parameters are exogenous only at first glance. In reality, the initial entanglement between subobjects results in intertwining between the remaining subobjects and the parameters. The interrelation between the two initial subobjects implies that the parameter space itself is a dynamic object in the effective theory. Untangling the constraints between this parameter space and the effective field implies an analogue of dynamical equations for the effective field. The dynamical characteristic of the parameter space translates into a relation between the metric characteristics of the space and the effective field. These geometric characteristics also includes an inert part due to the projected background of the projected subobject.

2 General set up and fields functional description

We describe the system as space of functional of an arbitrary field Ψ . This field is defined over some parameter space U . To a field is associated its set of realizations, i.e. the infinite set of values this field can take. Each of these values is a function defined over U . To such a realization Ψ_α we associate usually a weight given by $\exp(iS(\Psi_\alpha))$ where S is a given action functional. In the sequel, since we consider decomposition of a field into other fields, called subobjects of the field, we will work at some points with the realizations of the field rather than with the field itself. For this reason, we will consider the tensor product of the field $\Psi^{\otimes m}$ as an independent field defined on U^m including some constraint on this set. This amounts to consider multiple states as themselves

as basic elements. Actually, a realization of the product $\Psi^{\otimes m}$ is a is an infinite sum of products $\Psi_{\alpha_1} \dots \Psi_{\alpha_m}$, an intricate relation between the realization of $\Psi^{\otimes m}$ and those of Ψ , which justifies our choice.

Moreover, considering a functional for Ψ as a sum of linear functional of $\Psi^{\otimes m}$, we will restrict the functionals in the $\Psi^{\otimes m}$ to be linear. Any product of functional of the $\Psi^{\otimes m}$ can be itself considered as a series of liner functionals of the $\Psi^{\otimes m'}$.

2.1 States basis:

We consider some parameters spaces U, U^k, \dots with U^k is given by k copies of U . We introduce implicit relations:

$$U/c(U), \dots, U^k/c(U^k)$$

where the $c(U^k)$ are constraints.

This is leading to states combinations of:

$$|u_i\rangle, \dots |u_{i_1}\rangle \otimes \dots \otimes |u_{i_k}\rangle, \dots$$

plus some implicit constraint:

$$\begin{aligned} & |u_{i_1}\rangle \otimes \dots \otimes |u_{i_k}\rangle / c(u_{i_1}, \dots, u_{i_k}) \\ \rightarrow & |u_{i_1, \dots, u_{i_k}}\rangle \end{aligned}$$

so that states are:

$$\begin{aligned} & \sum_{k, i_1 \dots i_k} |u_{i_1}\rangle \otimes \dots \otimes |u_{i_k}\rangle / c((u_{i_1}, \dots, u_{i_k})_k) \\ \rightarrow & \sum_{k, i_1 \dots i_k} |u_{i_1, \dots, u_{i_k}}\rangle \end{aligned}$$

2.2 States and functionals

If we fix a basis $|u_i\rangle$, general states in this set up have form:

$$\sum \otimes_s \Psi(u_{i_s}) |u_{i_s}\rangle = \sum_{k, i_1 \dots i_k} \prod_s \Psi(u_{i_s}) |u_{i_1}\rangle \otimes \dots \otimes |u_{i_k}\rangle / c((u_{i_1}, \dots, u_{i_k})_k)$$

They are considered as particular realization $|\Psi\rangle$ of a field Ψ :

$$|\Psi\rangle = \sum \Psi^{\otimes k}(u_{i_1}, \dots, u_{i_k}) |u_{i_1, \dots, u_{i_k}}\rangle$$

with $\Psi^{\otimes k}(u_{i_1}, \dots, u_{i_k})$ represents the "wave function" of the state in the basis $|u_{i_1}, \dots, u_{i_k}\rangle$.

and more generally, products write:

$$\sum \otimes_l \Psi^{\otimes k_l}(u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)}) |u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)}\rangle / c((u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)})_l) = \sum_{k, (i_1 \dots i_k)_l} \Psi((u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)})_l) |(u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)})_l\rangle$$

In a perspective of second quantized formalism, we consider functionals:

$$\begin{aligned} & \sum \left\langle u_{i_1}^{(l')}, \dots, u_{i_{k'}^{(l')}} \right| \prod \Psi^{\dagger \otimes k'}(u_{i_1}^{(l')}, \dots, u_{i_{k'}^{(l')}}) F\left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}^{(l')}}\right)_{l'}, (u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)})_l\right) \prod \Psi^{\otimes k_l}(u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)}) |u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)}\rangle \\ = & \sum \prod \Psi^{\dagger \otimes k'}(u_{i_1}^{(l')}, \dots, u_{i_{k'}^{(l')}}) F\left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}^{(l')}}\right)_{l'}, (u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)})_l\right) \prod \Psi^{\otimes k_l}(u_{i_1}^{(l)}, \dots, u_{i_{k_l}}^{(l)}) \end{aligned}$$

that are thus linear combinations of:

$$\Psi^{\dagger \otimes \sum k'_{l'}} \left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}}^{(l')} \right)_{l'} \right) F \left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}}^{(l')} \right)_{l'}, \left(u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)} \right)_l \right) \Psi^{\otimes \sum k_l} \left(\left(u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)} \right)_l \right)$$

This expression will be compacted as:

$$F \left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}}^{(l')} \right)_{l'}, \left(u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)} \right)_l \right) \Psi^{\otimes [\sum k_l + \sum k'_{l'}]} \left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}}^{(l')} \right)_{l'}, \left(u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)} \right)_l \right)$$

so that we will consider a single field:

$$\Psi^{\otimes [kl+k'l']} \left(\left(u_{i_1}^{(l')}, \dots, u_{i_{k'}}^{(l')} \right)_{l'}, \left(u_{i_1}^{(l)}, \dots, u_{i_k}^{(l)} \right)_l \right) \rightarrow \Psi^{\otimes k} (U^k)$$

so that U stands for a product $\underline{U} \times U$.

The fields $\Psi^{\otimes k} (U^k)$ will be dealt with as if they were independent quantity. Actually, up to the constraints, a realization $\Psi^{\otimes k+l} (U^{k+l})$ is not the product of two realizations of $\Psi^{\otimes k} (U^k)$ and $\Psi^{\otimes l} (U^l)$, but rather a sum of products of such realizations:

$$\sum_{\alpha} \Psi_{\alpha}^{\otimes k} (U^k) \Psi_{\alpha}^{\otimes l} (U^l) \quad (1)$$

In coordinates it means that a realization $\Psi^{\otimes k+l} (u_{i_1}, \dots, u_{i_{k+l}})$ can be decomposed as sum of products of realization:

$$\Psi^{\otimes k+l} (u_{i_1}, \dots, u_{i_{k+l}}) \rightarrow \sum_{\alpha} \Psi_{\alpha}^{\otimes k} (u_{i_1}, \dots, u_{i_k}) \Psi_{\alpha}^{\otimes l} (u_{i_{k+1}}, \dots, u_{i_{k+l}})$$

The constraints are included by identifying some parameters:

$$\Psi (U^l) = \Psi^{\otimes l} (U^l) / f$$

and the tensor product of realizations of $\Psi (U)$ is a particular realization of $\Psi (U^l)$. In coordinates:

$$\Psi_1 (u_{i_1}) \otimes \dots \otimes \Psi_k (u_{i_k}) / f ((u_{i_1}, \dots, u_{i_k}))$$

is a realization of $\Psi (U^l) = \Psi (u_{i_1}, \dots, u_{i_k})$ where implicitly $(u_{i_1}, \dots, u_{i_k})$ stands for $(u_{i_1}, \dots, u_{i_k}) / f ((u_{i_1}, \dots, u_{i_k}))$. All realizations of the field $\Psi (U^l)$ are given by linear combinations of products of realizations $\Psi (u_{i_1})$.

Using the all sequence $\{\Psi^{\otimes k} (U^k)\}$ as variables allow to restrict the functionals that define the states to linear combinations:

$$F (\Psi (U)) = \sum_k F_{lin} (\Psi^{\otimes k} (U^k))$$

2.3 Remark

1. This decomposition will be generalized later to several collections of sets $\{U_j\}_j$ to write the decomposition in fields.

2. Since a realization of:

$$\Psi^{\otimes k+l} (U^{k+l})$$

writes as a sum:

$$\sum_{\alpha} \Psi_{\alpha}^{\otimes k} (U^k) \Psi_{\alpha}^{\otimes l} (U^l)$$

the product of the realizations of two fields:

$$\Psi^{\otimes k}(U^k) \Psi^{\otimes l}(U^l)$$

is a particular realization of $\Psi^{\otimes k+l}(U^{k+l})$.

3. In terms of states, i.e. functional, decomposition (1) becomes sums of terms of the type:

$$\sum F_{k,lin}(\Psi^{\otimes k}(U^k)) F_{l,lin}(\Psi^{\otimes l}(U^l))$$

3 Decomposition in two types of fields

3.1 Decomposition and states

We consider the decomposition of the previous system for field $\Psi^{\otimes k}(U^k)$ into two subsystems or subobjects, each of them defined by a field written $\Psi_j^{\otimes l}(U_j^l)$ and $\Psi_i^{\otimes k}(U_i^k)$ respectively along with their associated state space. The states generated by these two fields, modulo some constraints entangling subobjects, span the entire states' space of the system. This description generalizes the tensor product decomposition (1).

Assume a system described by fields:

$$\oplus \Psi^{\otimes m}(U^m / f_m)$$

with some constraint f_m on U^m and $\mathcal{H}((U))$ the spaces of linear functionals of $\oplus \Psi^{\otimes m}(U^m / f_m)$. We consider that the parameters:

$$\sum U^m / f_m$$

decompose into two different families in parameters by the following map:

$$\sum U^m / f_m \xrightarrow{g} V \left(\sum_{l_j, k_i} (U_j^{l_j} / f_{l_j} \times U_i^{k_i} / f_{k_i}) / f_{l_j, k_i} \right)$$

where:

$$\mathbf{V} = V \left(\sum_{l_j, k_i} (U_j^{l_j} / f_{l_j} \times U_i^{k_i} / f_{k_i}) / f_{l_j, k_i} \right)$$

denotes the set of subvarieties of:

$$U_{ij} = \sum_{l_j, k_i} (U_j^{l_j} / f_{l_j} \times U_i^{k_i} / f_{k_i}) / f_{l_j, k_i}$$

with implicit constraints. The previous decomposition is not a decomposition into the powers of subsets of U . Due to the arbitrary form of the constraint, we assume that the full series may enter in the decomposition of a given U^m / f_m . Associating to an element of U^m / f_m subvarieties of U_{ij} is analog to the description of some bound state in field theory in which such state is described by an infinite series of products of states, involving an infinite number of variable, later integrated to produce a state, the infinite series coming from some perturbative expansion. We will write for each $V \subset \mathbf{V}$, we write the decomposition:

$$V = \cup V_{l_i k_j}$$

with:

$$V_{l_i k_j} = V \cap \left(U_j^{l_j} / f_{l_j} \times U_i^{k_i} / f_{k_i} \right) / f_{l_j, k_i}$$

Previous formula leads thus to assume at the field level the corresponding decomposition in subobjects:

$$\Psi^{\otimes m} (U^m / f_m) \rightarrow \sum_{V \subset g(U^m / f_m)} \sum_{V = \cup V_{i,j}} \int_{V_{i,j}} h_{(k,l)}^m (U^m, U_j^l, U_i^k, V_{i,j}) \Psi_J^{\otimes l} (U_j^l) \otimes_{k,l} \Psi_I^{\otimes k} (U_i^k) \delta (f_{lk} (U_j^l, U_i^k)) dU_j^l dU_i^k$$

That formula has to be read is in terms of realization. The realizations $\Psi^{\otimes m} (U^m / f_m)$ write as series over products of realizations of fields $\Psi_J (U_j^l)$, $\Psi_I (U_i^k)$. Including the constraints, the tensor products represent series expansions:

$$\begin{aligned} & \Psi_J (U_j^l) \otimes_{k,l} \Psi_I (U_i^k) \delta (f_{lk} (U_j^l, U_i^k)) \\ \equiv & \sum_{\alpha} \Psi_{J,\alpha} (U_j^l) \Psi_{I,\alpha} (U_i^k) \delta (f_{lk} (U_j^l, U_i^k)) \end{aligned}$$

similar to that arising in the tensor product (1).

At the level of states, the functionals:

$$\sum_m \int a_m (U^m) \Psi^{\otimes m} (U^m) \quad (2)$$

expand as:

$$\sum_m \int a_m (U^m) \sum_{V \subset g(U^m / f_m)} \sum_{V = \cup V_{i,j}} \int_{V_{i,j}} h_{(k,l)}^m (U^m, U_j^l, U_i^k, V_{i,j}) \Psi_J^{\otimes l} (U_j^l) \otimes_{k,l} \Psi_I^{\otimes k} (U_i^k) \delta (f_{lk} (U_j^l, U_i^k)) dU_j^l dU_i^k$$

The sum:

$$\sum_{V \subset g(U^m / f_m)} \sum_{V = \cup V_{i,j}} \int_{V_{i,j}} h_{(k,l)}^m (U^m, U_j^l, U_i^k, V_{i,j})$$

can be replaced by an unconstrained integral:

$$\int h_{(k,l)}^m (U^m, U_j^l, U_i^k, V_{i,j}) \delta (f (U^m, U_j^l, U_i^k))$$

where the δ functions:

$$\delta (f (U^m, U_j^l, U_i^k))$$

implement the condition $\cup V_{i,j} \subset g (U^m / f_m)$. The integral:

$$\sum_m \int a_m (U^m) h_{(k,l)}^m (U^m, U_j^l, U_i^k, V_{i,j}) \delta (f (U^m, U_j^l, U_i^k))$$

is a function $a_{l,k} (U_j^l, U_i^k)$ and the functional (2) writes:

$$\sum_{\alpha} \int a_{l,k} (U_j^l, U_i^k) \Psi_{J,\alpha}^{\otimes l} (U_j^l) \Psi_{I,\alpha}^{\otimes k} (U_i^k) \delta (f_{lk} (U_j^l, U_i^k)) dU_j^l dU_i^k$$

which leads to consider a decomposition of spaces of state:

$$\mathcal{H}((U)) \subset \mathcal{H}(\{(U_j), (U_i)\})$$

Given that the space $\mathcal{H}((U))$ is an infinite series of tensor products, we assume in the sequel that:

$$\mathcal{H}((U)) = \mathcal{H}(\{(U_j), (U_i)\})$$

Note that the decomposition presented above represents the reverse path compared to the one leading to the formation of a composed states. We decompose a given state into a series of products of functionals associated with different subobjects. In the next sections, we will work directly with the products:

$$\Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes \Psi_{I,\alpha}^{\otimes k}(U_i^k) \delta(f_{lk}(U_j^l, U_i^k))$$

3.2 Partial states

Usually, a state for one single field, say $\{\Psi_J^{\otimes l}\}_l$ is defined by some functional of this field:

$$\sum \int s(U_j^l) \Psi_J^{\otimes l}(U_j^l) dU_j^l \quad (3)$$

that is, by the set of functions $\{s(U_j^l)\}$.

However, in the present context, both fields arising in the decomposition are subject to some constraints $\{f_{lk}(U_j^l, U_i^k)\}$. Thus, we have to consider that a state for one field is subject to some constraints in its integration variables.

We begin by defining the evaluation functional at (U_j^l, U_i^k) for two realizations of the field $\Psi_J^{\otimes l}$ and $\Psi_I^{\otimes k}$:

$$ev_{U_j^l, U_i^k} \left(\Psi_J^{\otimes l} \otimes_{l,k} \Psi_I^{\otimes k} \right) \rightarrow \Psi_J^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_I^{\otimes k}(U_i^k)$$

and impose the constraints $\delta(f_{lk}(U_j^l, U_i^k))$ between these two subobjects:

$$ev_{U_j^l, U_i^k} \left(\Psi_J^{\otimes l} \otimes_{l,k} \Psi_I^{\otimes k} \right) \delta(f_{lk}(U_j^l, U_i^k)) = \Psi_J^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_I^{\otimes k}(U_i^k) \delta(f_{lk}(U_j^l, U_i^k)) \quad (4)$$

The definition of a partial state for $\Psi_J^{\otimes l}$ or $\Psi_I^{\otimes k}$ should thus respect the constraints. We define a state for $\Psi_J^{\otimes l}$ by a set of functions:

$$s \equiv \left\{ s_{U_i^k}(U_j^l) \right\}_{(U_j^l, U_i^k)}$$

and a functional of $\Psi_J^{\otimes l} \otimes_{l,k} \Psi_I^{\otimes k}$ defined by the combinations of evaluations (4):

$$\begin{aligned} & \sum \int s_{U_i^k}(U_j^l) ev_{U_j^l, U_i^k} \left(\Psi_J^{\otimes l} \otimes_{l,k} \Psi_I^{\otimes k} \right) \delta(f_{lk}(U_j^l, U_i^k)) dU_j^l \\ &= \sum \int s_{U_i^k}(U_j^l) \Psi_J^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_I^{\otimes k}(U_i^k) \delta(f_{lk}(U_j^l, U_i^k)) \delta(f_{lk}(U_j^l, U_i^k)) dU_j^l \end{aligned}$$

Note that we will alternatively write $s \equiv \left\{ s_{U_i^k}(\Psi_J^{\otimes l}) \right\}$ or $\left\{ s_{U_j^l}(\Psi_I^{\otimes k}) \right\}$ to define a partial functional of $\Psi_J^{\otimes l}$ or $\Psi_I^{\otimes k}$ respectively.

4 Projection over functional minima and effective field

Once the system has been decomposed into subobjects $\Psi_J^{\otimes l}(U_j^l)$ and $\Psi_I^{\otimes k}(U_i^k)$, we examine the projection of $\Psi_I^{\otimes k}(U_i^k)$ onto the space generated by the saddle-point solutions of a functional S , analog to the background states in a typical action functional. Given the interdependence of subobjects through constraints and the initial system's dependence on products such as $\Psi_J(U_j^l) \otimes_{k,l} \Psi_I(U_i^k)$, we posit that the functional whose saddle points determine the projection depends on both fields $\Psi_J^{\otimes l}(U_j^l)$ and $\Psi_I^{\otimes k}(U_i^k)$, i.e. on both sets of realizations $\Psi_{J,\alpha}^{\otimes l}(U_j^l)$ and $\Psi_{I,\alpha}^{\otimes k}(U_i^k)$. The projection that defines the states minimizing S will consequently depend on $\Psi_J(U_j^l) \otimes_{k,l} \Psi_I(U_i^k)$ and potentially on some functional v of $\Psi_{J,\alpha}^{\otimes l}(U_j^l)$ that characterizes a state for $\Psi_{J,\alpha}^{\otimes l}(U_j^l)$. We thus write $S\left(\Psi_J(U_j^l) \otimes_{k,l} \Psi_I(U_i^k), v\right)$ for this functional or $S(v)$ for short.

4.1 States and projections

We consider the projection of states onto a background of the subspace defined by field $\Psi_I^{\otimes k}(U_i^k)$. Additionally, we assume the existence of a basis of states v for $\Psi_J^{\otimes l}$, i.e. functionals of the fields $\Psi_J^{\otimes l}$ so that:

$$\sum_v \prod_v$$

represents the identity. As a consequence, the states are projected onto spaces:

$$\sum_v \prod_v \otimes \prod_{\min S(v)}$$

by projecting the $\Psi_I^{\otimes k}$ onto the minima of $S(v)$ which depend on $\Psi_J^{\otimes l}$ through states v . This reflects that the decomposition is carried out for entangled states, that is, interacting fields.

To write a functional in the basis $\{v\}$, recall that each element of this basis is defined by a collection: $\{v_{U_i^k}(\Psi_J^{\otimes l})\}$. Starting with an arbitrary functional respecting the conservation $\delta(f_{lk}(U_j^l, U_i^k))$:

$$\int g(U_j^l, U_i^k) \Psi_J^{\otimes l}(U_j^l) \Psi_I^{\otimes k}(U_i^k) \delta(f_{lk}(U_j^l, U_i^k)) \quad (5)$$

that can be rewritten in the basis $\{v\}$ as:

$$\sum_{v, v_{U_i^k}} \int g(v, U_i^k) v_{U_i^k}(\Psi_J^{\otimes l}) ev_{U_i^k}(\Psi_I^{\otimes k}(U_i^k)) \delta(f_{lk}(U_j^l, U_i^k)) \quad (6)$$

where:

$$ev_{U_i^k}(\Psi_I^{\otimes k}(U_i^k)) = \Psi_I^{\otimes k}(U_i^k)$$

is the evaluation functional for $\Psi_I^{\otimes k}$. The states $v_{U_i^k}(\Psi_J^{\otimes l})$ on which the projection arise depend on the constraint $\delta(f_{lk}(U_j^l, U_i^k))$.

In coordinates, we can solve the constraints and $v_{U_i^k}(\Psi_J^{\otimes l})$ writes:

$$v_{U_i^k}(\Psi_J^{\otimes l}) = \int v(U_j^l/f_{lk}, \{U_i^k\}) \Psi_J^{\otimes l}(U_j^l/f_{lk}, \{U_i^k\}) d(U_j^l/f_{lk})$$

and $v(U_j^l/f_{lk}, \{U_i^k\})$ is the functional density associated to $v_{U_i^k}(\Psi_J^{\otimes l})$.

Given the infinite number of realizations, i.e. components involved in the products, we consider the multiple functional and replace:

$$v_{U_i^k} (\Psi_J^{\otimes l}) e v_{U_i^k} (\Psi_I^{\otimes k}) \rightarrow v_{U_i^k} (\Psi_{J,\alpha}^{\otimes l}) e v_{U_i^k} (\Psi_{I,\alpha}^{\otimes k})$$

for each realization α .

Note that we can consider these functionals:

$$v_{\{U_i^k\}} (\{\Psi_J^{\otimes l}\})$$

as eigenstates of some operator:

$$\Phi \left(\Psi_J^{\otimes k_l}, \frac{\delta}{\delta \Psi_J^{\otimes k_l}}, U_i^k, \Psi_I^{\otimes k} \right)$$

that depend on the U_i^k and $\Psi_I^{\otimes k}$ through the constraint and some operators involving interactions between both subobjects.

4.1.1 Remark

We will assume in general that fields are already chosen as eigenstates of Λ so that the \sum_v is performed for a collection:

$$\sum_{U_j^l/f_{lk}} v_{\{U_i^k\}}^{U_j^l/f_{lk}} \{\Psi_J^{\otimes l}\}$$

$$v_{\{U_i^k\}}^{U_j^l/f_{lk}} \{\Psi_J^{\otimes l}\} = \Psi_J^{\otimes l}(U_j^l) = \Psi_J^{\otimes l}(U_j^l/f_{lk}, \{U_i^k\})$$

which corresponds to sum over the evaluation at points U_j^l/f_{lk} and the decomposition is performed with respect to the values of U_j^l/f_{kl} . We can also consider restrict the projection to a subspace V of U_j^l/f_{lk} :

$$v_{\{U_i^k\}}^V \{\Psi_J^{\otimes l}\} = \Psi_J^{\otimes l}(V, \{U_i^k\})$$

which reduces the evaluation to $(V, \{U_i^k\})$.

4.2 Action functional saddle point and projection

We assume that the projection of $\Psi_{I,\alpha}^{\otimes k}(U_i^k)$ comes from the minimization:

$$\exp \left(-S \left(\left\{ \Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha}^{\otimes k}(U_i^k) \right\}_\alpha, \left\{ v_{\{U_i^k\}} (\Psi_{J,\alpha}^{\otimes k}) \otimes \Psi_{I,\alpha}^{\otimes k} \right\}_{\{U_i^k\}} \right) \right)$$

This form of S emphasizes that the primary object is the initial field $\Psi^{\otimes m}(U^m/f_m)$. Multiple realizations of the fields are implicated, as a consequence of our assumption about decomposition, indicating that realizations of the field $\Psi^{\otimes m}(U^m/f_m)$ involve multiple realizations $\Psi_{J,\alpha}^{\otimes l}, \Psi_{I,\alpha}^{\otimes k}$. We have assumed the functional depends on some specific intertwined states $v_{\{U_i^k\}}$ on which the field $\Psi_{J,\alpha}^{\otimes k}$ is projected to model that the projection of $\Psi_{I,\alpha}^{\otimes k}$ depends on a state for $\Psi_{J,\alpha}^{\otimes l}$. This reminiscent of current-current interaction involving some functionals (usually local) of the fields.

In appendix 1, we decompose the solutions of the saddle point equation as:

$$\left\{ \Psi_J^{\otimes l}(U_j^l) \bigoplus_{\substack{l \leq m \\ l' \leq m'}} \Psi_{I,0}^{\otimes k}(U_i^k) \right\} \equiv \left\{ \Psi_{I,0}^{\otimes k}(U_i^k) \right\}_{\Psi_J^{\otimes l}(U_j^l)}$$

and write a series expansion for $\Psi_{I,0}^{\otimes k}(U_i^k)$. We show that:

$$\begin{aligned} & \Psi_{I,0,\alpha}^{\otimes k}(U_i^k, \{\Psi_J^{\otimes l}\}) \\ = & \sum_{s,l_i,\dots,l_s} \int d\left(\{U_j^{l_i}\}_{l_i}\right) d\left(\{U_i^{k_i}/f_{k_i l_i}\}_{k_i, l_i}\right) \mathcal{K}_0^v\left(U_i^k, \{U_j^{l_i}\}_{l_i}, \{U_i^{k_i}/f_{k_i l_i}\}_i\right) \Psi_{J,\alpha}^{\otimes \sum_{i \leq n} l_i}\left(\{U_j^{l_i}\}\right) \end{aligned}$$

where:

$$\Psi_{J,\alpha}^{\otimes \sum_{i \leq n} l_i}\left(\{U_j^{l_i}\}\right) = \sum_{\{\alpha'_i\}_{i \leq n}} \prod_i \Psi_{J,\alpha'_i}^{\otimes l_i}\left(U_j^{l_i}\right)$$

and:

$$\mathcal{K}_0^v\left(U_i^k, \{U_j^{l_i}\}_{l_i}, \{U_i^{k_i}/f_{k_i l_i}\}_i\right) = \mathcal{K}_0\left(U_i^k, \{U_j^{l_i}\}_{l_i}, \{U_i^{k_i}/f_{k_i l_i}\}_i\right) \prod v\left(U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\right)$$

The dependency in α is justified by the fact that an element of $\{\alpha'_i\}_{i \leq n}$ is α or arises in the action in products involving α .

4.3 Degeneracy of saddle points

So far the formulas have been obtained for one saddle point. However, considering the presence of multiple realizations of the field in the functional, we can expect some symmetry permuting or combining these realizations to arise. We will assume certain symmetry groups that imply the degeneracy of the saddle point.

4.3.1 General formula

Consider the action:

$$S\left(\left\{\Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha}^{\otimes k}(U_i^k)\right\}_\alpha, \left\{v_{\{U_i^k\}}\left(\Psi_{J,\alpha}^{\otimes k}\right) \otimes \Psi_{I,\alpha}^{\otimes k}\right\}_{\{U_i^k\}}\right)$$

depending on a sequence of realizations:

$$\left\{\Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha}^{\otimes k}(U_i^k)\right\}_\alpha$$

that can also be written:

$$\left\{\Psi_{J,\alpha_i}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha_i}^{\otimes k}(U_i^k)\right\}_{i=1,\dots,n\dots}$$

We look for the invariance of S as a function of a given sequence $\left\{\Psi_{J,\alpha_i}^{\otimes l}(U_j^l)\right\}_i$

We assume the existence of sequences of groups of transformations $G_{\alpha_1, \dots, \alpha_n}^{k_1, \dots, k_n}\left(\left\{\Psi_{J,\alpha_i}^{\otimes l}(U_j^l)\right\}_{i=1, \dots, n}\right)$ with $n \geq 1$, written G^{k_1, \dots, k_n} for short, such that there is an invariance of S by action of G_{k_1, \dots, k_n} on:

$$\left\{\Psi_{I,\alpha_i}^{\otimes k_i}(U_i^{k_i})\right\}_{i \leq n}$$

The transformations for a set of realizations are parametrized by $\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}}[\Psi_J, \nu]$ with $\{k_i \leq n\} = (k_1, \dots, k_n)$ and $\alpha_1, \dots, \alpha_n$ is a n -uplet of realizations. These parameters span for example the symmetry groups of a series of the form:

$$\begin{aligned} & S\left(\left(\Psi_{I,\alpha_i}^{\otimes k_i}(U_i^{k_i})\right)\right) \tag{7} \\ = & \sum_m \sum_{\{k_i \leq m\} \subset \{k_i \leq n\}} \int f\left(\left\{U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\right\}_i\right) \prod_i \Psi_{J,\alpha'_i}^{\otimes l_i}\left(U_j^{l_i}/f_{k_i l_i}, U_i^{k_i}\right) \prod_{i=1}^m \Psi_{I,\alpha_i}^{\otimes k_i}\left(U_i^{k_i}\right) d\left(\left\{U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\right\}_i\right) \end{aligned}$$

where the transformation acts on n realizations:

$$G^{k_1, \dots, k_n} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \cdot \left(\prod_{i=1 \dots n} \left\{ \Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i} \right) \right\}_{i \leq n} \right) = \left(\prod_{i=1 \dots n} R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \Psi_{I, \alpha_i}^{\otimes k_i} \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right) \right)$$

with $R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}}$ acting on indices $\alpha_1, \dots, \alpha_n$:

$$R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \Psi_{I, \alpha_i}^{\otimes k_i} \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right) = \sum_{\alpha'_i} \left[R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \right]_{\alpha_i}^{\alpha'_i} \Psi_{I, \alpha'_i}^{\otimes k_i} \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right)$$

and:

$$g_{k_1, \dots, k_n} \left(\left(U_i^{k_i} \right) \right) = \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right)$$

acts on the coordinates $\left(U_i^{k_i} \right)$. The parameters $\left\{ \hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right\}$ describe the group elements. The transformation depends on the $\left\{ \Psi_{J, \alpha_i}^{\otimes l} \left(U_j^l \right) \right\}_{i=1, \dots, n}$ so that in the sequel, we will write:

$$\left\{ \hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left[\Psi_J, \nu \right] \right\}$$

for the set of parameters of the group G^{k_1, \dots, k_n} acting on several realizations $\alpha_1, \dots, \alpha_n$. The $k_i \leq n$ are integer variables.

More generally, we can assume that the transformations mix the elements of a collection of n -uplet of realizations. We write $\{\alpha_i \leq n\}$ for an arbitrary collection $\{\alpha_1, \dots, \alpha_n\}$ of such n -uplets. In this case, the transformation writes:

$$\begin{aligned} & G_{\alpha_1, \dots, \alpha_n}^{k_1, \dots, k_n} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \cdot \left(\prod_{i=1 \dots n} \left\{ \Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i} \right) \right\}_{i \leq n} \right) \\ &= \sum_{(\alpha'_1, \dots, \alpha'_n) \in \{\alpha_i \leq n\}} \left(\prod_{i=1 \dots n} \left[R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \right]_{(\alpha_1, \dots, \alpha_n)}^{(\alpha'_1, \dots, \alpha'_n)} \Psi_{I, \alpha'_i}^{\otimes k_i} \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right) \right) \end{aligned} \quad (8)$$

where the sum is over the collection considered, and with $\left[R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \right]_{(\alpha_1, \dots, \alpha_n)}^{(\alpha'_1, \dots, \alpha'_n)}$ acting on indices $\alpha_1, \dots, \alpha_n$:

$$\begin{aligned} & R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \Psi_{I, \alpha_i}^{\otimes k_i} \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right) \\ &= \sum_{(\alpha'_1, \dots, \alpha'_n) \in \{\alpha_i \leq n\}} \sum_{\alpha'_i \in (\alpha'_1, \dots, \alpha'_n)} \left[\left[R_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right) \right]_{(\alpha_1, \dots, \alpha_n)}^{(\alpha'_1, \dots, \alpha'_n)} \right]_{\alpha_i}^{\alpha'_i} \Psi_{I, \alpha'_i}^{\otimes k_i} \left(g_{k_1, \dots, k_n}^{k_i} U_i^{k_i} \right) \end{aligned} \quad (9)$$

the parameters $\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left[\Psi_J, \nu \right]$ are local coordinates for the transformation $G_{\alpha_1, \dots, \alpha_n}^{k_1, \dots, k_n} \left(\hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \right)$.

Considering all transformations, the full set of parameters is thus:

$$\left(\left\{ \hat{\Lambda}_{\alpha_1, \dots, \alpha_n}^{\{k_i \leq n\}} \left[\Psi_J, \nu \right] \right\} \right)_n$$

where $\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{\{k_i \leq n\}\}} [\Psi_J, \nu] \right\}$ describes all set of parameters for a given collection $\{\alpha_i \leq n\}$.

When all realizations $\{\alpha_i \leq n\}$ arise, the variables $\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{\{k_i \leq n\}\}} [\Psi_J, \nu] \right\}$ is independent from a given set $\alpha_1, \dots, \alpha_n$. As a consequence, considering the symetry groups for $\left\{ \Psi_{I, \alpha_i}^{\otimes k_i} (U_i^{k_i}) \right\}_{i \leq n}$ and assume that one of the index is a given realization α , the set $\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{\{k_i \leq n\}\}} [\Psi_J, \nu] \right\}$ only depends on α and we write:

$$\left\{ \hat{\Lambda}_{\alpha}^{\{\{k, k_i \leq n-1\}\}} [\Psi_J, \nu] \right\}$$

and:

$$\left(\left\{ \hat{\Lambda}_{\alpha}^{\{\{k, k_i \leq n-1\}\}} [\Psi_J, \nu] \right\} \right)_n$$

for n running from 1 to ∞ when all sets of parameters are taken into account. We show in appendix 2 that the fields solving saddle point equation write:

$$\Psi_{I,0,\alpha}^{\otimes k} \left(U_i^k, \left(\left\{ \hat{\Lambda}_{\alpha}^{\{\{k, k_i \leq n-1\}\}} [\Psi_J, \nu] \right\} \right)_n, \{\Psi_J^{\otimes l}\} \right)$$

We also show in appndix 2 that for transformation groups that satisfy:

$$G_{\alpha_1, \dots, \alpha_n}^{k_i, \dots, k_n} \subset G_{\alpha_1, \dots, \alpha_{n+1}}^{k_i, \dots, k_{n+1}}$$

the whole set of parameters:

$$\left(\left\{ \hat{\Lambda}_{\alpha}^{\{\{k, k_i \leq n-1\}\}} [\Psi_J, \nu] \right\} \right)_n$$

describes set of infinite dimensional flag manifolds, all starting with $\hat{\Lambda}_{\alpha}^{[k]} [\Psi_J, \nu]$.

In the sequel we note:

$$\begin{aligned} \hat{\Lambda}_{n,\alpha} [\Psi_J, \nu] &= \left\{ \hat{\Lambda}_{\alpha}^{\{\{k_i \leq n\}\}} [\Psi_J, \nu] \right\} \\ \hat{\Lambda}_{\infty,\alpha} [\Psi_J, \nu] &= \left(\left\{ \hat{\Lambda}_{\alpha}^{\{\{k_i \leq n\}\}} [\Psi_J, \nu] \right\} \right)_n \end{aligned}$$

and:

$$\begin{aligned} \hat{\Lambda}_{n,\alpha}^{[k]} [\Psi_J, \nu] &= \left\{ \hat{\Lambda}_{\alpha}^{\{\{k, k_i \leq n-1\}\}} [\Psi_J, \nu] \right\} \\ \hat{\Lambda}_{\infty,\alpha}^{[k]} [\Psi_J, \nu] &= \left(\left\{ \hat{\Lambda}_{\alpha}^{\{\{k, k_i \leq n-1\}\}} [\Psi_J, \nu] \right\} \right)_n \end{aligned}$$

When these states are independent from α , that index will be removed.

4.3.2 Example: Case of reparametrization invariance for original field

We start by considering general terms:

$$\sum_{\alpha_i^{(1)}, \dots, \alpha_i^{(l)}} \int f \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{1, \dots, l}, [\Psi_J] \right) \prod_{t=1, \dots, l} \Psi_{J, \alpha_i^{(t)}}^{\otimes l_i} \left(\left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right)_t \right) \Psi_{I, \alpha_i^{(t)}}^{\otimes k_i} \left(\left(U_i^{k_i} \right)_t \right) d \left(\left\{ \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right)_t \right\} \right) \quad (10)$$

arising from the power functions in the initial field theory:

$$\int f \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l}, [\Psi_J] \right) \prod_{i=1, \dots, l} \Psi^{\otimes(k_i+l_i)} \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l} \right) \quad (11)$$

or equivalently, from:

$$\int f \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l}, [\Psi_J] \right) \Psi^{\otimes l(k_i+l_i)} \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l} \right) \quad (12)$$

Starting with (12), we can consider some reparametrization invariance:

$$\begin{aligned} & g^{\{U_i^{k_i}\}} \cdot \Psi^{\otimes l(k_i+l_i)} \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l} \right) \\ &= \Psi^{\otimes l(k_i+l_i)} \left(g^{\{U_i^{k_i}\}} \cdot \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l} \right) \\ &= R \left(g^{\{U_i^{k_i}\}} \right) \Psi^{\otimes l(k_i+l_i)} \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l} \right) \end{aligned}$$

depending on the symmetries of the function

$$f \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l}, [\Psi_J] \right)$$

where $R \left(g^{\{U_i^{k_i}\}} \right)$ is a differential operator acting on $\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l}$.

In this case, the transformation for (10) is:

$$\begin{aligned} & G^{k_i, \dots, k_i} \left(\hat{\Lambda}^{[k_i, \dots, k_i]} [\Psi_J] \right) \cdot \left(\prod_{t=1, \dots, l} \Psi_{I, \alpha_i^{(t)}}^{\otimes k_i} \left((U_i^{k_i})_t \right) \right) \\ &= R \left(g^{\{U_i^{k_i}\}} \right) \left\{ \prod_{t=1, \dots, l} \Psi_{I, \alpha_i^{(t)}}^{\otimes k_i} \left(g^{\{U_i^{k_i}\}} \left((U_i^{k_i})_t \right) \right) \right\} \end{aligned}$$

Here, the parameters $\hat{\Lambda}^{[k_i, \dots, k_i]} [\Psi_J]$ for transformation $g^{\{U_i^{k_i}\}}$ are independent from the copies. The inclusion of groups $G_{\alpha_1, \dots, \alpha_n}^{k_i, \dots, k_i}$

$$G_{\alpha_1, \dots, \alpha_n}^{\overbrace{k_i, \dots, k_i}^{l'}} \left(\hat{\Lambda} \left[\overbrace{k_i, \dots, k_i}^{l'} \right] [\Psi_J] \right) \subset G_{\alpha_1, \dots, \alpha_n}^{\overbrace{k_i, \dots, k_i}^l} \left(\hat{\Lambda} \left[\overbrace{k_i, \dots, k_i}^l \right] [\Psi_J] \right)$$

comes from inclusion of symmetry groups of $f \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l'}, [\Psi_J] \right)$ and $f \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_{i=1, \dots, l}, [\Psi_J] \right)$.

4.3.3 Example 2: partial reparametrization symmetry

As an example we consider reparametrizations that do not originate from initial field. In (7) we look at the following second order term in $\Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i} \right)$:

$$\begin{aligned} & \sum_{\alpha_i, \alpha'_i} \int \Psi_{J, \alpha_i}^{\otimes l_i} \left((U_j^{l_i} / f_{k_i l_i}, U_i^{k_i})' \right) \Psi_{I, \alpha_i}^{\otimes k_i} \left((U_i^{k_i})' \right) \\ & \times f \left((U_j^{l_i} / f_{k_i l_i}, U_i^{k_i})', (U_j^{l_i} / f_{k_i l_i}, U_i^{k_i}), [\Psi_J] \right) \Psi_{J, \alpha'_i}^{\otimes l_i} \left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right) \Psi_{I, \alpha'_i}^{\otimes k_i} \left(U_i^{k_i} \right) d \left(\left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i \right) \end{aligned} \quad (13)$$

which arises from the decomposition of quadratic terms in the original field theory:

$$\int \left(\Psi^{\otimes k_i + l_i} \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \right) f \left(\left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right)', \left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right), [\Psi_J] \right) \Psi^{\otimes k_i + l_i} \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \quad (14)$$

The dependce in $[\Psi_J]$ being functional, i.e. arising from integrals of function in Ψ_J .

The term (13) is invariant for any transformation $G_{k_i, k_i} [\Psi_{J, \alpha_i}, \Psi_{J, \alpha'_i}]$ acting on $\left(\Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i} \right), \Psi_{I, \alpha'_i}^{\otimes k_i} \left(U_i^{k_i} \right) \right)$, preserving the form (13). Writing this action:

$$G_{\alpha_i, \alpha'_i}^{k_i, k_i} \left(\hat{\Lambda}_{\alpha_i, \alpha'_i}^{[k_i, k_i]} [\Psi_J] \right) \cdot \left(\Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i} \right), \Psi_{I, \alpha'_i}^{\otimes k_i} \left(U_i^{k_i} \right) \right) = \left(\prod_{\substack{\alpha_i, \alpha'_i \\ \alpha' = \alpha_i, \alpha'_i}} \sum_{\substack{\alpha = \alpha_i, \alpha'_i \\ \alpha' = \alpha_i, \alpha'_i}} \left[R_{\alpha_i, \alpha'_i}^{[k_i, k_i]} \left(\hat{\Lambda}_{\alpha_i, \alpha'_i}^{[k_i, k_i]} [\Psi_J] \right) \right]_{\alpha_i}^{\alpha'_i} \Psi_{I, \alpha'_i}^{\otimes k_i} \left(g_{k_i, k_i}^{k_i} U_i^{k_i} \right) \right)$$

The dependency in $[\Psi_J]$ arises from its realizations $\Psi_{J, \alpha_i}^{\otimes l_i}, \Psi_{J, \alpha'_i}^{\otimes l_i}$.

This symmetry is still present, if we assume a series expansion:

$$\begin{aligned} & S \left(\left(\Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i} \right) \right) \right) \\ &= \sum_n a_n \left(\int \left(\Psi^{\otimes k_i + l_i} \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \right) f \left(\left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right)', \left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right), [\Psi_J] \right) \Psi^{\otimes k_i + l_i} \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \right)^n \end{aligned}$$

In this case the symmetry groups are:

$$G_{k_i, k_i} [\Psi_J] \subset G_{k_i, k_i} [\Psi_J] \times G_{k_i, k_i} [\Psi_J] \subset \dots$$

More generally, considng expression (10), it is preserved for transformations $G_{\alpha_i, \dots, \alpha_l}^{k_i, \dots, k_l}$ for one set of rlztns:

$$\begin{aligned} & G_{\alpha_i, \dots, \alpha_l}^{k_i, \dots, k_l} \left(\hat{\Lambda}_{\alpha_i, \dots, \alpha_l}^{k_i, \dots, k_l} [\Psi_J] \right) \cdot \left(\prod_{t=1, \dots, l} \Psi_{I, \alpha_t}^{\otimes k_i} \left(\left(U_i^{k_i} \right)_t \right) \right) \\ &= \prod_{t=1, \dots, l} \left\{ \sum_{t'} \left[R_{\alpha_1, \dots, \alpha_n}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\alpha_i, \dots, \alpha_l}^{k_i, \dots, k_l} [\Psi_J] \right) \right]_{\alpha_i}^{\alpha_i^{(t')}} \Psi_{I, \alpha_i}^{\otimes k_i} \left(g^{\{U_i^{k_i}\}} \left(\left(U_i^{k_i} \right)_t \right) \right) \right\} \end{aligned}$$

compatible with the symmetries of functns $f \left(\left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right)', \left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right), [\Psi_J] \right)$. If a collection of realizations is involved in the transformation, the transformtn formula is rathr given by (8) and (9) with $R_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} \left(\hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} \right) \rightarrow R_{\{\alpha_i \leq n\}}^{k_1 \dots k_n} \left(\hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} \right)$.

The inclusion between groups arises if $G_{\{\alpha_i \leq n\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} \right)$ contains some subgroups:

$$\left(G_{\{\alpha_i \leq l\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq l\}}^{\{k_i \leq l\}} \right) \right)' \times \left(G_{\{\alpha_i \leq n-l\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq n-l\}}^{\{k_i \leq n-l\}} \right) \right)' \subset G_{\{\alpha_i \leq n\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} \right)$$

with the inclusion of subgroups:

$$\begin{aligned} & \left(G_{\{\alpha_i \leq l\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq l\}}^{\{k_i \leq l\}} \right) \right)' \subset G_{\{\alpha_i \leq l\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq l\}}^{\{k_i \leq l\}} \right) \\ & \left(G_{\{\alpha_i \leq n-l\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq n-l\}}^{\{k_i \leq n-l\}} \right) \right)' \subset G_{\{\alpha_i \leq n-l\}}^{k_i, \dots, k_i} \left(\hat{\Lambda}_{\{\alpha_i \leq n-l\}}^{\{k_i \leq n-l\}} \right) \end{aligned}$$

4.3.4 Remarks

1. Except for the case of reparametrization invariance of the original theory, the parameters $\hat{\mathbf{A}}^{\{k_i, \dots, k_i\}}_{\{\alpha_i^{(1)}, \dots, \alpha_i^{(p)}\}} [\Psi_J]$ depend on realizations $\{\alpha_i^{(1)}, \dots, \alpha_i^{(p)}\}$ of Ψ_J in general. In functionals, by change of variables in integrals, we may expect they can be identified with parameters independent from the realizations, i.e. that the various actions are copies of the same groups.

2. If $S\left(\left(\Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i}\right)\right)\right)$ depends only on the state ν , then:

$$\begin{aligned} & S\left(\left(\Psi_{I, \alpha_i}^{\otimes k_i} \left(U_i^{k_i}\right)\right)\right) \\ &= \sum_l a_l \sum_{\alpha_i^{(1)}, \dots, \alpha_i^{(l)}} \int f\left(\left\{U_i^{k_i}, \right\}_{1, \dots, l}\right) \\ & \times \prod_{t=1, \dots, l} \nu_{\{(U_i^{k_i})_t\}} \left(\Psi_{J, \alpha_i^{(t)}}^{\otimes l_i} \left(\left(U_j^l / f_{k_i l_i}, U_i^{k_i}\right)_t\right)\right) \Psi_{I, \alpha_i^{(t)}}^{\otimes k_i} \left(\left(U_i^{k_i}\right)_t\right) d\left(\left\{\left(U_i^{k_i}, U_j^l / f_{k_i l_i}\right)_t\right\}\right) \end{aligned}$$

and the dependence of the symmetry parameters writes:

$$\hat{\mathbf{A}}^{\{k_i, \dots, k_i\}}_{\{\alpha_i^{(1)}, \dots, \alpha_i^{(l)}\}} [\Psi_J] \rightarrow \hat{\mathbf{A}}^{\{k_i, \dots, k_i\}}_{\{\alpha_i^{(1)}, \dots, \alpha_i^{(l)}\}} [\Psi_J, \nu]$$

4.3.5 Projection over minima

Minimization of the action functional including degeneracy can be performed and yields the projected field:

$$\begin{aligned} & \Psi_{I, \alpha, 0}^{\otimes k} \left(U_i^k, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \{\Psi_J^{\otimes l}\}_l\right) \\ &= \sum_s \sum_{s, l_1, \dots, l_s, \{\alpha'_i\}_{i \leq s}} d\left(\left\{U_j^{l_i}\right\}_{l_i}\right) d\left(\left\{U_i^{k_i} / f_{k_i l_i}\right\}_{k_i, l_i}\right) \\ & \times \prod_i \Psi_{J, \alpha'_i}^{\otimes l_i} \left(U_j^{l_i}\right) \mathcal{K}_0^v \left(U_i^k, \left\{U_j^{l_i}\right\}_{l_i}, \left\{U_i^{k_i} / f_{k_i l_i}\right\}_i, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu]\right) \end{aligned} \quad (15)$$

where the kernel:

$$\begin{aligned} & \mathcal{K}_0^v \left(U_i^k, \left\{U_j^{l_i}\right\}_{l_i}, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu]\right) \\ &= \mathcal{K}_0 \left(U_i^k, \left\{U_j^{l_i}\right\}_{l_i}, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu]\right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i}\right) \end{aligned}$$

includes possibly derivatives in the various variables.

4.3.6 Restriction including constraints

Up until now, the derivation has been performed without taking into account the constraints between the two fields. Accounting for these constraints will reduce the symmetry group. As we need to consider the following terms in the functionals:

$$\Psi_J^{\otimes l} \left(U_j^l\right) \Psi_I^{\otimes k} \left(U_i^k\right) \delta \left(f_{lk} \left(U_j^l, U_i^k\right)\right)$$

we have to restrict saddle point solutions $\Psi_{I, 0}^{\otimes k} \left(U_i^k\right)$ by including a factor $\delta \left(f_{lk} \left(U_j^l, U_i^k\right)\right)$ and replace:

$$\Psi_I^{\otimes k} \left(U_i^k\right) \rightarrow \Psi_I^{\otimes k} \left(U_i^k\right) \delta \left(f_{lk} \left(U_j^l, U_i^k\right)\right)$$

This restriction reduces the symmetry group to a subgroup preserving the constraint. This also applies to the terms including $\prod_{i \leq s} \Psi_{J, \alpha'_i}^{\otimes l_i} (U_j^{l_i})$ in (15) which imposes the constraints $\delta(f_{l_i k_i}(U_i^{k_i}, U_j^{l_i}))$. We show in appendix 2 that locally, this amounts to reduce $\hat{\Lambda}_{\infty}^{[k]}[\Psi_J, \nu]$ to a set depending on $\{U_j^{l_i \leq s}\} \equiv \{U_j^{l_i}\}$ and rewrite the variables $\hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu]$ as functions of the set $\{U_j^{l_i}\}$:

$$\hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu] \rightarrow \hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}] = \left(\left\{ \hat{\Lambda}_{\alpha}^{[k, k_i \leq n-1]}[\Psi_J, \nu, \{U_j^{l_i}\}] \right\} \right)_n \quad (16)$$

and:

$$\hat{\Lambda}_{n, \alpha}^{[k]}[\Psi_J, \nu] \rightarrow \hat{\Lambda}_{n, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}] = \left\{ \hat{\Lambda}_{\alpha}^{[k, k_i \leq n-1]}[\Psi_J, \nu, \{U_j^{l_i}\}] \right\}$$

4.4 Projected functional and effective field

Once the saddle point has been found, we can compute the projected functional (6):

$$\sum_{v_{U_i^k}} \int g(v, U_i^k) v_{U_i^k} (\Psi_J^{\otimes l}) \prod_{\min S(v)} e_{v_{U_i^k}} (\Psi_I^{\otimes k}(U_i^k)) \delta(f_{lk}(U_j^l, U_i^k)) \quad (17)$$

by replacing $\Psi_I^{\otimes k}(U_i^k)$ with its saddle point value. This will enable to describe the effective theory after projection, as states of an effective field including the degrees of freedom of $\Psi_I^{\otimes k}(U_i^k)$.

The computation is presented in appendix 3 under the assumption:

$$v_{U_i^k}(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i}) = v(U_j^{l_i}) \delta(f_{lk}(U_j^l, U_i^k))$$

corresponding roughly to current-current interaction. As before, defining the realization α as a sum of products of realizations:

$$\Psi_{J, \alpha}^{\otimes \sum_{i \leq n} l_i}(\{U_j^{l_i}\}) = \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}, \alpha}^{\otimes l_i}(U_j^{l_i})$$

Defining also:

$$\begin{aligned} & \bar{g}\left(v, \{U_j^{l_i}\}_{i \leq n}, \hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}]\right) \\ &= \int g(v, U_i^k) \mathcal{K}_0\left(\{U_j^{l_i}\}_{l_i}, \{U_i^{k_i} / f_{k_i l_i}\}_{i \leq n}, \hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}]\right) dU_i^k / f_{kl} \prod dU_i^{k_i} / f_{k_i l_i} \end{aligned}$$

formula (17) writes:

$$\sum_{\alpha} \int \bar{g}\left(v, \{U_j^{l_i}\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}], v((U_j^{l_i}))\right) \left(\Psi_{J, \alpha}^{\otimes (\sum_i l_i)}(\{U_j^{l_i}\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}], v)\right)$$

with:

$$\Psi_{J, \alpha}^{\otimes (\sum_i l_i)}(\{U_j^{l_i}\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}], v) = \Psi_{J, \alpha}^{\otimes (\sum_i l_i)}(\{U_j^{l_i}\}_i) \Psi_{\alpha}(\hat{\Lambda}_{\infty, \alpha}^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}], v)$$

and:

$$v((U_j^{l_i})) = \sum_S \prod_{i \in S} v(U_i^{k_i})$$

Setting $\bar{g}_v \rightarrow \bar{g}$ since the fcnctls are arbitrary and the v dependency can be absorbed in the definition of \bar{g} . If the:

$$\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] = \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$$

are independent from α , we find the projected functional:

$$\int \bar{g} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \quad (18)$$

with the field given by a sum over realizations:

$$\begin{aligned} & \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \\ &= \sum_{\alpha} \Psi_{J, \alpha}^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i \right) \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \end{aligned}$$

If the $\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ are not independent of the copies $\Psi_{J, \alpha}^{\otimes k p_0}$, the identification is local. The fields $\Psi_J^{\otimes (\sum_i l_i)}$ are effective fields after projection of the field Ψ_I . They depend on some parameters that both keep track of the projected states, and the remaining object, through the field Ψ_J and the state ν . This dependency arises from the fact that the saddle point equations, and the symmetry groups depend on those quantities. We will detail below the form of the constraints and their dependency on the parameters $\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$, the field Ψ_J and state ν .

4.5 Averaging over cloud substratum and local field

We aim at describing the effective field:

$$\Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right)$$

involved in the functionals (18) as a more usual field defined at some given points. This is achieved by showing that $\hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ can be depicted as cloud of points surrounding these particular points.

4.5.1 Cloud of points

To explain this point, we assume, at least in first approximation that:

$$\hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] = \left(\hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \hat{\Lambda}_n \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right)_n$$

where $\left\{ U_j^{l_i} \right\} = \left\{ U_j^{l_i \leq s} \right\}$ when insertd in products $\prod_{i \leq s} \Psi_{J, \alpha_i}^{\otimes l_i} \left(U_j^{l_i} \right)$.

We have given above the description of $\hat{\Lambda}^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, U_j^l \right]$ in terms of set of flag manifolds starting with $\hat{\Lambda}^{[k]} \left[\Psi_J, \nu, U_j^l \right]$. As a consequence, the field:

$$\Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \left\{ \hat{\Lambda}^{\{k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} \right)_n, v \right)$$

describes both the integratd presenc of a cloud, that is the points of a space, with distinguished points $\hat{\Lambda}^{[k]} \left[\Psi_J, \nu, U_j^l \right]$. These points bear the physical quantities $\left\{ U_j^{l_i} \right\}_i$.

4.5.2 Series expansion of the field in cloud variables

To describe more precisely $\left\{ \hat{\Lambda}_\alpha^{\{k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}$, we use our two previous assumptions:

$$\hat{\Lambda}^{\{k_i \leq n\}} [\Psi_J, \nu, \{U_j^{l_i}\}] = \left\{ \hat{\Lambda}^{k_i} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}_{k_1, \dots, k_n}$$

and:

$$\hat{\Lambda}^{[k_i]} [\Psi_J, \nu, \{U_j^{l_i \leq n}\}] = \left(\hat{\Lambda}^{k_i} [\Psi_J, \nu, \{U_j^{l_i \leq n}\}] \right)$$

Thus, the set $\left\{ \hat{\Lambda}^{\{k_i \leq n-1\}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}$ consists of sequences:

$$\begin{aligned} & \hat{\Lambda}^{k_1} [\Psi_J, \nu, \{U_j^{l_i}\}], \left(\hat{\Lambda}^{k_{1,1}} [\Psi_J, \nu, \{U_j^{l_i}\}], \hat{\Lambda}^{k_{1,2}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right), \\ & \left(\hat{\Lambda}^{k_{2,1}} [\Psi_J, \nu, \{U_j^{l_i}\}], \hat{\Lambda}^{k_{2,2}} [\Psi_J, \nu, \{U_j^{l_i}\}], \hat{\Lambda}^{k_{2,3}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right) \dots \end{aligned}$$

Each $\hat{\Lambda}^{k_i} [\Psi_J, \nu]$ is described by an infinite number of representants. Consequently, if we consider the expansion of $\Psi_J^{\otimes (\sum_i l_i)}$ in terms of representations of the G_{k_i, \dots, k_n} :

$$\begin{aligned} & \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], \left\{ \hat{\Lambda}^{\{k_i \leq n-1\}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}_n \right), v \right) \\ = & \sum_{k_i} \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], \left\{ \hat{\Lambda}^{k_i} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}, v \right) \\ & + \sum_{k_{i,1/2}} \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], \left\{ \hat{\Lambda}^{k_{i,1}} [\Psi_J, \nu, \{U_j^{l_i}\}], \hat{\Lambda}^{k_{i,2}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}, v \right) + \dots \end{aligned}$$

where $\left\{ \hat{\Lambda}^{k_i} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}$ is the set of all the $\hat{\Lambda}^{k_i} [\Psi_J, \nu]$ for given k_i , and $\left\{ \hat{\Lambda}^{k_{i,1}} [\Psi_J, \nu, \{U_j^{l_i}\}], \hat{\Lambda}^{k_{i,2}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}$ the set of the $\left(\hat{\Lambda}^{k_{i,1}} [\Psi_J, \nu, \{U_j^{l_i}\}], \hat{\Lambda}^{k_{i,2}} [\Psi_J, \nu, \{U_j^{l_i}\}] \right)$ for given $k_{i,1}, k_{i,2}$.

The field is thus a series:

$$\begin{aligned} & \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_\infty^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], v \right) \tag{19} \\ = & \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], V, v \right) + \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], V^2, v \right) + \dots \end{aligned}$$

for:

$$V = \left\{ \left\{ \hat{\Lambda}^{k_i} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\} \right\}_{k_i}$$

gathering all the states $\left\{ \hat{\Lambda}^{k_i} [\Psi_J, \nu] \right\}$.

4.5.3 Averaging over cloud variables

Given that we have considered independently the tensor power $\Psi_J^{\otimes l} (U_j^l)$, we can decompose the effective field into series of products of realizations of several fields:

$$\begin{aligned} & \sum_{k_i} \Psi_{J, k_i}^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], V_{k_i}, v \right) \\ & + \sum_{k_{i,1/2}} \Psi_{J, k_{i,1}, \alpha}^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], V_{k_{i,1}}, v \right) \Psi_{J, k_{i,2}, \alpha}^{\otimes (\sum_i l_i - l_i')} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], V_{k_{i,2}}, v \right) + \dots \end{aligned}$$

where:

$$V_{k_i} = \left\{ \hat{\Lambda}^{k_i} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\}$$

By expanding the field as functional of one point, two points and so on:

$$\begin{aligned} & \Psi_{J,k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, U_j^l \right], V_{k_i}, v \right) \\ = & \sum_{\Lambda_{k_i}} \Psi_{J,k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, U_j^l \right], \Lambda_{k_i}, v \right) \\ & + \sum_{\Lambda_{k_i}^{(1)}, \Lambda_{k_i}^{(2)}} \Psi_{J,k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, U_j^l \right], \Lambda_{k_i}^{(1)}, \Lambda_{k_i}^{(2)}, v \right) + \dots \end{aligned}$$

where Λ_{k_i} are coordinates in V_{k_i} . Consequently the field can be seen as an average and expands in series:

$$\begin{aligned} & \Psi_{J,k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, U_j^l \right], V_{k_i}, v \right) \\ = & \sum_r \int_{(V_{k_i})^r} \Psi_{J,k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], (\Lambda_{k_i})^r, v \right) d(\Lambda_{k_i})^r \end{aligned} \quad (20)$$

The fields at stake are thus fields depending on one set of parameter variables $\hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$, but also include in an integrated manner, the internal space of points.

5 Variations of fields

5.1 Infinitesimal variation of the field

The field:

$$\Psi_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right)$$

encompasses the parameters $\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$. Since $\Psi_J^{\otimes \sum_i l_i}$ and $\Psi_I^{\otimes \sum_i k_i}$ are intertwined through the constraints, $\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ depends on Ψ_J and $\left\{ U_j^{l_i} \right\}$. A variation of Ψ_J modifies $\hat{\Lambda}_{\infty, \alpha}^{[k]}$. We aim at finding the conditions for parameters $\hat{\Lambda}_{\infty, \alpha}^{[k]}$ to be locally independent from Ψ_J , so that the field can be considered as function of the parameter space and that a field variation does not affect this parameter space.

We begin with the field decomposed as a sum over product of components:

$$\begin{aligned} & \Psi_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \\ = & \sum_{\alpha} \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) = \sum_{\alpha} \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(U_j^{l_i} \right) \Psi_{\alpha} \left(\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \end{aligned}$$

This field will be inserted in the state-functionals for one realization by introducing:

$$\delta \left(h_i \left(\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \left(\Psi_{J, \alpha}^{\otimes l_i}, v \right) \right) \right)$$

which represents the constraint defining $\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$. The field can thus also be written by including this constraint:

$$\begin{aligned} & \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \\ = & \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \delta \left(h_i \left(\hat{\Lambda}_{\alpha}^{\left[\left\{ k_i \leq n \right\} \right]} \left[\Psi_J, \nu, U_j^l \right], \left(\Psi_{J, \alpha}^{\otimes l_i}, v \right) \right) \right) \end{aligned}$$

Since $\Psi_{J,\alpha}^{\otimes \sum_i l_i}$ depends both directly in $\Psi_{J,\alpha}^{\otimes l_i}(U_j^{l_i})$ and indirectly via $\hat{\Lambda}_\alpha^{[k_i]}[\Psi_J, \nu, U_j^{l_i}]$, a set of variations $\{\delta\Psi_{J,\alpha}^{\otimes l_i}(U_j^{l_i})\}$ for each $U_j^{l_i}$, induces a variation:

$$\begin{aligned} & \frac{\delta\Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right)}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \\ &= \frac{\delta'\Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right)}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \\ &+ \frac{\delta\hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}]}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\hat{\Lambda}_{\infty,\alpha}^{[k]}} \Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right) \end{aligned}$$

with:

$$\frac{\delta\hat{\Lambda}_{\infty,\alpha}^{[k]}}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\hat{\Lambda}_{\infty,\alpha}^{[k]}} = \sum_{n, \{k, k_i \leq n-1\}} \frac{\delta\hat{\Lambda}_\alpha^{[\{k, k_i \leq n-1\}]}}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\hat{\Lambda}_\alpha^{[\{k, k_i \leq n-1\}]}}$$

and:

$$\frac{\delta'\Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right)}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)}$$

represents the variation of $\Psi_{J,\alpha}^{\otimes \sum_i l_i}$ for constant $\hat{\Lambda}$.

In turn, this associates the variation of the state v :

$$\begin{aligned} & \int v\left(\left(U_j^{l_i}\right)\right) \frac{\delta\Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right)}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \\ &= \int v\left(\left(U_j^{l_i}\right)\right) \frac{\delta'\Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right)}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \\ &+ \int v\left(\left(U_j^{l_i}\right)\right) \frac{\delta\hat{\Lambda}_{\infty,\alpha}^{[k]}}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\hat{\Lambda}_{\infty,\alpha}^{[k]}} \Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right) \end{aligned}$$

and invariance writes for this state:

$$\int v\left(\left(U_j^{l_i}\right)\right) \frac{\delta\hat{\Lambda}_{\infty,\alpha}^{[k]}}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\hat{\Lambda}_{\infty,\alpha}^{[k]}} \Psi_{J,\alpha}^{\otimes \sum_i l_i}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}], v\right) = 0 \quad (21)$$

If $\hat{\Lambda}_\alpha^{[k_i]}$ independent from copy, and if in first approximation $\frac{\delta\hat{\Lambda}_{\infty,\alpha}^{[k]}}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)}$ is independent from α ,

then writing:

$$\frac{\delta\hat{\Lambda}_{\infty,\alpha}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}]}{\delta\Psi_{J,\alpha}^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} = \frac{\delta\hat{\Lambda}_{\infty}^{[k]}[\Psi_J, \nu, \left\{U_j^{l_i}\right\}]}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)}$$

the variations can be summed over realizations and the invariance of parameters space with respect to field variations implies the effective field equation:

$$0 = \int v(U_i^l) \frac{\delta \hat{\Lambda}_\infty^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}]}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\Lambda}_\infty^{[k]}} \Psi_J^{\otimes (\sum_i l_i)} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}] \right) \quad (22)$$

5.2 Fields in local coordinates

When equation (22) is satisfied, the field can be rewritten:

$$\hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}, v \right) \delta \left(f \left(\hat{\Lambda}^{\{k, k_i \leq n-1\}}, \left(\Psi_{J,0,\alpha}^{\otimes l_i}, v \right) \right) \right) \quad (23)$$

with the same $\hat{\Lambda}^{[k_i]}$ for all fields $\hat{\Psi}_J^{\otimes \sum_i l_i}$. The field $\Psi_{J,0,\alpha}^{\otimes l_i}, v$ is some reference realization from which the $\hat{\Lambda}^{\{k, k_i \leq n-1\}}$ are defined and:

$$\hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}, v, \alpha \right) = \Psi_J^{\otimes (\sum_i l_i)} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}] \right)$$

is obtained by change of variable. This description implements locally the independence between field variation and parameter space, since in (23) the parameters depend on a fixed reference $\Psi_{J,0,\alpha}^{\otimes l_i}$ and remains inert with respect to the variations of $\hat{\Psi}_J^{\otimes \sum_i l_i}$.

The equation for $\hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}, v, \alpha \right)$ is:

$$0 = \int v(U_i^l) \Gamma \left([\Psi_J, \nu, U_j^l], (U_j^{l_i})' \right) \nabla_{\hat{\Lambda}_\infty^{[k]}} \hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}, v, \alpha \right) \quad (24)$$

where:

$$\Gamma \left([\Psi_J, \nu, U_j^l], (U_j^{l_i})' \right) = \frac{\delta \hat{\Lambda}_\infty^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}]}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \left(\frac{\delta \hat{\Lambda}_\infty^{[k]}[\Psi_J, \nu, \{U_j^{l_i}\}]}{\delta \hat{\Lambda}_\infty^{[k]}} \right)^{-1}$$

Equation (24), is a first order differential equation similar to a spinor equation, where the equivalent of the γ matrices are:

$$\hat{\Gamma} \left((U_j^{l_i})', U_j^l \right) = v(U_i^l) \Gamma \left([\Psi_J, \nu, U_j^l], (U_j^{l_i})' \right) \quad (25)$$

In (25); the variables U_i^l and $(U_j^{l_i})'$ act as spinor indices. In the case where these variable are discrete, a sum replaces the integral and we can rewrite (24):

$$0 = \sum_{U_j^l} \hat{\Gamma} \left((U_j^{l_i})', U_j^l \right) \nabla_{\hat{\Lambda}_\infty^{[k]}} \hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_\infty^{[k]}, v, \alpha \right) \quad (26)$$

which looks like massless spinor equation.

However, in (26), the parameters $\hat{\Lambda}_\infty^{[k]}$ represents the entire cloud of points, rather than single coordinates. Local variation of $\hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i \right)$ may locally modify the points of $\hat{\Lambda}_\infty^{[k]}$ without modifying globally the cloud. This point is studied in the next paragraph by considering the averaged field.

5.3 Averaged field variation

5.3.1 General form

Equation (22) can be rewritten for the averaged field (19). Considering only the first term in (19):

$$\Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right)$$

where V is a V_{k_i} , the variation equation becomes:

$$\begin{aligned} 0 &= \int v \left((U_j^{l_i}) \right) \frac{\delta \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\Lambda}^{[k]}} \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right) \\ &+ \int v \left((U_j^{l_i}) \right) \frac{\delta V}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\mathbf{v}} \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right) \end{aligned} \quad (27)$$

The computation of the variation is similar for other terms of (19). In (27), the first term corresponds to a local variation, while the second one describes global effect through a deformation of the manifold V .

Using the integral form (20), allows to rewrite (27):

$$\begin{aligned} &\int v \left((U_j^{l_i}) \right) \frac{\delta V}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\mathbf{v}} \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right) \\ &= \int v \left((U_j^{l_i}) \right) \sum_r \int_{(V_{k_i})^r} \frac{\delta (\Lambda_{k_i})^r}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{(\Lambda_{k_i})^r} \Psi_{J, k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], (\Lambda_{k_i})^r, v \right) d(\Lambda_{k_i})^r \\ &= \int v \left((U_j^{l_i}) \right) \sum_r \int_{B[(V_{k_i})^r]} \frac{\delta (\Lambda_{k_i})^r}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \Psi_{J, k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], (\Lambda_{k_i})^r, v \right) d(\Lambda_{k_i})^r \\ &\quad - \int v \left((U_j^{l_i}) \right) \sum_r \int_{(V_{k_i})^r} \nabla_{(\Lambda_{k_i})^r} h \left((\Lambda_{k_i})^r, \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right) \right) \\ &\quad \times \Psi_{J, k_i}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], (\Lambda_{k_i})^r, v \right) d(\Lambda_{k_i})^r \end{aligned} \quad (28)$$

with $B[(V_{k_i})^r]$ the boundary of $(V_{k_i})^r$ and:

$$h \left((\Lambda_{k_i})^r, \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right) \right) = \frac{\delta (\Lambda_{k_i})^r}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)}$$

This last equation depends on $(\Lambda_{k_i})^r$. Formula (28) can be rewritten as:

$$\begin{aligned} &\int v \left((U_j^{l_i}) \right) \frac{\delta V}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\mathbf{v}} \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right) \\ &= \int v \left((U_j^{l_i}) \right) \frac{\delta(B(V))}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], B(V), v \right) \\ &\quad - \int v \left((U_j^{l_i}) \right) h \left(V, \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right) \right) \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right) \end{aligned}$$

If we assume in average a proportionality factor ϵ between the field on the boundary $B[(V_{k_i})^r]$ and the field on the entire $(V_{k_i})^r$, we can write:

$$\begin{aligned} & \frac{\delta(B(V))}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], B(V), v\right) \\ &= \epsilon \frac{\delta V}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], V, v\right) \end{aligned}$$

So that, we have in average:

$$\begin{aligned} & \int v\left(\left(U_j^{l_i}\right)\right) \frac{\delta V}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\mathbf{v}} \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], V, v\right) \quad (29) \\ &= \int v\left(\left(U_j^{l_i}\right)\right) \left(\epsilon \frac{\delta V}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} - h\left(V, \Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)\right) \right) \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], V, v\right) \end{aligned}$$

where $h\left(V, \Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)\right)$ is the average of $h\left(\left(\Lambda_{k_i}\right)^r, \Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)\right)$ over V .

Given (29), equation (27) writes:

$$\begin{aligned} 0 &= \int v\left(\left(U_j^{l_i}\right)\right) \frac{\delta \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, U_j^{l_i}\right]}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} \nabla_{\hat{\Lambda}^{[k]}} \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], V, v\right) \quad (30) \\ &+ \int v\left(\left(U_j^{l_i}\right)\right) \left(\epsilon \frac{\delta V}{\delta\Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)} - h\left(V, \Psi_J^{\otimes l_i}\left(\left(U_j^{l_i}\right)'\right)\right) \right) \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], V, v\right) \end{aligned}$$

Considered as an equation where the parameter space is integrated, this looks like an equation for a local field. The variation in $\Psi_J^{\otimes(\sum_i l_i)}$ induces a modification of this parameter space at its border. This induces an analog of a mass term in (30), through its last contribution.

5.3.2 Field expansion and coherent states

Formula (30) can be computed if we expand the field $\Psi_J^{\otimes(\sum_i l_i)}$ as a function of V :

$$\begin{aligned} & \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], V, v\right) \quad (31) \\ &= \sum_r \int_{(V)^r} \Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], (V)^r, v\right) d(V)^r \end{aligned}$$

For coherent states, formula (31) reduces to:

$$\Psi_J^{\otimes(\sum_i l_i)}\left(\left\{U_j^{l_i}\right\}_i, \hat{\Lambda}^{[k]}\left[\Psi_J, \nu, \left\{U_j^{l_i}\right\}\right], v\right) \sum_r \frac{1}{r!} \prod \int_{V^r} \bar{\Psi}\left(V^{(r)}, v\right) d\left(V^{(r)}\right) \quad (32)$$

The computation leading to (29) can be performed directly in this case. Neglecting $h \left(V, \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right) \right)$, the variation of the second part in (32) is:

$$\begin{aligned}
& \frac{\delta V}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\mathbf{V}} \sum_r \frac{1}{r!} \prod \int_{V^r} \bar{\Psi} \left(V^{(r)}, v \right) d \left(V^{(r)} \right) \\
&= \sum_r \int_{B(V)} \frac{\delta V}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \bar{\Psi} \left(B(V), v \right) \frac{1}{(r-1)!} \prod \int_{V^{r-1}} \bar{\Psi} \left(V^{(r-1)}, v \right) d \left(V^{(r-1)} \right) \\
&= \int_{B(V)} c \left(B(V), \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right) \right) \bar{\Psi} \left(B(V), v \right) \sum_r \frac{1}{r!} \prod \int_{V^{(r)}} \bar{\Psi} \left(V^{(r)}, v \right) d \left(V^{(r)} \right)
\end{aligned}$$

Now, defining:

$$\begin{aligned}
& \int_{B(V)} c \left(B(V), \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right) \right) \bar{\Psi} \left(B(V), v \right) \\
&= C \left(\Psi_J^{\otimes (\sum_i l_i)} \left((U_j^{l_i})', \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], v \right) \right)
\end{aligned}$$

the invariance equation (30) becomes:

$$\begin{aligned}
0 &= \int v \left((U_j^{l_i}) \right) \frac{\delta \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right]}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\Lambda}^{[k]}} \Psi_J^{\otimes (\sum_i l_i)} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], V, v \right) \\
&+ C \left(\Psi_J^{\otimes (\sum_i l_i)} \left((U_j^{l_i})', \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], v \right) \right) \Psi_J^{\otimes (\sum_i l_i)} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], V, v \right)
\end{aligned}$$

After changing of variable for local coordinats, it also rewrites:

$$\begin{aligned}
0 &= \int v \left(U_i^{l_i} \right) \Gamma \left(\left[\Psi_J, \nu, U_j^{l_i} \right], (U_j^{l_i})' \right) \nabla_{\hat{\Lambda}_\alpha^{[k, k_i \leq n-1]}} \hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \left(\hat{\Lambda}^{[k, k_i \leq n-1]} \right) \right)_n, v, \alpha \tag{33} \\
&+ C \left(\Psi_J^{\otimes (\sum_i l_i)} \left((U_j^{l_i})', \hat{\Lambda}^{[k]}, v \right) \right) \hat{\Psi}_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \left(\hat{\Lambda}^{[k, k_i \leq n-1]} \right) \right)_n, v, \alpha
\end{aligned}$$

which is analogous to the dynamic equation of a massive field.

5.4 Generalization: projective invariance

More generally, rather than considering full invariance, we can consider only projective invariance with local additional contribution. In that case, equation (22) is not satisfied anymore, but includes an additional term:

$$\begin{aligned}
& \int v \left((U_j^{l_i}) \right) \frac{\delta \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right]}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\Lambda}_\alpha^{[k, k_i \leq n-1]}} \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) \\
&= \int v \left(\{ (U_j^{l_i}) \} \right) \frac{\delta V \left(\left\{ \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}_i, \hat{\Lambda}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) \right\} \right)}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left((U_j^{l_i})', \hat{\Lambda}^{[k, k_i \leq n-1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right)} \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right)
\end{aligned}$$

Appendix 5 shows that in this case, we can still define, at least locally a field depending on fixed set of parameters. In this case, the initial variation of the field rewrites:

$$\delta \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) = \delta' \hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left((U_j^{l_i}) \right) \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right)$$

which is performed at fixed $\hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$. As a consequence, even if the effective field $\Psi_{J,\alpha}^{\otimes \sum_i l_i}$ is not invariant, there is a related field $\hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i}$ that can be defined locally as a function of an invariant family of parametr $\hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$.

The fields $\hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i}$ writes:

$$\begin{aligned} & \hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \\ &= \hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty,\alpha}^{[k]}, v, \alpha \right) \delta \left(f \left(\hat{\Lambda}_{\infty,\alpha}^{[k]}, \left(\Psi_{J,0,\alpha}^{\otimes l_i}, v \right) \right) \right) \end{aligned}$$

If, as before, $\frac{\delta \hat{\Lambda}_{\alpha}^{\{k, k_i \leq n-1\}}}{\delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)}$ is independent from realizations, $\hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i}$ satisfies:

$$\begin{aligned} & \int \hat{v} \left((U_j^{l_i}) \right) \frac{\delta \hat{\Lambda}^{\{k, k_i \leq n-1\}}}{\delta \Psi_J^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\Lambda}^{\{k, k_i \leq n-1\}}} \hat{\Psi}_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \\ & \simeq v \left(\left((U_j^{l_i}) \right) \right) \frac{\delta' \hat{V} \left(\hat{\Psi}_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \right)}{\delta \Psi_J^{\otimes l_i} \left(\left((U_j^{l_i})' \right), \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right)} \end{aligned}$$

where:

$$\hat{v} \left((U_j^{l_i}) \right) = \int v \left((U_j^{l_i})'' \right) \frac{\delta \Psi_J^{\otimes \sum_i l_i} \left((U_j^{l_i})'' \right)}{\delta \hat{\Psi}_J^{\otimes \sum_i l_i} \left(U_j^{l_i} \right)}$$

6 Constraints and geometry of parameter space

So far, we have set aside the functional relation between the field $\Psi_J^{\otimes l}$ and the parameter space. We assumed that this space describes the parameters of some transformation groups among the realizations of the field $\Psi_I^{\otimes k}$. Considering that the generators of these groups satisfy some algebraic relations conditions the parameter space through some defining equatns. The equations defining ths space depend on the fields $\Psi_J^{\otimes l}$ since the action to minimize depends on these fields. This implies that the metric on the parameter space depends on $\Psi_J^{\otimes l}$, but also on the background for the fld $\Psi_I^{\otimes k}$.

6.1 Form of the constraint and metric

We assume that The constraint describes relations on the symmetry operators for $n \in \mathbb{N}$:

$$h_{kn} \left(\left\{ \mathbf{L}_{\{\alpha_i\}} \left(\Psi_J^{\otimes l} \right) \right\}_{i \leq n}, \left\{ \mathbf{U}_i^k \right\}_k, h_p \left((\Psi_J), U_j^l, \nu \right) \right) = 0 \quad (34)$$

where the functionals $h_l \left((\Psi_J), U_j^l, \nu \right)$ have the form:

$$h_p \left((\Psi_J), \left\{ U_j^{l_i} \right\}, \nu \right) = \left[\int h_p \left((U_j^{l_i}), \nu \right) \left(\prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \right) d \left(U_j^{l_i} \right) \right] \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \quad (35)$$

and \mathbf{U}_i^k acts on functions of U_i^k as the multiplication operator by U_i^k . Expression (35) corresponds to consider states $F\left(\Psi_{J,\alpha}^{\otimes \sum_i l_i}\right)$ localized at $U_j^{l_i}$.

We then evaluate (34) the projected states:

$$\prod_k \Psi_{I,\alpha,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right)$$

defined by the saddle point solutions $\Psi_{I,\alpha,0}^{\otimes k}$:

$$\begin{aligned} & \Psi_{I,\alpha,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right) \\ &= \sum_{l_i} \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) \mathcal{K}_0 \left(U_i^k, \{U_j^{l_i}\}_{l_i}, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu \right] \right) \end{aligned} \quad (36)$$

and equation (34) becomes a relation for each realization α :

$$\begin{aligned} 0 &= \int h_{k_n} \left(\left\{ \hat{\Lambda}_{n,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\}_k, \{U_i^k\}, h_p \left((\Psi_J), U_j^l, \nu \right) \right) \\ & \quad \prod_k \left| \Psi_{I,\alpha,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right) \right|^2 dU_i^k \end{aligned} \quad (37)$$

If we assume the independence of parameters from the realizations:

$$\hat{\Lambda}_{n,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] = \hat{\Lambda}_n^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right]$$

and:

$$\hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] = \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right]$$

we can sum over realizations before evaluation of (34) on the projected states. Defining:

$$\Psi_{I,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right) = \sum_{\alpha} \Psi_{I,\alpha,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right) \quad (38)$$

equation (34) becomes a relation:

$$\int h_{k_n} \left(\hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{U_i^k\}, h_p \left((\Psi_J), U_j^l, \nu \right) \right) \prod_k \left| \Psi_{I,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right) \right|^2 dU_i^k = 0 \quad (39)$$

We can expand h_{k_n} as a series of products:

$$\begin{aligned} & h_{k_n} \left(\hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{U_i^k\}, h_p \left((\Psi_J), U_j^l, \nu \right) \right) \\ &= \sum_m h_{k_n}^{(m)} \left(\hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{U_i^k\}, h_p \left((\Psi_J), U_j^l, \nu \right) \right) \bar{h}_m \left(\{U_i^k\} \right) \end{aligned}$$

and equation (39) writes:

$$0 = \sum_m h_{k_n}^{(m)} \left(\hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{U_i^k\}, h_p \left((\Psi_J), U_j^l, \nu \right) \right) \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \quad (40)$$

with:

$$\bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) = \int \bar{h}_m \left(\{U_i^k\} \right) \prod_k \left| \Psi_{I,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \{ \Psi_J^{\otimes l} \}_l \right) \right|^2 dU_i^k \quad (41)$$

Given the form of $h_p((\Psi_J), U_j^l, \nu)$, the constraint (40) depends on:

$$(\Psi_J), \nu, \left\{ \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right\}$$

The form of $\Psi_{I,\alpha,0}^{\otimes k}$ ((38) and (36)) and $\bar{H}_m \left(\Psi_{I,\alpha,0}^{\otimes k} \right)$ defined in (41) imply that the dependence of the constraint in $\left\{ \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \right\}, \bar{H}_m \left(\Psi_{I,\alpha,0}^{\otimes k} \right)$ can be described as function dependence in $\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0$ through:

$$\int \bar{h}_m(\{U_i^k\}) \prod_k \left| \Psi_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}, \hat{\Lambda}_\infty^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) \right|^2 \left| \mathcal{K}_0 \left(U_i^k, \{U_j^{l_i}\}_{l_i}, \hat{\Lambda}_\infty^{[k]} \left[\Psi_J, \nu \right] \right) \right|^2 dU_i^k$$

Remark that we can consider non localized states by replacing:

$$\begin{aligned} & \Psi_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}, \hat{\Lambda}_\infty^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) \\ \rightarrow & \nu \left(\Psi_J^{\otimes \sum_i l_i} \left(\hat{\Lambda}_\infty^{[k]} \left[\Psi_J, \nu \right] \right) \right) = \int \nu \left(\{U_j^{l_i}\} \right) \Psi_J^{\otimes \sum_i l_i} \left(\{U_j^{l_i}\}, \hat{\Lambda}_\infty^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) d \{U_j^{l_i}\} \end{aligned}$$

6.2 Lowest order expansion and metrics on projectd states

A second order expansion of the previous constraint enables to define a metric on the parameter space. It depends both on the field $\Psi_J^{\otimes \sum_i l_i}$ and the background $\Psi_{I,0}^{\otimes k}$.

6.2.1 Local functional

For a local functional at the lowest order:

$$\begin{aligned} 0 = & \gamma(\Psi_J, \nu) + N \left(\Psi_J, \nu, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \cdot \left(\left\{ \hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\} \right) \right. \\ & \left. + \left\{ \hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\} M \left(\Psi_J, \nu, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right) \left\{ \hat{\Lambda}_n \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\} \right) \end{aligned}$$

and redefining operators by translation, we find:

$$0 = \gamma(\Psi_J, \nu) + \left\{ \hat{\Lambda}_{n_1}^{[\mathbf{k}_1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\} M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \nu, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right) \left\{ \hat{\Lambda}_{n_2}^{[\mathbf{k}_2]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\} \quad (42)$$

with:

$$\begin{aligned} [\mathbf{k}_1] &= [k_{1,1} \dots k_{1,n_1}] \\ [\mathbf{k}_2] &= [k_{2,1}, \dots, k_{2,n_2}] \end{aligned}$$

The scalar functional has the form (we reintroduce the notation Ψ_J, Ψ_J^\dagger):

$$\alpha(\Psi_J) = \int \Psi_J^\dagger A \Psi_J + \int \Psi_J^\dagger \Psi_J^\dagger B \Psi_J \Psi_J + \dots$$

At the quadratic order the quadratic form defined in (42) becomes:

$$M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \nu, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right)$$

or equivalently:

$$M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \nu, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right)$$

that is, in the projected space, the metric is dynamically an object defined by the field:

$$\Psi_{I,\alpha,0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_\infty^{[k]} \left[\Psi_{J,\nu}, \left\{ U_j^{l_i} \right\} \right], \left\{ \Psi_J^{\otimes l} \right\}_l \right)$$

Ultimately, if (22), or its versions (30) of (33) are satisfied, the sytem is described by dynamical quantities $\Psi_J^{\otimes \sum_i l_i}$, $M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_{J,\nu}, \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \right), \bar{H}_m \left(\Psi_{I,\alpha,0}^{\otimes k} \right) \right)$ and is built on apparently exogeneous parameter space $\left\{ \hat{\Lambda}_{\infty,\alpha}^{[k]} \right\}$.

For one variable $\hat{\Lambda}^{[k]} \left[\Psi_{J,\nu}, \left\{ U_j^{l_i} \right\} \right]$ the metric tensor deriving from:

$$M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \nu, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right)$$

is written:

$$g \left(\{[k], [k]\}, \Psi_{J,\nu}, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right)$$

and depends on the projected field. This ones includes $\sum \Psi_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}, \hat{\Lambda}_\infty^{[k]} \left[\Psi_{J,\nu}, \left\{ U_j^{l_i} \right\} \right] \right)$ as needed and the background. $\mathcal{K}_0 \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\Lambda}_\infty^{[k]} \left[\Psi_{J,\nu} \right] \right)$. The metrics depends on the integrated background.

Considering the constraint for one variable $\hat{\Lambda}^{[k]} \left[\Psi_{J,\nu}, \left\{ U_j^{l_i} \right\} \right]$:

$$\left\{ \hat{\Lambda}^{[k]} \left[\Psi_{J,\nu}, \left\{ U_j^{l_i} \right\} \right] \right\} M \left(\{[k], [k]\}, \Psi_{J,\nu}, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right) \cdot \left\{ \hat{\Lambda}^{[k]} \left[\Psi_{J,\nu}, \left\{ U_j^{l_i} \right\} \right] \right\} + \gamma(\Psi_J) = 0$$

we recover some "usual" metric tensor:

$$g \left(\{[k], [k]\}, \Psi_{J,\nu}, \left\{ \left(\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right)$$

that also writes:

$$g \left(\{[k], [k]\}, \nu, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right)$$

The metric depends functionally of $\Psi_J^{\otimes \sum_i l_i}$, as needed, but also on the integrated field \mathcal{K}_0 . The metric component depends on physical characteristics of space.

6.2.2 Non local functional

For non local functional, the derivation is similar, where:

$$\nu \left(\Psi_J^{\otimes \sum_i l_i} \left(\hat{\Lambda}_\infty^{[k]} \left[\Psi_{J,\nu} \right] \right) \right) = \nu \left(\Psi_J^{\otimes \sum_i l_i} \right)$$

replaces $\Psi_J^{\otimes l_i} \left(U_j^{l_i} \right)$. We have for the quadratic form:

$$M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \left\{ \left(\nu \left(\Psi_J^{\otimes \sum_i l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right) = M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \left\{ \left(\nu \left(\Psi_J^{\otimes \sum_i l_i} \right) \mathcal{K}_0 \right) \right\} \right)$$

and for any variable, the form of the associated metric:

$$g \left(\{[k], [k]\}, \Psi_J, \left\{ \left(\nu \left(\Psi_J^{\otimes \sum_i l_i} \right), \bar{H}_m \left(\Psi_{I,0}^{\otimes k} \right) \right) \right\} \right) = g \left(\{[k], [k]\}, \left\{ \left(\nu \left(\Psi_J^{\otimes \sum_i l_i} \right) \mathcal{K}_0 \right) \right\} \right)$$

6.2.3 Expression in terms of averaged field

The constraints can also be rewritten in terms of the averaged fields. They were defined in (19) as:

$$\begin{aligned}
& \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{A}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \\
\equiv & \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \{V^k\}, v \right) \\
= & \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V, v \right) + \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], V^2, v \right) + \dots
\end{aligned}$$

where:

$$V = \left\{ \left\{ \hat{\mathbf{A}}^{k_i} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\}_{k_i} \right\}$$

depicts the entire set of parameters $\left\{ \hat{\mathbf{A}}^{k_i} \left[\Psi_J, \nu \right] \right\}$ and $\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ is a given parameter. Using the same decomposition for the constraints:

$$h_{k_n} \left(\left\{ \mathbf{L}_{\{\alpha_i\}} \left(\Psi_J^{\otimes l_i} \right) \right\}_{i \leq n}, \left\{ U_i^k \right\}_k, h_p \left((\Psi_J), U_j^l, \nu \right) \right) = 0$$

evaluated on a state of the field (36):

$$\int \nu \left(\left\{ U_j^{l_i} \right\}_i \right) \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \{V^k\}, v \right) \mathcal{K}_0 \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\mathbf{A}}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu \right] \right) d \left\{ U_j^{l_i} \right\}_i$$

yields:

$$\begin{aligned}
0 &= \int h_{k_n} \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \{V^k\}, \left\{ U_i^k \right\}_k, h_p \left((\Psi_J), U_j^l, \nu \right) \right) \\
&\quad \times \left| \nu \left(\left\{ U_j^{l_i} \right\}_i \right) \right|^2 \left| \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \{V^k\}, v \right) \right|^2 \\
&\quad \times \left| \mathcal{K}_0 \left(\left\{ U_i^k \right\}_k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\mathbf{A}}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu \right] \right) \right|^2 d \{V^k\} d \left\{ U_i^k \right\}_k
\end{aligned}$$

and this become an equation:

$$0 = H \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right], \nu \left(\Psi_J^{\otimes(\sum_i l_i)} \right), \mathcal{K}_0 \right)$$

This a manifold equation depending on the state of the system and the background kernel \mathcal{K}_0 . This means that the metric of this manifold:

$$g \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right], \nu \left(\Psi_J^{\otimes(\sum_i l_i)} \right), \mathcal{K}_0 \right)$$

is a series expansion:

$$\begin{aligned}
& g_0 \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right], \Psi_J^{\otimes(\sum_i l_i)} \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right), \mathcal{K}_0 \right) \\
& + \int g_1 \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right], \Psi_J^{\otimes(\sum_i l_i)} \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right), \mathcal{K}_0, \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)' \right) \int \Psi_J^{\otimes(\sum_i l_i)} \left(\left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)' \right) d \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)' \\
& + \int g_1 \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right], \Psi_J^{\otimes(\sum_i l_i)} \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right), \mathcal{K}_0, \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)', \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)'' \right) \\
& \times \Psi_J^{\otimes(\sum_i l_i)} \left(\left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)' \right) d \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)' \Psi_J^{\otimes(\sum_i l_i)} \left(\left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)'' \right) d \left(\hat{\mathbf{A}}^{[k]} \left[\Psi_J, \nu \right] \right)''
\end{aligned} \tag{43}$$

where the $\Psi_J^{\otimes(\sum_i l_i)} \left(\hat{\Lambda}^{[k]} [\Psi_J, \nu] \right)$ are some apparent local fields arising from expansions of:

$$\Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\Lambda}^{[k]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}], \{V^k\}, v \right)$$

The series (43) is similar to the one we would obtain by considering perturbative expansion for a system where metric and field interact, except that here, we have not tried to derive some equations relating those quantities. Our formula is only a general result without considerations about the way these quantities should interact. The main difference comes from the background kernel:

$$\mathcal{K}_0 \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\Lambda}_{\infty, \alpha}^{[k]} [\Psi_J, \nu] \right)$$

that does not appear directly, since the corresponding subobject has been projected and integrated in the fields, but this kernel should take part to the metric qtns.

6.3 Change of variable

We write for short:

$$\Psi_{I, \alpha, 0}^{\otimes k} \left(U_i^k, \hat{\Lambda}_{\infty}^{[k]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}], \left\{ \Psi_J^{\otimes l} \right\}_l \right) \rightarrow \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0$$

where:

$$\Psi_J^{\otimes \sum_i l_i} = \Psi_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}, \hat{\Lambda}_{\infty, \alpha}^{[k]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}] \right)$$

We can then consider a change of variable:

$$\begin{aligned} & M \left(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right) \\ &= U \left(\{[\mathbf{k}_1], [\mathbf{k}'_1]\} \right) N \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\}, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right) U^{-1} \left(\{[\mathbf{k}'_1], [\mathbf{k}_2]\} \right) \end{aligned}$$

where the vector of parameters is:

$$[\mathbf{k}'_1] = [k'_{1,1} \dots k'_{1,n_1}]$$

Under this change of variable:

$$\hat{\Lambda}_{n_1}^{[\mathbf{k}'_1]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}] = U \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\} \right) \sqrt{N \left(\{[\mathbf{k}_1], [\mathbf{k}_1]\}, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right)} \hat{\Lambda}_{n_1}^{[\mathbf{k}_1]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]$$

with:

$$\hat{U} \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\} \right) = U \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\} \right) \sqrt{N \left(\{[\mathbf{k}_1], [\mathbf{k}_1]\}, \left\{ \Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right\} \right)}$$

The variation of parameters in this change of variable is given by:

$$\begin{aligned} \frac{\partial \hat{\Lambda}_{n_1}^{[\mathbf{k}'_1]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]}{\partial \hat{\Lambda}_{n_2}^{[\mathbf{k}_2]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]} &= \frac{\partial \hat{U} \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\} \right)}{\partial \hat{\Lambda}_{n_2}^{[\mathbf{k}_2]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]} \hat{\Lambda}_{n_1}^{[\mathbf{k}_1]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}] + \hat{U} \left(\{[\mathbf{k}'_1], [\mathbf{k}_2]\} \right) \\ &= \frac{\partial \left(\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right)}{\partial \hat{\Lambda}_{n_2}^{[\mathbf{k}_2]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]} \frac{\partial \hat{U} \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\} \right)}{\partial \left(\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right)} \hat{\Lambda}_{n_1}^{[\mathbf{k}_1]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}] + \hat{U} \left(\{[\mathbf{k}'_1], [\mathbf{k}_2]\} \right) \end{aligned}$$

Any associated tensor transforms as

$$\begin{aligned} R_{[\mathbf{k}_1], [\mathbf{k}_2]} &= \frac{\partial \left(\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right)}{\partial \hat{\Lambda}_{n_2}^{[\mathbf{k}_2]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]} \frac{\partial \hat{U} \left(\{[\mathbf{k}'_2], [\mathbf{k}_3]\} \right)}{\partial \left(\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right)} \hat{\Lambda}_{n_3}^{[\mathbf{k}_{n_3}]} R'_{[\mathbf{k}'_1], [\mathbf{k}'_2]} \frac{\partial \hat{U}^{-1} \left(\{[\mathbf{k}_4], [\mathbf{k}'_1]\} \right)}{\partial \left(\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right)} \hat{\Lambda}_{n_4}^{[\mathbf{k}_{n_4}]} \frac{\partial \left(\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0 \right)}{\partial \hat{\Lambda}_{n_1}^{[\mathbf{k}_1]} [\Psi_J, \nu, \left\{ U_j^{l_i} \right\}]} \\ &+ \hat{U} \left(\{[\mathbf{k}'_1], [\mathbf{k}_1]\} \right) R'_{[\mathbf{k}'_1], [\mathbf{k}'_2]} \hat{U}^{-1} \left(\{[\mathbf{k}_2], [\mathbf{k}'_2]\} \right) \end{aligned}$$

it includes the inert contributions that depends on \mathcal{K}_0 through $\Psi_J^{\otimes \sum_i l_i} \mathcal{K}_0$. The tensor $R_{[\mathbf{k}_1], [\mathbf{k}_2]}$ can be seen as a generalized curvature, involving any set of points.

6.4 Invariance of constraint by change of parametrization

We consider the quadratic form M and assume the general form:

$$\begin{aligned} M & \left(\{[k], [k]\}, \Psi_J, \nu, \left\{ \left(\Psi_J^{\otimes l_i} (U_j^{l_i}), \bar{H}_m (\Psi_{I,0}^{\otimes k}) \right) \right\} \right) \\ & = \sum_{h_{l_i}} h_{l_i} \left((U_j^l), \{[\mathbf{k}_1], [\mathbf{k}_2]\}, \left\{ \int G (U_j^{l'}) \Psi_J^{\otimes l'} (U_j^{l'}) dU_j^{l'} \right\} \right) \Psi_J^{\otimes l_i} (U_j^{l_i}) \end{aligned}$$

where $\left\{ \int G (U_j^{l'}) \Psi_J^{\otimes l'} (U_j^{l'}) dU_j^{l'} \right\}$ is a set of arbitrary functionals of the $\Psi_J^{\otimes l'} (U_j^{l'})$

If this quadratic form is invariant through:

$$\Psi_J \rightarrow \mathbf{g} \cdot \Psi_J = \Psi_J (\mathbf{g} \cdot U_j^l)$$

then globally the entire set of parameters $\left\{ \hat{\Lambda}_{\infty, \alpha}^{[k_1]} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}$ is invariant under :

$$\mathbf{g} \cdot \left\{ \hat{\Lambda}_{\infty, \alpha}^{[k_1]} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\} = \left\{ \hat{\Lambda}_{\infty, \alpha}^{[k_1]} [\Psi_J, \nu, \{\mathbf{g} \cdot U_j^{l_i}\}] \right\} = \left\{ \hat{\Lambda}^{[k_1]} [\Psi_J, \nu, U_j^l] \right\}$$

Locally, assume that there is an action:

$$\Psi_J \rightarrow \mathbf{g} \cdot \Psi_J = \Psi_J (\mathbf{g} \cdot U_j^l)$$

such that:

$$\gamma (\Psi_J)$$

and the:

$$\left\{ \int G (U_j^{l'}) \Psi_J^{\otimes l'} (U_j^{l'}) dU_j^{l'} \right\}$$

If there are representations $R_{[\mathbf{k}'_1]}^{[\mathbf{k}_1]}$, $R_{[\mathbf{k}'_2]}^{[\mathbf{k}_2]}$, such that:

$$\begin{aligned} & h_{l_i} \left(\mathbf{g} \cdot (U_j^l), \{[\mathbf{k}_1], [\mathbf{k}_2]\}, \left\{ \int G (U_j^{l'}) \Psi_J^{\otimes l'} (U_j^{l'}) dU_j^{l'} \right\} \right) \mathbf{g} \cdot \Psi_J^{\otimes l_i} (U_j^{l_i}) \\ & = \sum_{[\mathbf{k}'_1], [\mathbf{k}'_2]} R_{[\mathbf{k}'_1]}^{[\mathbf{k}_1]} (\mathbf{g}) \times R_{[\mathbf{k}'_2]}^{[\mathbf{k}_2]} (\mathbf{g}) h_{l_i} \left((U_j^l), \{[\mathbf{k}_1], [\mathbf{k}_2]\}, \left\{ \int G (U_j^{l'}) \Psi_J^{\otimes l'} (U_j^{l'}) dU_j^{l'} \right\} \right) \Psi_J^{\otimes l_i} (U_j^{l_i}) \end{aligned}$$

we can rewrite the transformed constraints:

$$0 = \gamma (\Psi_J, \nu) + \left\{ R_{[\mathbf{k}'_1]}^{[\mathbf{k}_1]} \hat{\Lambda}_{n_1}^{[\mathbf{k}'_1]} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\} G (\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \Psi_J^{\otimes l} (U_j^l)) \left\{ R_{[\mathbf{k}'_2]}^{[\mathbf{k}_2]} \hat{\Lambda}_{n_2}^{[\mathbf{k}'_2]} [\Psi_J, \nu, \{U_j^{l_i}\}] \right\}$$

The representation $R_{[\mathbf{k}'_1]}^{[\mathbf{k}_1]}$ is a symmetry of parameter space:

$$\hat{\Lambda}_{n_1}^{[\mathbf{k}'_1]} [\Psi_J, \nu, \{U_j^{l_i}\}] \rightarrow R_{[\mathbf{k}'_1]}^{[\mathbf{k}_1]} \hat{\Lambda}_{n_1}^{[\mathbf{k}_1]} [\Psi_J, \nu, \{U_j^{l_i}\}]$$

and the function $G (\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \Psi_J^{\otimes l} (U_j^l))$ transforms as:

$$G (\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \Psi_J^{\otimes l} (U_j^l)) \rightarrow R_{[\mathbf{k}'_1]}^{[\mathbf{k}_1]} G (\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \Psi_J^{\otimes l} (U_j^l)) R_{[\mathbf{k}'_2]}^{[\mathbf{k}_2]}$$

7 Operators perspective: Average values on eigenstates of operators

We present an equivalent description to the one developed in the previous section. We begin with functional states and project onto the lowest eigenspace of operators acting on the state space of $\Psi_{I,\alpha}^{\otimes k}(U_i^k)$. This operator depends on the states of $\Psi_J^{\otimes l}(U_j^l)$ and we denote $\lambda(\Psi_J^{\otimes k})$ the eigenvalues. Degeneracies induce as before parameters $\left\{ \hat{\Delta}_{\infty,\alpha}^{[k_i]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\}_i$, and fields depending on these parameters. The projection amounts to compute averages over these eigenstates. The projection involves computing averages over these eigenstates. This approach has the particularity of dividing the state spaces into slices with respect to the eigenvalues. These spaces are not orthogonal to each other, as different eigenvalues correspond to different operators depending on different states from $\Psi_{J,\alpha}^{\otimes l}(U_j^l)$.

Then, we can define states and operators acting on these subspaces. For a given eigenvalue, we recover a similar formalism to the states and operators Hamiltonian formalism. For a given $\lambda(\Psi_J^{\otimes k})$, we can locally describe the degrees of freedom for a state and an operator by some functional of $\Psi_J^{\otimes k}$ and some $\left\{ \hat{\Delta}_{\infty,\alpha}^{[k_i]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\}_i$. States and operators have internal degrees of freedom defined on some parameter space at $\lambda(\Psi_J^{\otimes k})$. We can then define transitions between states for infinitesimally different eigenvalues $\lambda(\Psi_J^{\otimes k})$ and $\lambda(\Psi_J^{\otimes k}) + \delta\lambda(\Psi_J^{\otimes k})$, that is, transition operators between the different eigenspaces. Summing such transformation defines transitions between state spaces with different eigenvalues.

7.1 Principle

We start with the functionals without projection:

$$\sum_{\alpha} \int a_{l,k}(U_j^l, U_i^k) \Psi_{J,\alpha}^{\otimes l}(U_j^l) \Psi_{I,\alpha}^{\otimes k}(U_i^k) \delta(f_{lk}(U_j^l, U_i^k)) dU_j^l dU_i^k \quad (44)$$

and as above replace:

$$\Psi_{J,\alpha}^{\otimes l}(U_j^l)$$

by:

$$\Psi_{J,\alpha}^{\otimes l}(U_j^l/f_{lk}, \{U_i^k\})$$

to parametrize the functional by the $\{U_i^k\}$ residual parameters. Then, we will replace $\Psi_{I,\alpha}^{\otimes k}(U_i^k)$ in (44) by its the average on the minimal eigenstates of an operator:

$$\mathcal{H} \left(\left\{ \Psi_{I,\alpha}^{\otimes k}(U_i^k), \Pi_{\Psi_{I,\alpha}^{\otimes k}(U_i^k)} \right\}_k, \left\{ \Psi_{J,\alpha}^{\otimes l}(U_j^l/f_{lk}, \{U_i^k\}) \right\}_l \right)$$

with:

$$\Pi_{\Psi_{I,\alpha}^{\otimes k}(U_i^k)} = \frac{\delta}{\delta \Psi_{I,\alpha}^{\otimes k}(U_i^k)}$$

This is a similar approach to the previous one, but it replaces the formalism of saddle point by some projection over background eigenstates.

7.2 Average over eigenspaces

If the operator considered has only one eigenstate, this one has the form:

$$F_0 \left[\left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\} \right] \quad (45)$$

state functional of $\Psi_{I,\alpha}^{\otimes k}$ depending on a given $\Psi_{J,\alpha}^{\otimes l}$. The functional includes the constraints between $\Psi_{I,\alpha}^{\otimes k}$ and $\Psi_{J,\alpha}^{\otimes l}$.

In general including explicitly the constraint :

$$\Psi_I^{\otimes k} (U_i^k) \delta \left(flk \left(U_j^l, U_i^k \right) \right)$$

state (45) writes:

$$F_0 \left[\left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\} \right] \equiv \int F_0 \left[\left(\left\{ U_i^{k_i} \right\} \right), \left\{ \Psi_{J,\alpha}^{\otimes l_i}, U_i^{k_i} \right\} \right] \otimes_i \Psi_{I,\alpha}^{\otimes k_i} \left(U_i^{k_i} \right) \delta \left(flk_i \left(U_j^l, U_i^{k_i} \right) \right) dU_i^{k_i}$$

As before, the tensor \otimes stands for series of products of identical copies of fields, or in terms of states series of products of realizations. The dependency in $U_i^{k_i}$ depicts the constraints between $\left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\}$ and $\Psi_{I,\alpha}^{\otimes k_i} \left(U_i^{k_i} \right)$. This rewrites as series expansion:

$$\begin{aligned} & F_0 \left[\left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\} \right] \\ &= \sum_{s,l_1,\dots,l_s} d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right) d \left(\left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{k_i, l_i} \right) F_0^{(s)} \left[\left\{ U_j^{l_i} \right\}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\} \right] \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \otimes_i \Psi_{I,\alpha}^{\otimes k_i} \left(U_i^{k_i} \right) \end{aligned}$$

Including symetries and degeneracy amounts to introduce parameters $\left\{ \hat{\underline{\Lambda}}_{\infty,\alpha}^{[k_i]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\}_i$ and given a state ν for $\Psi_{J,\alpha}^{\otimes l}$, th states $F_0 \left[\left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\} \right]$ becomes:

$$\begin{aligned} & F_0^{\hat{\underline{\Lambda}}} \left[\left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\}, v \right] \tag{46} \\ & \equiv \sum_{s,l_1,\dots,l_s} d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right) d \left(\left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{k_i, l_i} \right) F_0^{(s)} \left[\left\{ U_j^{l_i} \right\}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\} \right] \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \otimes_i \Psi_{I,\alpha}^{\hat{\underline{\Lambda}}^{-1} \otimes k_i} \left(U_i^{k_i} \right) \end{aligned}$$

where $\Psi_{I,\alpha}^{\hat{\underline{\Lambda}}^{-1} \otimes k_i}$ are the transformd fields induced by the inverse transformation $\hat{\underline{\Lambda}}^{-1}$. The average values of $\Psi_I^{\otimes k} (U_i^k)$ in these states are:

$$\langle \Psi_I^{\otimes k} (U_i^k) \rangle_{\hat{\underline{\Lambda}}} = \int \Psi_I^{\otimes k} (U_i^k) \left| F_0^{\hat{\underline{\Lambda}}} \left[\left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\}, v \right] \right|^2 \prod \mathcal{D} \left\{ \Psi_{I,\alpha}^{\otimes k_i} \right\}$$

Then, using a change of variabl:

$$\Psi_{I,\alpha}^{\hat{\underline{\Lambda}}^{-1} \otimes k_i} \rightarrow \Psi_{I,\alpha}^{\otimes k_i}$$

we find ultimately the averages:

$$\langle \Psi_I^{\otimes k} (U_i^k) \rangle_{\hat{\underline{\Lambda}}} = \left\langle \Psi_I^{\hat{\underline{\Lambda}} \otimes k} (U_i^k) \right\rangle_{\hat{\underline{\Lambda}}} \equiv v_I^{\otimes k} \left(U_i^k, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\}, \hat{\underline{\Lambda}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \tag{47}$$

We rewitr the functional (44) using (47) in appendix 5. As before if the set $\hat{\underline{\Lambda}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ are independent of the realization, it becomes:

$$\begin{aligned} & F_{f,lin} \left(\left\{ \Psi_J^{\otimes l} (U_j^l) \right\}_l \right) \tag{48} \\ &= \int \bar{g} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\Lambda}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right) \right) \Psi_J^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\underline{\Lambda}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right) \end{aligned}$$

where the expression for $\Psi_J^{\otimes \sum l_i}$ and \bar{g} are given in appendix 5. The parametr $\lambda \left(\Psi_J^{\otimes k} \right)$ is the genval considered. The functn \bar{g} is an averaged functional.

Appendx 5 also provides exprssion for projected functionals over space of vector valued eign-
valuess. Functional (48) is similar to previous section and involves the effective field:

$$\Psi_J^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right) \quad (49)$$

the differenc with the previous sections is tht prmtrs $\hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ have been decomposd in $\hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$ nd $\lambda \left(\Psi_J^{\otimes k} \right)$.

7.3 Variations

Once the effective field (49) is obtained, the same principles as in the first approach apply. The independenc t frst rdr f prmtrs with rspct to $\Psi_J^{\otimes \sum l_i}$ yields the flwng qtn, smlr t (22) where the role of $\lambda \left(\Psi_J^{\otimes k} \right)$ is distinguishd:

$$\begin{aligned} 0 &= \int v \left(U_j^{l_i} \right) \frac{\delta \hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]}{\delta \Psi_J^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} \nabla_{\hat{\mathbf{\Lambda}}_\infty^{[k]}} \Psi_J^{\otimes \left(\sum_i l_i \right)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right) \\ &+ \int v \left(U_j^{l_i} \right) \frac{\delta \hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]}{\delta \Psi_J^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} \nabla_{\lambda \left(\Psi_J^{\otimes k} \right)} \Psi_J^{\otimes \left(\sum_i l_i \right)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right) \end{aligned} \quad (50)$$

and if averaged fields are considerd, we find an equatn similar to (30):

$$\begin{aligned} 0 &= \int v \left(\left(U_j^{l_i} \right) \right) \frac{\delta \hat{\mathbf{\Lambda}}^{[k]} \left[\Psi_J, \nu, U_j^{l_i} \right]}{\delta \Psi_J^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} \nabla_{\hat{\mathbf{\Lambda}}^{[k]}} \Psi_J^{\otimes \left(\sum_i l_i \right)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{\Lambda}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), \underline{V}, v \right) \\ &+ \int v \left(\left(U_j^{l_i} \right) \right) \frac{\delta \hat{\mathbf{\Lambda}}^{[k]} \left[\Psi_J, \nu, U_j^{l_i} \right]}{\delta \Psi_J^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} \nabla_{\lambda \left(\Psi_J^{\otimes k} \right)} \Psi_J^{\otimes \left(\sum_i l_i \right)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{\Lambda}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), \underline{V}, v \right) \\ &+ \int v \left(\left(U_j^{l_i} \right) \right) \left(\epsilon \frac{\delta \underline{V}}{\delta \Psi_J^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} - h \left(\underline{V}, \Psi_J^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right) \right) \right) \Psi_J^{\otimes \left(\sum_i l_i \right)} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\mathbf{\Lambda}}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \underline{V}, v \right) \end{aligned} \quad (51)$$

where \underline{V} is similar to V in the derivation of (30) but with the parameters restricted to $\hat{\mathbf{\Lambda}}_\infty^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right]$.

7.4 Fixed $\lambda \left(\Psi_J \right)$ slices

We consider the projection of the constraint on background states such that, at the lowest order:

$$\begin{aligned} 0 &= \left(\left\{ \hat{\mathbf{\Lambda}}_{n_1}^{[k_1]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} M \left(\left\{ \left[\mathbf{k}_1 \right], \left[\mathbf{k}_2 \right] \right\}, \Psi_J, \Psi_J^{\otimes l} \left(U_j^{l_i} \right) \right) \cdot \left\{ \hat{\mathbf{\Lambda}}_{n_2}^{[k_2]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} \\ &+ N \left\{ \left[\mathbf{k}_1 \right] \right\}, \Psi_J, \Psi_J^{\otimes l} \left(U_j^{l_i} \right) \cdot \left\{ \hat{\mathbf{\Lambda}}_{n_1}^{[k_1]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} + \alpha \left(\Psi_J \right) - \lambda \left(\Psi_J \right) \end{aligned}$$

implements $\lambda \left(\Psi_J \right)$ bounded from below. A change of variable leads to:

$$\left\{ \hat{\mathbf{\Lambda}}_{n_1}^{[k_1]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} M \left(\left\{ \left[\mathbf{k}_1 \right], \left[\mathbf{k}_2 \right] \right\}, \Psi_J, \Psi_J^{\otimes l} \left(U_j^{l_i} \right) \right) \cdot \left\{ \hat{\mathbf{\Lambda}}_{n_2}^{[k_2]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} + \alpha \left(\Psi_J \right) - \lambda \left(\Psi_J \right) = 0$$

This can be also written in a form where the parameters are not distinguished:

$$\left(\hat{\underline{\mathbf{A}}}_{n_1}^{[k_1]}, \lambda(\Psi_J) \right)$$

by a change of variable:

$$\begin{aligned} V_{[\mathbf{k}'_1]}^{[k_0]} \hat{\underline{\mathbf{A}}}_{n_1}^{[\mathbf{k}'_1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] &\rightarrow \lambda(\Psi_J) \\ V_{[\mathbf{k}'_1]}^{[k_1]/k_0} \hat{\underline{\mathbf{A}}}_{n_1}^{[\mathbf{k}'_1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] &\rightarrow \hat{\underline{\mathbf{A}}}_{n_1}^{[k_1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \end{aligned}$$

leading to:

$$\hat{\underline{\mathbf{A}}}_{n_1}^{[k_1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] M(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \Psi_J^{\otimes l} (U_j^l)) \cdot \hat{\underline{\mathbf{A}}}_{n_2}^{[k_2]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] + \alpha(\Psi_J) - \lambda(\Psi_J) = 0 \quad (52)$$

Recall that $\alpha(\Psi_J)$ is a functional of Ψ_J :

$$\alpha(\Psi_J) = \int \Psi_J^\dagger A \Psi_J + \int \Psi_J^\dagger \Psi_J^\dagger B \Psi_J \Psi_J + \dots$$

and note for later purpose that the number of states satisfying the constraint (52) increases with λ . since the quadratic relation involves tensor products of states. The higher the quadratic quantity, the higher the number of states satisfying the equation:

$$\# \left\{ \hat{\underline{\mathbf{A}}}_{n_1}^{[k_1]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] M(\{[\mathbf{k}_1], [\mathbf{k}_2]\}, \Psi_J, \Psi_J^{\otimes l} (U_j^l)) \cdot \hat{\underline{\mathbf{A}}}_{n_2}^{[k_2]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] + \alpha(\Psi_J) = \lambda \right\} \nearrow \text{ fr } \alpha(\Psi_J) \nearrow$$

8 States and operators

We detail the states on the subspaces defined by parameters $\left(\hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \lambda(\Psi_J^{\otimes k}) \right)$. This will allow to compute transitions between those spaces.

8.1 Functionals for projected fields

The states after projections are described by fields of the type:

$$\int v \left(\left(U_j^{l_i} \right) \right) \Psi_J^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \lambda(\Psi_J^{\otimes k}), v \right) d \left(U_j^{l_i} \right)$$

and the values $\hat{\underline{\mathbf{A}}}_{\infty}^{[k_i]}$ and $\lambda(\Psi_J^{\otimes k})$ r functionals of the fld and state ν . Note that:

$$\left(\hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \lambda(\Psi_J^{\otimes k}) \right) \equiv \hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right]$$

For functionals, we replace $v \left(\left(U_j^{l_i} \right) \right)$:

$$v \left(\left(U_j^{l_i} \right) \right) \rightarrow v \left(\left(U_j^{l_i}, \hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right), \lambda(\Psi_J^{\otimes k}) \right)$$

We assume that a change of variable in integrals leaves unchanged the boundary condition and we can replace:

$$\hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \rightarrow \hat{\underline{\mathbf{A}}}_{\infty}^{[k]} \left[\Psi_J, \nu \right]$$

so that a general functional writes:

$$\begin{aligned}
& \sum_v \int v \left((U_j^{l_i}), \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], \lambda(\Psi_J^{\otimes k}) \right) \\
& \times g \left(v, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], \lambda(\Psi_J^{\otimes k}) \right) \\
& \Psi_J^{\otimes \sum l_i} \left(\{U_j^{l_i}\}, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}], \lambda(\Psi_J^{\otimes k}), v \right) \times d\hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu, \{U_j^{l_i}\}] d(U_j^{l_i}) \\
= & \sum_v \int v \left((U_j^{l_i}), \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \lambda(\Psi_J^{\otimes k}) \right) \\
& g \left(v, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \lambda(\Psi_J^{\otimes k}) \right) \Psi_J^{\otimes \sum l_i} \left(\{U_j^{l_i}\}, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \lambda(\Psi_J^{\otimes k}), v \right) d\hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu] d(U_j^{l_i})
\end{aligned} \tag{53}$$

This formula describes functionals of the field:

$$\begin{aligned}
& \Psi_J^{\otimes \sum l_i} \left(\hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \lambda(\Psi_J^{\otimes k}), v \right) \\
= & \int v \left((U_j^{l_i}), \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \lambda(\Psi_J^{\otimes k}) \right) \Psi_J^{\otimes \sum l_i} \left(\{U_j^{l_i}\}, \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu], \lambda(\Psi_J^{\otimes k}), v \right) d(U_j^{l_i})
\end{aligned} \tag{54}$$

This effective field will be the bss to describe the states on subspaces defined by $\lambda(\Psi_J^{\otimes k})$.

8.2 States associated to functionals

This description can be translated in terms of states for the field (54). Functionals (53) are equivalent to states:

$$\left| \nu, \left\{ \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu] \right\}_k, \lambda(\nu) \right\rangle$$

with v is a functional state of Ψ_J . in the sequel, we omit the subscript k in $\left\{ \hat{\mathbf{A}}_\infty^{[k]} [\Psi_J, \nu] \right\}_k$.

Assume such state v can be parametrized through some parameters space $U^{(j)}$. The projected states become:

$$\left| U^{(j)}, \left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\}, \lambda(U^{(j)}) \right\rangle$$

where the parameters $U^{(j)}$ are analogous to some $\cup_j U_j^{l_j}$. For example, any functional:

$$v(\Psi_J) = \sum_t v_t(\Psi_J) = \sum_t \int a_t(U_{j_1}, \dots, U_{j_t}) \Psi_J(U_{j_1}, \dots, U_{j_t})$$

where the U_{j_k} er cmpnt f U_j , can be parametrized by $\{a_t(U_{j_1}, \dots, U_{j_t})\}_t$ and thus we cn chs:

$$U^{(j)} = \{a_t(U_{j_1}, \dots, U_{j_t})\}_t$$

Practically:

$$\left| U^{(j)}, \left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\}, \lambda(U^{(j)}) \right\rangle$$

are combination of products of states:

$$\sum \left| U^{(j)} \right\rangle_\gamma \left| \left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\}, \lambda(U^{(j)}) \right\rangle_\gamma$$

Since $U^{(j)}$ nd $\left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\}$ are related through the constraints, we will use these constraints in the sequel to reduce locally the number of parameters.

8.3 Constrained states

If some components $[U^{(j)}]$ of $U^{(j)}$ can be replaced by functions of $\hat{\mathbf{A}}_n^{[k]} [U^{(j)}]$ through the constraints, the states can be rewritten in the following manner. We start by writing the constraints:

$$h_{k_n} \left(\left\{ \hat{\mathbf{A}}_n^{[k]} [U^{(j)}] \right\}, h_p \left((\Psi_J), U^{(j)} \right) \right) = 0$$

so that if $h_p \left((\Psi_J), U_j^l, \nu \right)$ includes combinations of the $a_t (\Psi_J)$, some of them may be replaced by:

$$a_{t'} (\Psi_J) = f_t \left(\left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\} \right)$$

with:

$$\{t'\} \subset \{t\}$$

and the states write in a reduced form:

$$\sum \left| U^{(j)} / [U^{(j)}], [U^{(j)}] \left(\left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\}, \lambda \left(U^{(j)} \right) \right) \right\rangle_\gamma \left| \left\{ \hat{\mathbf{A}}_\infty^{[k]} [U^{(j)}] \right\}, \lambda \left(U^{(j)} \right) \right\rangle_\gamma$$

We will omit the subscript γ in the sequel. Moreover, if the $\hat{\mathbf{A}}^{[k]} [U^{(j)}]$ can be locally identified with fixed parameters $\hat{\mathbf{A}}^{[k]}$, we find the states:

$$\sum \left| U^{(j)} / [U^{(j)}], [U^{(j)}] \left(\left\{ \hat{\mathbf{A}}_\infty^{[k]} \right\}, \lambda \left(U^{(j)} \right) \right) \right\rangle \left| \left\{ \hat{\mathbf{A}}_\infty^{[k]} \right\}, \lambda \left(U^{(j)} \right) \right\rangle \quad (55)$$

In the sequel we replace $\left\{ \hat{\mathbf{A}}_\infty^{[k]} \right\} \rightarrow \left\{ \hat{\mathbf{A}}_\infty \right\}$, that stands for all realization of the $\hat{\mathbf{A}}_\infty^{[k]}$.

8.4 State space for a given λ

Until now, we have considered the state as depending on the states parametrized by $U^{(j)}$. We can reverse this point of view and regroup states such that $\lambda \left(U^{(j)} \right) = \lambda$, for each value of λ .

We first consider linear combinations of states (55) rewritten as:

$$\sum \left| \left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \left(U^{(j)} \right) \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \left(U^{(j)} \right) \right) \right\rangle_{U^{(j)}} \quad (56)$$

with:

$$\left| h \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \left(U^{(j)} \right) \right) \right\rangle_{U^{(j)}} = \left| U^{(j)} / [U^{(j)}], [U^{(j)}] \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \left(U^{(j)} \right) \right) \right\rangle$$

The subscript $U^{(j)}$ reminds that the identification is local, since the $\left\{ \hat{\mathbf{A}}_\infty \right\}$ are functionals of $U^{(j)}$.

Then, we restrict states $U^{(j)}$ to subspaces $U_\lambda^{(j)}$ such that $\lambda \left(U_\lambda^{(j)} \right) = \lambda$, and we write states (56) locally:

$$\left| \left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \right) \right\rangle_{U_\lambda^{(j)}} \equiv \left| \left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}_\infty \right\} \right) \right\rangle_{U_\lambda^{(j)}}$$

where the relation for λ is satisfied:

$$\lambda \left(h \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \right), U_\lambda^{(j)} \right) = \lambda$$

and the constraint writes:

$$h_{k_i} \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \left[\Psi_J, \left(h \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \right), U_\lambda^{(j)} \right) \right] \right\}_i, h_p \left((\Psi_J), \left(h \left(\left\{ \hat{\mathbf{A}}_\infty \right\}, \lambda \right), U_\lambda^{(j)} \right) \right) \right) = 0$$

We also define the space spanned by states:

$$\mathcal{H}_{(\lambda, \{\hat{\underline{\mathbf{A}}}_\infty(U_\lambda^{(j)})\}, U_\lambda^{(j)})} = \left\{ \left| \left\{ \hat{\underline{\mathbf{A}}}_\infty \right\}, \lambda \right\rangle \left| h \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty \right\} \right) \right\rangle_{U_\lambda^{(j)}} \right\}_{U_\lambda^{(j)}}$$

which is also written \mathcal{H}_λ for short. States belonging to this space are series expansions:

$$\sum_{n, U_n^{(j)}} \alpha_n \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty \right\}, \lambda, U_n^{(j)} \right) \delta \left(\lambda \left(h \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty \right\}, \lambda \right), U_n^{(j)} \right) - \lambda \right) \left| \left\{ \hat{\underline{\mathbf{A}}}_\infty \right\}, \lambda \right\rangle \left| h_n \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty \right\} \right) \right\rangle_{U_n^{(j)}}$$

Remark:

1. locally, these states can be generated by a field:

$$\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}_\infty \right\}, \lambda \right)$$

2. Covariantly the states are functionals $\alpha(\Sigma)$ (standing for $\alpha(\Sigma(U^{(j)}))$).

8.5 Operators

In this context, using a basis of states defined by some parameters A , operators write,:

$$[A] \Xi [A']$$

with the notation:

$$[A] \Xi [A'] = \int |A\rangle \Xi(A, A') \langle A'|$$

where:

$$\begin{aligned} [A] &= \left[U^{(j)} / [U^{(j)}], [U^{(j)}] \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda(U^{(j)}) \right), \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda(U^{(j)}) \right] \\ &= \left[h \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda(U^{(j)}) \right)_{U^{(j)}}, \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda(U^{(j)}) \right] \end{aligned}$$

and:

$$\begin{aligned} [A'] &= \left[U^{(j')} / [U^{(j')}], [U^{(j')}] \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]'} \right\}, \lambda(U^{(j')}) \right), \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]'} \right\}, \lambda(U^{(j')}) \right] \\ &= \left[h \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]'} \right\}, \lambda(U^{(j')}) \right)_{U^{(j')}}, \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]'} \right\}, \lambda(U^{(j')}) \right] \end{aligned}$$

The $\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}$ stand for any collection of $\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}$.

In terms of eigenvalues λ and λ' and with the decomposition in terms of $U_\lambda^{(j)}$ this also reduces to:

$$\left[U_j / [U_j], h \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda \right)_{U_\lambda^{(j)}}, \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda \right] \Xi \left[(U_j / [U_j])', h' \left(\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]'} \right\}, \lambda' \right)_{U_{\lambda'}^{(j')}}, \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]'} \right\}, \lambda' \right]$$

The same forms are recovered in part 3 starting directly from states and operators formalism. In the sequel, we write $\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}$ for $\left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}$. Conclude by noting that locally, these states and operator are states build from fields:

$$\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}_\infty^{[k_i]} \right\}, \lambda \right)$$

9 Transitions between spaces \mathcal{H}_λ

In this section we consider the state transitions between states due to transition prtr btwn spaces $\mathcal{H}_{\lambda_0} \rightarrow \mathcal{H}_\lambda$.

9.1 Transformation $\mathcal{H}_{\lambda_0} \rightarrow \mathcal{H}_\lambda$

Consider two eigenvalues λ_0 and λ . We assume that there are isomorphism of spaces:

$$\mathcal{H}_{(\lambda_0, \{\hat{\mathbf{A}}^{[k_i]}(U_{\lambda_0}^{(j)}), U_{\lambda_0}^{(j)}\})} \simeq \mathcal{H}_{(\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)}\})}$$

where:

$$\mathcal{H}_{(\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)}\})} = \left\{ \left| \lambda, \{\hat{\mathbf{A}}^{[k_i]}\} \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U_\lambda^{(j)}} \right\}_{U_\lambda^{(j)}}$$

given by:

$$T_{\lambda_0\lambda} : \mathcal{H}_{(\lambda_0, \{\hat{\mathbf{A}}^{[k_i]}(U_{\lambda_0}^{(j)}), U_{\lambda_0}^{(j)}\})} \rightarrow \mathcal{H}_{(\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)}\})}$$

so that:

$$T_{\lambda_0\lambda} \left| \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_0 \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_0 \right) \right\rangle_{U_0^{(j)}} \in \left\{ \left| \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U^{(j)}} \right\}$$

Remark

The transformation can be considered covariantly by replacng:

$$\left| \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \left| H \left(\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U_0^{(j)}} \rightarrow \left| \Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_0 \right\rangle \left| H \left(\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U_0^{(j)}}$$

where $\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}$ are hypersurface of $\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}$. The transformation rewrtts:

$$T_{\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_0 \Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}} \left| \Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U^{(j)}} \in \left\{ \left| \Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_0 \right\rangle \left| h \left(\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U_0^{(j)}} \right\}$$

9.2 Amplitudes

To describe the amplitudes of transitions between states, we assume that infinitesimlly, the transformation has the form:

$$\delta T_{\lambda\lambda+\delta\lambda} = \delta\lambda \bar{V} \left(U_\lambda^{(j)}, U_{\lambda+\delta\lambda}^{(j)} \right)$$

where operator $\delta T_{\lambda\lambda+\delta\lambda}$ is a transformation:

$$\delta T_{\lambda\lambda+\delta\lambda} : \mathcal{H}_{(\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)}\})} \rightarrow \mathcal{H}_{(\lambda+\delta\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_{\lambda+\delta\lambda}^{(j)}), U_{\lambda+\delta\lambda}^{(j)}\})}$$

To find the transformation $\delta T_{\lambda\lambda+\delta\lambda}$ recall that the states of $\mathcal{H}_{(\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)}\})}$ can be built from fields:

$$\Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right)$$

The amplitudes:

$$U_j \left\langle \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) \left| \delta T_{\lambda\lambda+\delta\lambda} \right| \Psi_J \left((U_j)' / [(U_j)'], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right) \right\rangle_{(U_j)'}$$

are computed by pulling back the field state:

$$\left\langle \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) \right\rangle \rightarrow \left\langle \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \right\rangle$$

from $\mathcal{H}_{(\lambda+\delta\lambda, \{\hat{\mathbf{\Delta}}^{[k_i]}(U_{\lambda+\delta\lambda}^{(j)}), U_{\lambda+\delta\lambda}^{(j)}\})}$ to $\mathcal{H}_{(\lambda, \{\hat{\mathbf{\Delta}}^{[k_i]}(U_{\lambda}^{(j)}), U_{\lambda}^{(j)}\})}$ by some parallel transport operator $P_{\lambda\lambda+\delta\lambda}$ and then computing the matrix elements of the pulled-backed transformation:

$$T_{\lambda} : \mathcal{H}_{(\lambda, \{\hat{\mathbf{\Delta}}^{[k_i]}(U_{\lambda}^{(j)}), U_{\lambda}^{(j)}\})} \rightarrow \mathcal{H}_{(\lambda, \{\hat{\mathbf{\Delta}}^{[k_i]}(U_{\lambda}^{(j)}), U_{\lambda}^{(j)}\})}$$

so that the transition is the compositn:

$$\delta T_{\lambda\lambda+\delta\lambda} = P_{\lambda\lambda+\delta\lambda} T_{\lambda}$$

9.2.1 Global identification

We assume in first approximation that the parameters $\{\hat{\mathbf{\Delta}}^{[k_i]}\}$ are global. The pull back of the fld state $P_{\lambda\lambda+\delta\lambda}$ is trivial: we can identify the various spaces, the transport involves standard "derivatives". Formally, it is generated by the operatr:

$$\begin{aligned} & \left(\Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) - \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \right) \frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)} \\ &= \frac{d}{d\lambda} \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)} \end{aligned}$$

that can be exponentiated:

$$P_{\lambda\lambda+\delta\lambda} = \exp \left(\int i\delta\lambda \frac{d}{d\lambda} \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)} \mathcal{D} \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \right)$$

Assuming a standard form for the transformation part $\delta T_{\lambda\lambda+\delta\lambda}$ with $\frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)}$ separable from other variables:

$$\begin{aligned} \delta T_{\lambda\lambda+\delta\lambda} &= V \left(\Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right), \nabla_{\left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}} \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \right) \\ &+ V_2 \left(\frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)} \right) \end{aligned}$$

the composition:

$$P_{\lambda\lambda+\delta\lambda} \delta T_{\lambda\lambda+\delta\lambda}$$

can be simplified by the computation of the matrices element of:

$$\begin{aligned} & \exp \left(\int i\delta\lambda \frac{d}{d\lambda} \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)} \mathcal{D} \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right) \right) \\ & \times \exp \left(iV_2 \left(\frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{\Delta}}^{[k_i]} \right\}, \lambda \right)} \right) \right) \end{aligned}$$

so that the transition is generated by:

$$\begin{aligned}
& S(\Psi_J) \\
&= S\left(\frac{d}{d\lambda}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right), \nabla_{\{\hat{\Delta}^{[k_i]}\}}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right), \Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right)\right)
\end{aligned}$$

9.2.2 Local identification

When the identifications are local, the transport is non trivial. Changing the variables, modifies the states. The transport is generated by some covariant derivatives. We show in appendix 6 that the general form for the amplitudes of this operator are generated by some functional:

$$\begin{aligned}
& S(\Psi_J) \\
&= S\left(\nabla_{\lambda}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right), \nabla_{\{\hat{\Delta}^{[k_i]}\}}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right), \Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right)\right)
\end{aligned}$$

with:

$$\begin{aligned}
& \nabla_{\{\hat{\Delta}^{[k_i]}\}}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right) \\
&= \nabla_{\{\hat{\Delta}^{[k_i]}\}}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right) + \left(\left(A_{\{\hat{\Delta}^{[k_i]}\}}\right)_k^{k'}(U_j/[U_j])\Psi_J\left((U_j/[U_j])_{k'}, \{\hat{\Delta}^{[k_i]}\}, \lambda\right)\right)_k
\end{aligned}$$

and:

$$\begin{aligned}
& \delta\lambda\nabla_{\lambda}\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right) \\
&= \frac{\partial\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right)}{\partial\lambda} + \left((A_{\lambda})_k^{k'}(U_j/[U_j])\Psi_J\left((U_j/[U_j])_{k'}, \{\hat{\Delta}^{[k_i]}\}, \lambda\right)\right)_k
\end{aligned}$$

are some covariant derivatives with connection matrices:

$$\left(A_{\{\hat{\Delta}^{[k_i]}\}}, (A_{\lambda})\right)$$

connecting the states with different values of λ .

The matrix elements of transitions $T_{\lambda\lambda'}$ are obtained by exponentiation of S . Actually:

$$T_{\lambda\lambda'} = \prod (1 + \delta T_{\lambda\lambda + \delta\lambda}) \tag{57}$$

so that:

$$\begin{aligned}
& \langle\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda'\right) | T_{\lambda\lambda'} | \Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right)\rangle \\
&= \langle\Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda'\right) | \exp\left(i \int \delta\lambda S(\Psi_J) \delta\lambda\right) | \Psi_J\left(U_j/[U_j], \{\hat{\Delta}^{[k_i]}\}, \lambda\right)\rangle
\end{aligned} \tag{58}$$

9.2.3 Remark 1

These amplitudes between states can also be rewritten covariantly:

$$\left\langle \Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left\langle H \left(\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right|_{U_0^{(j)}} \exp \left(iS \left(\Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right) \right) \left| \Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \left| H \left(\Sigma \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right) \right\rangle_{U_0^{(j)}}$$

with local invariance with respect to transformation of $\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}$.

9.2.4 Remark 2

The connection

$$\Gamma \equiv \left(\left(A_{\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}} \right) (U_j / [U_j]), (A_\lambda) (U_j / [U_j]) \right)$$

is itself a field dependent object, since it is derived from the constraint. Given the form of the constraint, it has the form:

$$R(\Gamma) = F \left(\Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right), \nabla_{\hat{\mathbf{A}}^{[k_i]}}^\Gamma \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right) \right)$$

and some transition amplitude should be associated to this quantity.

9.2.5 Non uniqueness of connection and modification of amplitude

Even if connections are considered as inert, there should be several possible connections corresponding to the transport:

$$\left\langle \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) \right| \rightarrow \left\langle \Psi_J \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right) \right|$$

since there should be multiple way to send a state with parameter $\left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda + \delta\lambda \right)$ to $\left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right)$. This corresponds to the fact that some relative dimension may arise between these two spaces of parameters:

$$\dim \left(\left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) / \left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right) \right) > 0$$

and that there are many maps sending $\left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right)$ to $\left(U_j / [U_j], \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda + \delta\lambda \right)$.

The amplitudes thus should modify (58). Actually, we first consider that $\delta T_{\lambda\lambda+\delta\lambda}$ is replaced by the set:

$$\left(\delta T_{\lambda\lambda+\delta\lambda}^{((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})} \right)_{(U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda}}$$

accounting for all possible maps from one space to the other. Then (57) is modified by taking into account of these multiple paths, so that we replace:

$$1 + \delta T_{\lambda\lambda+\delta\lambda} \rightarrow \sum_{(U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda}} 1 + \delta T_{\lambda\lambda+\delta\lambda}^{((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})} \quad (59)$$

and the transitions are:

$$T_{\lambda\lambda'} = \prod \left(1 + \sum_{(U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda}} \delta T_{\lambda\lambda+\delta\lambda}^{((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})} \right)$$

In terms of amplitudes this becomes:

$$\begin{aligned}
& \langle \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda') | T_{\lambda\lambda'} | \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) \rangle \\
&= \langle \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda + \delta\lambda) | \\
& \quad \times \prod_{\delta\lambda} \prod_{(U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda}} \exp \left(i\delta\lambda S \left(\Psi_J, \left(A_{\{\hat{\mathbf{A}}^{[k_i]}\}}, (A_\lambda) \right)^{((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})} \right) \right) | \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) \rangle
\end{aligned} \tag{60}$$

The upperscript $((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})$ reminds that the connections $\left(A_{\{\hat{\mathbf{A}}^{[k_i]}\}}, (A_\lambda) \right)$ depend on the path chosen. Assuming that we can replace in average:

$$\left(A_{\{\hat{\mathbf{A}}^{[k_i]}\}}, (A_\lambda) \right)^{((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})} \rightarrow \left(A_{\{\hat{\mathbf{A}}^{[k_i]}\}}, (A_\lambda) \right)$$

the transition becomes:

$$\begin{aligned}
& \langle \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda') | T_{\lambda\lambda'} | \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) \rangle \\
&= \langle \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda + \delta\lambda) | \\
& \quad \times N((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda}) \prod_{\delta\lambda} \exp \left(i\delta\lambda S \left(\Psi_J, \left(A_{\{\hat{\mathbf{A}}^{[k_i]}\}}, (A_\lambda) \right) \right) \right) | \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) \rangle
\end{aligned}$$

where $N((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})$ is the number, or the volume of set of maps $((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+d\lambda})$.

Ultimately, the transitions become:

$$\begin{aligned}
& \langle \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda') | T_{\lambda\lambda'} | \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) \rangle \\
&= \langle \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda + \delta\lambda) | \\
& \quad \times N((U_j)_\lambda \leftrightarrow (U_j)_{\lambda'}) \exp \left(i \int \delta\lambda S \left(\Psi_J, \left(A_{\{\hat{\mathbf{A}}^{[k_i]}\}}, (A_\lambda) \right) \right) \right) | \Psi_J (U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) \rangle
\end{aligned} \tag{61}$$

with $N((U_j)_\lambda \leftrightarrow (U_j)_{\lambda'})$ the number, or the volume of set of maps $((U_j)_\lambda \leftrightarrow (U_j)_{\lambda'})$.

9.3 Particular states

We consider a particular case in which a state can be decomposed into two separate sets:

$$\begin{aligned}
U_0^{(j/p)} &= (U_0^{(j/p)}, U_0^{(jp)}) \\
U^{(j/p)} &= (U^{(j/p)}, U^{(jp)})
\end{aligned}$$

where $\lambda(U_0^{(j)}) = \lambda(U_0^{(j/p)})$ and $\lambda(U^{(j)}) = \lambda(U^{(j/p)})$. It describes a decomposition in which $U_0^{(j/p)}$ is much larger than $U^{(jp)}$ and determines the parameter space, while the stes defined by $U^{(jp)}$ consists in some system evolving in these parameter space. We consider the states:

$$\left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \lambda \right), \hat{U}^{(j)} \right\rangle_{U^{(j)}} = \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{/p}, \lambda \right\rangle \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_p \right\rangle \left| h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_p, \lambda \right) \right\rangle_{U^{(j)}}$$

This corresponds to sum tensor products of fields states. The parameter space decomposes into a large set $\{\hat{\mathbf{A}}^{[k_i]}\}_{/p}$ and a small set $\{\hat{\mathbf{A}}^{[k_i]}\}_p$ describing the system studied.

$$\begin{aligned} |\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}\rangle |h(\{\hat{\mathbf{A}}^{[k_i]}\})\rangle_{U^{(j)}} &= |\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}_{/p}\rangle |h(\{\hat{\mathbf{A}}^{[k_i]}\}_{/p})\rangle_{U^{(j)}} |\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}_p\rangle |h(\{\hat{\mathbf{A}}^{[k_i]}\}_p)\rangle_{U^{(j)}} \\ &= |\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}\rangle |h(\{\hat{\mathbf{A}}^{[k_i]}\}_{/p})\rangle_{U^{(j)}} |h(\{\hat{\mathbf{A}}^{[k_i]}\}_p)\rangle_{U^{(j)}} \end{aligned}$$

whr:

$$V(\{\hat{\mathbf{A}}^{[k_i]}\}_p) \ll V(\{\hat{\mathbf{A}}^{[k_i]}\}_{/p})$$

and:

$$\Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda) = \left\{ \Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}_p, \lambda), \Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}_{/p}, \lambda) \right\}$$

and:

$$S(\Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}, \lambda)) = S_p(\Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}_p, \lambda)) + S_{/p}(\Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}_{/p}, \lambda))$$

Then we can factor:

$$\left\langle \Sigma \{\hat{\mathbf{A}}^{[k_i]}\}_p \left| \left\langle H(\Sigma \{\hat{\mathbf{A}}^{[k_i]}\}_p) \right|_{U_0^{(j)}} \exp\left(i S_p(\Psi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[k_i]}\}_p, \lambda))\right) \right| \Sigma \{\hat{\mathbf{A}}^{[k_i]}\}_p \right| H(\Sigma \{\hat{\mathbf{A}}^{[k_i]}\}_p) \right\rangle_{U_0^{(j)}}$$

and interaction with full field arises through Γ .

9.4 Transitions for operators:

We consider some operator:

$$\mathcal{H}(\lambda, \{\hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)}\}) \simeq \mathcal{H}(\lambda', \{\hat{\mathbf{A}}^{[k_i]}(U_{\lambda'}^{(j)}), U_{\lambda'}^{(j)}\})$$

with matrices elements:

$$\left[\lambda', \{\hat{\mathbf{A}}'^{[k_i]}\}, h(\{\hat{\mathbf{A}}'^{[k_i]}\}) \right]_{U^{(j)'}} \Phi \left[\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}, h(\{\hat{\mathbf{A}}^{[k_i]}\}) \right]_{U^{(j)}}$$

We consider that these matrices elements can be decomposed between $/p$ and p states:

$$\left[\lambda', \{\hat{\mathbf{A}}'^{[k_i]}\}, h(\{\hat{\mathbf{A}}'^{[k_i]}\}_{/p}), h(\{\hat{\mathbf{A}}'^{[k_i]}\}_p) \right]_{U^{(j)'}} \Phi \left[\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}, h(\{\hat{\mathbf{A}}^{[k_i]}\}_{/p}), h(\{\hat{\mathbf{A}}^{[k_i]}\}_p) \right]_{U^{(j)}}$$

for $\lambda(U^{(j)}) \simeq \lambda((U^{(j)})')$. Here $|U^{(j)p}\rangle$ is at scale of some system. We assume that $\dim U^{(j)p} \ll \dim U^{(j)}$, so that $\lambda^{(i)}(U^{(j)})$ and $\{\hat{\mathbf{A}}^{[k_i]}\}$ can be considered independent from $U^{(j)p}$ and $\lambda^{(i)}(U^{(j)}) = \lambda^{(i)}$. For operators depending only on the $U^{(j)p}$, we can discard the $U^{(j)/p}$, $(U^{(j)/p})'$

Rstricting to:

$$\left[\lambda', \{\hat{\mathbf{A}}'^{[k_i]}\}, h(\{\hat{\mathbf{A}}'^{[k_i]}\}_p) \right]_{U^{(j)'}} \Phi \left[\lambda, \{\hat{\mathbf{A}}^{[k_i]}\}, h(\{\hat{\mathbf{A}}^{[k_i]}\}_p) \right]_{U^{(j)}}$$

One can also write the states as function of some state $|U^{(j)p}\rangle$ by using:

$$\left| h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_p \right) \right\rangle = \int g^{(j)} \left(U^{(j)p}, h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_p \right) \right) |U^{(j)p}\rangle dU^{(j)p}$$

and:

$$\begin{aligned} & \left[\lambda', \left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\}, U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[\lambda, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, U^{(j)p} \right]_{U^{(j)}} \\ &= \left| \left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\} \right\rangle \left[\lambda', U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[\lambda, U^{(j)p} \right]_{U^{(j)}} \left\langle \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\} \right| \\ &= \left| \left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\} \right\rangle \int g^{(j)\dagger} \left(\left(U^{(j)p} \right)', h \left(\left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\}_p \right) \right) \\ &\quad \times \left[\lambda', U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[\lambda, U^{(j)p} \right]_{U^{(j)}} g^{(j)} \left(U^{(j)p}, h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_p \right) \right) dU^{(j)p} d \left(U^{(j)p} \right)' \left\langle \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\} \right| \end{aligned}$$

writing:

$$\begin{aligned} & \left[\lambda_0, U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[\lambda_0, U^{(j)p} \right]_{U^{(j)}} = \left[U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[U^{(j)p} \right]_{U^{(j)}} \\ &= \left| \left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\} \right\rangle \int g^\dagger \left(\left(U^{(j)p} \right)', h \left(\left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\}_p \right) \right) \\ &\quad \exp \left(-i \left(\lambda \left(U^{(j)'} \right) - \lambda_0 \right) V \right) \left[U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[U^{(j)p} \right]_{U^{(j)}} \exp \left(i \left(\lambda \left(U^{(j)} \right) - \lambda_0 \right) V \right) \\ &\quad \times g \left(U^{(j)p}, h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_p \right) \right) dU^{(j)p} d \left(U^{(j)p} \right)' \left\langle \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\} \right| \end{aligned}$$

The usual transformation corresponds to the case where V is diagonal in the basis $U^{(j)p}$ and $\lambda(U^{(j)'}) = \lambda(U^{(j)})$:

$$\begin{aligned} & \exp \left(-i \left(\lambda \left(U^{(j)'} \right) - \lambda_0 \right) V \right) \left[U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[U^{(j)p} \right]_{U^{(j)}} \exp \left(i \left(\lambda \left(U^{(j)} \right) - \lambda_0 \right) V \right) \\ &= \exp \left(-i \left(\lambda \left(U^{(j)} \right) - \lambda_0 \right) \left(V \left(\left(U^{(j)p} \right)' \right) - V \left(U^{(j)p} \right) \right) \right) \left[U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[U^{(j)p} \right]_{U^{(j)}} \end{aligned}$$

and:

$$\begin{aligned} &= \left| \left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\} \right\rangle \int g^\dagger \left(\left(U^{(j)p} \right)', h \left(\left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\}_p \right) \right) g \left(U^{(j)p}, h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_p \right) \right) \\ &\quad \exp \left(-i \left(\lambda \left(U^{(j)} \right) - \lambda_0 \right) \left(V \left(\left(U^{(j)p} \right)' \right) - V \left(U^{(j)p} \right) \right) \right) \left[U^{(j)p'} \right]_{U^{(j)'}} \Phi \left[U^{(j)p} \right]_{U^{(j)}} \\ &\quad \times dU^{(j)p} d \left(U^{(j)p} \right)' \left\langle \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\} \right| \end{aligned} \tag{62}$$

9.4.1 Example

For operators:

$$\left[\lambda^{(i)}, U_j^{p'} \right] \Psi \left[\lambda^{(i)}, U_j^p \right] = \left| U_j^{p+k} \right\rangle \langle U_j^p |$$

formula (62) writes:

$$\left| \left\{ \hat{\underline{\mathbf{A}}}^{\prime[k_i]} \right\} \right\rangle \int g^\dagger \left(U_j^{p+k}, h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{p+k} \right) \right) \left| U_j^{p+k} \right\rangle \langle U_j^p | g \left(U_j^p, h \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_p \right) \right) dU_j^p dU_j^{p+k} \left\langle \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\} \right|$$

In the usual set up, we can consider transltn invarianc:

$$g^\dagger \left(U_j^{p+k}, h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{p+k} \right) \right) g \left(U_j^p, h \left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_p \right) \right) = \exp \left(i U_j^{p+k/p} \cdot \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{p+k/p} \right)$$

where $p + k/p$ stands for the set $\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{p+k}$, parameters $\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_p$ excluded, so that the operator is gvn by:

$$\left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \int \exp \left(i U_j^{p+k/p} \cdot \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{p+k/p} \right) \left| U_j^{p+k} \right\rangle \langle U_j^p | dU_j^p dU_j^{p+k/p} \left\langle \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right|$$

This is similar to a usual field description in terms of creatn and annihilation operators.

10 Number of states and connections amplitude

10.1 Number of states

We write the constraint:

$$\left\{ \hat{\mathbf{A}}^{[k_i]} [\Psi_J, \nu, U_j^l] \right\}_i M(\{[k_i], [k_{i'}]\}, \Psi_J, \Psi_J^{\otimes l}(U_j^l)) \cdot \left\{ \hat{\mathbf{A}}^{[k_{i'}]} [\Psi_J, \nu, U_j^l] \right\}_i + \alpha(\Psi_J) - \lambda(\Psi_J) = 0$$

where:

$$\alpha(\Psi_J) = \int \Psi_J^\dagger A \Psi_J + \int \Psi_J^\dagger \Psi_J^\dagger B \Psi_J \Psi_J + \dots$$

We assume that the number of states satisfying $\alpha(\Psi_J) = \lambda$ increases with λ :

$$\#N(U_\lambda^{(j)}) = \# \{ S(\Psi_J), \alpha(S(\Psi_J)) = \lambda \} \nearrow \text{ fr } \alpha(\Psi_J) \nearrow$$

with $S(\Psi_J)$ denoting any state of Ψ_J . This is justified since the constraint:

$$\lambda(\Psi_J) = \left\{ \hat{\mathbf{A}}^{[k_i]} [\Psi_J, \nu, U_j^l] \right\}_i M(\{[k_i], [k_{i'}]\}, \Psi_J, \Psi_J^{\otimes l}(U_j^l)) \cdot \left\{ \hat{\mathbf{A}}^{[k_{i'}]} [\Psi_J, \nu, U_j^l] \right\}_i + \alpha(\Psi_J)$$

involves a quadratic quantity depending on the states. Since this is computed on tensor products of states, increasing $\lambda(\Psi_J)$ implies a larger number of states satisfying this equation.

10.2 States combinations:

We consider some smeared transitions combining the transitions defined in the previous sections.

For a set decomposed into global system and local one through decomposition $(U^{(j)/p}, U^{(j)p})$, we define the smeared state by α as:

$$\begin{aligned} & \sum_{U^{(j)}, \lambda(U^{(j)})=\lambda} \alpha(U^{(j)}) \left| U^{(j)/p}, U^{(j)p}, \lambda(U^{(j)}), \left\{ \hat{\mathbf{A}}^{[k_i]}(U^{(j)}) \right\} \right\rangle \\ &= \sum_{U^{(j)}, MU^{(j)}=\alpha^{-1}(\lambda)} \alpha(U^{(j)}) \left| U^{(j)/p}, U^{(j)p}, \lambda(U^{(j)}), \left\{ \hat{\mathbf{A}}^{[k_i]}(U^{(j)}) \right\} \right\rangle \\ &\equiv \left| \alpha, U^{(j)p}, \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \end{aligned}$$

We can consider the map:

$$\left| \alpha, U^{(j)p}, \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle_{\{\alpha\}} \xrightarrow{\delta} \left| \alpha', (U^{(j)p})', \lambda', \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle_{\{\alpha'\}}$$

for $\lambda' > \lambda$ since $\#N(U_{\lambda'}^{(j)}) > \#N(U_\lambda^{(j)})$ with $\lambda' > \lambda$. The corresponding amplitude are computed through (61).

10.3 Connection

We consider the set of infinitesimal maps

$$\delta_k : \mathcal{H}\left(\lambda, \left\{ \hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)} \right\}\right) \rightarrow \mathcal{H}\left(\lambda + \delta_k \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)} \right\}\right)$$

summed up by following graph:

$$\begin{array}{ccc} & |\lambda + \delta_1 \lambda\rangle & \\ & \uparrow & \\ |\lambda\rangle + \delta_2 |\lambda\rangle & \leftarrow \begin{array}{c} |\lambda\rangle \\ \downarrow \\ |\lambda\rangle + \delta_k |\lambda\rangle \end{array} & \rightarrow |\lambda\rangle + \delta_n |\lambda\rangle \\ & \searrow & \nearrow \end{array}$$

where $|\lambda\rangle$ represents $\mathcal{H}\left(\lambda, \left\{ \hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)} \right\}\right)$ and $|\lambda\rangle + \delta_k |\lambda\rangle$ stands for $\mathcal{H}\left(\lambda + \delta_k \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]}(U_\lambda^{(j)}), U_\lambda^{(j)} \right\}\right)$. Our aim is to sum transitions over all these maps and compute an average $\delta\lambda$. In the sequel, we assume that the parameter $\lambda(U^{(j)})$ is defined by the "surrounding" part of the system:

$$\lambda(U^{(j)}) \simeq \lambda(U^{(j)/p})$$

We also consider the number of maps between $|\lambda\rangle$ and $|\lambda\rangle + \delta |\lambda\rangle$:

$$N\left((U_j)_\lambda \leftrightarrow (U_j)_{\lambda+\delta\lambda}\right)$$

counting the number of maps from $(U_j)_\lambda$ to $(U_j)_{\lambda+\delta\lambda}$.

$$\delta T \left| \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_\lambda, U^{(j)p} \right\rangle = \sum_{(U_{\lambda+\delta\lambda}^{(j)})'} \frac{N\left(U_\lambda^{(j)} \rightarrow (U_{\lambda+\delta\lambda}^{(j)})'\right)}{\sum_{(U_{\lambda+\delta\lambda}^{(j)})'} N\left(U_\lambda^{(j)} \rightarrow (U_{\lambda+\delta\lambda}^{(j)})'\right)} T_{U_\lambda^{(j)}, (U_{\lambda+\delta\lambda}^{(j)})'} \left| \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_\lambda, U^{(j)p} \right\rangle$$

$$N_{k_0} > N_k, k \neq k_0$$

$$\delta T = \sum_{(U_{\lambda+\delta\lambda}^{(j)})'} \frac{N\left(U_\lambda^{(j)} \rightarrow (U_{\lambda+\delta\lambda}^{(j)})'\right)}{\sum_{(U_{\lambda+\delta\lambda}^{(j)})'} N\left(U_\lambda^{(j)} \rightarrow (U_{\lambda+\delta\lambda}^{(j)})'\right)} T_{U_\lambda^{(j)}, (U_{\lambda+\delta\lambda}^{(j)})'}$$

We will consider the transitions for states composed from:

$$|U^{(j)}\rangle = |U^{(j)/p}\rangle |U^{(j)p}\rangle$$

and smeared by coefficients α :

$$\left| \alpha, U^{(j)p}, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle = \sum_{U^{(j)}, \lambda(U^{(j)})=\lambda_0} \alpha(U^{(j)}) \left| U^{(j)/p}, U^{(j)p}, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle$$

And a sequence of infinitesimal transitions will write:

$$\left| \alpha^{(1)}, \lambda_0^{(1)}, U^{(j)p}, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0^{(1)}} \right\rangle \rightarrow \left| \alpha^{(2)}, \lambda_0^{(2)}, U^{(j)p}, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0^{(2)}} \right\rangle \rightarrow \dots \rightarrow \left| \alpha^{(n)}, \lambda_0^{(n)}, U^{(j)p}, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0^{(n)}} \right\rangle$$

with:

$$\lambda_0^{(l)} > \lambda_0^{(l-1)}$$

Covariantly, this sequence writes also:

$$\left| \Sigma_1(U_1^{(j)}) \right\rangle \rightarrow \left| \Sigma_2(U_2^{(j)}) \right\rangle \rightarrow \left| \Sigma_3(U_3^{(j)}) \right\rangle \rightarrow \dots$$

with:

$$\lambda(U_n^{(j)}) > \dots > \lambda(U_1^{(j)})$$

10.4 Mechanism:

10.4.1 Global principle

Recall that:

$$U_\lambda^{(j)} = U^{(j)} \text{ s.t. } \lambda(U^{(j)}) = \lambda$$

The transition for a state $(U^{(j)/p}, U^{(j)p})$ writes:

$$\begin{aligned} & \delta_{\lambda_0 \rightarrow \lambda} \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ = & \sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} \frac{N(U_{\lambda_0}^{(j)} \rightarrow U_\lambda^{(j)}) \exp\left(i\left(\lambda(U_{\lambda_0}^{(j)}) - \lambda(U_\lambda^{(j)})\right) V(U_\lambda^{(j)}, U_{\lambda_0}^{(j)})\right)}{\sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} N(U_{\lambda_0}^{(j)} \rightarrow U_\lambda^{(j)})} \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \end{aligned}$$

which rewrites after smearing:

$$\begin{aligned} & \delta_{\lambda_0 \rightarrow \lambda} \left| \alpha, \left(U^{(j)p} \right)_0, \lambda_0, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ = & \sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} \sum_{U_{\lambda_0}^{(j)}, \lambda(U_{\lambda_0}^{(j)}) = \lambda_0} \frac{N(U_{\lambda_0}^{(j)} \rightarrow U_\lambda^{(j)}) \exp\left(i(\lambda_0 - \lambda) V(U_\lambda^{(j)}, U_{\lambda_0}^{(j)})\right) \alpha(U_{\lambda_0}^{(j)})}{\sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} N(U_{\lambda_0}^{(j)} \rightarrow U_\lambda^{(j)})} \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \end{aligned}$$

As before, we assume that we can decompose the degrees of freedom $U_\lambda^{(j)p}$ and $U_\lambda^{(j)/p}$:

$$V(U_\lambda^{(j)}, U_{\lambda_0}^{(j)}) = V(U_\lambda^{(j)p}, U_{\lambda_0}^{(j)p}) + V(U_\lambda^{(j)/p}, U_{\lambda_0}^{(j)/p})$$

and:

$$\begin{aligned} & \exp\left(i(\lambda_0 - \lambda) V(U_\lambda^{(j)}, U_{\lambda_0}^{(j)})\right) \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ = & \sum_{U^{(j)p}} \gamma\left(\left(U^{(j)p} \right)_{\lambda_0}, \left(U^{(j)p} \right)\right) \left| \left(U^{(j)/p}, \left(U^{(j)p} \right) \right)_\lambda, \lambda, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_\lambda \right\rangle \end{aligned}$$

As a consequence, the statistic transition is:

$$\begin{aligned} & \delta_{\lambda_0 \rightarrow \lambda} \left| \alpha, U_{\lambda_0}^{(j)p}, \lambda_0, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ = & \sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} \left(\frac{N(U_{\lambda_0}^{(j)} \rightarrow U_\lambda^{(j)})}{\sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} N(U_{\lambda_0}^{(j)} \rightarrow U_\lambda^{(j)})} \right. \\ & \left. \times \sum_{\substack{U_{\lambda_0}^{(j)}, \lambda(U_{\lambda_0}^{(j)}) = \lambda_0, \\ \delta_{\lambda_0 \rightarrow \lambda} U_{\lambda_0}^{(j)} = U_\lambda^{(j)}}} \alpha(U_{\lambda_0}^{(j)}) \gamma\left(\left(U^{(j)p} \right)_{\lambda_0}, \left(U^{(j)p} \right)\right) \left| \left(U^{(j)/p}, \left(U^{(j)p} \right) \right)_\lambda, \lambda, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_\lambda \right\rangle \right) \\ \equiv & \sum_{U_\lambda^{(j)}, \lambda(U_\lambda^{(j)}) = \lambda} \alpha' \left(U_\lambda^{(j)}, \alpha, \left(U^{(j)p} \right)_{\lambda_0} \right) \left| \left(U^{(j)/p}, \left(U^{(j)p} \right) \right)_\lambda, \lambda, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_\lambda \right\rangle = \left| \alpha', U_\lambda^{(j)p}, \lambda, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}_\lambda \right\rangle \end{aligned}$$

which writes in a compact form:

$$\delta_{\lambda_0 \rightarrow \lambda} \left| \alpha, U_{\lambda_0}^{(j)p}, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle_{\{\alpha\}} = \left| \alpha', U_{\lambda}^{(j)p}, \lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda} \right\rangle_{\{\alpha'\}}$$

As before, locally, we can assume that:

$$\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} = \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda} = \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}$$

and that the change in manifold:

$$\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda}, \lambda$$

is included in the transition V through the sum over maps.

10.4.2 Infinitesimal transitions

$$\begin{aligned} & \delta \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ &= \sum_{\delta\lambda} \sum_{U_{\lambda_0}^{(j)} + \delta U_{\lambda_0}^{(j)} | \delta\lambda (U_{\lambda_0}^{(j)}) = \delta\lambda} \frac{N \left(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0}^{(j)} + \delta U_{\lambda_0}^{(j)} \mid \delta\lambda \left(\delta U_{\lambda_0}^{(j)} \right) = \delta\lambda \right)}{\sum_{\delta\lambda} \sum_{U_{\lambda_0}^{(j)} + \delta U_{\lambda_0}^{(j)}} N \left(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0}^{(j)} + \delta U_{\lambda_0}^{(j)} \right)} \\ & \times \exp \left(i\delta\lambda V \left(U_{\lambda}^{(j)}, U_{\lambda_0}^{(j)} \right) \right) \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \end{aligned}$$

with $\delta U_{\lambda_0}^{(j)}$ elementary modification. thht s:

$$\# \left(\delta U_{\lambda_0}^{(j)} \right)' \neq \delta U_{\lambda_0}^{(j)} : U_{\lambda_0}^{(j)} \rightarrow \left(\delta U_{\lambda_0}^{(j)} \right)' \rightarrow \delta U_{\lambda_0}^{(j)}$$

$$\begin{aligned} & \delta \left| \alpha, \left(U^{(j)p} \right)_0, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ &= \sum_{U_{\lambda_0}^{(j)}, \lambda (U_{\lambda_0}^{(j)}) = \lambda_0} \sum_{\delta\lambda} \sum_{U_{\lambda_0}^{(j)} + \delta U_{\lambda_0}^{(j)} | \delta\lambda (\delta U_{\lambda_0}^{(j)}) = \delta\lambda} \\ & \frac{N \left(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0}^{(j)} + \delta\lambda \right) \exp \left(i\delta\lambda V \left(U_{\lambda_0 + \delta\lambda}^{(j)}, U_{\lambda_0}^{(j)} \right) \right)}{\sum_{\delta\lambda} \sum_{U_{\lambda}^{(j)}, \lambda (U_{\lambda}^{(j)}) = \lambda_0 + \delta\lambda} N \left(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0 + \delta\lambda}^{(j)} \right)} \alpha \left(U_{\lambda_0}^{(j)} \right) \left| \left(U^{(j)/p}, U^{(j)p} \right)_{\lambda_0}, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0} \right\rangle \\ & \equiv \sum_{\delta\lambda} \sum_{U_{\lambda_0}^{(j)} + \delta U_{\lambda_0}^{(j)} | \delta\lambda (\delta U_{\lambda_0}^{(j)}) = \delta\lambda} \alpha' \left(U_{\lambda}^{(j)}, \alpha, \delta\lambda, U_{\lambda_0}^{(j)p} \right) \left| U_{\lambda_0 + \delta\lambda}^{(j)/p}, U_{\lambda}^{(j)p}, \lambda_0 + \delta\lambda, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0 + \delta\lambda} \right\rangle \\ & \equiv \left| \bar{\alpha}', \left(U^{(j)p} \right), \lambda_0 + \delta\bar{\lambda}, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda_0 + \delta\bar{\lambda}} \right\rangle \end{aligned}$$

If we consider the $U^{(j)p}$ states and assuming that locally we can identify the $\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda}$ with a constant st f prmtrs:

$$\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}_{\lambda} \rightarrow \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}$$

then the transition becomes

$$\begin{aligned} \delta \left| \left(U^{(j)p} \right)_0, \lambda_0, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle &= \left| \left(U^{(j)p} \right)_0, \lambda_0 + \delta\bar{\lambda}, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \\ &= \exp(i\delta\bar{\lambda}V) \left| \left(U^{(j)p} \right)_0, \lambda_0 + \delta\bar{\lambda}, \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right\rangle \end{aligned}$$

where:

$$\exp(i\delta\bar{\lambda}V) = \sum_{\delta\lambda} \sum_{U_{\lambda_0}^{(j)}, \lambda(U_{\lambda_0}^{(j)})=\lambda_0} \sum_{U_{\lambda_0+\delta\lambda}^{(j)}, \lambda(U_{\lambda_0+\delta\lambda}^{(j)})=\lambda_0+\delta\lambda} \frac{N(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0+\delta\lambda}^{(j)}) \exp(i\delta\lambda V(U_{\lambda_0+\delta\lambda}^{(j)}, U_{\lambda_0}^{(j)}))}{\sum_{\delta\lambda} \sum_{U_{\lambda}^{(j)}, \lambda(U_{\lambda}^{(j)})=\lambda_0+\delta\lambda} N(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0+\delta\lambda}^{(j)})} \alpha(U_{\lambda_0}^{(j)})$$

and $\delta\bar{\lambda}$ is given by a first order expansion:

$$\delta\bar{\lambda} = \sum_{U_{\lambda_0}^{(j)}, \lambda(U_{\lambda_0}^{(j)})=\lambda_0} \sum_{\delta\lambda} \sum_{U_{\lambda_0+\delta\lambda}^{(j)}, \lambda(U_{\lambda_0+\delta\lambda}^{(j)})=\lambda_0+\delta\lambda} N(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0+\delta\lambda}^{(j)}) \delta\lambda$$

under our previous hypothesis $N(U_{\lambda_0}^{(j)} \rightarrow U_{\lambda_0+\delta\lambda}^{(j)}) \nearrow$ for $\delta\lambda \nearrow$. Then the averaged parameter is positive:

$$\delta\bar{\lambda} > 0$$

As a consequence the connection between the different spaces $\mathcal{H}_{\left(\lambda, \left\{ \hat{\mathbf{A}}^{[k_i]}(U_{\lambda}^{(j)}), U_{\lambda}^{(j)} \right\}\right)}$ are weighted such that in average $\delta\lambda > 0$. In other word, the amplitudes:

$$\mathcal{H}_{\left(\lambda, \left\{ \hat{\mathbf{A}}^{[k_i]}(U_{\lambda}^{(j)}), U_{\lambda}^{(j)} \right\}\right)} \rightarrow \mathcal{H}_{\left(\lambda', \left\{ \hat{\mathbf{A}}^{[k_i]}(U_{\lambda'}^{(j)}), U_{\lambda'}^{(j)} \right\}\right)}$$

are greater for $\lambda' > \lambda$.

Part II General formalism

We extend the previous formalism in the following manner. We assume that decompositions can involve an arbitrary set of subobjects, and they can be multiple; that is, several possible decompositions may be considered simultaneously, akin to a superposition of states. Additionally, decompositions can have an impact on subobjects. To achieve this, we consider that each state space involved may itself include subobjects, representing some bound states that concentrate a series of states in the considered space. Furthermore, any subobject may itself be decomposed by the decomposition of the space to which it belongs. In other words, an initial subobject may be decomposed into subobjects of this subobject. This is achieved by introducing products and compositions of decompositions.

11 Fields formulation

11.1 General set up

We keep the formalism and notations of the first part and consider some parameters spaces U, U^k, \dots with U^k is given by k copies of U . We also consider implicit relations:

$$U/c(U), \dots, U^k/c(U^k)$$

where the $c(U^k)$ are constraints and states on $\cup U^k/c(U^k)$ are tensor products plus constraints:

$$\begin{aligned} & \sum_{k, i_1 \dots i_k} |u_{i_1}\rangle \otimes \dots \otimes |u_{i_k}\rangle / c((u_{i_1}, \dots, u_{i_k})_k) \\ \rightarrow & \sum_{k, i_1 \dots i_k} |u_{i_1}, \dots, u_{i_k}\rangle \end{aligned}$$

As before, the states are built from fields $\Psi^{\otimes k}(U^k)$ and up to the constraints, a realization $\Psi^{\otimes k+l}(U^{k+l})$ is a sum of products of such realizations:

$$\sum_{\alpha} \Psi_{\alpha}^{\otimes k}(U^k) \Psi_{\alpha}^{\otimes l}(U^l)$$

so that a realization $\Psi^{\otimes k+l}(u_{i_1}, \dots, u_{i_{k+l}})$ can be decomposed as sum of products of realization:

$$\Psi^{\otimes k+l}(u_{i_1}, \dots, u_{i_{k+l}}) \rightarrow \sum_{\alpha} \Psi_{\alpha}^{\otimes k}(u_{i_1}, \dots, u_{i_k}) \Psi_{\alpha}^{\otimes l}(u_{i_{k+1}}, \dots, u_{i_{k+l}})$$

The constraints are included by identifying some parameters:

$$\Psi(U^l) = \Psi^{\otimes l}(U^l) / f$$

ndrstd tht U_j^k stnds fr $U_j^k/c(U_j^k)$. and the tensor product is a particular realization of $\Psi(U^l)$:

$$\Psi(u_{i_1}, \dots, u_{i_k}) = \Psi(u_{i_1}) \otimes \dots \otimes \Psi(u_{i_k}) / f((u_{i_1}, \dots, u_{i_k}))$$

11.2 Subspaces, subobjects

We generalize the notion of decomposition. To do so we detail the notion of subobjects and maps of subobjects.

11.2.1 Subobjects maps

For the set:

$$U_j^{(p)} \subset \oplus_k U_j^k$$

We consider a collection of subjects, that is a st f mps:

$$\{U_j\}_p = \left\{ U_j^{(p)} \xrightarrow{p} \mathcal{V}(\oplus_k U_j^k) \right\}_p$$

The set $\mathcal{V}(\oplus_k U_j^k)$ is formally a collection of submanifold of $\oplus_k U_j^k$. t includes th snl pnt * fr $k = 0$. Maps p are running over a set of indices. For a given collection (p_l) , we describe the maps:

$$\begin{aligned} p_l & : (U_j)^{(p_l)} \rightarrow \mathcal{V}(\oplus_k U_j^k) \\ u^{(p_l)} & \rightarrow \oplus_k \left\{ (u_{i_1}, \dots, u_{i_k}) (u^{(p_l)}) \right\}, f_k(u_{i_1}, \dots, u_{i_k}, u^{(p_l)}) = 0 \end{aligned}$$

where $\{(u_{i_1}, \dots, u_{i_k}) (u^{(p_l)})\}$ is the subvariety defined by several equation:

$$f_k(u_{i_1}, \dots, u_{i_k}, u^{(p_l)}) = 0$$

These maps have a translation in terms of states:

$$s_{p_l} : H \left((U_j)^{(p_l)} \right) \hookrightarrow \oplus_k H^{\otimes k} (U_j)$$

To detail s_{p_l} we dfn:

$$i_k : \begin{array}{l} (U_j)^k \rightarrow H^{\otimes k} (U_j) \\ (u_{i_1}, \dots, u_{i_k}) \rightarrow |u_{i_1}, \dots, u_{i_k}\rangle \end{array}$$

and we define s_{p_l} as

$$s_{p_l}(u) = \int_{(U_j)^k \cap p_l(u^{(p_l)})} g_{k,p_l}(u^{(p_l)}, u) i_k(u)$$

The maps g_{k,p_l} characterizes th ppctn p_l . To shorten the notations and since k and p_l rrs explicitly through $u^{(p_l)}$ nd u , we will write g for g_{k,p_l} .

In coordinates:

$$s_{p_l} : |u^{(p_l)}\rangle \rightarrow \sum_k \int_{(u_{i_1}, \dots, u_{i_k}) \in p_l(u^{(p_l)})} g(u^{(p_l)}, u_{i_1}, \dots, u_{i_k}) |u_{i_1}, \dots, u_{i_k}\rangle$$

it includes $|1\rangle$ fr $k = 0$.

Then we can define S_{p_l} that translates p_l at states level:

$$S_{p_l} : \oplus H^{\otimes k_l} \left((U_j)^{(p_l)} \right) \hookrightarrow \oplus_k H^{\otimes k} (U_j)$$

which is defined by its components $s_{p_l}^{\otimes k_l}$:

$$s_{p_l}^{\otimes k_l} : H^{\otimes k_l} \left((U_j)^{(p_l)} \right) \hookrightarrow \oplus_k H^{\otimes k} (U_j)$$

obtained by tensor products

$$s_{p_l}^{\otimes k_l} : |u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)}\rangle \rightarrow \sum_k \sum_{(\alpha_m)_{m \leq l} \in P(k)} \int_{((u_{i_e})_{e \in \alpha_m}) \in p_l(u_t^{(p_l)})} \prod_{\alpha_m} g_{\alpha_m}(u_m^{(p_l)}, (u_{i_e})_{e \in \alpha_m}) |u_{i_1}, \dots, u_{i_k}\rangle$$

11.2.2 Fields and functionals

The states map allow to consider associated field, whose functional generate the states. Defining:

$$\Psi_{J, s_{p_l}}^{\otimes k_l} \left((U_j)^{(p_l)} \right) = \sum_k g \left(u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)}, u_{i_1}, \dots, u_{i_k} \right) \Psi_J^{\otimes k} (U_j^k)$$

field $\left\{ \Psi_{J, s_{p_l}}^{\otimes k_l} \right\}_k$ define the subject field define by a map p_l of the collection. We consider this fields as independent quantity to model the emerging agregate object that may arise within the initial stt space. As a cosequence, we consider that the states in terms of functional are given by sums f fcnlns over subjects. The fcnln t cnsdr ar ths:

$$\sum F_{lin} \left(\Psi_J^{\otimes k} (U_j^k) / f \right) + \sum_{(p_l)} \sum F_{lin}^{(p_l)} \left(\Psi_{J, s_{p_l}}^{\otimes k_l} \left((U_j)^{(p_l)} \right) / f \right)$$

11.3 Composed subobject

More genrally, we may expect sequences of sbbjcts, i.e. sbbjct f sbbjct nd s n. W ths hv t cmps sbbjcts mps. Ths dn n tw stps

11.3.1 Subobjects of subobjects

We first define subobjects of subobjects in a similar manner as in previous section. We thus consider:

$$\left\{ \left((U_j)^{(p)} \right)^{(p')} \xrightarrow{p'} \mathcal{V} \oplus_k \left((U_j)^{(p)} \right)^k \right\}_{p'}$$

and collections of subobjects of subobjects are given by maps:

$$\begin{aligned} p_{l,u} &: \left((U_j)^{(p_u)} \right)^{(p_l)} \rightarrow \mathcal{V} \left(\oplus_k \left((U_j)^{(p_u)} \right)^k \right) \\ u^{(p_l,u)} &\rightarrow \oplus_k \left(u_{i_1}^{(u)}, \dots, u_{i_k}^{(u)} \right) \left(u^{(p_l,u)} \right), f_k \left(u_{i_1}^{(u)}, \dots, u_{i_k}^{(u)}, u^{(p_l,u)} \right) = 0 \end{aligned}$$

The map $s_{p_l,u}$, and its states translations $s_{p_l,u}^{\otimes k_l}$ and $S_{p_l,u}$ are given by:

$$\begin{aligned} s_{p_l,u} &: H \left(\left((U_j)^{(p_u)} \right)^{(p_l)} \right) \hookrightarrow \oplus_k H^{\otimes k} \left((U_j)^{(p_u)} \right) \\ s_{p_l,u}^{\otimes k_l} &: H^{\otimes k_l} \left(\left((U_j)^{(p_u)} \right)^{(p_l)} \right) \hookrightarrow \oplus_k H^{\otimes k} \left((U_j)^{(p_u)} \right) \end{aligned}$$

and:

$$S_{p_l,u} : \oplus H^{\otimes k_l} \left(\left((U_j)^{(p_u)} \right)^{(p_l,u)} \right) \hookrightarrow \oplus_k H^{\otimes k} \left((U_j)^{(p_u)} \right)$$

we also write:

$$\left((U_j)^{(p_u, p_l u)} \right)$$

for:

$$\left(\left((U_j)^{(p_u)} \right)^{(p_l, u)} \right)$$

11.3.2 "Composed Maps:

We aim at defining the composition of subobjects maps at the level of stats:

$$S_{p_{(l),u}} = S_{p_u} \circ S_{p_l,u} : \oplus H^{\otimes k_l} \left(\left((U_j)^{(p_u)} \right)^{(p_l, u)} \right) \hookrightarrow \oplus_k H^{\otimes k} (U_j)$$

To do so we first define the product of maps:

Products of maps We consider products of sbbjcts:

$$p_{l_1} \times p_{l_2} : (U_j)^{(p_{l_1})} \times (U_j)^{(p_{l_2})} \rightarrow \mathcal{V} \left(\oplus_k U_j^k \right)$$

Define:

$$p_l \left(u_{p_l} \right) = \cup_k V_k \left(p_l \left(u_{p_l} \right) \right)$$

with the U_j^k are the components of p_l .

The product is defined by:

$$p_{l_1} \times p_{l_2} \left(u_{p_{l_1}} \times u_{p_{l_2}} \right) = \cup_k \left(V_{k_1} \left(p_{l_1} \left(u_{p_{l_1}} \right) \right) \times V_{k-k_1} \left(p_{l_2} \left(u_{p_{l_2}} \right) \right) \right)$$

For exmple, considering:

$$U_{j_1} \times U_{j_2} / c (U_{j_1}, U_{j_2})$$

the image of $p_{l_1} \times p_{l_2}$ is:

$$\left\{ p_{l_1} \left(u_{p_{l_1}} \right) \times p_{l_2} \left(u_{p_{l_2}} \right) \right\}_{u_{p_{l_1}} \in (U_j)^{(p_{l_1})}, u_{p_{l_2}} \in (U_j)^{(p_{l_2})}}$$

At level of hilbert spacs, we have the map:

$$S_{p_{l_1}} \otimes S_{p_{l_2}} : \oplus_{k_{l_1}, k_{l_2}} \left(H^{\otimes k_{l_1}} \left((U_j)^{(p_{l_1})} \right) \otimes H^{\otimes k_{l_2}} \left((U_j)^{(p_{l_2})} \right) \right) \hookrightarrow \oplus_k H^{\otimes k} (U_j)$$

written in coordinates:

$$\begin{aligned} S_{p_l} & : s_{p_{l_1}}^{\otimes k_{l_1}} \otimes s_{p_{l_2}}^{\otimes k_{l_2}} : \left| u_1^{(p_{l_1})}, \dots, u_{k_{l_1}}^{(p_{l_1})} \right\rangle \left| u_1^{(p_{l_2})}, \dots, u_{k_{l_2}}^{(p_{l_2})} \right\rangle \\ & \rightarrow \sum_{k=k_1+k_2} \sum_{(\alpha_{m_i})_{m_i \leq l_i} \in P(k_i)} \prod_{i=1}^2 \int_{\left((u_{ie})_{e \in \alpha_{m_i}} \right) \in \left(p_{l_i} \left(u_t^{(p_{l_i})} \right) \right)} \prod_{\alpha_{m_i}} g_{\alpha_{m_i}} \left(u_{m_i}^{(p_{l_i})}, (u_{ie})_{e \in \alpha_{m_i}} \right) |u_{i_1}, \dots, u_{i_k}\rangle \end{aligned}$$

as before the maps $g_{\alpha_{m_i}}$ characterize $p_{l_1} \times p_{l_2}$.

Compositn of mps We can now compose maps of subbjct to define sequences of subobjects. Starting with:

$$p_l : (U_j)^{(p_l)} \rightarrow \mathcal{V} \left(\oplus_k U_j^k \right)$$

and:

$$p_{l,l'} : \left((U_j)^{(p_l)} \right)^{(p_{l,l'})} \rightarrow \mathcal{V} \left(\oplus_k \left((U_j)^{(p_l)} \right)^k \right)$$

these maps can be composed:

$$p_l \circ p_{l,l'} \rightarrow \mathcal{V} \left(\oplus_k U_j^k \right)$$

by considering first:

$$\left((U_j)^{(p_l)} \right)^{(p_{l,l'})} \rightarrow \mathcal{V} \left(\oplus_{k_l} \left((U_j)^{(p_l)} \right)^{k_l} \right) \rightrightarrows_{V \in \sqcup \mathcal{V} \left(\oplus_k \left((U_j)^{(p_l)} \right)^k \right)} \sqcup \{u \in V\} \rightarrow \mathcal{V} \left(\oplus_k \left((U_j)^{(p_l)} \right)^k \right)$$

that transforms the parameters of the sub-subobject $u_{p_l, p_{l,l'}}$ into:

$$u_{p_l, p_{l,l'}} \rightarrow p_{l,l'} \left(u_{p_l, p_{l,l'}} \right) \rightrightarrows \left\{ u_{p_l} \in p_{l,l'} \left(u_{p_l, p_{l,l'}} \right) \right\} \rightarrow \prod_{u_{p_l} \in p_{l,l'} \left(u_{p_l, p_{l,l'}} \right)} p_l \left(u_{p_l} \right)$$

with as before:

$$p_l \left(u_{p_l} \right) = \cup_k V_k \left(p_l \left(u_{p_l} \right) \right)$$

and:

$$p_l \left(u_{p_l} \right) \times p_l \left(u'_{p_l} \right) = \cup_k \left(V_{k_1} \left(p_l \left(u_{p_l} \right) \right) \times V_{k-k_1} \left(p_l \left(u'_{p_l} \right) \right) \right)$$

For states, the composition leads to:

$$S_{p_l, p_{l,l'}} = S_{p_l} \circ S_{p_{l,l'}} : \oplus H^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{(p_{l,l'})} \right) \hookrightarrow \oplus_k H^{\otimes k} (U_j)$$

On the 1 states, it is given by $s_{p_l, l'}^{\otimes k_{l'}}$:

$$\begin{aligned} s_{p_l, l'}^{\otimes k_{l'}} &: \left| u_1^{(p_l, l')}, \dots, u_{k_{l'}}^{(p_l, l')} \right\rangle \\ &\rightarrow \sum_{k_l} \sum_{(\alpha_{m'})_{m' \leq k_{l'}} \in P(k_l)} \int_{\left((u_{i_e}^{(p_l)})_{e \in \alpha_{m'}} \right) \in p_l \left(u_t^{(p_l, l')} \right)} \prod_{\alpha_{m'}} g_{\alpha_{m'}} \left(u_m^{(p_l, l')}, (u_{i_e}^{(p_l)})_{e \in \alpha_{m'}} \right) \left| u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right\rangle \end{aligned}$$

with s_{p_l} dfnd s bfr:

$$s_{p_l} : \left| u^{(p_l)} \right\rangle \rightarrow \sum_k \int_{(u_{i_1}, \dots, u_{i_k}) \in (p_l(u^{(p_l)}))} g_k \left(u^{(p_l)}, u_{i_1}, \dots, u_{i_k} \right) \left| u_{i_1}, \dots, u_{i_k} \right\rangle$$

Then, on the full states space, we have:

$$S_{p_l} = s_{p_l}^{\otimes k_l} : \left| u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right\rangle \rightarrow \sum_k \sum_{(\alpha_m)_{m \leq l} \in P(k)} \int_{\left((u_{i_e})_{e \in \alpha_m} \right) \in (p_l(u_t^{(p_l)}))} \prod_{\alpha_m} g_{\alpha_m} \left(u_m^{(p_l)}, (u_{i_e})_{e \in \alpha_m} \right) \left| u_{i_1}, \dots, u_{i_k} \right\rangle$$

and:

$$\begin{aligned} S_{p_l, p_l, l'}^{k_{l'}} &= S_{p_l} \circ s_{p_l, l'}^{\otimes k_{l'}} : \left| u_1^{(p_l, l')}, \dots, u_{k_{l'}}^{(p_l, l')} \right\rangle \\ &\rightarrow \sum_k \sum_{k_l} \sum_{\substack{(\alpha_{m'})_{m' \leq k_{l'}} \in P(k_l) \\ (\alpha_m)_{m \leq l} \in P(k)}} \int_V \left(u_m^{(p_l, l')}, (u_{i_e}^{(p_l)})_{e \in \alpha_{m'}}, (u_{i_e})_{e \in \alpha_m} \right) \\ &\quad \times \prod_{\alpha_{m'}, \alpha_m} g_{\alpha_{m'}, \alpha_m} \left(u_m^{(p_l, l')}, (u_{i_e}^{(p_l)})_{e \in \alpha_{m'}}, (u_{i_e})_{e \in \alpha_m} \right) \left| u_{i_1}, \dots, u_{i_k} \right\rangle \end{aligned}$$

with:

$$V \left(u_m^{(p_l, l')}, (u_{i_e}^{(p_l)})_{e \in \alpha_{m'}}, (u_{i_e})_{e \in \alpha_m} \right) = \left\{ \left((u_{i_e}^{(p_l)})_{e \in \alpha_{m'}} \right) \in \left(p_l \left(u_t^{(p_l, l')} \right) \right) \right\} \times \left\{ \left((u_{i_e})_{e \in \alpha_m} \right) \in \left(p_l \left(u_t^{(p_l)} \right) \right) \right\}$$

and the functions:

$$g_{\alpha_{m'}, \alpha_m} \left(u_m^{(p_l, l')}, (u_{i_e}^{(p_l)})_{e \in \alpha_{m'}}, (u_{i_e})_{e \in \alpha_m} \right) = g_{\alpha_{m'}} \left(u_m^{(p_l, l')}, (u_{i_e}^{(p_l)})_{e \in \alpha_{m'}} \right) g_{\alpha_m} \left(u_m^{(p_l)}, (u_{i_e})_{e \in \alpha_m} \right)$$

characterizing the compositions of mps.

12 Decomposition in subspaces

We generalize the notion of decompositions to an arbitrary set of arbitrary number of subobjects. We will assume that these decompositions are compatible downward for subobjects.

12.1 Decomposition and maps

We consider the assumption that there is a collection:

$$p_l : (U_j)^{(p_l)} \rightarrow \mathcal{V}(\oplus_k U_j^k)$$

such that:

$$\begin{aligned} \oplus_k H^{\otimes k}(U_j) &= \oplus_{(k_l)} \otimes_{l=1}^m s_{p_l}^{\otimes k_l} \left(\left(H \left((U_j)^{(p_l)} \right) \right)^{\otimes k_l} \right) / \prod_{p_1, \dots, p_m} f_{p_1 \dots p_m} \\ &\equiv \oplus_{(k_l)} \otimes_{l=1}^m \left(H \left((U_j)^{(p_l)} \right) \right)^{\otimes k_l} / \prod_{p_1, \dots, p_m} f_{p_1 \dots p_m} \end{aligned}$$

$$\begin{aligned} |u_{i_1}, \dots, u_{i_k}\rangle &= \sum_{(k_l)} h_{s_{p_l}^{\otimes k_l}} \left(\left(u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right)_l, u_{i_1}, \dots, u_{i_k} \right) \otimes_l s_{p_l}^{\otimes k_l} \left(|u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)}\rangle \right) / \prod_{p_1, \dots, p_m} f_{p_1 \dots p_m} \\ &\equiv \sum_{(k_l)} h_{s_{p_l}^{\otimes k_l}} \left(\left(u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right)_l, u_{i_1}, \dots, u_{i_k} \right) \otimes_l \left(|u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)}\rangle \right) / \prod_{p_1, \dots, p_m} f_{p_1 \dots p_m} \end{aligned}$$

The constraints $f_{p_1 \dots p_m}$ are assumed to arise from operator invariance:

$$K |u_{i_1}, \dots, u_{i_k}\rangle = 0$$

that is, written in term of the previous decomposition:

$$\sum_{(k_l)} h_{s_{p_l}^{\otimes k_l}} \left(\left(u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right)_l, u_{i_1}, \dots, u_{i_k} \right) \sum_l K \left(|u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)}\rangle \right) \otimes_{l' \neq l} \left| u_1^{(p_{l'})}, \dots, u_{k_l}^{(p_{l'})} \right\rangle = 0$$

We may assume that implies an equation on the parameters:

$$\sum_l f_l \left(\left(u_i^{(p_l)} \right) \right) = 0$$

and:

$$|u_{i_1}, \dots, u_{i_k}\rangle = \sum_{(k_l)} h_{s_{p_l}^{\otimes k_l}} \left(\left(u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right)_l, u_{i_1}, \dots, u_{i_k} \right) \prod \delta \left(f_{p_l} \left(\left(u_i^{(p_l)} \right) \right) \right) \otimes_l \left| u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right\rangle$$

We extend this relations to decompose subobject into subobject of subobjects:

$$p_{l,l'} : \left((U_j)^{(p_l)} \right)^{(p_{l,l'})} \rightarrow \mathcal{V} \left(\oplus_{k_l} \left((U_j)^{(p_l)} \right)^{k_l} \right)$$

Using the notation:

$$\left((U_j)^{(p_l)} \right)^{(p_{l,l'})} \equiv (U_j)^{(p_l, p_{l,l'})}$$

we hv:

$$\begin{aligned} \oplus_k H^{\otimes k} \left(\left((U_j)^{(p_l)} \right) \right) &= \otimes_{l'=1}^m S_{p_{l',l}} \left(\oplus_{k_{l'}} H^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l,l'})} \right) \right) \right) / \prod_{p_1, l, \dots, p_m, l} f_{p_1, l \dots p_m, l} \\ &\equiv \otimes_{l'=1}^m \left(\oplus_{k_{l'}} H^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l,l'})} \right) \right) \right) / \prod_{p_1, l, \dots, p_m, l} f_{p_1, l \dots p_m, l} \\ &\equiv \oplus_{(k_{l'})} \otimes_{l'=1}^m \left(H^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l,l'})} \right) \right) \right) / \prod_{p_1, l, \dots, p_m, l} f_{p_1, l \dots p_m, l} \end{aligned}$$

Then at the level of entire state space, defining:

$$\begin{aligned}\mathcal{H}(U_j) &= \oplus_k H^{\otimes k}(U_j) \\ \mathcal{H}\left(\left\{(U_j)^{(p_l)}\right\}_{l \leq m}\right) &= \oplus_{(k_l)} \otimes_{l=1}^m s_{p_l}^{\otimes k_l} \left(\left(H \left((U_j)^{(p_l)} \right) \right)^{\otimes k_l} \right) / \prod_{p_1, \dots, p_m} f_{p_1 \dots p_m}\end{aligned}$$

we have the decomposition:

$$\begin{aligned}\mathcal{H}\left(\left((U_j)^{(p_l)}\right)\right) &= \oplus_k H^{\otimes k} \left(\left((U_j)^{(p_l)} \right) \right) \\ \mathcal{H}\left(\left\{(U_j)^{(p_l, p_{l'})}\right\}_{l' \leq m}\right) &= \oplus_{(k_{l'})} \otimes_{l'=1}^m S_{p_{l'}, l} \left(H^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right) \right) \right) / \prod_{p_1, l, \dots, p_m, l} f_{p_1, l, \dots, p_m, l} \\ \mathcal{H}\left(\left\{(U_j)^{(p_l, p_{l'})}\right\}_{\substack{l \leq m \\ l' \leq m'}}\right) &= \otimes_{l=1, l'=1}^{m, m'} S_{p_{l'}, l} \left(\oplus H^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right) \right) \right) / \prod_{p_1, l, \dots, p_{m'}, l} f_{p_1, l, \dots, p_{m'}, l}\end{aligned}$$

12.2 States decomposition

12.2.1 General formulation: Fields decomposition

Writing:

$$\begin{aligned}\left((U_j^{\otimes k})^{(p_l)} \hookrightarrow U_j^{\otimes k} \right)^{s_{p_l}^{\otimes k_l}} &: H(U_j^{\otimes k}) \rightarrow H\left((U_j^{\otimes k})^{(p_l)} \right) \\ |u_{i_1} \otimes \dots \otimes u_{i_k}\rangle &\rightarrow \sum_t |u_1^{(p_l)} \otimes \dots \otimes u_t^{(p_l)}\rangle\end{aligned}$$

we translate the dcmpstn n trms f flds. We assume that stts $F(\Psi_J(U_j))$ can be decomposed along the field decomposition:

$$\begin{aligned}\Psi_J^{\otimes k}(U_j^k) &\rightarrow \bigoplus_m \bigoplus_{D_j^{p_l, m}} \bigoplus_{s_{p_l}} h_{\otimes k_l} \left(\left(\left((U_j)^{(p_l)} \right)^{k_l} \right)_l, U_j^k \right) \\ &\times \otimes_{p_l} \delta \left(f_{p_l} \left(\left(u_i^{(p_l)} \right) \right) \right) \Psi_{J, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right)\end{aligned}$$

with $D_j^{p_l, m}$ standing for the possible decompositions:

$$\mathcal{H}(U_j) = \mathcal{H}\left(\left\{(U_j)^{(p_l)}\right\}_{l \leq m}\right)$$

where we define

$$\left((U_j)^{(p_l)} \right)^{k_l} = \left(u_1^{(p_l)}, \dots, u_{k_l}^{(p_l)} \right)$$

and:

$$U_j^k = (u_1, \dots, u_k) / c(u_1, \dots, u_k)$$

The $u_i^{(p_l)}$ are coordinates on $(U_j)^{(p_l)}$, with u_1 the coordinates on U_j . We also define the subobject fields:

$$\Psi_{J, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right) = s_{p_l}^{\otimes k_l} \left(\Psi_J^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right) \right)$$

In a way that is similar to the states transformation:

$$s_{p_l}^{\otimes k_l} \left(\Psi_J^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right) \right) = \sum_k \sum_{(\alpha_m)_{m \leq l} \in P(k)} \int_{((u_{ie})_{e \in \alpha_m}) \in p_l} \prod_{\alpha_m} g_{\alpha_m} \left(u_m^{(p_l)}, (u_{ie})_{e \in \alpha_m} \right) \Psi_J^{\otimes k} (u_{i_1}, \dots, u_{i_k})$$

The tensor products \otimes in the decomposition stands for series expansion of products of independent copies of the $\Psi_{J, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right)$. As in the first part, any field realization of $\Psi_J^{\otimes k} (U_j^k)$ is decomposed as a series expansion of products of realizations of the fields $\Psi_{J, s_{p_{l'}}}^{\otimes k_{l'}} \otimes_{\otimes k_{l'}}^{\otimes k_{l'}}$. The constraints are taken into account through some Dirac functions:

$$\begin{aligned} & \otimes_{p_l} \delta \left(f_{p_l} \left(\left(u_i^{(p_l)} \right) \right) \right) \Psi_{J, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right) \\ & \equiv \sum_{\alpha} \delta \left(f_{p_l} \left(\left(u_i^{(p_l)} \right) \right) \right) \prod_{p_l} \Psi_{J, \alpha, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right) \end{aligned}$$

Note that the $h_{s_{p_l}}^{\otimes k_l} \left(\left(\left((U_j)^{(p_l)} \right)^{k_l} \right)_l, U_j^k \right)$ include constraint between objects of the type:

$$\delta \left(f_{\gamma} \left(\left\{ \left((U_j)^{(p_l)} \right)^{k_l} \right\}_l \right) \right)$$

where γ is an index for the several constraints.

12.2.2 Subobjects decomposition

We also assume that subobject are decomposed similarly:

$$\begin{aligned} \Psi_{J, s_{p_u}}^{\otimes k_u} \left(\left((U_j)^{(p_u)} \right)^{k_u} \right) & \rightarrow \bigoplus_{m, m'} \bigoplus_{D_{j, p_u}^{(p_l, p_{l'}), m, m'}} h_{s_{p_{l'}}}^{(p_u)} \left(\left(\left((U_j)^{(p_l, p_{l'})} \right) \right)^{k_{l'}}, \left((U_j)^{(p_u)} \right)^{k_u} \right) \\ & \times \otimes_{p_{l'}, l} \delta \left(f_{p_{l'}, l} \left(\left(u_i^{(p_{l'}, l)} \right) \right) \right) \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left(\left(\left((U_j)^{(p_l, p_{l'})} \right) \right)^{k_{l'}} \right) \end{aligned}$$

where the symbols:

$$D_{j, p_u}^{(p_l, p_{l'}), m, m'}$$

stand for the possible decompositions:

$$\mathcal{H} \left(\left((U_j)^{(p_u)} \right) \right) = \mathcal{H} \left(\left\{ (U_j)^{(p_l, p_{l'})} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right)$$

and:

$$\left(\left((U_j)^{(p_l, p_{l'})} \right) \right)^{k_{l'}} = \left(u_1^{(p_{l'}, l)}, \dots, u_{k_l}^{(p_{l'}, l)} \right)^{k_{l'}}$$

where $u_i^{(p_{l'}, l)}$ are coordinates on $(U_j)^{(p_l, p_{l'})}$. We define the subobject fields:

$$\Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left(\left(\left((U_j)^{(p_l, p_{l'})} \right) \right)^{k_{l'}} \right) = \otimes_{s_{p_{l'}, l}}^{\otimes k_{l'}} \left(\Psi_J^{\otimes k_{l'}} \left(\left(\left((U_j)^{(p_l, p_{l'})} \right) \right)^{k_{l'}} \right) \right)$$

As before, the products \otimes stands for series expansion of products of independent copies of the $\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}}$ $\left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right)$. For the fields realizations, the tensor products represent series expansions:

$$\begin{aligned} & \otimes_{p_{l',l}} \delta \left(f_{p_{l',l}} \left(\left(u_i^{(p_{l',l})} \right) \right) \right) \Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right) \\ \equiv & \sum_{\alpha} \delta \left(f_{p_{l',l}} \left(\left(u_i^{(p_{l',l})} \right) \right) \right) \prod_{p_{l',l}} \Psi_{J, \alpha, s_{p_{l',l}}}^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right) \end{aligned}$$

Remrks 1. As before that the:

$$h_{\otimes_{s_{p_{l',l}}}^{\otimes k_{l'}}}^{(p_u)} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right)_l, \left((U_j)^{(p_u)} \right)^{k_u}$$

include constraint between objects:

$$\delta \left(f_{\gamma} \left(\left\{ \left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right\}_l \right) \right)$$

where γ is an index for the several constraints.

2. We can identify:

$$\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right)$$

and:

$$\begin{aligned} S_{p_l} \circ \left(\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right) \right) & \equiv S_{p_{l',l}}^{\otimes k_{l'}} \left(\Psi_J^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right) \right) \\ & \equiv \Psi_{J, S_{p_{l',l}}}^{\otimes k_{l'}} \left(\left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \right) \end{aligned}$$

12.2.3 fields and subobjects decomposition

We can gather the decompositions for fields and subobjects. Writing:

$$\begin{aligned} [p_u]^{k_u} & \equiv \left((U_j)^{(p_u)} \right)^{k_u} \\ [p_l, p_{l'}]^{k_{l'}} & \equiv \left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}} \end{aligned}$$

and expressing the constraints as:

$$\begin{aligned} \delta(f_{p_l}) & = \delta \left(f_{p_l} \left(\left(u_i^{(p_l)} \right) \right) \right) \\ \delta(f_{p_{l',l}}) & = \delta \left(f_{p_{l',l}} \left(\left(u_i^{(p_{l',l})} \right) \right) \right) \end{aligned}$$

we can rewrite the decomposition is:

$$\begin{aligned}
& \Psi_J^{\otimes k}(U_j^k) \bigoplus_{(U_j)^{(p_u)}} \Psi_{J, s_{p_u}}^{\otimes k_u}([p_u]^{k_u}) \\
\rightarrow & \bigoplus_m \bigoplus_{D_j^{p_l, m}} \left(h_{s_{p_l}}^{\otimes k_l} \left(([p_l]^{k_l} \right)_l, U_j^k \right) \otimes_{p_l} \delta(f_{p_l}) \Psi_{J, s_{p_l}}^{\otimes k_l}([p_l]^{k_l}) \\
& \bigoplus_{m'} \bigoplus_{D_j^{(p_l, p_{l'}), m, m'}} h_{s_{p_{l'}, l}}^{(p_u)} \left(([p_l, p_{l'}]^{k_{l'}} \right)_l, [p_u]^{k_u} \right) \otimes_{p_{l'}, l} \delta(f_{p_{l'}, l}) \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)_l
\end{aligned}$$

It translates that $U_j^{\otimes k}$ is terminal, the relation is "aggregated" on the $\Psi_J^{\otimes k}(U_j^{\otimes k})$ side. However, expressing $\Psi_J^{\otimes k}(U_j^{\otimes k})$ as a function of $\Psi_{J, s_{p_l}}^{\otimes k_l} \left((U_j^{\otimes k})^{(p_l)} \right)$ yields intricate relations.

12.2.4 States functional decomposition

The decomposition of a state functional has the form:

$$F_{lin}(\Psi_J^{\otimes k}(U_j^{\otimes k})) = \int f \left(U_j^{\otimes k}, \hat{\Pi}_{U_j^{\otimes k}} \right) \Psi_J^{\otimes k}(U_j^{\otimes k}) dU_j^{\otimes k} + \sum_{(p_u)} F_{lin} \left(\Psi_{J, s_{p_u}}^{\otimes k_u} \left((U_j)^{(p_u)} \right)^{k_u} \right) \quad (63)$$

where subobjects are included. Note that this can be expanded recursively:

$$\begin{aligned}
& F_{lin}(\Psi_J^{\otimes k}(U_j^{\otimes k})) \\
= & \int f \left(U_j^{\otimes k}, \hat{\Pi}_{U_j^{\otimes k}} \right) \Psi_J^{\otimes k}(U_j^{\otimes k}) dU_j^{\otimes k} \\
& + \sum_{(p_u)} \int f_1 \left((U_j)^{(p_u)} \right)^{k_u}, \hat{\Pi}_{(U_j)^{(p_u)} \right) \Psi_{J, s_{p_u}}^{\otimes k_u} \left((U_j)^{(p_u)} \right)^{k_u} \\
& + \sum_{(p_{u'}, u)} \int f_2 \left((U_j)^{(p_{u'}, u)} \right)^{k_{u'}, u}, \hat{\Pi}_{(U_j)^{(p_{u'}, u)} \right) \Psi_{J, s_{p_{u'}, u}}^{\otimes k_{u'}, u} \left((U_j)^{(p_{u'}, u)} \right)^{k_{u'}, u} \\
& + \dots
\end{aligned}$$

Using the decomposition, states (63) can be rewritten in the following manner. The First term is generated by sums:

$$\begin{aligned}
F_{lin}(\Psi_J(U_j^{\otimes k})) & = F_{lin}(i_1 \dots i_m) \left(\otimes_l \Psi_{J, s_{p_l}}^{\otimes k_l} \left((U_j)^{(p_l)} \right)^{k_l} \right) \\
& = \sum_m \sum_{D_j^{p_l, m}} g_{s_{p_l}}^{\otimes k_l} \left(\left\{ (U_j)^{(p_l)} \right\}_{l \leq m} \right) \otimes_{l \leq m} \Psi_{J, s_{p_l}}^{\otimes k_l} \left((U_j)^{(p_l)} \right)^{k_l}
\end{aligned} \quad (64)$$

where:

$$g_{s_{p_l}}^{\otimes k_l} \left(\left\{ (U_j)^{(p_l)} \right\}_{l \leq m} \right) = \int f \left(U_j^{\otimes k}, \hat{\Pi}_{U_j^{\otimes k}} \right) h_{s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right)_l, U_j^k \right) d(U_j^k)$$

The second term describes subobjects terms and includes in the decomposition:

$$\begin{aligned}
& Flin \left(\Psi_{J, s_{p_u}}^{\otimes k_u} \left(\left((U_j)^{(p_u)} \right)^{k_u} \right) \right) \\
&= Flin^{(p_u)} \left(\otimes_l \Psi_{J, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right) \right) \\
&= \sum_m \sum_{D_j^{p_l, m}} g_{s_{p_l}}^{(p_u)} \left(\left\{ \left((U_j)^{(p_l)} \right)^{k_l} \right\}_{l \leq m} \right) \otimes_{p_l} \delta(f_{p_l}) \prod_{l \leq m} \Psi_{J, s_{p_l}}^{\otimes k_l} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right)
\end{aligned} \tag{65}$$

with:

$$g_{s_{p_l}}^{(p_u)} \left(\left\{ \left((U_j)^{(p_l)} \right)^{k_l} \right\}_{l \leq m} \right) = \int f \left(U_j^{\otimes k}, \hat{\Pi}_{U_j^{\otimes k}} \right) h_{s_{p_l}}^{(p_u)} \left(\left((U_j)^{(p_l)} \right)^{k_l} \right)_l, \left((U_j)^{(p_u)} \right)^{k_u} \right) d \left(\left((U_j)^{(p_u)} \right)^{k_u} \right)$$

plus contributions from subobject:

$$Flin \left(\Psi_{J, s_{p_u}}^{\otimes k_u} \left([p_u]^{k_u} \right) \right) = \sum_m \sum_{D_j^{p_l, m}} \left(\sum_{m'} \sum_{D_{j, p_u}^{(p_l, p_{l'})}, m, m'} g \left(\left([p_l, p_{l'}]^{k_{l'}} \right)_{l, l'} \right) \otimes_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)$$

where we have:

$$\begin{aligned}
& g \left(\left([p_l, p_{l'}]^{k_{l'}} \right)_{l, l'} \right) \\
&= \int f \left(\left((U_j)^{(p_u)} \right)^{k_u}, \hat{\Pi}_{\left((U_j)^{(p_u)} \right)^{k_u}} \right) h_{s_{p_{l'}, l}}^{(p_u)} \left(\left([p_l, p_{l'}]^{k_{l'}} \right)_l, [p_u]^{k_u} \right) \prod_{p_{l'}, l} \delta(f_{p_{l'}, l}) d \left([p_u]^{k_u} \right)
\end{aligned}$$

Both terms can be regrouped by writing:

$$\Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l]^{k_l} \right) \bigoplus_{(p_{l'})} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \equiv \bigoplus_{(p_{l'})} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

with:

$$[p_l, p_{l'}]^{k_{l'}} = \left((U_j)^{(p_l, p_{l'})} \right)^{k_{l'}}$$

where the *id* map:

$$\begin{aligned}
p_{l, l} &: \left((U_j)^{(p_l)} \right) \rightarrow \mathcal{V} \left(\bigoplus_k \left((U_j)^{(p_l)} \right)^k \right) \\
(U_j)^{(p_l)} &\mapsto (U_j)^{(p_l)}
\end{aligned}$$

is added to the set of (p_l, l') . The full functional writes:

$$Flin \left(\Psi_{J, s_{p_u}}^{\otimes k_u} \left([p_u]^{k_u} \right) \right) = \sum_m \sum_{D_j^{p_l, m}} \left(\sum_{m'} \sum_{D_{j, p_u}^{(p_l, p_{l'})}, m, m'} g \left(\left([p_l, p_{l'}]^{k_{l'}} \right)_{l, l'} \right) \otimes_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right) \tag{66}$$

12.2.5 Partial states

As in part one we can define partial states for one of the subobjects, written 0 to distinguish it from others subobjects. A partial functional for this objects and its subobjects characterized by their fields $\Psi_{J,s_{p_{l'}},0}^{\otimes k_{l'}}$ is a collection:

$$\left\{ v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J,s_{p_{l'}},0}^{\otimes k_{l'}} \right) \right\}_{[p_l, p_{l'}]^{k_{l'}}} = \left\{ v_{[p_l, p_{l'}]^{k_{l'}}} \left([p_0, p_{l'}]^{k_{l'}} \right) \right\}_{[p_0, p_{l'}]^{k_{l'}}, [p_l, p_{l'}]^{k_{l'}}} \quad (67)$$

of densities that can be inserted in integrals of tensor products, so that a partial functional is given by the collection:

$$\int v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J,s_{p_{l'}},0}^{\otimes k_{l'}} \right) \otimes ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J,s_{p_{l'}},l}^{\otimes k_{l'}} \right) \delta \left(f_\gamma \left([p_0, p_{l'}]^{k_{l'}}, \{ [p_l, p_{l'}]^{k_{l'}} \} \right) \right) d \left([p_0, p_{l'}]^{k_{l'}}, \{ [p_l, p_{l'}]^{k_{l'}} \} \right) \quad (68)$$

where $ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J,s_{p_{l'}},l}^{\otimes k_{l'}} \right)$ is the evaluation functional for $\Psi_{J,s_{p_{l'}},l}^{\otimes k_{l'}} :$

$$ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J,s_{p_{l'}},l}^{\otimes k_{l'}} \right) = \Psi_{J,s_{p_{l'}},l}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

13 Projection on one particular subobject

13.1 Decomposition including a given subobject

We will consider particular decomposition:

$$D_j^{p_0, p_l, m}$$

standing for:

$$\mathcal{H}(U_j) = \mathcal{H} \left(\left((U_j)^{(p_0)} \right) \right) \otimes \mathcal{H} \left(\left\{ (U_j)^{(p_l)} \right\}_{l \leq m} \right)$$

and project the subobject corresponding to the (p_l) on some particular states. That is, we project on $\mathcal{H} \left(\left((U_j)^{(p_0)} \right) \right)$.

Since we add subobjects, we have to include also decomposition:

$$D_{j, p_u}^{(p_0, p_{l'}_0, p_l, p_{l'}_l), m, m'}$$

standing for th decomposition:

$$\mathcal{H} \left(\left((U_j)^{(p_u)} \right) \right) = \mathcal{H} \left(\left((U_j)^{(p_0, p_{l'}_0)} \right)^{k_{l'}} \right) \otimes \mathcal{H} \left(\left\{ (U_j)^{(p_l, p_{l'}_l)} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right)$$

Considering again the *id* map we write:

$$\Psi_{J,s_{p_0}}^{\otimes k_{p_0}} \left([p_0]^{k_{p_0}} \right) \bigoplus_{(p_{l'}_0)} \Psi_{J,s_{p_{l'}_0}}^{\otimes k_{l'}} \left([p_0, p_{l'}_0]^{k_{l'}} \right) \equiv \bigoplus_{(p_{l'}_0)} \Psi_{J,s_{p_{l'}_0}}^{\otimes k_{l'}} \left([p_0, p_{l'}_0]^{k_{l'}} \right)$$

and:

$$\Psi_{J,s_{p'l',l}}^{\otimes k_{l'}} \left([p_l]^{k_l} \right) \bigoplus_{(p_{l'l})} \Psi_{J,s_{p'l',l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}} \right) \equiv \bigoplus_{(p_{l'l})} \Psi_{J,s_{p'l',l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}} \right)$$

The full functional then writes for this decomposition:

$$\begin{aligned} & F_{f,lin} \left(\left\{ \Psi_{J,s_{p'l',0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right\}_{(p_{l'0})} \right) \\ &= \sum_m \sum_{D_j^{p_0, p_{l'}, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'0}, p_{l'}, p_{l'l}), m, m'}} g \left(\left\{ [p_l, p_{l'l}]^{k_{l'}} \right\}_{l, l'}, \{([p_0, p_{l'0}])\}_{0, l'} \right) \\ & \times \left(\otimes_{l'} \Psi_{J,s_{p'l',0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right) \left(\otimes_{l'} \Psi_{J,s_{p'l',l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}} \right) \right) \end{aligned} \quad (69)$$

13.2 Projection on $\mathcal{H} \left(\left((U_j)^{(p_0)} \right) \right)$

Projection on the states of $\mathcal{H} \left(\left((U_j)^{(p_0)} \right) \right)$ leads to an effective functional integrating the degrees of freedom for the projected states of $\bigoplus_{(p_{l'l})} \Psi_{J,s_{p'l',l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}} \right)$.

As in the first part, we will by expressing the functionals of $\Psi_{J,s_{p'l',0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right)$ in the basis of some functionals. The remaining degrees of freedom defined by $\Psi_{J,s_{p'l',l}}^{\otimes k_{l'}}$ will then be projected on some background, depending on the states considered.

13.2.1 Particular basis of states $\Psi_{J,s_{p'l',0}}^{\otimes k_{l'}}$

We first describe a basis of states for $\Psi_{J,s_{p'l',0}}^{\otimes k_{l'}}$. To do so, we assume that the state space for $\Psi_{J,s_{p'l',0}}^{\otimes k_{l'}}$:

$$H \left(\left\{ \Psi_{J,s_{p'l',0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right\}_{(p_{l'0})} \right)$$

decomposes as a sum of projection

$$id = \sum_v \prod_v$$

where the partial states $\{v\}$ form an arbitrary basis defined by densities, as in (67):

$$v \left\{ \Psi_J^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right\} = \left\{ v_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J,s_{p'l',0}}^{\otimes k_{l'}} \right) \right\}_{[p_l, p_{l'l}]^{k_{l'}}$$

The states $\{v\}$ can be eigenstates of some operator:

$$\Phi \left(\Psi_J^{\otimes k_{l'}}, \frac{\delta}{\delta \Psi_J^{\otimes k_{l'}}} \right)$$

As before we write:

$$\left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \left([p_0, p_{l'0}]^{k_{l'}} \right) \otimes \Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}} \right) = ev_{[p_0, p_{l'0}]^{k_{l'}}, [p_l, p_{l'l}]^{k_{l'}}} \left(\left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \otimes \Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \right)$$

where the evaluation $ev_{[p_0, p_{l'0}]^{k_{l'}}, [p_l, p_{l'l}]^{k_{l'}}$ computes:

$$\left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \otimes \Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}}$$

at a point defined by the coordinates $[p_0, p_{l'0}]^{k_{l'}}, [p_l, p_{l'l}]^{k_{l'}}$.

The constraints $\delta \left(f_\gamma \left([p_0, p_{l'0}]^{k_{l'}}, [p_l, p_{l'l}]^{k_{l'}} \right) \right)$ are kept implicit for the sake of simplicity. The decomposition for functionals with respect to v is performed by multiplying the evaluation by density $v_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right)$ (see 68):

$$\prod_v \rightarrow v_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \otimes ev_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \right) \delta \left(f_\gamma \left([p_0, p_{l'0}]^{k_{l'}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right) \right) \quad (70)$$

In local coordinats, the constraints:

$$\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right)$$

can be solved and $\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right)$ is replaced by:

$$\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right)$$

with:

$$\overline{[p_0, p_{l'0}]^{k_{l'}}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \equiv [p_0, p_{l'0}]^{k_{l'}} / \prod_{p_{1,l}, \dots, p_{m',l}} f_{p_{1,l} \dots p_{m',l}}, f_{p_{1,l} \dots p_{m',l}}^{-1} \{ [p_l, p_{l'l}]^{k_{l'}} \}$$

Implicitly $\Psi_J^{\otimes k_l}$ and $\Psi_J^{\otimes k_{p_0}}$ stand for $\Psi_{J, s_{p_0}}^{\otimes k_{p_0}}$ and $\Psi_{J, s_{p_l}}^{\otimes k_l}$.

As a consequence, (70) writes:

$$\begin{aligned} & v_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \otimes ev_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \right) \delta \left(f_\gamma \left([p_0, p_{l'0}]^{k_{l'}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right) \right) \\ &= \int v \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right) \\ & \quad \times \Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right) d \left(\overline{[p_0, p_{l'0}]^{k_{l'}}} \right) \otimes ev_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \right) \\ &= v_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \otimes ev_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',l}}}^{\otimes k_{l'}} \right) \end{aligned}$$

where we have:

$$\begin{aligned} & v_{[p_l, p_{l'l}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \right) \\ &= \int v \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right) \times \Psi_{J, s_{p_{l',0}}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \{ [p_l, p_{l'l}]^{k_{l'}} \} \right) d \left(\overline{[p_0, p_{l'0}]^{k_{l'}}} \right) \end{aligned} \quad (71)$$

We assume below that fields are chosen eigenstates of Λ so that the \sum_v is performed for a collection
 As an example, we can consider a particular case, we can choose for $v_{[p_l, p_{l'}]}^{k_{l'}}$:

$$v_{[p_l, p_{l'}]}^{k_{l'}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) = \frac{\sum_{[p_0, p_{l'0}]} \overline{v_{[p_0, p_{l'0}]}^{k_{l'}}}}{[p_0, p_{l'0}]^{k_{l'}}} \left\{ \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right\}$$

where

$$\frac{\sum_{[p_0, p_{l'0}]} \overline{v_{[p_0, p_{l'0}]}^{k_{l'}}}}{[p_0, p_{l'0}]^{k_{l'}}} \left\{ \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right\} = \Psi_J^{\otimes l} \left([p_0, p_{l'0}]^{k_{l'}} \right)$$

This formula amounts to sum over the evaluation at points $\overline{[p_0, p_{l'0}]^{k_{l'0}}}$ and the decomposition is performed with respect to the values of $\overline{[p_0, p_{l'0}]^{k_{l'0}}}$. We can also consider to restrict the projection to a subspace $V^{\subset \overline{[p_0, p_{l'0}]^{k_{l'0}}}}$ of $\overline{[p_0, p_{l'0}]^{k_{l'0}}}$:

$$\frac{\sum_{[p_0, p_{l'0}]} \overline{v_{[p_0, p_{l'0}]}^{k_{l'0}}}}{[p_0, p_{l'0}]^{k_{l'0}}} \left\{ \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'0}} \right\} = \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'0}} \left(V^{\subset \overline{[p_0, p_{l'0}]^{k_{l'0}}}}, \{ [p_l, p_{l'}]^{k_{l'}} \} \right)$$

which reduces the evaluation to $V^{\subset \overline{[p_0, p_{l'0}]^{k_{l'0}}}}, \{ [p_l, p_{l'}]^{k_{l'}} \}$.

13.2.2 Projection along the basis

Once the partial states $\{v\}$ are chosen, the states functionals of $\bigoplus_{(p_{l'})} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$ are projected by operator:

$$\sum_v \prod_v \otimes \prod_{\min S(v)}$$

Start with projection \prod_v . This is done by first rewriting the functional (69) with the decomposition:

$$\begin{aligned} & \sum_m \sum_{D_j^{p_0, p_l, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'0}, p_l, p_{l'})}, m, m'} g \left(\{ [p_l, p_{l'}]^{k_{l'}} \}_{l, l'}, \{ ([p_0, p_{l'0}]) \}_{0, l'} \right) \\ & \times \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right) \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right) \end{aligned} \quad (72)$$

and express this functional in the basis of states $\{v\}$, each of them defined by the collection $\left\{ v_{[p_l, p_{l'}]}^{k_{l'}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \right\}$ given in (71). To do so, we insert in (72) the identity operator $\sum_v \prod_v$

so that the functional becomes:

$$\begin{aligned}
& \sum_v \prod_v \sum_m \sum_{D_j^{p_0, p_l, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'0}, p_l, p_{l'})}, m, m'} g \left(\left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{l, l'}, \{([p_0, p_{l'0}])\}_{0, l'} \right) \quad (73) \\
& \times \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right) \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right) \\
& \rightarrow \sum_v \prod_v \sum_m \sum_{D_j^{p_0, p_l, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'0}, p_l, p_{l'})}, m, m'} g \left(\left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{l, l'}, v \right) \\
& \otimes v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \left(\Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)
\end{aligned}$$

and this becomes a functional of:

$$v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \left(\Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)$$

We have thus performed a change of basis in (73), by replacing the evaluation functionals for the field $\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}}$ by the basis of states $v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right)$. The functns $g \left(\left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{l, l'}, v \right)$ are the coefficients of the functional in this new basis.

13.2.3 Projection of the degrees of freedom of $\Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}}$

As in the first part, we consider that the projection comes from the minimization of a given functional of the form:

$$\exp \left(-S \left(\left\{ \otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \otimes_{l', l} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right\}, v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \otimes_{l', l} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right) \right)$$

where the indices denoting the realizations are implicit.

The projection of functionals by:

$$\sum_v \prod_v \otimes \prod_{\min S(v)}$$

will be:

$$\begin{aligned}
& \sum_v \sum_m \sum_{D_j^{p_0, p_l, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'0}, p_l, p_{l'})}, m, m'} g \left(\left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{l, l'}, v \right) \\
& v_{\{[p_l, p_{l'}]^{k_{l'}}\}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \otimes \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \exp(-S) \prod \mathcal{D} \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)
\end{aligned}$$

Minimizing S leads to $\Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$ with multiplication with a background:

$$\Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

and the functional reduces to:

$$\sum_m \sum_{D_j^{p_0, p_l, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'0}, p_l, p_{l'})}, m, m'} g \left(\left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{l, l'}, v \right) \\ \otimes v_{\{[p_l, p_{l'}]^{k_{l'}}\}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \left(\Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)$$

The part:

$$g \left(\left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{l, l'}, v \right) \times v_{\{[p_l, p_{l'}]^{k_{l'}}\}} \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \left(\otimes_{l'} \Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)$$

is a functional of:

$$v_{\{[p_l, p_{l'}]^{k_{l'}}\}} \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \left(\otimes_{l'} \Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)$$

13.2.4 Saddle point solutions without degeneracy

The background $\otimes_{l'} \Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$ is obtained by minimisation:

$$S \left(\left\{ \otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \otimes_{l, l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right\}, v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \otimes \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right)$$

The tensor:

$$\otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \otimes_{l, l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

stands for set of realizations:

$$\left\{ \otimes_{l'} \Psi_{J, \alpha, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \otimes_{l, l'} \Psi_{J, \alpha, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right\}$$

and:

$$v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \otimes \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

is evaluated at the considered realization:

$$\Psi_{J, \alpha, s_{p_{l'}, 0}}^{\otimes k_{l'}} \otimes \Psi_{J, \alpha, s_{p_{l'}, l}}^{\otimes k_{l'}}$$

We will write the functional $v_{[p_l, p_{l'}]^{k_{l'}}$ in components:

$$v_{[p_l, p_{l'}]^{k_{l'}}} \left(\otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \\ = \int v \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, [p_l, p_{l'}]^{k_{l'}} \right) \times \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, [p_l, p_{l'}]^{k_{l'}} \right) d \left(\overline{[p_0, p_{l'0}]^{k_{l'}}} \right)$$

As before, including the *id* map, we have:

$$\left\{ \Psi_{J,0,s_{p_l}^{\otimes k_l}}^{\otimes k_l}([p_l, k_l]) \bigoplus_{(p_{l'})} \Psi_{J,0,s_{p_{l'},l}^{\otimes k_{l'}}}^{\otimes k_{l'}}([p_l, p_{l'}]^{k_{l'}}) \right\}_{\substack{l \leq m \\ l' \leq m'}} \equiv \left\{ \bigoplus_{(p_{l'})} \Psi_{J,0,s_{p_{l'},l}^{\otimes k_{l'}}}^{\otimes k_{l'}}([p_l, p_{l'}]^{k_{l'}}) \right\}_{\substack{l \leq m \\ l' \leq m'}}$$

We also write:

$$v \{ \Psi_J^{\otimes k_{l'}} \} = \sum \int v([p_0, p_{l'0}]^{k_{l'}}) \Psi_J^{\otimes k_{l'}}([p_0, p_{l'0}]^{k_{l'}}) d[p_0, p_{l'0}]^{k_{l'}}$$

As in the first part, the series expansion of the saddle point solution is:

$$\begin{aligned} & \Psi_{J,0,s_{p_{l'},l}^{\otimes k_{l'}}}^{\otimes k_{l'}}([p_l, p_{l'}]^{k_{l'}}, v) \\ &= \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_{1,2}}} d \left(\overline{\{ [p_0, p_{l'0}]^{k_{l'}} \}_{l \leq s, l' \leq s'}} \right) d \left(\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1, l_1 \neq l \\ l'_1 \leq m'_1}} \left\{ [p_{l_2}, p_{l'_2 l_2}]^{k_{l'_2}} \right\}_{\substack{l_2 \leq m_2 \\ l'_2 \leq m'_2}} \right) \\ & \times \mathcal{K}_0 \left(\left\{ \overline{\{ [p_0, p_{l'0}]^{k_{l'}} \}_{l \leq s, l' \leq s'}} \right\}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ [p_{l_2}, p_{l'_2 l_2}]^{k_{l'_2}} \right\}_{\substack{l_2 \leq m_2 \\ l'_2 \leq m'_2}} \right) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J,s_{p_{l'},0}^{\otimes k_{l',i}}}^{\otimes k_{l',i}} \left(\overline{\{ [p_0, p_{l'0}]^{k_{l'}} \}_{l \leq s, l' \leq s'}} \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \prod_{\substack{l_2 \leq m_2 \\ l'_2 \leq m'_2}} v_{[p_{l_2}, p_{l'_2 l_2}]^{k_{l'_2}}} \{ \Psi_J^{\otimes k_{l'}} \} \end{aligned} \quad (74)$$

where:

$$\overline{\{ [p_0, p_{l'0}]^{k_{l'}} \}_{l \leq s, l' \leq s'}} \equiv \overline{\left((U_j)^{(p_0, p_{l'0})} \right)^{k_{l'}}} = \left((U_j)^{(p_0, p_{l'0})} \right)^{k_{l'}} / \prod_{p_{1,l}, \dots, p_{m',l}} f_{p_{1,l}, \dots, p_{m',l}}$$

and where the $f_{p_{l',0}, p_{1, \dots, p_m}}$ describe the constraint involving $(U_j)^{(p_0, p_{l'0})}$ and the $(U_j)^{(p_l)}$ and their subobjects. Note that:

$$\left\{ \overline{\{ [p_0, p_{l'0}]^{k_{l'}} \}_{l \leq s, l' \leq s'}} \right\}$$

are s copies of:

$$\left\{ \overline{\{ [p_0, p_{l'0}]^{k_{l'}} \}_{l' \leq s'}} \right\}$$

We have also defined:

$$[p, p', k'] = \{ p_0, p_{l'0}, k_{l'} \}_{l \leq s, l' \leq s'}$$

and:

$$(m, m', [p, p', k'])_{1,2} = \left\{ \left\{ p_{l_1}, p_{l'_1 l_1}, k_{l'_1} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ p_{l_2}, p_{l'_2 l_2}, k_{l'_2} \right\}_{\substack{l_2 \leq m_2 \\ l'_2 \leq m'_2}} \right\}$$

Gathering indices p_{l_1}, p_{l_2}, \dots as p_{l_1} and defining the following partitions of $\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} :$

$$\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} = \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}^c}$$

we find a more compact formulation for the saddle point solution without degeneracy:

$$\begin{aligned}
& \Psi_{J,0,s_{p_{l',l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, v \right) \\
&= \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} d \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1, l_1 \neq l \\ l'_1 \leq m'_1}} \right) \\
& \times \mathcal{K}_0 \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\
& \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J,s_{p_{l',0}}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \left\{ \Psi_J^{\otimes k_{l'}} \right\}
\end{aligned} \tag{75}$$

13.2.5 Including degeneracy: general formula

As in part one, we consider transformations parametrized by:

$$\begin{aligned}
& \left\{ \left\{ \left(\hat{\mathbf{\Lambda}}^{(p_l)} \left(\Psi_J^{\otimes k_{p_0}}, v \right) \right) \right\}_{p_l}, \left\{ \left(\hat{\mathbf{\Lambda}}^{(p_l, p_{l', l}, k_{l'})} \left(\Psi_J^{\otimes k_{p_0, l'}}, v \right) \right) \right\}_{p_{l', l}} \right\} \\
& \rightarrow \left\{ \left\{ \left(\hat{\mathbf{\Lambda}}^{(p_l, p_{l', l}, k_{l'})} \left(\Psi_J^{\otimes k_{p_0, l'}}, v \right) \right) \right\}_{p_{l', l}} \right\}
\end{aligned}$$

with in general $\left(\left(\hat{\mathbf{\Lambda}}^{(p_l, p_{l', l}, k_{l'})} \left(\Psi_J^{\otimes k_{p_0, l'}}, v \right) \right) \right) \simeq \left(\hat{\mathbf{\Lambda}}^{(p_l, p_{l', l})} \left(\Psi_J^{\otimes k_{p_0, l'}}, v \right) \right)^{k_{l'}}$.

In the sequel, we will write the parameters in a more compact form:

$$\left\{ \left\{ \left(\hat{\mathbf{\Lambda}}^{(p_l, p_{l', l}, k_{l'})} \left(\Psi_J^{\otimes k_{p_0, l'}}, v \right) \right) \right\}_{p_{l', l}} \right\} \rightarrow \left[\begin{array}{c} (p_l, p_{l', l}, k_{l'}) \\ \hat{\mathbf{\Lambda}} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \rightarrow \left[\begin{array}{c} (p_l, p_{l', l}, k_{l'}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \tag{76}$$

The upper indices account for the decomposition of the initial field, indicating that the degeneracy parameters depend on the subobjects decomposition. This dependency involves the mappings $(p_l, p_{l', l})$ and the tensor power $k_{l'}$ of the subobject arising in the saddle-point solution. The lower indices account for the dependence of symmetries on the remaining non-projected field. The symmetry parameters are functions of the states in which this effective system is considered¹.

In (76), the arrow indicates that we will omit $\hat{\mathbf{\Lambda}}$, which represented the parameters in the previous sections, to retain only the parameters main determinants—the mappings between subobjects.

¹In fact, note that in this formula, taking account of subobject implies that the parameters implicitly stand for:

$$\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ \hat{\mathbf{\Lambda}} \\ [\{\Psi_J[l', 0]\}, [p_0, p_{l'0}]^{k_{l'}}] \end{array} \right] = \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ \hat{\mathbf{\Lambda}} \\ \{\Psi_J[l', 0], \Psi_{J, \alpha, s_{p_{l', 0}}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right)\} \end{array} \right]$$

If the subobjects are included, the generators of transformations between realizations of $\Psi_{J,0,s_{p_{l'},l}}^{\otimes k_{l'}}$ are:

$$\begin{aligned} \mathbf{L}(\Psi_J^{\otimes k_{p_0}}, v) &= \sum_{\substack{(s,s',[p,p',k']) \\ (m_1,m'_1,[p,p',k'])}} d\left(\overline{\{[p_0,p_{l'0}]^{k_{l'}}\}}_{l' \leq s'}\right) d\left(\left\{\left[p_{l_1},p_{l'_1 l_1}\right]^{k_{l'_1}}\right\}_{l'_1 \leq m'_1}\right) \\ &\quad \mathbf{I}\left(\mathbf{U}_{[p_l,p_{l'l}]^{k_{l'}}}, \overline{\Pi}_{[p_l,p_{l'l}]^{k_{l'}}}, \overline{\{[p_0,p_{l'0}]^{k_{l'}}\}}_{l' \leq s}, \left\{\left[p_{l_1},p_{l'_1 l_1}\right]^{k_{l'_1}}\right\}_{l'_1 \leq m'_1}, v\right) v \left\{\left[p_{l_1},p_{l'_1 l_1}\right]^{k_{l'_1}}\right\}_{l'_1 \leq m'_1} \left\{\Psi_J^{\otimes k_{l'}}\right\} \end{aligned} \quad (77)$$

We will wrt:

$$(m_1, m'_1, [p, p', k']) = (m, m', [p, p', k'])_1$$

and \mathbf{I} has components dual to $\left[\begin{array}{c} (p_l, p_{l',l}, k_{l'}) \\ \{[\Psi_J[l', 0]], v\} \end{array} \right]$ so that, we write the solution is given by grp ctn:

$$\begin{aligned} &\Psi_{J,0,s_{p_{l'},l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}}, \left[\begin{array}{c} (p_l, p_{l',l}, k_{l'}) \\ \{[\Psi_J[l', 0]], v\} \end{array} \right], v \right) \\ &= \exp\left(i\mathbf{L}(\Psi_J^{\otimes k_{p_0}}, \nu) \cdot \left[\begin{array}{c} (p_l, p_{l',l}, k_{l'}) \\ \{[\Psi_J[l', 0]], v\} \end{array} \right]\right) \Psi_{J,0,s_{p_{l'},l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}}, \left[\begin{array}{c} (p_l, p_{l',l}, k_{l'}) \\ \{[\Psi_J[l', 0]], v\} \end{array} \right]\right) \end{aligned}$$

and the dependency in $(\Psi_J^{\otimes k_{p_0}})$ is kept implicit. Inserting this result in the saddle point equation (75) yields:

$$\begin{aligned} &\Psi_{J,0,s_{p_{l'},l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ \{[\Psi_J[l', 0]], v\} \end{array} \right] \right\}, v \right) \\ &= \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} d\left(\overline{\{[p_0,p_{l'0}]^{k_{l'}}\}}_{l' \leq s}, \left\{\left[p_{l_1},p_{l'_1 l_1}\right]^{k_{l'_1}}\right\}_{l'_1 \leq m'_1, l'_1 \neq l}\right) \\ &\quad \times \mathcal{K}_0 \left(\overline{\{[p_0,p_{l'0}]^{k_{l'}}\}}_{l' \leq s}, \left\{\left[p_{l_1},p_{l'_1 l_1}\right]^{k_{l'_1}}\right\}_{l'_1 \leq m'_1}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ \{[\Psi_J[l', 0]], v\} \end{array} \right] \right\}_{l'_1 \leq m'_1} \right) \\ &\quad \times \prod_{\substack{l' \leq s \\ l' \leq s'}} \Psi_{J,s_{p_{l'},0}}^{\otimes k_{l'}} \left(\overline{\{[p_0,p_{l'0}]^{k_{l'}}\}}_{l' \leq s}, \left\{\left[p_{l_1},p_{l'_1 l_1}\right]^{k_{l'_1}}\right\}_{l'_1 \leq m'_1} \right) \prod_{\mathcal{P}^c} v_{[p_{l_1},p_{l'_1 l_1}]^{k_{l'_1}}} \left\{\Psi_J^{\otimes k_{l'}}\right\} \end{aligned} \quad (78)$$

with:

$$\begin{aligned} & \mathcal{K}_0 \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m \\ l'_1 \leq m'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ &= \exp \left(i \mathbf{L} \left(\Psi_J^{\otimes k_{p_0}}, v \right) \cdot \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \right) \mathcal{K}_0 \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m \\ l'_1 \leq m'}} \right) \end{aligned}$$

and the operator $\mathbf{L} \left(\Psi_J^{\otimes k_{p_0}} \right)$ acts on the $\overline{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_{i_1}}}}$ degrees of freedm. We will also wrt:

$$\Psi[l', 0] = \otimes_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \quad (79)$$

13.2.6 Projected functional

To each state:

$$\Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}, v \right)$$

we associate coefficient $\Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}, v \right)$ to obtain a projected background state, and this leads to define the generic projected functional (see appendix 7):

$$\begin{aligned} & \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} \int g^{\mathcal{K}} \left(\overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \quad (80) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'_i}} \left(\overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P}^c \neq \emptyset} v_{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \left\{ \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'_1}} \right\} \\ & \times \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) d \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) dv \end{aligned}$$

13.2.7 State dependent effective field

Reintroducing the realization index α , this is a functional of the state:

$$\begin{aligned}
& v \left\{ \left[p_{l_1}, p_{l'_1 l_1} \right]^{k_{l'_1}} \right\} \Psi_{J, \alpha, s_{p_{l'_1, 0}}, s_{p_{l'_1, 0}}}^{\otimes \sum k_{l'_i} + \sum k_{l'_1}} \left(\overline{\left[p_0, p_{l'_i 0} \right]^{k_{l'_i}}} \right) \Psi_\alpha \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_{J, \alpha} [l', 0]] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) \\
&= \int \sum_{\alpha \in \{\alpha_i\} \cup \{\alpha'_i\}} \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, \{\alpha_i\}, s_{p_{l', 0}}}^{\otimes k_{l'_i}} \left(\overline{\left[p_0, p_{l'_i 0} \right]^{k_{l'_i}}} \right), \left\{ \left[p_{l_1}, p_{l'_1 l_1} \right]^{k_{l'_1}} \right\}_{\mathcal{P}} \\
&\times \prod_{\mathcal{P}^c \neq \emptyset} v \left\{ \left[p_{l_1}, p_{l'_1 l_1} \right]^{k_{l'_1}} \right\} \Psi_{J, \{\alpha'_i\}, s_{p_{l'_1, 0}}}^{\otimes k_{l'_1}} \left(\overline{\left[p_0, p_{l'_i 0} \right]^{k_{l'_i}}} \right) \Psi_\alpha \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_{J, \alpha} [l', 0]] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) d \left\{ \left[p_{l_1}, p_{l'_1 l_1} \right]^{k_{l'_1}} \right\}
\end{aligned}$$

If the coordinates are independent of α , we can sum over components and the functional depends on the effective field:

$$\begin{aligned}
& v \left\{ \left[p_{l_1}, p_{l'_1 l_1} \right]^{k_{l'_1}} \right\} \Psi_{J, s_{p_{l', 0}}, s_{p_{l', 0}}}^{\otimes \sum k_{l'_i} + \sum k_{l'_1}} \left(\overline{\left[p_0, p_{l'_i 0} \right]^{k_{l'_i}}} \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J [l', 0]] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) \quad (81) \\
&= \sum_{\alpha} v \left\{ \left[p_{l_1}, p_{l'_1 l_1} \right]^{k_{l'_1}} \right\} \Psi_{J, \alpha, s_{p_{l', 0}}, s_{p_{l', 0}}}^{\otimes \sum k_{l'_i} + \sum k_{l'_1}} \left(\overline{\left[p_0, p_{l'_i 0} \right]^{k_{l'_i}}} \right) \Psi_\alpha \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J [l', 0]] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right)
\end{aligned}$$

13.2.8 Local effective field

In appendix 7, we rewrite the generic functional (80) by summing over ν and by introducing series of realizations, as in part one. It yields for the generic functional:

$$\begin{aligned}
& \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} \int \sum_{\alpha} \bar{g}^{\mathcal{K}} \left(\left\{ \left[p_0, p_{l' 0} \right]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J [l', 0]] \end{array} \right], [p_0, p_{l' 0}]^{k_{l'}}, v \right\} \right) \\
&\times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, \alpha, s_{p_{l', 0}}}^{\otimes \sum k_{l'_i}} \left(\overline{\left[p_0, p_{l'_i 0} \right]^{k_{l'_i}}} \right) \Psi_\alpha \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J [l', 0]] \end{array} \right], [p_0, p_{l' 0}]^{k_{l'}}, v \right\}, v \right) d \left(\left\{ \left[p_0, p_{l' 0} \right]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right)
\end{aligned}$$

If the set $\left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\}$ is independent of the copies $\Psi_{J, \alpha, s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}}$, this becomes:

$$\begin{aligned} & \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} \int \bar{g}^{\mathcal{K}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) \\ & \times \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) d \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}} \right) \\ & \equiv F_{f, lin} \left(\left\{ \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) \right\}_{l'} \right) \end{aligned} \quad (82)$$

where the local field is defined by:

$$\begin{aligned} & \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) \\ & = \sum_{\alpha} \prod_{\substack{l' \leq s \\ l' \leq s'}} \Psi_{J, \alpha, s_{p_{l'0}}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \Psi_{\alpha} \left(\left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) \end{aligned} \quad (83)$$

If the $\left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\}$ are not independent of the copies $\Psi_{J, \alpha}^{\otimes k_{p_0}}$, identification is local.

14 Variations

14.1 Local invariance

The principle is similar to part one. Starting with the effective field after projection:

$$\begin{aligned} & \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) \\ & = \sum_{\alpha} \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right), v, \alpha \\ & = \sum_{\alpha} \prod_{\substack{l' \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} , \alpha \right) \Psi_{\alpha} \left(\left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) \end{aligned} \quad (84)$$

We request that variations of fields states does not affect the parameter space. More generally we will consider the state dependent field (81). Using (84), it can be rewritten more compactly as:

$$\int v \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}} \right) \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l' \leq s} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}, \left\{ \left[\left[\Psi_J [l', 0] \right], [p_0, p_{l'0}]^{k_{l'}} , v \right] \right\} \right) d \left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l' \leq s \\ l' \leq s'}}$$

where:

$$v \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right)$$

includes some delta functions to account for the partition $\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}, \overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}}$ in (81). The integral element $d \left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}$ will be omitted.

The variation of $\Psi_{J, s_{p_{l'0}}}^{\otimes k_{l'}} \left([\bar{p}_0, \bar{p}_{l'0}]^{k_{l'}}, \beta \right)$ in the previous expression leads to:

$$\begin{aligned} & \int v \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, v \right)}{\delta \Psi_{J, s_{p_{l'0}}}^{\otimes k_{l'}} \left([\bar{p}_0, \bar{p}_{l'0}]^{k_{l'}}, \beta \right)} \\ = & \int v \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta' \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, v, \alpha \right)}{\delta' \Psi_{J, s_{p_{l'0}}}^{\otimes k_{l'}} \left([\bar{p}_0, \bar{p}_{l'0}]^{k_{l'}}, \beta \right)} \\ & + \int v \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}}{\delta \Psi_{J, s_{p_{l'0}}}^{\otimes k_{l'}} \left([\bar{p}_0, \bar{p}_{l'0}]^{k_{l'}}, \beta \right)} \\ & \times \nabla \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\} \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, v, \alpha \right) \end{aligned}$$

where δ' is the variation with constnt prmtrs. The first order independence of $\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right]$ wrts:

$$\begin{aligned} 0 & = \int v \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}}{\delta \Psi_{J, s_{p_{l'0}}}^{\otimes k_{l'}} \left([\bar{p}_0, \bar{p}_{l'0}]^{k_{l'}}, \beta \right)} \\ & \times \nabla \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\} \Psi_{J, \Pi_{l'} s_{p_{l'0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, v, \alpha \right) \end{aligned}$$

Or in local coordinates, as in the first part:

$$\begin{aligned} & \int \sum_{\alpha} \left[\Xi^{(k_i)} \left(\left\{ \{\Psi_J[l', 0]\}, \hat{\Lambda}^{(k_{l'_1}), \alpha} \right\} \right) \right]_{\beta, [\bar{p}_0, \bar{p}_{l'_0}]^{k_{l'_1}}}^{\alpha, v} \nabla_{\hat{\Lambda}^{(k_i)}} \hat{\Psi}_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ \left\{ [p_0, p_{l'_0}]^{k_{l'_1}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right\}, \hat{\Lambda}^{(k_{l'_1}), \alpha} \right) \Big|_i \Big) \\ &= \frac{\delta}{\delta \Psi_{J, \alpha, s_{p_{l'_0}}}^{\otimes k_{l'_1}} (\bar{p}_0, \bar{p}_{l'_0})^{k_{l'_1}}, \beta} V \left(\left\{ \hat{\Psi}_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ \left\{ [p_0, p_{l'_0}]^{k_{l'_1}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right\}, \hat{\Lambda}^{(k_{l'_1}), \alpha} \right) \right\} \right) \end{aligned}$$

where:

$$\begin{aligned} & \left[\Xi^{(k_i)} \left(\left\{ \{\Psi_J[l', 0]\}, \hat{\Lambda}^{(k_{l'_1}), \alpha} \right\} \right) \right]_{\beta, [\bar{p}_0, \bar{p}_{l'_0}]^{k_{l'_1}}}^{\alpha, v} \\ &= \int v \left(\left\{ [p_0, p_{l'_0}]^{k_{l'_1}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}}{\delta \Psi_{J, s_{p_{l'_0}}}^{\otimes k_{l'_1}} (\bar{p}_0, \bar{p}_{l'_0})^{k_{l'_1}}, \beta} \left(\frac{\delta \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right]}{\delta \hat{\Lambda}^{(k_{l'_1}), \alpha}} \right)^{-1} \end{aligned}$$

14.2 Averaged field and global invariance

As in the first part, we can rather require a global invariance for an averaged field. This one is defined as a local field, defined in one point, with cloud of parameters that are considered in an integrated manner.

14.2.1 Averaged field

To explain the forementioned claim, we assume that for a sequence of included group of symmetry:

$$\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} = \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\infty}$$

with:

$$\left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\infty} = \left(\left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}]_{i \leq n} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right)_n$$

and:

$$\left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}]_{i \leq n} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\infty} = \left(\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}]_{i \leq n} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right)_n$$

with $(p_{l_1}, p_{l'_1 l_1}, k_{l'_1})$ an initial set of parameters. That is the parameter spaces is a set of infinite flag manifold starting with one set of parameters:

$$\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right]$$

and the field can be described by:

$$\Psi_{J, \Pi_{l'} s_{p_{l'_0}}}^{\otimes \sum_{l'} k_{l'_1}} \left(\left\{ [p_0, p_{l'_0}]^{k_{l'_1}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right), \left(\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}]_{i \leq n} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right)_n, v$$

which accounts both the integrated presenc of a cloud, that is the points of a space, with distinguished points $\left[\begin{array}{c} (p_{l_1}, p'_{l_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right]$. These points bear the physical quantities dependency.

14.2.2 Series expansion of the field in cloud variables

As in part one, we can expand the field in series of the cloud variables. Given that:

$$\left(\left(\left[\begin{array}{c} [p_{l_i}, p'_{l'_i l_i}, k_{l'_i}]_{i \leq n} \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right) \right)_n$$

is a set of sequences:

$$\left\{ \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, \left\{ \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}), (p_{l_2}, p'_{l'_2 l_2}, k_{l'_2}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\} \dots$$

and given the hypothesis of group inclusions, we can decompose the parameters depending on several indices as:

$$\left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}), (p_{l_2}, p'_{l'_2 l_2}, k_{l'_2}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] = \left(\left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right], \left[\begin{array}{c} (p_{l_2}, p'_{l'_2 l_2}, k_{l'_2}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right)$$

and, as a consequenc, we can expand $\Psi_{J, \Pi_{l'} s_{p'_{l', 0}}}^{\otimes \sum_{l'} k_{l'}}$ as series of one point, two points...:

$$\begin{aligned} & \Psi_{J, \Pi_{l'} s_{p'_{l', 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} , \left(\left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} [p_{l_i}, p'_{l'_i l_i}, k_{l'_i}]_{i \leq n} \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\} \right)_n , v \right) \\ &= \sum \Psi_{J, \Pi_{l'} s_{p'_{l', 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} , \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\} , v \right) \\ &+ \sum \Psi_{J, \Pi_{l'} s_{p'_{l', 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} , \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} (p_{l_{i_1}}, p'_{l'_{i_1} l_{i_1}}, k_{l'_{i_1}}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right], \left[\begin{array}{c} (p_{l_{i_2}}, p'_{l'_{i_2} l_{i_2}}, k_{l'_{i_2}}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\} , v \right) \\ &+ \dots \end{aligned}$$

In this sum, the set:

$$\left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right]$$

appears through an infinite number of representants, since there are an infinite number of flags arising in the sum. As a consequence, the field is thus a series:

$$\begin{aligned}
& \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left(\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} [(p_{l_i}, p_{l'_i l_i}, k_{l'_i})]_{i \leq n} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_n \right) \right), v \\
&= \sum \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], v \right) \\
&+ \sum \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} (p_{l_{i_1}}, p_{l'_{i_1} l_{i_1}}, k_{l'_{i_1}}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}, \left[\begin{array}{c} (p_{l_{i_2}}, p_{l'_{i_2} l_{i_2}}, k_{l'_{i_2}}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}, v \\
&+ \dots
\end{aligned}$$

and this series enables to rewrite $\Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}}$ as the following functional:

$$\begin{aligned}
& \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], V_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}, v \right) \\
&+ \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], V_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}^2, v \right) + \dots
\end{aligned} \tag{85}$$

for the set of points:

$$V_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})} = \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}$$

14.2.3 Averaging over cloud variables

As in the first part, this can be rewritten as an average over cloud variables:

$$\begin{aligned}
& \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left(\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left\{ \left[\begin{array}{c} [(p_{l_i}, p_{l'_i l_i}, k_{l'_i})]_{i \leq n} \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_n \right) \right), v \\
&= \sum_r \int_{V_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}^r} \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \left(\Lambda_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})} \right)^r, v \right) d \left(\Lambda_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})} \right)^r
\end{aligned} \tag{86}$$

where the $\Lambda_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}$ are local coordinates for $V_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}$.

14.2.4 Invariance for averaged field

For the averaged field, the local invariance of the parameter space with respect to the fluctuations of $\Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}}$ may be disregarded if we consider that only the entire cloud should not be affected. This corresponds to impose that the variations at the border are cancelled. Starting with the equation

(85) and imposing the invariance (we only write the first term of (86), the computation is similar for the higher order terms):

$$\begin{aligned}
0 &= \int v \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \left\{ \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}}{\delta \Psi_{J, s_{p'_{l'_1, 0}}}^{\otimes k_{l'}}} \left([\bar{p}_0, \bar{p}'_{l'_1 0}]^{k_{l'}}, \beta \right)} \\
&\times \nabla \left\{ \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \Psi_{J, \Pi_{l'} s_{p'_{l'_1, 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}, v \right) \\
&+ \int v \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}}{\delta \Psi_{J, s_{p'_{l'_1, 0}}}^{\otimes k_{l'}}} \left([\bar{p}_0, \bar{p}'_{l'_1 0}]^{k_{l'}}, \beta \right)} \\
&\times \nabla V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})} \Psi_{J, \Pi_{l'} s_{p'_{l'_1, 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}, v \right)
\end{aligned}$$

A derivation similar to (30) leads to the following relation:

$$\begin{aligned}
0 &= \int v \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \left\{ \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}}{\delta \Psi_{J, s_{p'_{l'_1, 0}}}^{\otimes k_{l'}}} \left([\bar{p}_0, \bar{p}'_{l'_1 0}]^{k_{l'}}, \beta \right)} \\
&\times \nabla \left\{ \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \Psi_{J, \Pi_{l'} s_{p'_{l'_1, 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}, v \right) \\
&+ \int v \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \left(\epsilon \frac{\delta V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}}{\delta \Psi_{J, s_{p'_{l'_1, 0}}}^{\otimes k_{l'}}} \left([\bar{p}_0, \bar{p}'_{l'_1 0}]^{k_{l'}}, \beta \right) - h \left(V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}, \Psi_{J, \Pi_{l'} s_{p'_{l'_1, 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \right) \right) \\
&\times \Psi_{J, \Pi_{l'} s_{p'_{l'_1, 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left[\begin{array}{c} (p_{l_1}, p'_{l'_1 l_1}, k_{l'_1}) \\ [\hat{\mathbf{\Lambda}} \\ \{\Psi_J[l', 0]\}, v] \end{array} \right], V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}, v \right)
\end{aligned}$$

where:

$$h \left(V_{(p_{l_i}, p'_{l'_i l_i}, k_{l'_i})}, \Psi_{J, \Pi_{l'} s_{p'_{l'_1, 0}}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p'_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \right)$$

is the average of:

$$h \left(\left(\Lambda_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})} \right)^r, \Psi_{J, \Pi_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \right) = \frac{\delta \left(\Lambda_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})} \right)^r}{\delta \Psi_{J, \Pi_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right)}$$

over $V_{(p_{l_i}, p_{l'_i l_i}, k_{l'_i})}$.

15 Constraints

15.1 General dependency

So far we left aside the form of the constraints. The treatment is very similar to the particular case of the first part, so that we only generalize the results. The constraint describes relations on the smtr operators for $n \in \mathbb{N}$:

$$h_{k_n} \left(\{ \mathbf{L}_{\{\alpha_i\}} (\Psi_J^{\otimes l}) \}_{i \leq n}, \{ U_i^k \}_i, h_p ((\Psi_J), U_j^l, \nu) \right) = 0 \quad (87)$$

where the $\{ U_i^k \}_i$ are the initial parameters for the projected fields, the $\mathbf{L}_{\{\alpha_i\}} (\Psi_J^{\otimes l})$ act on the set of projected fields and where the functionals $h_l ((\Psi_J), U_j^l, \nu)$ have the form:

$$h_p ((\Psi_J), \{ U_j^{l_i} \}, \nu) = \left[\int h_p ((U_j^{l_i}), \nu) \left(\prod_i \Psi_J^{\otimes l_i} (U_j^{l_i}) \right) d(U_j^{l_i}) \right] \prod_i \Psi_J^{\otimes l_i} (U_j^{l_i})$$

Evaluated on the projectd states, and assuming the independence of parameters from the realizations, this becomes a relation:

$$0 = \int h_{k_n} \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, \{ U_i^k \}_i, h_p ((\Psi_J), U_j^l, \nu) \right) \prod_{k_{l'}} \left| \Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, v \right) \right|^2 d \{ U_i^k \}_i \quad (88)$$

where $\Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}}$ has been defined in (78). A series expansion of the constnt in the $\{ U_i^k \}_i$ shows that the constraint become functional relations between the parameters, and:

$$\{ \Psi [l', 0] \}, \nu, \{ \Psi [l', 0] \}, \bar{H}_m \left(\Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \right)$$

where the fld $\Psi [l', 0]$ defined in (79) and:

$$\bar{H}_m = \int \bar{h}_m (\{ U_i^k \}_i) \prod_{k_{l'}} \left| \Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'l}]^{k_{l'}}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}, v \right) \right|^2 d \{ U_i^k \}_i$$

for some functions $\bar{h}_m (\{ U_i^k \}_i)$.

Alternatively, the constraint depends on the kernel defined in (78):

$$\left| \mathcal{K}_0 \left(\left\{ \overline{[p_0, p'_{l'}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p'_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \right|^2$$

that keeps track of the projected states in the definition of the constraint.

15.2 Lowest order expansion and Metrics on projected states

A second order expansion of the previous constraint enables to define a metric on the parameter space. It depends both on the field $\{\Psi[l', 0]\}$, and the background $\Psi_{J,0,s_{p'_{l',l}}}^{\otimes k_{l'}}$ through $\bar{H}_m \left(\Psi_{J,0,s_{p'_{l',l}}}^{\otimes k_{l'}} \right)$.

For a local functional at the lowest order we find:

$$0 = \gamma(\{\Psi[l', 0]\}, \nu) + \left\{ \left[\begin{array}{c} [p_{l_i}, p'_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right]_{i \leq n_1} \right\} \quad (89)$$

$$\times M \left(\left\{ [p_{l_i}, p'_{l'_i l_i}, k_{l'_i}], [p_{l_j}, p'_{l'_j l_j}, k_{l'_j}] \right\}, \nu, \left\{ \{\Psi[l', 0]\}, \bar{H}_m \right\} \right) \left\{ \left[\begin{array}{c} [p_{l_j}, p'_{l'_j l_j}, k_{l'_j}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right]_{j \leq n_2} \right\}$$

The scalar functional is a series expansion of integrals for the fields $\{\Psi[l', 0]\}$.

Formula (89) implies that in the projected space, the metric is dynamically an object defined by the projected field:

$$\Psi_{J,0,s_{p'_{l',l}}}^{\otimes k_{l'}} \left([p_l, p'_{l'l}]^{k_{l'}}, \left\{ \left[\begin{array}{c} [p_{l_i}, p'_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}, v \right)$$

and that the metric tensor for one point is a functional of:

$$g \left(\left\{ (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}), (p_{l_j}, p'_{l'_j l_j}, k_{l'_j}) \right\}, \nu, \{\Psi[l', 0] \mathcal{K}_0\} \right)$$

See discussion in part one.

16 Constraints, and reparametrization

16.1 Constraints defining the parameters

Assuming that the dependency in $\Psi_J^{\otimes k_p}$ can be described by the constraints:

$$H \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', \eta]\}, [v]] \end{array} \right] \right\}, \left\{ \Psi_J^{\otimes k_p}, [p_0, p'_{l'0}] \right\} \right) = 0$$

We implement this condition in functionals by including $\delta \left(H \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', \eta]\}, [v]] \end{array} \right] \right\}, \Psi_J^{\otimes k_p} \right) \right)$:

$$\int \bar{g} \left(\{[p_0, p'_{l'0}]\}_{l'}, \left\{ \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \end{array} \right] \right\} \right)$$

$$\Psi_{J, \prod_i s_{p'_{l',0}}}^{\otimes \sum_i k_{l'_i}} \left(\{[p_0, p'_{l'0}]\}_{l'}, \left[(p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \right] \right) \delta \left(H \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \end{array} \right] \right\}, \Psi_J^{\otimes k_p} \right) \right)$$

where the $\left(\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right)$ are local coordinates. The relation $H\left(\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right), \Psi_J^{\otimes k_p}\right)$ is global. In coordinates, we assume a relation involving the linear function:

$$\mathbf{h} \cdot \left(\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right) = h_i^j \left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)$$

and relation:

$$0 = H\left(\mathbf{h} \cdot \left(\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right), \Psi_J^{\otimes k_p} \left(\left\{\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right\}\right)\right)$$

where $\Psi_J^{\otimes k_p} \left(\left\{\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right\}\right)$ is an averaged field over independent degrees of freedom:

$$\begin{aligned} & \Psi_J^{\otimes k_p} \left(\left\{\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right\}\right) \\ &= \int v \left(\overline{[p_0, p_{l'0}]^{k_{l'0}}}\right) \Psi_J^{\otimes k_{l'0}} \left(\overline{[p_0, p_{l'0}]^{k_{l'0}}}, \left\{\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right\}\right) d \left(\overline{[p_0, p_{l'0}]^{k_{l'0}}}\right) \end{aligned}$$

for some function $v \left(\overline{\left(\left(U_j\right)^{\left(p_0, p_{l'0}\right)}\right)^{k_{l'0}}}\right)$. The constraint expresses parameters of $\left(\mathbf{L} \left(\Psi_J^{\otimes k_p}\right)\right)$ as functions of $\Psi_J^{\otimes k_p}$. The $\left\{\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)\right\}$ become metric spaces, with metrics, functional of $\Psi_J^{\otimes k_p}$.

16.2 Reparametrization and field transformation

We do not include subobject, for the sake of simplicity.

16.2.1 Changes of variables in the parameters

We assume that in the sets:

$$V_{\left(p_{l_i}, p_{l'_i}, k_{l'_i}\right)} = \left\{ \left[\begin{array}{c} \left(p_{l_i}, p_{l'_i}, k_{l'_i}\right) \\ \left[\left\{\Psi_J[l', 0]\right\}, v\right] \end{array} \right] \right\}$$

the variables $\left[\begin{array}{c} \left(p_{l_i}, p_{l'_i}, k_{l'_i}\right) \\ \left[\left\{\Psi_J[l', 0]\right\}, v\right] \end{array} \right]$ transform as the initial parameters U_i in a change of variables.

Under these assumptions, we consider some transformations described by some group elements \mathbf{g} acting on the field $\Psi_{J, s_{p_{l', 0}}}^{\otimes k_{l', i}}$. We consider the transformation:

$$\begin{aligned} \Psi_{J, s_{p_{l', 0}}}^{\otimes k_{l', i}} \left(\overline{[p_0, p_{l'0}]^{k_{l', i}}}\right) &\rightarrow \mathbf{g} \cdot \Psi_{J, s_{p_{l', 0}}}^{\otimes k_{l', i}} \left(\overline{[p_0, p_{l'0}]^{k_{l', i}}}\right) \\ &= \Psi_{J, s_{p_{l', 0}}}^{\otimes k_{l', i}} \left(\mathbf{g}^{-1} \cdot \overline{[p_0, p_{l'0}]^{k_{l', i}}}\right) \end{aligned}$$

We also assume that the kernel \mathcal{K}_0 arising in the saddle point equation:

$$\begin{aligned} & \mathcal{K}_0 \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l', i}}}\right\}_{l', i}, \left\{ \overline{[p_{l_i}, p_{l'_i}]^{k_{l'_i}}}, \left[\begin{array}{c} \left(p_{l_i}, p_{l'_i}, k_{l'_i}\right) \\ \left[\left\{\Psi_J[l', 0]\right\}, v\right] \end{array} \right] \right\}_i \right) \\ &= \exp \left(i \mathbf{L} \left(\Psi_J^{\otimes k_{p_0}} \right) \cdot \left\{ \left[\begin{array}{c} \left(p_{l_i}, p_{l'_i}, k_{l'_i}\right) \\ \left[\left\{\Psi_J[l', 0]\right\}, v\right] \end{array} \right] \right\} \right) \mathcal{K}_0 \left(\left\{ \overline{[p_0, p_{l'0}]^{k_{l', i}}}\right\}_{l', i}, \left\{ \overline{[p_{l_i}, p_{l'_i}]^{k_{l'_i}}}\right\}_i \right) \end{aligned}$$

transforms in the following way under \mathbf{g} :

$$\begin{aligned} & \mathbf{g} \cdot \mathcal{K}_0 \left(\left\{ [p_0, p'_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right) \\ \rightarrow & \mathcal{K}_0 \left(\left\{ \mathbf{g}^{-1} \cdot [p_0, p'_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right) \end{aligned}$$

These kernel arise in integrals while computing the saddle points. This involves products of variables $\left\{ [p_0, p'_{l'0}]^{k_{l',i}} \right\}_{l',i}$ and $\overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}$ in the kernel. We can assume a translational invariance, so that by change of variable:

$$\begin{aligned} & \mathbf{g} \cdot \mathcal{K}_0 \left(\left\{ [p_0, p'_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right) \\ \rightarrow & \mathcal{K}_0 \left(\left\{ [p_0, p'_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \mathbf{g} \cdot \overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p'_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right) \end{aligned} \quad (90)$$

16.2.2 Transformations for degeneracy generators

We rewrite (90) by considering the transformation properties of the degeneracies operators. We define the operators:

$$\mathbf{U}_{[p_l, p'_{l'}]^{k_{l'}}}, \Pi_{[p_l, p'_{l'}]^{k_{l'}}$$

of multiplication and derivation by coordinates of $\left((U_j)^{(p_l, p'_{l'})} \right)^{k_{l'}}$.

To study the invariance properties of the states, we assume the general form for the generators $\mathbf{L} \left(\Psi_J^{\otimes k_{p_0}} \right)$:

$$\begin{aligned} \mathbf{L} \left(\Psi_J^{\otimes k_{p_0}} \right) &= \sum_{i, k_{p_i}} \mathbf{1} \left(\mathbf{U}_{[p_l, p'_{l'}]^{k_{l'}}}, \Pi_{[p_l, p'_{l'}]^{k_{l'}}}, \overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}}, \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) \\ &\times \left\{ \prod \int d \overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}} d \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right. \\ &\times v \left(\left(\overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}}, \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) \Psi_J^{\otimes k_{p_i}} \left(\overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}}, \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) \right) \end{aligned} \quad (91)$$

where the $\mathbf{1}$ are defined in (154):

$$\mathbf{1} \left(\mathbf{U}_{[p_l, p'_{l'}]^{k_{l'}}}, \Pi_{[p_l, p'_{l'}]^{k_{l'}}}, \overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}}, \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right)$$

If these operators satisfy also some linear transformations properties:

$$\begin{aligned} & \mathbf{g} \cdot \mathbf{1} \left(\mathbf{U}_{[p_l, p'_{l'}]^{k_{l'}}}, \Pi_{[p_l, p'_{l'}]^{k_{l'}}}, \overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}}, \mathbf{g} \cdot \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) \\ &= \mathbf{1} \left(\mathbf{g} \cdot \mathbf{U}_{[p_l, p'_{l'}]^{k_{l'}}}, \Pi_{\mathbf{g} \cdot [p_l, p'_{l'}]^{k_{l'}}}, \overline{[p_0, p'_{l'0}]^{k_{p_0}^{(i)}}}, \left\{ [p_l, p'_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) \end{aligned}$$

and if this action of \mathbf{g} is given by the action of some operators:

$$M_{\mathbf{g}} \left(\mathbf{U}_{\overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}}, \hat{\Pi}_{\overline{[p_{l_i}, p'_{l'_i l_i}]^{k_{l'_i}}}} \right) \quad (92)$$

16.2.3 Transformations for the kernel

Using (92), the transformation (90) for the kernel is thus:

$$\begin{aligned}
& \mathbf{g} \cdot \mathcal{K}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right) \\
&= \mathcal{K}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \mathbf{g} \cdot \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right) \\
&= \exp \left(i M_{\mathbf{g}} \left(\mathbf{U}_{\overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}}, \hat{\Pi}_{\overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}} \right) \right) \mathcal{K}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right\}_i \right)
\end{aligned}$$

The linear transformation on the $[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}$ translates on the operator generating degeneracies:

$$\begin{aligned}
& \exp \left(i M_{\mathbf{g}} \left(\mathbf{U}_{\overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}}, \hat{\Pi}_{\overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}} \right) \right) \tag{93} \\
& \times \exp \left(i \mathbf{L} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right) \cdot \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right) \right) \exp \left(-i M_{\mathbf{g}} \left(\mathbf{U}_{\overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}}, \hat{\Pi}_{\overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}} \right) \right) \\
&= \exp \left(i \mathbf{L}^{\mathbf{g}^{-1}} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right) \cdot \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right) \right)
\end{aligned}$$

where $\mathbf{L}^{\mathbf{g}} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right)$ is the operator with generator:

$$\mathbf{1} \left(\mathbf{U}_{\mathbf{g} \cdot [p_l, p_{l'l}]^{k_{l'}}}, \Pi_{\mathbf{g} \cdot [p_l, p_{l'l}]^{k_{l'}}}, \overline{[p_0, p_{l'0}]^{k_{p_0}^{(i)}}}, \left\{ \mathbf{g} \cdot [p_l, p_{l'l}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right)$$

Given that we assumed dual linear transformation for the parameters $\left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right)$, we find:

$$\begin{aligned}
& \exp \left(i \mathbf{L}^{\mathbf{g}} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right) \cdot \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right) \right) \tag{94} \\
& \equiv \exp \left(i \mathbf{L} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right) \cdot \left(\mathbf{g} \cdot \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J[l', 0]\}, v] \end{array} \right] \right) \right)
\end{aligned}$$

Then, using (93) and (94), we can deduce the transformation property for the kernel:

$$\begin{aligned}
& \mathcal{K}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \mathbf{g} \cdot \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J [l', 0]\}, v] \end{array} \right] \right\}_i \right) \\
&= \exp \left(i M_{\mathbf{g}} \left(\mathbf{U} \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \hat{\Pi} \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}} \right) \right) \\
& \exp \left(i \mathbf{L} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right) \cdot \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J [l', 0]\}, v] \end{array} \right] \right) \right) \exp \left(-i M_{\mathbf{g}} \left(\mathbf{U} \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \hat{\Pi} \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}} \right) \right) \\
& \times \exp \left(i M_{\mathbf{g}} \left(\mathbf{U} \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}, \hat{\Pi} \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}} \right) \right) \mathcal{K}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l',i}} \right\}_{l',i}, \left\{ \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}} \right\}_i \right) \\
&= \exp \left(i \mathbf{L} \left(\mathbf{g} \cdot \Psi_J^{\otimes k_{l_i}} \right) \cdot \left(\mathbf{g} \cdot \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J [l', 0]\}, v] \end{array} \right] \right) \right) \mathcal{K}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l',i}} \right\}_{l',i}, \mathbf{g} \cdot \left\{ \overline{[p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}} \right\}_i \right)
\end{aligned}$$

16.2.4 Transformation for functionals and effective fld

Inserting previous formula in (80), the function characterizing the generic formula is transformed as:

$$\int g^{\mathcal{K}} \left(\mathbf{g} \cdot \overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}}, \mathbf{g} \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J [l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right)$$

where the actions $\mathbf{g} \cdot \overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}}$, $\mathbf{g} \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}$ take into account the initial change of variables.

Since we have assumed some invariance, we can assume that the integral of $g^{\mathcal{K}}$ over $\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}$ is invariant, so that we can replace:

$$\mathbf{g} \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\} \rightarrow \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}$$

A second change of variable:

$$\overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}} \rightarrow \cdot \overline{[p_0, p_{l'_i 0}]^{k_{l'_i}}}$$

As a consequence, in (82), $\bar{g}^{\mathcal{K}}$ is invariant, and the effective field (83) is replaced by:

$$\Psi_{J, \prod_i^s p_{l', 0}}^{\otimes \sum_i k_{l', i}} \left(\left\{ \mathbf{g}^{-1} \cdot [p_0, p_{l'0}]^{k_{l', i}}, \mathbf{g} \cdot \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J [l', 0]\}, v] \end{array} \right] \right\}_i \right)$$

and this implies the transformation property of the effective field:

$$\mathbf{g} \cdot \Psi_{J, \prod_i^s p_{l', 0}}^{\otimes \sum_i k_{l', i}} \left(\left\{ [p_0, p_{l'0}]^{k_{l', i}}, \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J [l', 0]\}, v] \end{array} \right] \right\}_i \right) \equiv \Psi_{J, \prod_i^s p_{l', 0}}^{\otimes \sum_i k_{l', i}} \left(\left\{ \mathbf{g}^{-1} \cdot [p_0, p_{l'0}]^{k_{l', i}}, \mathbf{g} \cdot \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\{\mathbf{g} \cdot \Psi_J [l', 0]\}, v] \end{array} \right] \right\}_i \right)$$

16.3 Constraints and transformation

If the parameters $\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right]$ satisfy equations of the form:

$$0 = f \left(\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \{h_p((\Psi_J), [p_0, p'_{l_0}])\} \right) \quad (95)$$

where the vector functions $h_p((\Psi_J))$ have the following functional form:

$$h_p((\Psi_J), [p_0, p'_{l_0}]) = \left[\int h_p([\bar{p}_0, \bar{p}'_{l_0}]^{k'_{l'}}) \Psi_{J, s_{\bar{p}'_{l_0}, 0}}^{\otimes k'_{l'}, i}([\bar{p}_0, \bar{p}'_{l_0}]^{k'_{l'}}) d[\bar{p}_0, \bar{p}'_{l_0}]^{k'_{l'}} \right] \Psi_{J, s_{p'_{l_0}, 0}}^{\otimes k'_{l'}, i}([p_0, p'_{l_0}]^{k'_{l'}, i})$$

We can write the transformation property for these functionals:

$$\begin{aligned} \mathbf{g}.h_p((\Psi_J)) &= h_p(\mathbf{g}.(\Psi_J)) \\ &= \left[\int h_p(\mathbf{g}.[\bar{p}_0, \bar{p}'_{l_0}]^{k'_{l'}}) \mathbf{g}.\Psi_{J, s_{\bar{p}'_{l_0}, 0}}^{\otimes k'_{l'}, i}([\bar{p}_0, \bar{p}'_{l_0}]^{k'_{l'}}) d[\bar{p}_0, \bar{p}'_{l_0}]^{k'_{l'}} \right] \mathbf{g}.\Psi_{J, s_{p'_{l_0}, 0}}^{\otimes k'_{l'}, i}([p_0, p'_{l_0}]^{k'_{l'}, i}) \end{aligned}$$

Consider first a set of functions:

$$\left\{ h_p(\mathbf{g}.[p_0, p'_{l_0}]^{k'_{l'}, i}) \mathbf{g}.\Psi_{J, s_{p'_{l_0}, 0}}^{\otimes k'_{l'}, i}([p_0, p'_{l_0}]^{k'_{l'}, i}) \right\}$$

belonging to some representation $\mathcal{R}(\mathbf{g})$ of \mathbf{g} , and transforming as:

$$\left\{ h_p(\mathbf{g}.[p_0, p'_{l_0}]^{k'_{l'}, i}) \mathbf{g}.\Psi_{J, s_{p'_{l_0}, 0}}^{\otimes k'_{l'}, i}([p_0, p'_{l_0}]^{k'_{l'}, i}) \right\} = \mathcal{R}(\mathbf{g}) \left\{ h_p([p_0, p'_{l_0}]^{k'_{l'}, i}) \Psi_{J, s_{p'_{l_0}, 0}}^{\otimes k'_{l'}, i}([p_0, p'_{l_0}]^{k'_{l'}, i}) \right\}$$

so that:

$$\mathbf{g}.h_p((\Psi_J)) = \mathcal{R}(\mathbf{g}) h_p((\Psi_J))$$

Consider also that at the lowest order approxmtn, the function f involves some scalar products:

$$f \left(\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \{h_p((\Psi_J))\} \right) = \bar{f} \left(\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \{h_p((\Psi_J))\} \right)$$

Under these assumptions, the constraints (95) satisfy some invariance property:

$$f \left(\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \mathcal{R}(\mathbf{g}) \{h_p((\Psi_J))\} \right) = f \left(\mathcal{R}(\mathbf{g}^{-1}) \left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \{h_p((\Psi_J))\} \right)$$

This relation is satisfied if we consider the lowest order constraint:

$$f \left(\sum \left(\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] G \left(\left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \Psi_J \right) \left[\begin{array}{c} (p_{l_i}, p'_{l_i}, k'_{l_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] - \gamma(\Psi_J) \right)^2 \right)$$

and if we also assume that $\gamma(U^{(j)})$ arising in the constraints are invariant under transformation of the parameters.

$$\gamma(\mathbf{g} \cdot \Psi_J) = \gamma(\Psi_J)$$

As a consequence, function $G \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \Psi_J \right)$ transforms as:

$$G \left(\mathcal{R}_0(\mathbf{g}) \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \mathbf{g} \cdot \Psi_J \right) = (\mathcal{R}_0^t(\mathbf{g}))^{-1} G \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \Psi_J \right) \mathcal{R}_0^{-1}(\mathbf{g})$$

or written differently:

$$\mathcal{R}_0(\mathbf{g}) \cdot G \left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \Psi_J \right) = (\mathcal{R}_0^t(\mathbf{g}))^{-1} G \left(\mathcal{R}_0^{-1}(\mathbf{g}) \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right], \mathbf{g}^{-1} \cdot \Psi_J \right) \mathcal{R}_0^{-1}(\mathbf{g})$$

17 Several components

17.1 General set up

The same procedure applies if there are several fields corresponding to several types of non projected subbjcts. We replac:

$$U_j^{(p_0)} \xrightarrow{p_0} \mathcal{V}(\oplus_k U_j^k)$$

with the collection:

$$\left\{ U_j^{(p_\eta)} \xrightarrow{p_\eta} \mathcal{V}(\oplus_k U_j^k) \right\}_{p_\eta}$$

The corresponding field for subobject η writes as a vector component:

$$\Psi_{J, \alpha, s_{p_{\eta' \eta}}}^{\otimes k_{\eta' \eta}} \left([p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}} \right) = \left[\Psi_{J, \alpha, s_{p_{l' \eta}}}^{\otimes k_{\eta' \eta}} \left([p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}} \right) \right]_\eta$$

The whole set of remaining subobject is thus:

$$\left[\Psi_{J, \alpha, s_{p_{\eta' \eta}}}^{\otimes k_{\eta' \eta}} \left([p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}} \right) \right]$$

Similarly, we define the vectors:

$$\left[[p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}} \right] \text{ and } \left[\Psi_{J, \alpha, s_{p_{\eta' \eta}}}^{\otimes k_{\eta' \eta}} \left(\overline{[p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}}}, \{ [p_{l_i}, p_{l'_i}]^{k_{l'_i}} \} \right) \right]$$

with components:

$$\left[[p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}} \right]_\eta = [p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}}$$

and:

$$\left[\Psi_{J, \alpha, s_{p_{\eta' \eta}}}^{\otimes k_{\eta' \eta}} \left(\overline{[p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}}}, \{ [p_{l_i}, p_{l'_i}]^{k_{l'_i}} \} \right) \right]_\eta = \Psi_{J, \alpha, s_{p_{\eta' \eta}}}^{\otimes k_{\eta' \eta}} \left(\overline{[p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}}}, \{ [p_{l_i}, p_{l'_i}]^{k_{l'_i}} \} \right)$$

The sets $\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}}$ are $[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}$ quotiented by constraints imposed by $\{[p_l, p_{l'}]^{k_{l'}}$.

We will also consider the product of remaining fields:

$$\prod_\eta \Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \left([p_\eta, p_{\eta'}]^{k_{\eta' \eta}} \right)$$

and for their subobjects:

$$\prod_\eta \Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \left(\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right)$$

17.2 Projected states

The states are decomposed as:

$$1 = \sum_{\{v_\eta\}} \prod_{\{v_\eta\}} = \sum_{[v]} \prod_{[v]}$$

which accounts for a decomposition relative to multiple states.

Including the projection on background states for some subobjects yields the projection:

$$\sum_{[v]} \prod_{[v]} \otimes \prod_{\min S([v])}$$

We start with the projection \prod_v . The basis of states is:

$$v \left(\left[\Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \right] \right) = \prod_\eta v_\eta \left(\Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \left([p_\eta, p_{\eta'}]^{k_{\eta' \eta}} \right) \right)$$

and including the evaluation functional on other fields yields as before:

$$\begin{aligned} & \left[v \left(\Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \right) \right] \otimes ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'} l}}^{\otimes k_{l'}} \right) \delta \left(f_\gamma \left([p_\eta, p_{\eta'}]^{k_{\eta' \eta}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right) \right) \\ &= \int \prod_\eta v_\eta \left(\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right) \\ & \times \left[\Psi_{J, \alpha, s_{p_{l'} \eta}}^{\otimes k_{l' \eta}} \left(\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right) \right] d \left(\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}} \right) \otimes ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'} l}}^{\otimes k_{l'}} \right) \\ &= \left[v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \right) \right] \otimes ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'} l}}^{\otimes k_{l'}} \right) \end{aligned}$$

with the implicit notation:

$$\left[\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right] \equiv \left[\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}} \right] / \prod_{p_{1,l}, \dots, p_{m',l}} f_{p_{1,l}, \dots, p_{m',l}}, f_{p_{1,l}, \dots, p_{m',l}}^{-1} \{[p_l, p_{l'}]^{k_{l'}}\}$$

As a consequence, projection on the states $[v]$, amounts to include the following contributions in the functional :

$$\prod_{[v]} \rightarrow \left[v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, \alpha, s_{p_{\eta'} \eta}}^{\otimes k_{\eta' \eta}} \right) \right] \otimes ev_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'} l}}^{\otimes k_{l'}} \right) \prod_\eta \delta \left(f_\eta \left(\overline{[p_\eta, p_{\eta'}]^{k_{\eta' \eta}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right) \right)$$

where:

$$\begin{aligned} & \left[v_{[p_l, p_{l'}]}^{k_{l'}} \left(\Psi_{J, \alpha, s_{p_{l'}}, 0}^{\otimes k_{\eta'} \otimes k_{\eta'}} \right) \right] \\ &= \int \prod_{\eta} v_{\eta} \left(\overline{[p_{\eta}, p_{\eta'}]}^{k_{\eta'}}, \{ [p_l, p_{l'}]^{k_{l'}} \} \right) \left[\Psi_{J, \alpha, s_{p_{l'}}, 0}^{\otimes k_{l'}} \left(\overline{[p_{\eta}, p_{\eta'}]}^{k_{\eta'}}, \{ [p_l, p_{l'}]^{k_{l'}} \} \right) \right] d \left(\prod_{\eta} \overline{[p_{\eta}, p_{\eta'}]}^{k_{\eta'}} \right) \end{aligned}$$

Once the decomposition with respect to the states $[v]$ is performed, we consider the projection coming from:

$$\begin{aligned} & \exp \left(-S \left(\left\{ \otimes \left[\Psi_{J, \alpha, s_{p_{l'}}, 0}^{\otimes k_{\eta'} \otimes k_{\eta'}} \left([p_{\eta}, p_{\eta'}]^{k_{\eta'}} \right) \right] \otimes_{l'} \Psi_{J, \alpha, s_{p_{l'}}, l}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right\} \right. \right. \\ & \left. \left. \left[v_{[p_l, p_{l'}]}^{k_{l'}} \left(\Psi_{J, \alpha, s_{p_{l'}}, 0}^{\otimes k_{\eta'} \otimes k_{\eta'}} \right) \right] \left(\Psi_{J, \alpha, s_{p_{l'}}, l}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right) \right) \right) \end{aligned}$$

17.3 Projected functionals and effective fields

Computations similar to the previous sections leads to the projected functionals (for one realization α). Assume a collection $\{\eta_i\}$ among the remaining subobjects. Choosing the valuation for v_{η_i} :

$$v_{\eta_i} \rightarrow [p_{\eta_i}, p_{\eta'_i}, \eta_i]$$

the projected functional is:

$$\begin{aligned} & \sum_{\left\{ \left(s, s', [p_{\eta}, p_{\eta'}]_{i, i'} \right) \right\}} \int \bar{g}^{\mathcal{K}} \left(\left[\left\{ [p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right], \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\Psi_J [p_{\eta}, p_{\eta'}], v \end{array} \right] \right\}, [v] \right) \\ & \times \left(\prod_{\substack{i \leq s \\ i' \leq s'}} \left[\Psi_{J, \alpha, s_{p_{\eta'_i}}, \eta_i}^{\otimes k_{\eta'_i}, \eta_i} \left([p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i} \right) \right] \right) \Psi \left(\left(\left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\Psi_J [p_{\eta}, p_{\eta'}], v \end{array} \right] \right), [v] \right) d \left(\left\{ [p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \end{aligned}$$

with:

$$([p, p', k'])_i = \left\{ p_{\eta_i}, p_{\eta'_i}, k_{\eta'_i} \right\}_{\substack{i \leq s \\ i \leq s'}}$$

and:

$$(m, m', [p, p', k']) = \left\{ p_l, p_{l'}, k_{l'} \right\}_{\substack{l \leq m \\ l' \leq m'}}$$

and:

$$[v] = \{v_{\eta_i}\}$$

Constraints between components coming from $\overline{[p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i}}$, $\{ [p_l, p_{l'}]^{k_{l'}} \}$ and the variables $\hat{\Lambda}^{(p_l)} \left(\Psi_J^{\otimes k_{p_0}}, [v] \right)$ are implicit. Considering general states:

$$v \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i}, \eta_i \right]^{k_{\eta'_i}, \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) = \prod_{\substack{i \leq s \\ i' \leq s'}} v_{\eta_i} \left(\left[p_{\eta_i}, p_{\eta'_i}, \eta_i \right]^{k_{\eta'_i}, \eta_i} \right)$$

the effective fields have the form:

$$\int d \left\{ \left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } \right\}_{\substack{i \leq s \\ i' \leq s'}} v \left(\left\{ \left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \quad (96)$$

$$\times \Psi_{\substack{\otimes \sum_{i'} k_{\eta'_{i'} \eta_i} \\ \otimes k_{\eta'_{i'} \eta_i} \\ J, \alpha_i, \prod_{i'} s_{p_{\eta'_{i'} \eta_i}}} }} \left(\left\{ \left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } \right\}_{\substack{i \leq s \\ i' \leq s'}} , \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_{J, \alpha_i} [p_{\eta}, p_{\eta'} \eta]] \end{array} \right] \right\}, [v] \right) \quad (97)$$

We can define a global field by summing over realiations if the variables $\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_{J, \alpha_i} [p_{\eta}, p_{\eta'} \eta]] \end{array} \right]$ are independent from realizations. In this case, we find a single composed field:

$$\begin{aligned} & \Psi_{\substack{\otimes \sum_{i'} k_{\eta'_{i'} \eta_i} \\ \otimes k_{\eta'_{i'} \eta_i} \\ J, \prod_{i'} s_{p_{\eta'_{i'} \eta_i}}} }} \left(\left\{ \left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } \right\}_{\substack{i \leq s \\ i' \leq s'}} , \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_J [p_{\eta}, p_{\eta'} \eta]] \end{array} \right] \right\}, [v] \right) \\ &= \sum_{\alpha_i} \left(\prod_{i, i'} \left[\Psi_{\substack{\otimes k_{\eta'_{i'} \eta_i} \\ \otimes k_{\eta'_{i'} \eta_i} \\ J, \alpha_i, s_{p_{\eta'_{i'} \eta_i}}} }} \left(\left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } \right) \right] \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_{J, \alpha_i} [p_{\eta}, p_{\eta'} \eta]] \end{array} \right] \right\}, [v], \alpha_i \right) \end{aligned}$$

The projection has led to a single composed field. However, when the transformation group are, at least in first approximation, independent and can be written $\mathbf{L}_{\eta} \left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right)$ with a formula similar to (91), we can assume that the parameters $\left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_{J, \alpha_i} [p_{\eta}, p_{\eta'} \eta]] \end{array} \right]$ are independent, and that, the effective field becomes:

$$\begin{aligned} & \prod_i \sum_{\alpha_i} \left(\left[\Psi_{\substack{\otimes k_{\eta'_{i'} \eta_i} \\ \otimes k_{\eta'_{i'} \eta_i} \\ J, \alpha_i, s_{p_{\eta'_{i'} \eta_i}}} }} \left(\left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } \right) \right] \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_{J, \alpha} [p_{\eta}, p_{\eta'} \eta]] \end{array} \right] \right\}, [v] \right) \quad (98) \\ &= \prod_i \Psi_{\substack{\otimes k_{\eta'_{i'} \eta_i} \\ \otimes k_{\eta'_{i'} \eta_i} \\ J, \alpha_i, s_{p_{\eta'_{i'} \eta_i}}} }} \left(\left[p_{\eta_i}, p_{\eta'_{i'}} \right]^{k_{\eta'_{i'} \eta_i} } , \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_i}) \\ [\Psi_{J, \alpha} [p_{\eta}, p_{\eta'} \eta]] \end{array} \right] \right\}, v_{\eta} \right) \end{aligned}$$

and the system is described by several field $\Psi_{\substack{\otimes k_{l' \eta} \\ \otimes k_{l' \eta} \\ J, s_{p_{l' \eta}, \eta}}} }$.

In general this decomposition is not ensured. We may assume that the constraints defining the generators $\mathbf{L}_{\eta} \left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right)$ and thus the parameters depend on all the $\Psi_{J, \eta}^{\otimes k_{p_0}}$. Expanding the series in constraints and assuming the independency in first approximation, we can assume the constraints to have the frm:

$$\begin{aligned} 0 &= \sum_{\eta} h_{k_n}^{(\eta)} \left(\left\{ \mathbf{L}_{\eta, \{\alpha_i\}} \left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right) \right\}_{i \leq n}, \{U_i^k\}_i, h_p \left(\left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right), U_{j, \eta}^l, \nu_{\eta} \right) \right) \\ &+ \sum_{[\eta]} h_{k_n}^{[\eta]} \left(\left\{ \left\{ \mathbf{L}_{\eta, \{\alpha_i\}} \left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right) \right\}_{i \leq n} \right\}_{\eta}, \{U_i^k\}_i, \left\{ h_p \left(\left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right), U_{j, \eta}^l, \nu_{\eta} \right) \right\}_{\eta} \right) \end{aligned}$$

where the sum is over any arbitrary set $[\eta]$ of nn prjctd stts. If the $h_{k_n}^{(\eta)}$ are positive, we may expect that the parameters are defined such that:

$$h_{k_n}^{(\eta)} \left(\left\{ \mathbf{L}_{\eta, \{\alpha_i\}} \left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right) \right\}_{i \leq n}, \{U_i^k\}_i, h_p \left(\left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right), U_{j, \eta}^l, \nu_\eta \right) \right) = 0$$

so that we recover flds $\Psi_{J, s_{p_{l'}}, \eta}^{\otimes k_{l'} \eta} \left([p_\eta, p_{\eta'}]^{k_{\eta'} \eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_{J, \alpha} [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}, v_\eta \right)$. The remaining relations:

$$h_{k_n}^{[\eta]} \left(\left\{ \left\{ \mathbf{L}_{\eta, \{\alpha_i\}} \left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right) \right\}_{i \leq n} \right\}_\eta, \{U_i^k\}_i, \left\{ h_p \left(\left(\Psi_{J, \eta}^{\otimes k_{p_0}} \right), U_{j, \eta}^l, \nu_\eta \right) \right\}_\eta \right) = 0$$

are constraints between some parametrs. Changing of variables leads to reinterpret this constraints as δ functions in the functionls that becomes series involvng products:

$$\prod_i \Psi_{J, s_{p_{l'}}, \eta_i}^{\otimes k_{l'} \eta_i} \left([p_{\eta_i}, p_{\eta'_i}]^{k_{\eta'_i} \eta_i}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_{J, \alpha} [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}, v_{\eta_i} \right) \prod_\eta \delta \left(f \left(\left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_{J, \alpha} [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}_\eta \right) \right)$$

Ths constraints imply that some localty in this functionl has to be introduced.

17.4 Variation and averaged field

Writing the variation of the effective fld:

$$0 = \int v \left(\left\{ [p_{\eta_i}, p_{\eta'_i}]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \frac{\delta \Psi_{J, \prod_{i'} s_{p_{l'}}, \eta_i}^{\otimes \sum_{i'} k_{\eta'_i} \eta_i} \left(\left\{ [p_{\eta_i}, p_{\eta'_i}]^{k_{\eta'_i} \eta_i} \right\}_{i \leq s}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}, [v] \right)}{\delta \Psi_{J, s_{p_{l'}}, \eta}^{\otimes \sum_{i'} k_{\eta'_i} \eta} \left([\bar{p}_\eta, \bar{p}_{\eta'}]^{k_{\eta'} \eta} \right)}$$

Imposing that the variables $\left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_{J, \alpha} [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}$ are independent at the first order from the field variations, we are led to the field equation:

$$0 = \int v \left(\left\{ [p_{\eta_i}, p_{\eta'_i}]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \frac{\delta \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}}{\delta \Psi_{J, s_{p_{l'}}, \eta}^{\otimes \sum_{i'} k_{\eta'_i} \eta} \left([\bar{p}_\eta, \bar{p}_{\eta'}]^{k_{\eta'} \eta} \right)}$$

$$\times \nabla \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_{J, \alpha} [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\} \Psi_{J, \prod_{i'} s_{p_{l'}}, \eta_i}^{\otimes \sum_{i'} k_{\eta'_i} \eta_i} \left(\left\{ [p_{\eta_i}, p_{\eta'_i}]^{k_{\eta'_i} \eta_i} \right\}_{i \leq s}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i}, k_{l'_i}) \\ [\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}_{i' \leq s'} \right)$$

which becomes in local coordinates $\left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), \eta \right]$:

$$0 = \int \left[\Xi^{(k_i)} \left(\left\{ \Psi_J [p_\eta, p_{\eta' \eta}] \right\}, \left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), \eta \right] \right) \right]_{[\bar{p}_\eta, \bar{p}_{\eta' \eta}]}^{k_{\eta' \eta}} \\ \times \nabla \left\{ \left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), \eta \right] \right\} \Psi_{J, \prod_{i'} s_{p_{\eta'_i \eta_i}}}^{\otimes \sum_{i'} k_{\eta'_i \eta_i}} \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i \eta_i} \right]^{k_{\eta'_i \eta_i}} \right\}_{\substack{i \leq s \\ i' \leq s'}}, \left\{ \left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), \eta \right] \right\}, v \right)$$

where:

$$\left[\Xi^{(k_i)} \left(\left\{ \Psi_J [l', 0] \right\}, \left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), \eta \right] \right) \right]_{[\bar{p}_\eta, \bar{p}_{\eta' \eta}]}^{k_{\eta' \eta}} \\ = \int v \left(\left\{ \left[p_\eta, p_{l' \eta} \right]^{k_{l' \eta}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \frac{\delta \left\{ \left[\left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right) \right] \right] \right\}}{\delta \Psi_{J, s_{p_{l' \eta}}}^{\otimes k_{l' \eta}} \left([\bar{p}_\eta, \bar{p}_{l' \eta}]^{k_{l' \eta}} \right)} \left(\frac{\delta \left[\left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right) \right] \right]}{\delta \left[\left\{ \Psi_{J, \alpha} [p_\eta, p_{\eta' \eta}] \right\}, v \right]} \right)^{-1}$$

with identifications:

$$v \rightarrow [p_\eta, p_{\eta' \eta}]^{k_{\eta' \eta}}$$

and this equation mixes components due to the constraints.

17.5 Averaged field variation

As before considered successive inclusions of grps of transformations, the effective fld is a series:

$$\Psi_{J, \prod_{i'} s_{p_{\eta'_i \eta_i}}}^{\otimes \sum_{i'} k_{\eta'_i \eta_i}} \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i \eta_i} \right]^{k_{\eta'_i \eta_i}} \right\}_{\substack{i \leq s \\ i' \leq s'}}, \left[\left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right) \right] \right], V_{\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), v} \right) \quad (99) \\ + \Psi_{J, \prod_{i'} s_{p_{\eta'_i \eta_i}}}^{\otimes \sum_{i'} k_{\eta'_i \eta_i}} \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i \eta_i} \right]^{k_{\eta'_i \eta_i}} \right\}_{\substack{i \leq s \\ i' \leq s'}}, \left[\left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right) \right] \right], V_{\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right), v}^2 \right) + \dots$$

for:

$$V_{\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right)} = \left\{ \left[\left[\left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right) \right] \right] \right\}$$

and the variation equation for this averaged field is:

$$\begin{aligned}
0 &= \int d \left\{ \left[p_{\eta_i}, p_{\eta'_i} \right]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} v \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i} \right]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \frac{\delta \left\{ \left[\left[\Psi_J \left[p_{\eta}, p_{\eta'} \right] \right], v \right] \right\}}{\delta \Psi_{J, s_{p_{l'}, \eta}}^{\otimes k_{l'} \eta}} \left(\left[\bar{p}_{\eta}, \bar{p}'_{\eta} \right]^{k_{l'} \eta} \right)} \\
&\times \nabla \left\{ \left[\left[\Psi_J \left[p_{\eta}, p_{\eta'} \right] \right], v \right] \right\} \Psi_{J, \prod_{i'} s_{p_{\eta'_i} \eta_i}}^{\otimes \sum_{i'} k_{\eta'_i} \eta_i} \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i} \right]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right), \left[\left[\Psi_J \left[p_{\eta}, p_{\eta'} \right] \right], v \right], V_{\left(p_{l_i}, p_{l'_i}, k_{l'_i} \right), v} \\
&+ \int d \left\{ \left[p_{\eta_i}, p_{\eta'_i} \right]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} v \left(\left\{ \left[p_{\eta_i}, p_{\eta'_i} \right]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \\
&\times \left(\epsilon \frac{\delta V_{\left(p_{l_i}, p_{l'_i}, k_{l'_i} \right)}}{\delta \Psi_{J, s_{p_{l'}, \eta}}^{\otimes k_{l'} \eta}} \left(\left[\bar{p}_{\eta}, \bar{p}'_{\eta} \right]^{k_{l'} \eta} \right) - h \left(V_{\left(p_{l_i}, p_{l'_i}, k_{l'_i} \right)}, \Psi_{J, \prod_{l'} s_{p_{l'}, 0}}^{\otimes \sum_{l'} k_{l'}} \left(\left\{ \left[p_{l_0}, p_{l'_0} \right]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \right) \right) \\
&\times \Psi_{J, \prod_{i'} s_{p_{\eta'_i} \eta_i}}^{\otimes \sum_{i'} k_{\eta'_i} \eta_i} \left(\left\{ \left[p_{l_0}, p_{l'_0} \right]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right), \left[\left[\Psi_J \left[p_{\eta}, p_{\eta'} \right] \right], v \right], V_{\left(p_{l_i}, p_{l'_i}, k_{l'_i} \right), v}
\end{aligned}$$

18 Average values on eigenstates of operators

This is an equivalent description already presented in the first part. We start with the functionals without path integration:

$$\begin{aligned}
&F_{f, lin} \left(\left\{ \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left[p_{l_0}, p_{l'_0} \right]^{k_{l'}} \right\}_{l'} \right) \\
&= \sum_m \sum_{D_j^{p_0, p_1, m}} \sum_{m'} \sum_{D_{j, p_u}^{(p_0, p_{l'_0}, p_1, p_{l'_1}), m, m'}} g \left(\{ [p_l, p_{l'}] \}_{l, l'}, \{ [p_0, p_{l'_0}] \}_{l'} \right) \\
&\prod_{l'} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'_0}]^{k_{l'}} \right) \prod_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)
\end{aligned}$$

and replace:

$$\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'_0}]^{k_{l'}} \right)$$

by:

$$\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'_0}]^{k_{l'}} / \prod_{p_{1, l}, \dots, p_{m', l}} f_{p_{1, l}, \dots, p_{m', l}}, f_{p_{1, l}, \dots, p_{m', l}}^{-1} \left\{ [p_l, p_{l'}]^{k_{l'}} \right\}_{\substack{l \leq m \\ l' \leq m'}} \right)$$

Implicitly $\Psi_J^{\otimes k_{l'}}$ and $\Psi_J^{\otimes k_{p_0}}$ stand for $\Psi_{J, s_{p_0}}^{\otimes k_{p_0}}$ and $\Psi_{J, s_{p_l}}^{\otimes k_{l'}}$.

We will consider the average of the field:

$$\prod_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

on minimal eigenstates of an operator:

$$\mathcal{H} \left(\left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right), \Pi_{\Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)} \right\}_{l, l'} , \left\{ \Psi_{J, s_{p_{l', 0}}}^{\otimes k_{l'}} \left([p_0, p_{l'}]^{k_{l'}} \right) \right\}_{l'} \right)$$

with:

$$\Pi_{\Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)} = \frac{\delta}{\delta \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)}$$

18.1 Average over one eigenspas, one remaining subobject

If there is only one eigenstate, the projectd functional has the follwing form (we assume one remaining subobject, the general case will be studied below):

$$\begin{aligned} & F_0 \left[\left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \right\} \right] \\ & \equiv \int F_0 \left[[p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right] \right] \otimes \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) d \left([p_l, p_{l'}]^{k_{l'}} \right) \\ & = \int F_0 \left[[p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right] \right] \otimes \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) d \left([p_l, p_{l'}]^{k_{l'}} \right) \end{aligned}$$

The dependence $\left[\Psi_J^{\otimes k_{p_0}} \right]$ is through a functional of the type $v \left(\Psi_J^{\otimes k_{p_0}} \right)$, as in the previous sections. As before, the tensor \otimes stands for series of products of identical copies.

ncldng symtris parmtrized by $\left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ \hat{\Lambda} \\ \{ \Psi_J[l', 0], v \} \end{array} \right]$, th functnls bcms:

$$F_0 \left[\left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \right\}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ \{ \Psi_J[l', 0], v \} \end{array} \right] \right\} \right]$$

average values in this state:

$$\left\langle \prod_{l'} \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \right\rangle = \prod_{l'} \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ \{ \Psi_J[l', 0], v \} \end{array} \right] \right\} \right)$$

where:

$$\begin{aligned} & \prod_{l'} \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ \{ \Psi_J[l', 0], v \} \end{array} \right] \right\} \right) \\ & = \int \prod_{l'} \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \left| F_0 \left[\left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \right\}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ \{ \Psi_J[l', 0], v \} \end{array} \right] \right\} \right] \right|^2 \prod \mathcal{D} \left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \right\} \end{aligned}$$

nd th measure is invariant for this transformation. Then:

$$\begin{aligned} & \prod_{l'} \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ \{ \Psi_J[l', 0], v \} \end{array} \right] \right\} \right) \\ & = \int \prod_{l'} \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right) \left| F_0 \left[\left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \right\}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ \{ \Psi_J[l', 0], v \} \end{array} \right] \right\} \right] \right|^2 \prod \mathcal{D} \left\{ \Psi_{J, s_{p_{l', l}}}^{\otimes k_{l'}} \right\} \end{aligned}$$

As a consequence, replacing:

$$\prod_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}} \right)$$

by its average over the eigenstate considered in the functional, and considering linear combinations over these states:

$$\int \Psi \left(\hat{\mathcal{O}}_i^{(p_l)} \left(\Psi_J^{\otimes k_{p_0}} \right) \right) \prod_{l'} \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \right)$$

and:

$$\Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, \left[\Psi_J^{\otimes k_{p_0}} \right], \left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \right)$$

has an expansion:

$$\begin{aligned} & \Psi_{J, s_{p_{l'}, l}}^{\otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, v, \left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \right) \\ &= \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} d \left(\left\{ \overline{[p_0, p_{l'}]^{k_{l'}}}_{l' \leq s'} \right\}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1, l_1 \neq l \\ l'_1 \leq m'_1}} \right) \\ & \times \mathcal{V}_0 \left(\left\{ \overline{[p_0, p_{l'}]^{k_{l'}}}_{l' \leq s'} \right\}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \right) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1}]}^{k_{l'_1}} \left\{ \Psi_J^{\otimes k_{l'}} \right\} \end{aligned}$$

and the functionals decompose for one realization:

$$\begin{aligned} & F_{f, lin} \left(\left\{ \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left([p_0, p_{l'}]^{k_{l'}} \right) \right\}_{(p_{l'}^0)} \right) \tag{100} \\ &= \sum_m \sum_{m'} \int \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} d \left(\left\{ \overline{[p_0, p_{l'}]^{k_{l'}}}_{l' \leq s'} \right\}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) d \left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \\ & \times g \left(\left\{ [p_l, p_{l'}] \right\}_{l, l'}, \left\{ \overline{[p_0, p_{l'}]^{k_{l'}}}_{l' \leq s'} \right\} \right) \mathcal{V}_0 \left(\left\{ \overline{[p_0, p_{l'}]^{k_{l'}}}_{l' \leq s'} \right\}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \right) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1}]}^{k_{l'_1}} \left\{ \Psi_J^{\otimes k_{l'}} \right\} \Psi \left(\left\{ \left[\begin{array}{c} [p_l, p_{l'}]^{k_{l'}} \\ [\Psi_J [l', 0]] \end{array} \right], v \right\} \right) \end{aligned}$$

where $D_j^{p_0, p_l, m}$ stand for decomposition

$$\mathcal{H}(U_j) = \mathcal{H} \left(\left((U_j)^{(p_0)} \right) \right) \otimes \mathcal{H} \left(\left\{ (U_j)^{(p_l)} \right\}_{l \leq m} \right)$$

and $D_{j,p_u}^{(p_0,p_{l'0},p_l,p_{l'})^{m,m'}}$ for decomposition:

$$\mathcal{H}\left(\left((U_j)^{(p_u)}\right)\right) = \mathcal{H}\left(\left((U_j)^{(p_0,p_{l'0})}\right)^{k_{l'}}\right) \otimes \mathcal{H}\left(\left\{\left(U_j\right)^{(p_l,p_{l'})}\right\}_{\substack{l \leq m \\ l' \leq m'}}\right)$$

By changing variables in (100), we show in appendix 7 that:

$$\begin{aligned} & F_{f,lin} \left(\left\{ \Psi_{J, s_{p_{l'},0}}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right\}_{(p_{l'0})} \right) \\ &= \sum_m \sum_{m'} \int \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} d \left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} d \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_J[l',0]\}, v] \end{array} \right] \right\} \\ & \times \bar{g} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_J[l',0]\}, v] \end{array} \right] \right\} \right) \Psi_{J, \prod_{l'} s_{p_{l'},0}}^{\otimes \sum_{l'} k_{l'}} \left(\prod_{l'} [p_0, p_{l'0}]^{k_{l',i}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_J[l',0]\}, v] \end{array} \right] \right\} \right) \end{aligned}$$

Introducing explicitly the eigenvalue on which the projection occurs, the effective field is similar to the one component case:

$$\begin{aligned} & \Psi_{J, \prod_{l'} s_{p_{l'},0}}^{\otimes \sum_{l'} k_{l'}} \left(\prod_{l'} [p_0, p_{l'0}]^{k_{l',i}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_{J,\alpha}[l',0]\}, v] \end{array} \right] \right\}, \lambda([\{\Psi_J[l',0]\}, v]) \right) \\ &= \sum_{\mathcal{P}, \alpha} \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'},0}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1}, k_{l'_1}] \right\}_{\mathcal{P}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_{J,\alpha}[l',0]\}, v] \end{array} \right] \right\} \right) \\ & \times \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1}, k_{l'_1}]^{k_{l'_1}}} \left\{ \Psi_J^{\otimes k_{l'}} \right\} \Psi_{\alpha} \left(\left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_{J,\alpha}[l',0]\}, v] \end{array} \right] \right\} \right) d \left\{ \left[\begin{array}{c} [p_l, p_{l'_l}, k_{l'_l}] \\ [\{\Psi_{J,\alpha}[l',0]\}, v] \end{array} \right] \right\} \end{aligned}$$

18.2 Average over eigenspaces, several remaining subobjects

This case is similar to the case one remaining subobject. If the remaining subobjects cannot be separated, the projection yields the composed effective field similar to (96):

$$\begin{aligned} & \int d \left\{ [p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} v \left(\left\{ [p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \\ & \times \Psi_{J, \prod_{i'} s_{p_{\eta'_i}, \eta_i}}^{\otimes \sum_{i'} k_{\eta'_i}, \eta_i} \left(\left\{ [p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i} \right\}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1}, k_{l'_1}) \\ [\{\Psi_J[p_{\eta}, p_{\eta'}]\}, v] \end{array} \right] \right\}, [v], \lambda([\{\Psi_J[p_{\eta}, p_{\eta'}]\}, v]) \right) \end{aligned}$$

If, on the contrary several components can be isolated (see (82)), the system is described by the fields, indexed by η , the remaining objects:

$$\Psi_{J, s_{p_{\eta'_i}, \eta_i}}^{\otimes k_{\eta'_i}, \eta_i} \left([p_{\eta_i}, p_{\eta'_i}, \eta_i]^{k_{\eta'_i}, \eta_i}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1}, k_{l'_1}) \\ [\{\Psi_J[p_{\eta}, p_{\eta'}]\}, v] \end{array} \right] \right\}, [v], \lambda([\{\Psi_J[p_{\eta}, p_{\eta'}]\}, v]) \right) \quad (101)$$

18.3 Transitions

As in part one, we can redefine the fields (101) if the remaining subobject can be disentangled:

$$\Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta} \left([p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [l' \eta, \eta]\}, v] \end{array} \right] \right\}, [v], \lambda \right)$$

where $[p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta}$ describes the remaining free parameters of $[p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta}$ when the constraints with the parameters $\left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [l' \eta, \eta]\}, v] \end{array} \right] \right\}$ are solved. Equivalently, effective field write:

$$v \left([p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} \right) \Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta} \left([p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [l' \eta, \eta]\}, v] \end{array} \right] \right\}, [v], \lambda \right)$$

If the subobjects remain composite, an effective field for a collection of subobjects η_i, η'_i rewrites

$$\int v \left(\left\{ \overline{[p_{\eta_i}, p_{\eta'_i} \eta_i]^{k_{\eta'_i} \eta_i}} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right) \Psi_{J, \prod_{i'} s_{p_{\eta'_i} \eta_i}}^{\otimes \sum_{i'} k_{\eta'_i} \eta_i} \left(\left\{ \overline{[p_{\eta_i}, p_{\eta'_i} \eta_i]^{k_{\eta'_i} \eta_i}} \right\}_{\substack{i \leq s \\ i' \leq s'}} \right), \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_{\eta}, p_{\eta'} \eta]\}, v] \end{array} \right] \right\}, [v] \right) \quad (102)$$

with:

$$\overline{[p_{\eta_i}, p_{\eta'_i} \eta_i]^{k_{\eta'_i} \eta_i}} = [p_{\eta_i}, p_{\eta'_i} \eta_i]^{k_{\eta'_i} \eta_i} / \left\{ [p_{\eta_i}, p_{\eta'_i} \eta_i]^{k_{\eta'_i} \eta_i} \right\}_{\substack{i \leq s \\ i' \leq s'}}$$

The transitions for the fields (101) (similar formula are obtained for (102)) are generated by a functional:

$$S \left(\Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta}, \frac{\delta}{\delta \Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta}}, \nabla \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_{\eta}, p_{\eta'} \eta]\}, v] \end{array} \right] \nabla_{\lambda} \right)$$

with $\Psi_{J, s_{p_{l'} \eta}}^{\otimes k_{l'} \eta}$ defined in (101), and the covariant derivatives are:

$$\begin{aligned} & \nabla_{\{\hat{\mathbf{A}}^{[k_i]}\}} \Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta} \left([p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_{\eta}, p_{\eta'} \eta]\}, v] \end{array} \right] \right\}, [v], \lambda \right) \\ = & \nabla_{\{\hat{\mathbf{A}}^{[k_i]}\}} \Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta} \left([p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_{\eta}, p_{\eta'} \eta]\}, v] \end{array} \right] \right\}, [v], \lambda \right) \\ & + \left(\mathbf{A}_k^{k'} \Psi_{J, s_{p_{\eta'} \eta}}^{\otimes k_{\eta'} \eta} \left(([p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta} / [p_{\eta}, p_{\eta'} \eta]^{k_{\eta'} \eta})_{k'}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_{\eta}, p_{\eta'} \eta]\}, v] \end{array} \right] \right\}, [v], \lambda \right) \right) \end{aligned}$$

with:

$$\mathbf{A} = A \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_{\eta}, p_{\eta'} \eta]\}, v] \end{array} \right] \right\}$$

and:

$$\begin{aligned}
& \delta\lambda \nabla_\lambda \Psi_{J, s_{p_{\eta'}\eta}}^{\otimes k_{\eta'}\eta} \left([p_\eta, p_{\eta'}]^{k_{\eta'}\eta} / [p_\eta, p_{\eta'}]^{k_{\eta'}\eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}, [v], \lambda \left([\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \right) \right) \\
&= \frac{\partial}{\partial \lambda} \Psi_{J, s_{p_{\eta'}\eta}}^{\otimes k_{\eta'}\eta} \left([p_\eta, p_{\eta'}]^{k_{\eta'}\eta} / [p_\eta, p_{\eta'}]^{k_{\eta'}\eta}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [l'\eta, \eta]\}, v] \end{array} \right] \right\}, [v], \lambda \right) \\
&+ \left((A\lambda)_k^{k'} \Psi_{J, s_{p_{\eta'}\eta}}^{\otimes k_{\eta'}\eta} \left(([p_\eta, p_{\eta'}]^{k_{\eta'}\eta} / [p_\eta, p_{\eta'}]^{k_{\eta'}\eta})_{k'}, \left\{ \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \right\}, [v], \lambda \right) \right)
\end{aligned}$$

are some covariant derivatives. Indices k are some local coordinates in the space of maps:

$$([p_\eta, p_{\eta'}]^{k_{\eta'}\eta} / [p_\eta, p_{\eta'}]^{k_{\eta'}\eta})$$

so that the connections are matrices.

The transitions generalizes (61). For two states A and A' , it writes:

$$\begin{aligned}
& \langle A', \lambda' | T_{\lambda\lambda'} | \Psi_J (U_j / [U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda) \rangle \\
&= \langle \Psi_J (U_j / [U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda + \delta\lambda) | \\
&\quad \times N \left(\left(\{ [p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \} \right)_\lambda \leftrightarrow \left(\{ [p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \} \right)_{\lambda'} \right) \\
&\quad \times \exp \left(i \int \delta\lambda S \left(\Psi_{J, s_{p_{\eta'}\eta}}^{\otimes k_{\eta'}\eta}, \frac{\delta}{\delta \Psi_{J, s_{p_{\eta'}\eta}}^{\otimes k_{\eta'}\eta}}, \nabla_\lambda \left[\begin{array}{c} (p_{l_i}, p_{l'_i l_i}, k_{l'_1}) \\ [\{\Psi_J [p_\eta, p_{\eta'}]\}, v] \end{array} \right] \nabla_\lambda \right) \right) | A, \lambda \rangle
\end{aligned} \tag{103}$$

with $N \left(\left(\{ [p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \} \right)_\lambda \leftrightarrow \left(\{ [p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \} \right)_{\lambda'} \right)$ the number, or the volume of set of maps between the two sets of maps:

$$\left(\{ [p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \} \right)_\lambda \leftrightarrow \left(\{ [p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \} \right)_{\lambda'}$$

Note these maps do not always factor as:

$$\prod \left(\left([p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \right)_\lambda \leftrightarrow \left([p_\eta, p_{\eta'}]^{k_{\eta'}\eta} \right)_{\lambda'} \right)$$

Depending on the values of λ some fields may be independent or glued as one global object (see comments after (96)).

Part III States and operators approach

We introduce an alternative approach, which is less general than the field formalism described previously but retains the key aspect of decomposing state space into entangled spaces. Beginning with a direct description of states and operators, we retrieve the essential elements of the field formalism. Notably, spaces of parameters dependent on the states emerge as a characteristic feature.

19 States formulation

We begin by delineating states resulting from the decomposition of a parameter space and the associated state space into entangled subspaces. Projecting onto one of these subspaces allows us to describe states as dependent on certain seemingly exogenous parameters. In fact, the entanglement of the initial states is still present, by the way of constraints conditioning these parameters. This description retains the main features introduced in the preceding sections of this work.

19.1 States and projection on partial states

We start with the decomposition of a state space into two subspaces entangled through some constraints. These constraints arise as eigenvalues of operators defined on the entire initial states space.

19.1.1 Set up

Consider U divided in two different collections (plus identification constraints) $(U^{(i)})$ and $(U^{(j)})$. The $U^{(i)}$ and $U^{(j)}$ stand for the previous $\cup U_i^k$, $\cup U_j^l$ in the field approach. In terms of states they encompass tensor products.

Disregarding first the identifications, one is left with a collection of parameters:

$$\left\{ \left(U^{(i)} \mid U^{(j)} \right) \right\}$$

with associated states:

$$\left(\hat{U}^{(i)} \mid \hat{U}^{(j)} \right) \rightarrow \left\{ \begin{array}{l} \left| \hat{U}^{(i)} \mid \hat{U}^{(j)} \right\rangle \\ \left\langle \hat{U}^{(i)} \mid \hat{U}^{(j)} \right| \end{array} \right. \quad (104)$$

which is a short cut for some combinations, up to constraints, of basis states:

$$\sum_{k,l} a \left(u_1^{(i)} \dots u_k^{(i)}, u_1^{(j)} \dots u_l^{(j)} \right) \left| u_1^{(i)} \dots u_k^{(i)} \mid u_1^{(j)} \dots u_l^{(j)} \right\rangle$$

The states (104) form a basis of an internal space.

The whole space being set of linear combinations:

$$\sum a \left(\hat{U}^{(i)}, \hat{U}^{(j)} \right) \left| \hat{U}^{(i)} \mid \hat{U}^{(j)} \right\rangle$$

We assume that the states $\left| \hat{U}^{(i)} \mid \hat{U}^{(j)} \right\rangle$ can be divided in sectors through some operator M eigenvalues (or equivalently, they are defined as eigenstates of this operator). This implies

$$\prod_{M=m} \left| \hat{U}^{(i)} \mid \hat{U}^{(j)} \right\rangle$$

Assume also that M is a combination of two operators $M = f(M_i, M_j)$ acting on the $\hat{U}^{(i)}$ and $\hat{U}^{(j)}$ part. Thus:

$$\sum_m \prod_{M=m} \left| \hat{U}^{(i)} \mid \hat{U}^{(j)} \right\rangle = \sum_m \prod_{\delta(f(M_i, U^{(j)}))=m} \left| \hat{U}^{(i)} \right\rangle \left| \hat{U}^{(j)} \right\rangle$$

For a basis $\left| \hat{U}^{(j)} \right\rangle$ of eigenstates of M_j , the constraint becomes $\delta \left(f \left(M_i, \hat{U}^{(j)} \right) - m \right)$ and:

$$\sum_m \prod_{\delta(f(M_i, U^{(j)}))=m} \equiv \prod_{\hat{U}^{(j)}}^{(i)}$$

19.1.2 Partial diagonalization of projection operator

We will project the states space on the eigenstates of some operator. This projection accounts for constraints between the two states spaces, so that it follows several steps. An example base on some basic model is given in appendix 8.

We define:

$$\hat{H}_{i,j} = \sum_l \hat{H}_j^{(l)} \hat{H}_i^{(l)}$$

where $\hat{H}_j^{(l)}$ and $\hat{H}_i^{(l)}$ act on the $|\hat{U}^{(i)}\rangle$ and $|\hat{U}^{(j)}\rangle$ respectvl. We assume $\hat{H}_j^{(l)}$ commute with M_j and have common eigenstates $|U^{(j)}\rangle$:

$$\begin{aligned} \prod_{\hat{U}^{(j)}}^{(i)} \sum_l |U^{(j)}\rangle \langle U^{(j)}| \hat{H}_j^{(l)} |U^{(j)}\rangle \langle U^{(j)}| \hat{H}_i^{(l)} \prod_{\hat{U}^{(j)}}^{(i)} &= |U^{(j)}\rangle \prod_{U^{(j)}}^{(i)} \sum_l \langle U^{(j)}| \hat{H}_j^{(l)} |U^{(j)}\rangle \hat{H}_i^{(l)} \prod_{U^{(j)}}^{(i)} \langle U^{(j)}| \\ &\equiv |U^{(j)}\rangle \prod_{U^{(j)}}^{(i)} \hat{H}_i(U^{(j)}) \prod_{U^{(j)}}^{(i)} \langle U^{(j)}| \end{aligned}$$

We consider the operators $\prod_{U^{(j)}}^{(i)} \hat{H}_i(U^{(j)}) \prod_{U^{(j)}}^{(i)}$ and their eigenstates:

$$\prod_{U^{(j)}}^{(i)} \hat{H}_i(U^{(j)}) \prod_{U^{(j)}}^{(i)} = \prod_{U^{(j)}}^{(i)} \sum_l \langle U^{(j)}| \hat{H}_j^{(l)} |U^{(j)}\rangle \hat{H}_i^{(l)} \prod_{U^{(j)}}^{(i)} \rightarrow |\lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}_i\rangle \quad (105)$$

The $\Lambda_i(U^{(j)})$ eigenvalues of set of operators $\{\hat{\Lambda}_i(U^{(j)})\}$. Operators $\{\hat{\Lambda}_i(U^{(j)})\}$ describe some degeneracy. As an example we can consider:

$$\{\Lambda_i(U^{(j)})\}_i = \cup_n \{\Lambda_i^n(U^{(j)})\}$$

for some fixed set of parameters $\Lambda_i(U^{(j)})$. This corresponds to a constraint

$$f(H, \{\hat{\Lambda}_i(U^{(j)})\}) \quad (106)$$

and the state space decomposing accordingly to:

$$\oplus \left| \left(\Lambda_i(U^{(j)}) \right)_1 \dots \left(\Lambda_i(U^{(j)}) \right)_n \right\rangle$$

with:

$$\lambda(U^{(j)}) = \hat{f} \left(\left(\Lambda_i(U^{(j)}) \right)_1 \dots \left(\Lambda_i(U^{(j)}) \right)_n \right)$$

solving (106) on state:

$$\left| \left(\Lambda_i(U^{(j)}) \right)_1 \dots \left(\Lambda_i(U^{(j)}) \right)_n \right\rangle$$

We can write the states arising in (105) as a linear combination:

$$\left| \lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}_i \right\rangle = \sum \psi(\Lambda^i(U^{(j)})) \left| \lambda(U^{(j)}), \Lambda^i(U^{(j)}) \right\rangle \quad (107)$$

As a consequence:

$$\left| \lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}_i \right\rangle = \left| \lambda(U^{(j)}), \mathbf{s}(\Lambda(U^{(j)})) \right\rangle$$

where $\mathbf{S}(\mathbf{\Lambda}(U^{(j)}))$ is the entire space defined by the $\mathbf{\Lambda}(U^{(j)})$. The states arising in (107) are the of the same type as the ones defined in the first part. They represent states that are series depending on increasing sequences of parameters, forming a vld of pnts.

In terms of functionals, this translates in states:

$$\psi\left(\lambda\left(U^{(j)}\right), \mathbf{S}\left(\mathbf{\Lambda}\left(U^{(j)}\right)\right)\right)$$

a functional defined on the space $\mathbf{S}(\mathbf{\Lambda}(U^{(j)}))$.

The $\mathbf{\Lambda}_i(U^{(j)})$ are defined on projected space $\prod_{U^{(j)}}^{(i)}$. The $\mathbf{\Lambda}_i(U^{(j)})$ are function of the constraint:

$$\mathbf{\Lambda}_i(U^{(j)}) \equiv \mathbf{\Lambda}_i([U^{(j)}])$$

with $[U^{(j)}]$ denotes the components of $U^{(j)}$ that define $\prod_{U^{(j)}}^{(i)}$.

As an exemple, we can consider in first approximation that the constraint at the quadratic rdr:

$$\begin{aligned} & H \left(\left(\left(\left\{ \hat{\mathbf{\Lambda}}_i(U^{(j)}) \right\} \cdot \left\{ \hat{\mathbf{\Lambda}}_i(U^{(j)}) \right\} - \lambda^2(U^{(j)}) - \alpha(U^{(j)}) \right)^2 \right)^{\otimes n} \right) \\ \rightarrow & \left(\left\{ \mathbf{\Lambda}_i(U^{(j)}) \right\} \cdot \left\{ \mathbf{\Lambda}_i(U^{(j)}) \right\} - \lambda^2(U^{(j)}) - \alpha(U^{(j)}) \right)^2 = 0 \end{aligned}$$

where the constraint has one realization on each state $|(\mathbf{\Lambda}_i(U^{(j)}))_1 \dots (\mathbf{\Lambda}_i(U^{(j)}))_n\rangle$:

$$\left(\left((\mathbf{\Lambda}_i(U^{(j)}))_1, \dots, (\mathbf{\Lambda}_i(U^{(j)}))_n \right)^2 - \lambda^2(U^{(j)}) - \alpha(U^{(j)}) \right)^2 = 0$$

19.1.3 Projected states

We assume that the projectn $\prod_{U^{(j)}}^{(i)}$ are defined as some eigenstates of an operator M_i :

$$\prod_{U^{(j)}}^{(i)} = \prod_{M_i=m(U^{(j)})}^{(i)} = \prod_{M_i=m([U^{(j)}])}^{(i)}$$

The $\hat{\mathbf{\Lambda}}_i(U^{(j)})$ commute with M_i and thus there is a relation:

$$f\left(\left\{\mathbf{\Lambda}_i\left([U^{(j)}]\right)\right\}\right) = m\left([U^{(j)}]\right)$$

we assume that this relation may be inverted, which implies that:

$$\text{vect} \left\{ \left| U^{(j)} \right\rangle \left| \lambda(U^{(j)}), \left\{ \mathbf{\Lambda}_i([U^{(j)}]) \right\}_i \right\rangle \right\}$$

can also be written:

$$\text{vect} \left\{ \left| U^{(j)} / [U^{(j)}], [U^{(j)}] (\{\mathbf{\Lambda}_i\}) \right\rangle \left| \lambda(U^{(j)}), \{\mathbf{\Lambda}_i\} \right\rangle \right\} = \text{vect} \left\{ \left| U^{(j)/\{\mathbf{\Lambda}_i\}}, [U^{(j)}] (\{\mathbf{\Lambda}_i\}) \right\rangle \left| \lambda(U^{(j)}), \{\mathbf{\Lambda}_i\} \right\rangle \right\}$$

for any given set of $\{\mathbf{\Lambda}_i\}$, whereas as in part one $[U^{(j)}] (\{\mathbf{\Lambda}_i\})$ is the subspace of parameters that is expressed as a function of the $\{\mathbf{\Lambda}_i\}$. The remaining independent parameters $U^{(j)} / [U^{(j)}]$ will be also written $U^{(j)/\{\mathbf{\Lambda}_i\}}$.

We gather the $|U^{(j)/\{\mathbf{\Lambda}_i\}}, U^{(j)}(\{\mathbf{\Lambda}_i\})\rangle |\lambda(U^{(j)}), \mathbf{\Lambda}_i\rangle$ with a given $\lambda(U^{(j)})$ and consider:

$$\text{vect} \left\{ \left| U^{(j)/\{\mathbf{\Lambda}_i\}}, U^{(j)}(\{\mathbf{\Lambda}_i\}) \right\rangle \left| \lambda(U^{(j)}), \{\mathbf{\Lambda}_i\} \right\rangle \right\}_{\lambda(U^{(j)})=\lambda}$$

Thus any states in this space writes:

$$\begin{aligned} & \int_{\lambda(U^{(j)})=\lambda} \psi \left(\lambda, U^{(j)/\{\mathbf{\Lambda}_i\}}, [U^{(j)}(\{\mathbf{\Lambda}_i\})] \right) \left| U^{(j)/\{\mathbf{\Lambda}_i\}}, U^{(j)}(\{\mathbf{\Lambda}_i\}) \right\rangle |\lambda, \mathbf{\Lambda}_i\rangle d\mathbf{\Lambda}_i \\ &= \sum \int_{\lambda(U^{(j)})=\lambda} \psi \left(\lambda, U^{(j)/\mathbf{\Lambda}^i}, [U^{(j)}(\{\mathbf{\Lambda}_i\})] \right) \left| U^{(j)/\mathbf{\Lambda}^i}, [U^{(j)}(\{\mathbf{\Lambda}_i\})] \right\rangle |\lambda, \mathbf{\Lambda}^i\rangle d\mathbf{\Lambda}_i \end{aligned}$$

which leads to describe such state by the functional:

$$\psi \left(\lambda, U^{(j)/\mathbf{\Lambda}^i}, [U^{(j)}(\mathbf{\Lambda})], \mathbf{S}(\mathbf{\Lambda}) \right)$$

a functional of the function $U^{(j)/\mathbf{\Lambda}^i}(\mathbf{\Lambda})$ over $\mathbf{S}(\mathbf{\Lambda})$.

19.1.4 Remark: inverting the relation

If we assume the $|\lambda(U^{(j)}), \{\mathbf{\Lambda}_i\}\rangle$ can be written in terms of a common basis $|\bar{\mathbf{\Lambda}}_i\rangle$ independent of $U^{(j)}$:

$$\left| \lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}) \right\rangle = \int h \left(\lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right) |\bar{\mathbf{\Lambda}}_i\rangle d\bar{\mathbf{\Lambda}}_i \quad (108)$$

$$\left| U^{(j)/\{\hat{\mathbf{\Lambda}}_i\}}, [U^{(j)}(\{\mathbf{\Lambda}_i\})] \right\rangle \left| \lambda(U^{(j)}), \{\mathbf{\Lambda}_i\} \right\rangle = \sum_{\mathbf{\Lambda}_i} \left| U^{(j)/\hat{\mathbf{\Lambda}}_i}, [U^{(j)}(\{\mathbf{\Lambda}_i\})] \right\rangle h \left(\lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right) |\bar{\mathbf{\Lambda}}_i\rangle$$

where as before $[U^{(j)}(\{\bar{\mathbf{\Lambda}}_i\})]$ represents the degrees of freedom of $U^{(j)}$ that can be expressed as function of $\{\bar{\mathbf{\Lambda}}_i\}$.

The $\mathbf{\Lambda}_i$ are of the form $\mathbf{\Lambda}^{\otimes i}$ and $[U^{(j)}]$ denotes the components of $U^{(j)}$ that define $\prod_{U^{(j)}}^{(i)}$. If $h \left(\lambda(U^{(j)}), \hat{\mathbf{\Lambda}}_i(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right)$ is invertible, we have:

$$\begin{aligned} & \sum_{\mathbf{\Lambda}_i(U^{(j)})} h^\dagger \left(\lambda(U^{(j)}), \{\mathbf{\Lambda}_i(U^{(j)})\}, \{\bar{\mathbf{\Lambda}}_i\} \right) \left| U^{(j)/\mathbf{\Lambda}_i}, [U^{(j)}(\{\bar{\mathbf{\Lambda}}_i\})] \right\rangle \left| \lambda(U^{(j)}), \{\bar{\mathbf{\Lambda}}_i\} \right\rangle \\ &= \sum_{\mathbf{\Lambda}_i(U^{(j)})} \sum_{\bar{\mathbf{\Lambda}}'_i} \left| U^{(j)/\bar{\mathbf{\Lambda}}_i}, \bar{\mathbf{\Lambda}}_i \right\rangle h^\dagger \left(\lambda(U^{(j)}), \hat{\mathbf{\Lambda}}_i(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right) h \left(\lambda(U^{(j)}), \hat{\mathbf{\Lambda}}_i(U^{(j)}), \{\bar{\mathbf{\Lambda}}'_i\} \right) |\{\bar{\mathbf{\Lambda}}'_i\}\rangle \\ &= \left| U^{(j)/\mathbf{\Lambda}_i}, [U^{(j)}(\{\bar{\mathbf{\Lambda}}_i\})] \right\rangle \left| \lambda(U^{(j)}), \{\bar{\mathbf{\Lambda}}_i\} \right\rangle \end{aligned}$$

However, if $h \left(\lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right)$ is not invertible, we can write only on some subspace:

$$\begin{aligned} & \sum_{\mathbf{\Lambda}_i(U^{(j)})} h^\dagger \left(\lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}), o_i \right) \left| U^{(j)/\mathbf{\Lambda}_i}, \bar{\mathbf{\Lambda}}_i \right\rangle \left| \lambda(U^{(j)}), \hat{\mathbf{\Lambda}}_i \right\rangle \\ &= \sum_{\mathbf{\Lambda}_i(U^{(j)})} \sum_{\mathbf{\Lambda}'_i(U^{(j)})} \left| U^{(j)/\mathbf{\Lambda}_i}, \bar{\mathbf{\Lambda}}_i \right\rangle h^\dagger \left(\lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right) h \left(\lambda(U^{(j)}), \mathbf{\Lambda}_i(U^{(j)}), \bar{\mathbf{\Lambda}}'_i \right) |\bar{\mathbf{\Lambda}}'_i\rangle \\ &= \sum_{\bar{\mathbf{\Lambda}}_i(U^{(j)})} \left| U^{(j)/\mathbf{\Lambda}_i}, (\mathbf{\Lambda}_i) \right\rangle \left| \lambda(U^{(j)}), \bar{\mathbf{\Lambda}}_i \right\rangle |\bar{\mathbf{\Lambda}}_i\rangle \end{aligned}$$

19.2 States Covariantly

The degeneracy for the projected prtr wrts:

$$\left[\left[\left(U^{(i)} \right)' \right] \left(U^{(j)} \right) H_i^{(l)} \left(U^{(j)} \right)_{\times \bar{V}_{U^{(i)}, (U^{(i)})', U^{(j)}}^l} \left[U^{(i)} \right], \mathbf{\Lambda}_i \left(U^{(j)} \right) \right] = 0$$

for some of the $\{\mathbf{\Lambda}_i(U^{(j)})\}$ arises for the more compact form (173)

$$H_i \left(U^{(j)} \right) = H \left[U^{(j)}, \mathbf{\Lambda}_i \left(U^{(j)} \right) \right]$$

We rewrite this relation as:

$$G \left(H_i \left(U^{(j)} \right), \left\{ \mathbf{\Lambda}_i \left(U^{(j)} \right) \right\} \right) = 0$$

and consider that this arises from more general constraint on operators:

$$G \left(\left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right) = 0 \quad (109)$$

where prtrs $\mathbf{\Lambda}_\alpha(U^{(j)})$ are gvn b:

$$\mathbf{\Lambda}_\alpha \left(U^{(j)} \right) = \mathbf{\Lambda}_0 \left(U^{(j)} \right), \mathbf{\Lambda}_i \left(U^{(j)} \right)$$

Assuming that thr s trnsfrmtn $M^{(i)} \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\}$ prsrvn:

$$G \left(M^{(i)} \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right) = 0$$

This trnsfrmtn m rs frm:

$$M^{(i)} \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} = \left\{ \mathbf{\Lambda}_\alpha \left(\left(M^{(i)} \right)^{-1} U^{(j)} \right) \right\}$$

We assume that thr trnsfrmtn trnslt n st:

$$\left| M^{(i)} \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle = R \left(M^{(i)} \right) \left| \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle$$

so that we have:

$$\left| \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle = R \left(M^{(i)} \right) \left| \left(M^{(i)} \right)^{-1} \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle$$

This means that the states $\left| \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle$ dpnds n th ntr spc (109):

$$\left| \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle = \left| \Sigma \left(U^{(j)} \right) \right\rangle$$

$$\left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} = \cup \left\{ \left(\Lambda_\alpha \left(U^{(j)} \right) \right)^k \right\}$$

$$\left| \left\{ \mathbf{\Lambda}_\alpha \left(U^{(j)} \right) \right\} \right\rangle = \sum \int_{G(\{\mathbf{\Lambda}_\alpha(U^{(j)})\})=0} g \left(\Lambda_{\alpha,1} \left(U^{(j)} \right), \dots, \Lambda_{\alpha,k} \left(U^{(j)} \right) \right) \left| \left(\Lambda_\alpha \left(U^{(j)} \right) \right)^k \right\rangle$$

where $\left| \left(\Lambda_\alpha \left(U^{(j)} \right) \right)^k \right\rangle$ is defined for given values of the variables. The invariant state is thus an averaged state.

Considering rather representation $R \left(M^{(i)} \right)$, we are led to consider components:

$$|\Sigma(U^{(j)})\rangle_a = \sum \int_{G(\{\Lambda_\alpha(U^{(j)})\})=0} g_a(\Lambda_{\alpha,1}(U^{(j)}), \dots, \Lambda_{\alpha,k}(U^{(j)})) |\{\Lambda_\alpha(U^{(j)})\}\rangle$$

where $R(M^{(i)})$ acts on these components:

$$\begin{aligned} & (R(M^{(i)})|\Sigma(U^{(j)})\rangle)_a \\ &= R(M^{(i)})_a^b \sum \int_{G(\{\Lambda_\alpha(U^{(j)})\})=0} g_b(\Lambda_{\alpha,1}(U^{(j)}), \dots, \Lambda_{\alpha,k}(U^{(j)})) |\{\Lambda_\alpha(U^{(j)})\}\rangle \end{aligned}$$

assuming a transformation:

$$(M^{(i)})^{-1} \{\Lambda'_\alpha(U^{(j)})\} = \lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}$$

$$|\Sigma(U^{(j)})\rangle = |\{\Lambda'_\alpha(U^{(j)})\}\rangle = |(M^{(i)})^{-1} \{\Lambda'_\alpha(U^{(j)})\}\rangle = |\lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}\rangle$$

$$|\{\Lambda_\alpha(U^{(j)})\}\rangle = \sum \int g(\lambda(U^{(j)}), \Lambda_{i,1}(U^{(j)}) \dots \Lambda_{i,k}(U^{(j)})) |\lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}\rangle = |\lambda(U^{(j)}), \{\Lambda_i(U^{(j)})\}\rangle$$

and we recover our previous formulation by choice of coordinates.

19.3 Operators induced by states

The states $|U^{(j)/\Lambda_i}, (\Lambda_i)\rangle$ and spaces:

$$\text{vect} \left\{ |U^{(j)/\Lambda_i}, \{\Lambda_i\}\rangle \right\}_{\lambda(U^{(j)})=\lambda}$$

can be derived from some field theory, if we consider the field operator:

$$|U^{(i')}\rangle |\hat{U}^{(j')}\rangle \Psi(\hat{U}^{(j)}, \hat{U}^{(j')}, U^{(i)}, U^{(i')}) \langle \hat{U}^{(j)} | \langle U^{(i)} |$$

that can be projected through operator $\prod_{U^{(j)}, \lambda(U^{(j)})}$. This projection yields:

$$\prod_{U^{(j)}, \lambda(U^{(j)})} \rightarrow \Psi(U^{(j)/\Lambda_i}(\Lambda_i)) |\lambda, \Lambda_i\rangle \rightarrow |\lambda, \Lambda_i\rangle \Psi(U^{(j)/\Lambda_i}(\Lambda_i), \lambda) \langle \lambda, \Lambda_i |$$

The spcc $(\hat{\delta}_i)$ ndwd wt $G(\lambda(U^{(j)}), \hat{\Lambda}_i(U^{(j)}))$ depends on $\Psi(U^{(j)/\Lambda_i}(\Lambda_i))$:

$$G(\lambda(U^{(j)}), \hat{\Lambda}_i(U^{(j)})) \rightarrow G(\Psi(U^{(j)/\Lambda_i}(\Lambda_i)))$$

20 Operator formalism

In this section, we rather start from an operator formalism and describe directly the constraints and projections in terms of operators.

20.1 Operators and constraints

Keeping the decomposition (104) presented in the previous section, we write an operator under the following form:

$$\left[(\hat{U}^{(i)})' \mid (\hat{U}^{(j)})' \right] \Psi \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] \equiv \int \left| (\hat{U}^{(j)})' \right\rangle \left\langle (\hat{U}^{(i)})' \right| \Psi \left((\hat{U}^{(i)})', (\hat{U}^{(j)})', \hat{U}^{(i)}, \hat{U}^{(j)} \right) \langle \hat{U}^{(i)} \mid \langle \hat{U}^{(j)} \mid \quad (110)$$

where the $\left| (\hat{U}^{(j)})' \right\rangle \left\langle (\hat{U}^{(i)})' \right|$ represent a basis (104). Operator (110) can be decomposed as a sum:

$$\left[\hat{U}' \right] \Psi \left[\hat{U} \right] \rightarrow \sum_{\alpha} \int \left[(\hat{U}^{(i)})' \mid (\hat{U}^{(j)})' \right] \left(\Psi_i^{(V)} \right)_{\times V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}}^{\alpha} \left(\Psi_j^{(V)} \right) \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] dV_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}}^{\alpha}$$

where the $V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}^{\alpha}$ model some constraints and are defined by equations:

$$V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}^{\alpha} : f^{\alpha} \left(\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})' \right) = 0$$

The sum is over the various constraints, or functions f^{α} (α describing any set of constraints).

For example assuming constraints factoring as:

$$g_i \left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \right) = g_j \left(\hat{U}^{(j)}, (\hat{U}^{(j)})' \right)$$

where g_i and g_j may be multicomponents and take values $v \in V$, one has:

$$\left[\hat{U}' \right] \Psi \left[\hat{U} \right] \rightarrow \int_{v \in V} \left[(\hat{U}^{(i)})' \mid (\hat{U}^{(j)})' \right]_{g_i(\hat{U}^{(i)}, (\hat{U}^{(i)})') = g_j(\hat{U}^{(j)}, (\hat{U}^{(j)})')} \left(\Psi_i^{(V)} \right) \times \left(\Psi_j^{(V)} \right) \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] dv$$

Usually we can think the constraints as some conservation relations. If $\hat{U}^{(i)}$ and $\hat{U}^{(j)}$ depend on some variables k_i and k_j , the constraints are of the form: $k_i - k'_i = k_j - k'_j$.

The constraints may be written using δ functions:

$$\begin{aligned} \left[\hat{U}' \right] \Psi \left[\hat{U} \right] &= \sum_{\alpha} \int \left[(\hat{U}^{(i)})' \mid (\hat{U}^{(j)})' \right] \left(\Psi_i^{(V)} \right) \delta \left(V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}^{\alpha} \right) \left(\Psi_j^{(V)} \right) \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] \\ &\quad \times d \left(\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})' \right) \end{aligned}$$

More is said about the constraints in appendix 9. We describe these constraints as equations satisfied by operators, and recover the states resulting from these constraints.

20.2 Projection operators

20.2.1 Global projection operator

The operator defining the prjctn are defined following the previous decompostn: $\Psi_i^{(V)} \rightarrow \hat{H}_i^{(V)}$, $\Psi_j^{(V)} \rightarrow \hat{H}_j^{(V)}$:

$$\sum_V \left[(\hat{U}^{(i)})' \mid (\hat{U}^{(j)})' \right] \left(\hat{H}_i^{(V)} \right)_{\times V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}} \left(\hat{H}_j^{(V)} \right) \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] \quad (111)$$

We assume that for all $\hat{U}^{(i)}, (\hat{U}^{(i)})'$, prtrs $\delta \left(V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}^\alpha \right) \hat{H}_j^{(V)}$ have common eigenstates $|U^{(j)}\rangle$ with eigenvalues:

$$h_\alpha \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right) \delta \left(\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha \right)$$

where $\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha$ are possible remaining constraints such as $\left\{ g_i \left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \right) = v \in \bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha \right\}$.

Disregarding $\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha$ we also assume that the eigenvalue $h_\alpha \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right)$ is build from operators $\Lambda_i(U^{(j)})$ commuting with $\hat{H}_j^{(V)}$ written as $\hat{H}_j^{(V)}(\Lambda_i(U^{(j)}))$ (similarly to a band hamiltonian) including some degeneracy. Operator (111) writes:

$$\begin{aligned} & \int |U^{(j)}\rangle \langle U^{(j)}| \left[\left((\hat{U}^{(i)})' \mid (\hat{U}^{(j)})' \right) \left(\hat{H}_i^{(V)} \right) \left(H_j^{(V)} \right) \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] |U^{(j)}\rangle \langle U^{(j)}| \right. \\ & \quad \times \delta \left(V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}^\alpha \right) d \left(\hat{U}^{(j)}, (\hat{U}^{(j)})' \right) \\ & = \int |U^{(j)}\rangle \left[(\hat{U}^{(i)})' \right] \left(\hat{H}_i^{(V)} \right) \left[\hat{U}^{(i)} \right] h_\alpha^\vee \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right) \langle U^{(j)}| \\ & \equiv \int |U^{(j)}\rangle \left[(\hat{U}^{(i)})' \right] \left(\hat{H}_i(U^{(j)}) \right) \left[\hat{U}^{(i)} \right] \langle U^{(j)}| \end{aligned}$$

The operator $\left[(\hat{U}^{(i)})' \right] \left(\hat{H}_i(U^{(j)}) \right) \left[\hat{U}^{(i)} \right]$ may include constraints $\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha$ so that (111) writes:

$$\equiv \sum_\alpha \left[(\hat{U}^{(i)})' \mid U^{(j)} \right] \left(\hat{H}_i^{(V)} \right)_{\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha} \times \left(H_j^{(V)} \right) \left[\hat{U}^{(i)} \mid U^{(j)} \right] \quad (112)$$

where the $h_\alpha \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right)$ are eigenvalues of the $H_j^{(V)}$ in stts $|U^{(j)}\rangle$.

20.2.2 Generalization: local diagonalization

The previous set up can be generalized. If $\delta \left(V_{\hat{U}^{(i)}, (\hat{U}^{(i)})', \hat{U}^{(j)}, (\hat{U}^{(j)})'}^\alpha \right) \hat{H}_j^{(V)}$ have common eigenstates:

$$\left| U^{(j)} \right\rangle_{(\hat{U}^{(i)}, (\hat{U}^{(i)})')} \equiv \left| U^{(j)} \right\rangle_i$$

depending on $\left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \right)$ with eigenvalues:

$$h_\alpha \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right) \delta \left(\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha \right)$$

where $\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha$ are the possible remaining constraints such as:

$$\left\{ g_i \left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \right) = v \in \bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^\alpha \right\}$$

and the decomposition (111) becomes:

$$\begin{aligned}
& \sum_{\alpha} \int |U^{(j)}\rangle_i \langle U^{(j)}|_i \left[(\hat{U}^{(i)})' | (\hat{U}^{(j)})' \right] \left(\hat{H}_i^{(V)} \right)_{V_{\hat{U}^{(i)},(\hat{U}^{(i))}',\hat{U}^{(j)},(\hat{U}^{(j))}'}}^{\alpha} \times \left(H_j^{(V)} \right) \left[\hat{U}^{(i)} | \hat{U}^{(j)} \right] | U^{(j)}\rangle_i \langle U^{(j)}|_i \quad (113) \\
&= \sum_{\alpha} \int |U^{(j)}\rangle_i \left[(\hat{U}^{(i)})' \right] \left(\hat{H}_i^{(V)} \right) \left[\hat{U}^{(i)} \right] h_{\alpha} \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right) \delta \left(V_{\hat{U}^{(i)},(\hat{U}^{(i))}',U^{(j)}}^{\alpha} \right) \langle U^{(j)}|_i d \left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \right) \\
&\equiv \sum_{\alpha} \left[(\hat{U}^{(i)})' | U_{(\hat{U}^{(i)},(\hat{U}^{(i))}')}^{(j)} \right] \left(\hat{H}_i^{(V)} \right)_{\bar{V}_{\hat{U}^{(i)},(\hat{U}^{(i))}',U^{(j)}}^{\alpha}} \times \left(H_j^{(V)} \right) \left[\hat{U}^{(i)} | U_{(\hat{U}^{(i)},(\hat{U}^{(i))}')}^{(j)} \right]
\end{aligned}$$

Writing $F_{\hat{U}^{(i)},(\hat{U}^{(i))}'}$ the fibre of $V_{\hat{U}^{(i)},(\hat{U}^{(i))}',\hat{U}^{(j)},(\hat{U}^{(j))}'}$ over a given $\hat{U}^{(i)}, (\hat{U}^{(i)})'$, we can decompose:

$$\left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' \right\} = \cup_{F^{\alpha}} \left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' \right\}_{F^{\alpha}}$$

with:

$$\left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' \right\}_{F^{\alpha}} = \left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' / F_{\hat{U}^{(i)},(\hat{U}^{(i))}' = F^{\alpha}}^{\alpha} \right\}$$

Along a basis $\left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' \right\}_{F^{\alpha}}$, the operator:

$$\delta \left(V_{\hat{U}^{(i)},(\hat{U}^{(i))}',\hat{U}^{(j)},(\hat{U}^{(j))}' }^{\alpha} \right) \hat{H}_j^{(V)}$$

writes:

$$\delta \left(\left(\hat{U}^{(j)}, (\hat{U}^{(j)})' \right) \subset F^{\alpha} \right) \hat{H}_j^{(V)}$$

We can assume that the $\delta \left(\left(\hat{U}^{(j)}, (\hat{U}^{(j)})' \right) \subset F^{\alpha} \right) \hat{H}_j^{(V)}$ have common eigenstates $|U^{(j)}\rangle_{F^{\alpha}}$. Decomposing the measure of integration along the fibre:

$$d \left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \right) = d \left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' \right\}_{F^{\alpha}} dF^{\alpha}$$

formula (113) becomes ultimately:

$$\begin{aligned}
& \sum_{\alpha} \int |U^{(j)}\rangle_{F^{\alpha}} \left[(\hat{U}^{(i)})' \right] \left(\hat{H}_i^{(V)} \right) \left[\hat{U}^{(i)} \right] h_{\alpha} \left(U^{(j)}, \hat{U}^{(i)}, (\hat{U}^{(i)})' \right) d \left\{ \hat{U}^{(i)}, (\hat{U}^{(i)})' \right\}_{F^{\alpha}} \langle U^{(j)}|_{F^{\alpha}} dF^{\alpha} \\
&\equiv \sum_{\alpha} \int \left[(\hat{U}^{(i)})' | U_{F^{\alpha}}^{(j)} \right] \left(\hat{H}_{iF^{\alpha}}^{(V)} \right) \left(H_j^{(V)} \right) \left[\hat{U}^{(i)} | U_{F^{\alpha}}^{(j)} \right] dF^{\alpha}
\end{aligned}$$

with:

$$\hat{H}_{iF^{\alpha}}^{(V)} = \hat{H}_i^{(V)} \delta \left(\hat{U}^{(i)}, (\hat{U}^{(i)})' \subset F^{\alpha} \right)$$

20.3 Projection on partial states: Bass f eigenstates depending on $U^{(j)}$

Consider the sum of projections along the eigenstates $U^{(j)}$:

$$\sum_j \prod_j \oplus \prod_j = 1$$

and decompose the operator (112) along this decomposition. For each eigenstate, we obtain an operator acting on the $(U^{(i)})$ degrees of freedom of the states space:

$$\sum_{\alpha} \left[\left(\hat{U}^{(i)} \right)' \mid U^{(j)} \right] \left(\hat{H}_i^{(V)} \right)_{\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^{\alpha}} \times \left(H_j^{(V)} \right) \left[\hat{U}^{(i)} \mid U^{(j)} \right]$$

$$\Pi_j \oplus \Pi_j \sum_{\alpha} \left[\left(\hat{U}^{(i)} \right)' \right] (U_j) \left(\hat{H}_i^{(l)} \right)_{\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^{\alpha}} \times (U_j) \left[\hat{U}^{(i)} \right]$$

As before, we assume that:

$$\left[\left(\hat{U}^{(i)} \right)' \right] (U_j) \left(\hat{H}_i^{(l)} \right)_{\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^{\alpha}} \times (U_j) \left[\hat{U}^{(i)} \right]$$

commute with a family of operators $\hat{\Lambda}_i (U^{(j)})$ depending on $U^{(j)}$ and that the relation:

$$f \left(H \left(U^{(j)} \right), \Lambda_i \left(U^{(j)} \right), \alpha \left(U^{(j)} \right) \right) = 0$$

stands for each of the U_j . As a consequence, the diagonalization of:

$$\sum_{\alpha} \left[\left(\hat{U}^{(i)} \right)' \right] (U_j) \left(\hat{H}_i^{(l)} \right)_{\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^{\alpha}} \times (U_j) \left[\hat{U}^{(i)} \right]$$

yields degenerated spaces:

$$\sum_{\alpha} \left[\left(\hat{U}^{(i)} \right)' \right] (U_j) \left(\hat{H}_i^{(l)} \right)_{\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^{\alpha}} \times (U_j) \left[\hat{U}^{(i)} \right] \rightarrow \left[\left(\hat{U}^{(i)} \right)' \right] (U_j) H_i (U_j) \left[\hat{U}^{(i)} \right] \rightarrow \left| \lambda (U_j), \Lambda_i \left(U^{(j)} \right) \right\rangle$$

The constraints $\bar{V}_{\hat{U}^{(i)}, (\hat{U}^{(i)})', U^{(j)}}^{\alpha}$ imply that the states $\left| \lambda (U_j), \Lambda_i \left(U^{(j)} \right) \right\rangle$ do not generate the entire initial states space. The relation

$$f \left(H \left(U^{(j)} \right), \hat{\Lambda}_i \left(U^{(j)} \right), \alpha \left(U^{(j)} \right) \right) = 0 \quad (114)$$

allows thus to replace some parameters $[U_j]$ as functional of the eigenvalues $\{ \Lambda_i \left(U^{(j)} \right), \lambda \left(U_j \right) \}$ and states:

$$\left| (U_j) \right\rangle \left| \lambda \left(U_j \right), \Lambda_i \left(U^{(j)} \right) \right\rangle$$

are linear combinations of states of the form:

$$\left| \left(U_j / [U_j], [U_j] \left(\{ \Lambda_i \left(U^{(j)} \right) \} \right), \lambda \left(U_j \right) \right) \right\rangle \left| \lambda \left(U_j \right), \Lambda_i \left(U^{(j)} \right) \right\rangle \quad (115)$$

The parameters $U_j / [U_j]$ represent the remaining parameters after using the constraint (114) has been imposed and $[U_j] \left(\{ \Lambda_i \left(U^{(j)} \right) \} \right)$ describes the part of U_j written as a function of $\{ \Lambda_i \left(U^{(j)} \right) \}$. Note that this replacement depends on (114) and thus on $\lambda \left(U_j \right), \Lambda_i \left(U^{(j)} \right)$. This change of variable is local: the parameters $U^{(j)}$ describing some fundamental state, determines the eigenstates defined by $(\lambda \left(U_j \right), \Lambda_i \left(U^{(j)} \right))$ and thus the decomposition. The entanglement between the $U^{(i)}$ and $U^{(j)}$ degrees of freedom translate in this interdependence between the parameters and the apparent system's degrees of freedom $U_j / [U_j]$.

Remark also that the states (115) are similar to the states described in the first and second part of this work. There is local relation between the apparent degrees of freedom and the effective parameter space described by $(\lambda \left(U_j \right), \Lambda_i \left(U^{(j)} \right))$.

20.4 Projected operators for fixed value of λ

As in the first part, we can consider the linear span of the states such that $\lambda(U_j) = \lambda$ and project operators along the basis (115). Starting with an operator:

$$\left[\left(\hat{U}^{(i)} \right)' \mid \left(\hat{U}^{(j)} \right)' \right] \Psi \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right]$$

it can be expressed in a basis corresponding to the projection operator. Introducing the change of basis:

$$\begin{aligned} & \left| \left(U_j / [U_j], [U_j] \left(\{ \mathbf{\Lambda}_i (U^{(j)}) \} \right), \lambda(U_j) \right) \right\rangle \left| \lambda(U_j), \mathbf{\Lambda}_i (U^{(j)}) \right\rangle \\ &= \int g \left(U_j, \lambda(U_j), \mathbf{\Lambda}_i (U^{(j)}), \hat{U}^{(i)}, \hat{U}^{(j)} \right) \left| \hat{U}^{(i)} \mid \hat{U}^{(j)} \right\rangle d\hat{U}^{(i)} d\hat{U}^{(j)} \end{aligned}$$

we can write:

$$\begin{aligned} & \left[\left(\hat{U}^{(i)} \right)' \mid \left(\hat{U}^{(j)} \right)' \right] \Psi \left[\hat{U}^{(i)} \mid \hat{U}^{(j)} \right] \tag{116} \\ &= \int g \left(U_j, \lambda, \mathbf{\Lambda}_i (U^{(j)}), \left(\hat{U}^{(i)} \right)', \left(\hat{U}^{(j)} \right)' \right) \\ & \quad \times \left[\left\{ \mathbf{\Lambda}_i (U^{(j')}) \right\}, \lambda \mid U_{j'} / [U_{j'}], \left[U_{j'} \left(\{ \mathbf{\Lambda}_i (U^{(j')}) \} \right) \right] \right] \Psi \left[\left\{ \mathbf{\Lambda}_i (U^{(j)}) \right\}, \lambda \mid U_j / [U_j], \left[U_j \left(\{ \mathbf{\Lambda}_i (U^{(j)}) \} \right) \right] \right] \\ & \quad \times \int g^\dagger \left(U_{j'}, \lambda(U_{j'}), \mathbf{\Lambda}_i (U^{(j)}), \hat{U}^{(i)}, \hat{U}^{(j)} \right) d \left(\hat{U}^{(i)} \right)' d \left(\hat{U}^{(j)} \right)' d\hat{U}^{(i)} d\hat{U}^{(j)} d\lambda \end{aligned}$$

Formula (116) defines an operator:

$$\left[\left\{ \mathbf{\Lambda}_i (U^{(j')}) \right\}, \lambda \mid U_{j'} / [U_{j'}], \left[U_{j'} \left(\{ \mathbf{\Lambda}_i (U^{(j')}) \} \right) \right] \right] \Psi \left[\left\{ \mathbf{\Lambda}_i (U^{(j)}) \right\}, \lambda \mid U_j / [U_j], \left[U_j \left(\{ \mathbf{\Lambda}_i (U^{(j)}) \} \right) \right] \right] \tag{117}$$

acting on stts such that $\lambda(U_j) = \lambda$.

In the sequel, we will consider the notation:

$$U_j / [U_j], \left[U_j \left(\{ \mathbf{\Lambda}_i (U^{(j)}) \} \right) \right] = U_j / [U_j], [U_j]$$

to depict the decomposition of U_j in which $[U_j (\{ \mathbf{\Lambda}_i (U^{(j)}) \})]$ is the part of U_j rewritten using the constrnts as a fnction of $\{ \mathbf{\Lambda}_i (U^{(j)}) \}$ and $U_j / [U_j]$ are the remaining free parameters. As a consequence, the operator rewrites:

$$\left[\left\{ \mathbf{\Lambda}_i (U^{(j')}) \right\}, \lambda \mid U_{j'} / [U_{j'}], [U_{j'}] \right] \Psi \left[\left\{ \mathbf{\Lambda}_i (U^{(j)}) \right\}, \lambda \mid U_j / [U_j], [U_j] \right]$$

Consider that the eigenstates of operators $\hat{\mathbf{\Lambda}}_i (U^{(j)})$ generate:

$$\text{vect} \left\{ \left| \lambda(U_j), \mathbf{\Lambda}_i (U^{(j)}) \right\rangle \right\}$$

so that a change of variable

$$\begin{aligned} \left\{ \mathbf{\Lambda}_i (U^{(j')}) \right\} & \rightarrow \{ \mathbf{\Lambda}'_i \} \\ \left\{ \mathbf{\Lambda}_i (U^{(j)}) \right\} & \rightarrow \{ \mathbf{\Lambda}_i \} \end{aligned}$$

allows to consider the parameters $\{\Lambda'_i\}$ and $\{\Lambda_i\}$ as exogeneous variables. The change of variable is performed by integrals:

$$\begin{aligned}
& \left[\{\Lambda'_i\}, \lambda \mid U_{j'}/[U_{j'}], [U_{j'}] \right] \Psi \left[\{\Lambda_i\}, \lambda \mid U_j/[U_j], [U_j] \right] \\
= & \int h^\dagger \left(\lambda \left(U^{(j')} \right), \Lambda_i \left(U^{(j')} \right), \Lambda'_i \right) \\
& \times \left[\left\{ \Lambda_i \left(U^{(j')} \right) \right\}, \lambda \mid U_{j'}/[U_{j'}], [U_{j'}] \right] \Psi \left[\left\{ \Lambda_i \left(U^{(j)} \right) \right\}, \lambda \mid U_j/[U_j], [U_j] \right] \\
& \times h \left(\lambda \left(U^{(j)} \right), \Lambda_i \left(U^{(j)} \right), \Lambda_i \right)
\end{aligned} \tag{118}$$

where $h \left(\lambda \left(U^{(j)} \right), \Lambda_i \left(U^{(j)} \right), \Lambda_i \right)$ performs the change of basis, see (108).

20.5 Remark:

Similarly to part one, the previous approach can be generalized to a multiple decomposition of the initial parameters space: $[U^{(i)} \mid U^{(j)} \mid U^{(k)} \dots]$. Some stats are prjctd, and some constraints V_{ik} between $U^{(i)}$, $U^{(k)}$ translates on the projected states between the remaining degrees of freedom and the parameters arising in the projection.

21 Transformation $\left| U_0^{(j)} \right\rangle \rightarrow \left| U^{(j)} \right\rangle$ and amplitudes

21.1 States description

We start with the states where part of the $U^{(i)}$ are written as functions of $\{\bar{\Lambda}_i\}$:

$$\left| \lambda \left(U^{(j)} \right), \{\bar{\Lambda}_i\} \right\rangle_{U^{(j)}/\{\bar{\Lambda}_i\}, [U^{(j)}(\{\Lambda_i\})]} = \left| \lambda \left(U^{(j)} \right), \{\bar{\Lambda}_i\} \right\rangle_{h(\{\bar{\Lambda}_i\})_{U^{(i)}}} \tag{119}$$

where as before $[U^{(j)}(\{\bar{\Lambda}_i\})]$ represents the degrees of freedom of $U^{(j)}$ that can be expressd as function of $\{\bar{\Lambda}_i\}$. The subscript $U^{(i)}$ reminds that

$$\left| h(\{\bar{\Lambda}_i\})_{U^{(i)}} \right\rangle = \left| U^{(j)}/\{\bar{\Lambda}_i\}, [U^{(j)}(\{\Lambda_i\})] \right\rangle$$

depends on $U^{(j)}$ through the constraints and the free parameters $U^{(j)}/\{\bar{\Lambda}_i\}$.

Note that the identification is only local in general, due to the constraints sets $\{\bar{\Lambda}_i\}$ and states $h(\{\bar{\Lambda}_i\})$ are not independent. This impacts the amplitudes between two states with different parameters. We will use this remark below.

Gathering states such that $\lambda(U^{(j)}) = \lambda$ the state writes:

$$\mathcal{H}_{(\lambda, \{\bar{\Lambda}_i\}(U_\lambda^{(j)}), U_\lambda^{(j)})} = \left\{ \left| \lambda, \{\bar{\Lambda}_i\} \right\rangle_{h(\{\bar{\Lambda}_i\})_{U^{(i)}}} \right\}_{\lambda=\lambda(U^{(j)})}$$

For later purpose, we consider a decomposition:

$$U^{(j)} \rightarrow \left(U^{(j)/p}, U^{(j)p} \right)$$

which is preserved for the eigenstates parametrs:

$$\{\bar{\Lambda}_i\} = \left(\{\bar{\Lambda}_{i,p}\}, \{\bar{\Lambda}_i\}_{/p} \right)$$

so that the state $|h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}}$ of this space decomposes as:

$$|\lambda, \{\bar{\Lambda}_i\}\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} = |\lambda, \{\bar{\Lambda}_i\}_{/p}\rangle |h_{/p}(\{\bar{\Lambda}_i\}_{/p})\rangle_{U^{(i)}} |\{\bar{\Lambda}_{i,p}\}\rangle |h_p(\{\bar{\Lambda}_{i,p}\})\rangle_{U^{(i)}} \quad (120)$$

Subspace spanned by states (120) are series expansion:

$$\int g\left(\lambda, \{\bar{\Lambda}_i\}_{/p}, \bar{\Lambda}_{i,p}, h(\{\bar{\Lambda}_i\}_{/p}), h(\{\bar{\Lambda}_{i,p}\})\right) |\lambda, \{\bar{\Lambda}_i\}_{/p}\rangle |h(\{\bar{\Lambda}_i\}_{/p})\rangle_{U^{(i)}} |\{\bar{\Lambda}_{i,p}\}\rangle |h(\{\bar{\Lambda}_{i,p}\})\rangle_{U^{(i)}} \quad (121)$$

States in (120) decompose the system between few degrees of freedom and backgrnd.

21.2 Transformation $|U_0^{(j)}\rangle \rightarrow |U^{(j)}\rangle$

As in first and second part, we aim at consider transitions between two stts wth different values of λ . Our assumption is that

for $U_0^{(j)}$ and $U^{(j)}$ there exists:

$$T_{\lambda_0\lambda} : \mathcal{H}_{(\lambda_0, \{\{\bar{\Lambda}_i\}_{U_{\lambda_0}^{(j)}}, U_{\lambda_0}^{(j)}\})} \simeq \mathcal{H}_{(\lambda, \{\{\bar{\Lambda}_i\}_{U_{\lambda}^{(j)}}, U_{\lambda}^{(j)}\})}$$

that is, the transformation $T_{\lambda_0\lambda}$:

$$T_{\lambda_0\lambda} : \left\{ |\lambda(U_0^{(j)}), \bar{\Lambda}_i\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U_0^{(j)}} \right\}_{\lambda_0} \rightarrow \left\{ |\lambda(U^{(j)}), \bar{\Lambda}_i\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} \right\}_{\lambda}$$

21.3 Amplitudes of transitions

Given that the states for which we compute the transitions are analog to those in the first part, the derivation follows the same stp.

We consider the infinitesimal transition $\delta T_{\lambda'\lambda}$.

Given that $\delta T_{\lambda'\lambda}$ transforms states from one spaces to another space, the transition include some transport operator. In addition, the transition depends on the operator acting on the set wth gnvl λ . As a consequence, the transition depends on two parts.

The transport, involves "derivatives" $\{\hat{\Lambda}_i\}_{\lambda'} - \{\hat{\Lambda}_i\}_{\lambda} + A(\{\hat{\Lambda}_i\}_{\lambda})$, the connectn $A(\{\hat{\Lambda}_i\}_{\lambda})$ correcting the fact that $\{\hat{\Lambda}_i\}_{\lambda'}$ nd $\{\hat{\Lambda}_i\}_{\lambda}$ do not act on the same space.

The second part of the transition depends on operators $\{\hat{\Lambda}_i\}$ and $\{\Pi_{\bar{\Lambda}_i}\}$ acting with multiplication by $\bar{\Lambda}_i$ and differentiation by $\bar{\Lambda}_i$:

$$\begin{aligned} \hat{\Lambda}_i(\lambda) |\lambda, \{\bar{\Lambda}_i\}\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} &= \bar{\Lambda}_i |\lambda, \{\bar{\Lambda}_i\}\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} \\ \Pi_{\bar{\Lambda}_i}(\lambda) |\lambda, \{\bar{\Lambda}_i\}\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} &= \nabla_{\bar{\Lambda}_i} |\lambda, \{\bar{\Lambda}_i\}\rangle |h(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} \end{aligned}$$

However, the second action also has to be corrected due to the constraints. Modifying $\{\bar{\Lambda}_i\} \rightarrow \{\bar{\Lambda}_i\} + \delta\{\bar{\Lambda}_i\}$ through $\nabla_{\bar{\Lambda}_i}$ modifies the state $h(\{\bar{\Lambda}_i\})$ and should be modified including a covariant derivative.

We present the technical derivation of the transition in the next paragraphs, starting with the case where the parameters are global, and then consider the implications of only local identification.

21.3.1 Global identification

We assume in first approximation that the parameters $\bar{\Lambda}_i$ are global. The amplitudes

$$\langle \lambda + \delta\lambda, \{\bar{\Lambda}_i\}' \big|_{U^{(j)}} \langle h'(\{\bar{\Lambda}_i\}') \big| \delta T_{\lambda\lambda+\delta\lambda} | \lambda, \{\bar{\Lambda}_i\} \rangle | h(\{\bar{\Lambda}_i\}) \rangle_{U^{(j)}} \quad (122)$$

are computed by pulling back the field state:

$$\langle \lambda + \delta\lambda, \{\bar{\Lambda}_i\}' \big|_{U^{(j)}} \langle h'(\{\bar{\Lambda}_i\}') \big| \rightarrow \langle \lambda, \{\bar{\Lambda}_i\}' \big|_{U^{(j)}} \langle h(\{\bar{\Lambda}_i\}') \big|$$

from $\mathcal{H}_{(\lambda+\delta\lambda, \{\bar{\Lambda}_i\}'(U_{\lambda+\delta\lambda}^{(j)}, U_{\lambda+\delta\lambda}^{(j)})}$ to $\mathcal{H}_{(\lambda, \{\bar{\Lambda}_i\}'(U_\lambda^{(j)}, U_\lambda^{(j)})}$ by some parallel transport operator $P_{\lambda\lambda+\delta\lambda}$ and then computing the matrix elements of:

$$P_{\lambda\lambda+\delta\lambda} \delta T_{\lambda\lambda+\delta\lambda}$$

Assuming that the parameters $\{\bar{\Lambda}_i\}$ are global, the transport is trivial: we can identify the various spaces, the transport involves only derivatives.

Formally, it transports $h(\{\bar{\Lambda}_i\}(\lambda + \delta\lambda))$ to $h(\{\bar{\Lambda}_i\}(\lambda))$ and the parallel transport is generated by the operator:

$$(\{\bar{\Lambda}_i(\lambda + \delta\lambda)\} - \{\bar{\Lambda}_i(\lambda)\}) \frac{\delta}{\delta \{\bar{\Lambda}_i(\lambda)\}} = \delta\lambda \frac{\partial \{\bar{\Lambda}_i(\lambda)\}}{\partial \lambda} \frac{\delta}{\delta \{\bar{\Lambda}_i(\lambda)\}}$$

and by exponentiation, the transport is:

$$P_{\lambda\lambda+\delta\lambda} = \exp \left(\int i\delta\lambda \nabla_\lambda \{\bar{\Lambda}_i(\lambda)\} \frac{\delta}{\delta \{\bar{\Lambda}_i(\lambda)\}} \mathcal{D} \{\bar{\Lambda}_i(\lambda)\} \right)$$

Inserting this operator in the matrix elements of an arbitrary expression $\exp(iF(\Pi_{\{\bar{\Lambda}_i(\lambda)\}}))$, the saddle point computation allows to replace:

$$\Pi_{\{\bar{\Lambda}_i(\lambda)\}} \rightarrow \frac{\partial \{\bar{\Lambda}_i(\lambda)\}}{\partial \lambda}$$

As a consequence the transition operator involves the variables $\bar{\Lambda}_i(\lambda)$ and $\frac{\partial \{\bar{\Lambda}_i(\lambda)\}}{\partial \lambda}$. By exponentialization of infinitesimal matrix elements leads to transitions:

$$\begin{aligned} & \langle \lambda, \{\bar{\Lambda}_i\}' \big|_{U^{(j)}} \langle h(\{\bar{\Lambda}_i\}') \big| \exp \left(i \int_{\lambda(U^{(j)})_0}^{\lambda(U^{(j)})} L \left(\{\bar{\Lambda}_i(\lambda)\}, \left\{ \frac{\partial \{\bar{\Lambda}_i(\lambda)\}}{\partial \lambda} \right\} \right) d\lambda \right) | \lambda_0, \{\bar{\Lambda}_i\} \rangle | h_0(\{\bar{\Lambda}_i\}) \rangle_{U_0^{(j)}} \\ &= \int_{U^{(j)}} \langle h(\{\bar{\Lambda}_i\}') \big| \int \exp \left(i \int_{\{\bar{\Lambda}_i(\lambda_0)\}=\{\bar{\Lambda}_i\}}^{\{\bar{\Lambda}_i(\lambda)\}=\{\bar{\Lambda}_i\}'} L \left(\{\bar{\Lambda}_i(\lambda)\}, \left\{ \frac{\partial \{\bar{\Lambda}_i(\lambda)\}}{\partial \lambda} \right\} \right) d\lambda \right) \\ & \quad \times \mathcal{D} \bar{\Lambda}_i(\lambda) | h_0(\{\bar{\Lambda}_i\}) \rangle_{U_0^{(j)}} d \{\bar{\Lambda}_i\}' d \{\bar{\Lambda}_i\} \end{aligned} \quad (123)$$

21.3.2 Local identification

Again the amplitudes:

$$\langle \lambda + \delta\lambda, \{\bar{\Lambda}_i\}' \big|_{U^{(j)}} \langle h'(\{\bar{\Lambda}_i\}') \big| \delta T_{\lambda\lambda+\delta\lambda} | \lambda, \{\bar{\Lambda}_i\} \rangle | h(\{\bar{\Lambda}_i\}) \rangle_{U^{(j)}} \quad (124)$$

are computed by composing some parallel transport operator $P_{\lambda\lambda+\delta\lambda}$ and then computing the matrix elements of:

$$P_{\lambda\lambda+\delta\lambda}\delta T_{\lambda\lambda+\delta\lambda}$$

As in the global case, the transport $P_{\lambda\lambda+\delta\lambda}$ is performed through generator of translation operator:

$$\left(\{\bar{\Lambda}_i(\lambda + \delta\lambda)\} - \{\bar{\Lambda}_i(\lambda)\}\right) \frac{\delta}{\delta\{\bar{\Lambda}_i(\lambda)\}}$$

However, since $\bar{\Lambda}'_i(\lambda + \delta\lambda)$ and $\bar{\Lambda}_i(\lambda)$ do not act on the same space, their difference is not the derivative of a set of variable $\bar{\Lambda}_i(\lambda)$ that be defined for every λ . This derivative has to be corrected to account for the change in spaces the modification of λ accounts for.

Given the constraints:

$$\begin{aligned} & \left(\{\bar{\Lambda}'_i(\lambda + \delta\lambda)\} - \{\bar{\Lambda}_i(\lambda)\}\right) \\ &= \delta\lambda \left(\frac{\partial\{\bar{\Lambda}_i(\lambda)\}}{\partial\lambda} + (M\{\bar{\Lambda}_i(\lambda)\}) \right) = \delta\lambda \underline{\nabla}_\lambda \{\bar{\Lambda}_i(\lambda)\} \end{aligned}$$

and the transport writes:

$$P_{\lambda\lambda+\delta\lambda} = \exp\left(\int i\delta\lambda \underline{\nabla}_\lambda \{\bar{\Lambda}_i(\lambda)\} \frac{\delta}{\delta\{\bar{\Lambda}_i(\lambda)\}} \mathcal{D}\{\bar{\Lambda}_i(\lambda)\}\right)$$

The matrix elements of $\frac{\delta}{\delta\{\bar{\Lambda}_i(\lambda)\}}$ are computed using dual basis, involving amplitudes of the form:

$$\exp\left(i\left(\{\bar{\Lambda}_i(\lambda)\}' - \{\bar{\Lambda}_i(\lambda)\}\right) \Pi_{\{\bar{\Lambda}_i(\lambda)\}}\right)$$

where $\Pi_J(U_j/[U_j], \{\hat{\mathbf{A}}^{[ki]}\}, \lambda)$ are elements of dual basis. By parallel transport back to $\lambda + \delta\lambda$ we can write:

$$\{\bar{\Lambda}_i(\lambda)\}' \rightarrow \{\bar{\Lambda}_i(\lambda + \delta\lambda)\}'$$

so that matrices elements involving $\frac{\delta}{\delta\bar{\Lambda}_i(\lambda)}$ write:

$$\exp\left(i\delta\lambda \underline{\nabla}_\lambda \{\bar{\Lambda}_i(\lambda)\} \Pi_{\{\bar{\Lambda}_i(\lambda)\}}\right)$$

The matrix contribution of any operator depending on $\Pi_{\{\bar{\Lambda}_i(\lambda)\}}$ is:

$$\exp\left(i\delta\lambda \underline{\nabla}_\lambda \{\bar{\Lambda}_i(\lambda)\} \Pi_{\{\bar{\Lambda}_i(\lambda)\}}\right) \exp\left(iF\left(\Pi_{\{\bar{\Lambda}_i(\lambda)\}}\right)\right)$$

obtained by saddle point equation:

$$\Pi_{\{\bar{\Lambda}_i(\lambda)\}} = \underline{\nabla}_\lambda \{\bar{\Lambda}_i(\lambda)\} + \text{something including the field} \quad (125)$$

Moreover, matrix elements of $\hat{\mathbf{A}}_i(\lambda)$ involve terms:

$$\{\bar{\Lambda}_i + \delta\bar{\Lambda}_i\} - \{\bar{\Lambda}_i\}$$

since in amplitudes $\hat{\mathbf{A}}_i(\lambda)$ acts on different values of $\hat{\mathbf{A}}_i(\lambda)$, see (124). Due to constraints this variation in parameters $\{\bar{\Lambda}_i\}$ involves a change in the state, so that the matrix element $\hat{\mathbf{A}}_i(\lambda)$ between two states evaluated at $\{\bar{\Lambda}_i + \delta\bar{\Lambda}_i\}$ and $\{\bar{\Lambda}_i\}$ induces a contribution:

$$\begin{aligned}
h(\{\bar{\Lambda}_i + \delta\bar{\Lambda}_i\}, \lambda) - h(\{\bar{\Lambda}_i\}, \lambda) &= \delta\bar{\Lambda}_i \left(\frac{\delta}{\delta\{\bar{\Lambda}_i(\lambda)\}} h(\{\bar{\Lambda}_i\}, \lambda) + A_{\bar{\Lambda}_i(\lambda)} h(\{\bar{\Lambda}_i\}, \lambda) \right) \quad (126) \\
&= \delta\bar{\Lambda}_i \nabla_{\{\bar{\Lambda}_i(\lambda)\}} h(\{\bar{\Lambda}_i\}, \lambda)
\end{aligned}$$

Gathering (125) and (126), we find that the infinitesimal transitions have the form:

$$L\left(\{\bar{\Lambda}_i(\lambda)\}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}}\right\}\right)$$

where the $\{\bar{\Lambda}_i(\lambda)\}$ are independent sts of variables $\{\bar{\Lambda}_i\}$, one for each λ and $(\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}})$ are defined in (125) and (126). Exponentiating and composing the infinitesimal transformations yields the amplitudes:

$$\begin{aligned}
&\langle \lambda, \{\bar{\Lambda}_i\}' |_{U^{(j)}} \langle h(\{\bar{\Lambda}_i\}') | \exp\left(i \int_{\lambda((U^{(j)})_0)}^{\lambda(U^{(j)})} L\left(\{\bar{\Lambda}_i(\lambda)\}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}}\right\}\right) d\lambda \right) \rangle_{U^{(j)}} \quad (127) \\
&|\lambda_0, \{\bar{\Lambda}_i\}\rangle |h_0(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} \\
&= \int_{U^{(j)}} \langle h(\{\bar{\Lambda}_i\}') | \int \exp\left(i \int_{\{\bar{\Lambda}_i(\lambda_0)\}=\{\bar{\Lambda}_i\}}^{\{\bar{\Lambda}_i(\lambda)\}=\{\bar{\Lambda}_i\}'} L\left(\{\bar{\Lambda}_i(\lambda)\}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}}\right\}\right) d\lambda \right) \\
&\quad \times \mathcal{D}\bar{\Lambda}_i(\lambda) |h_0(\{\bar{\Lambda}_i\})\rangle_{U^{(j)}} d\{\bar{\Lambda}_i\}' d\{\bar{\Lambda}_i\}
\end{aligned}$$

For states (120) the system decomposes as:

$$\begin{aligned}
&\langle \lambda, \{\bar{\Lambda}_i\}'_{/p} |_{U^{(j)}} \langle h(\{\bar{\Lambda}_{i,p}\}') | \exp\left(i \int_{\lambda((U^{(j)})_0)}^{\lambda(U^{(j)})} L\left(\{\bar{\Lambda}_i(\lambda)\}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}}\right\}\right) d\lambda \right) \rangle_{U^{(j)}} \quad (128) \\
&|\lambda_0, \{\bar{\Lambda}_i\}'_{/p}\rangle |h_0(\{\bar{\Lambda}_{i,p}\}')\rangle_{U^{(j)}} \\
&= \int_{U^{(j)}} \langle \lambda, \{\bar{\Lambda}_i\}'_{/p} |_{U^{(j)}} \int \exp\left(i \int_{\{\bar{\Lambda}_i(\lambda_0)\}=\{\bar{\Lambda}_i\}}^{\{\bar{\Lambda}_i(\lambda)\}=\{\bar{\Lambda}_i\}'} L\left(\{\bar{\Lambda}_i(\lambda)\}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}}\right\}\right) d\lambda \right) |\lambda_0, \{\bar{\Lambda}_i\}'_{/p}\rangle \\
&\quad \times \mathcal{D}\bar{\Lambda}_i(\lambda) d\{\bar{\Lambda}_i\}'_{/p} d\{\bar{\Lambda}_{i,p}\}' d\{\bar{\Lambda}_{i,p}\} d\{\bar{\Lambda}_i\} \quad (129)
\end{aligned}$$

and:

$$\begin{aligned}
&L\left(\{\bar{\Lambda}_i(\lambda)\}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}}\right\}\right) \\
&= L_{/p}\left(\{\bar{\Lambda}_i\}'_{/p}, \left\{\nabla_\lambda \{\bar{\Lambda}_i(\lambda)\}, \nabla_{\{\bar{\Lambda}_i\}'_{/p}}\right\}\right) + L_p\left(\{\bar{\Lambda}_i\}_p, \left\{\nabla_\lambda \{\bar{\Lambda}_{i,p}(\lambda)\}, \nabla_{\{\bar{\Lambda}_i\}_p}\right\}\right) \\
&\quad + L_{/p,p}\left(\{\bar{\Lambda}_i\}'_{/p}, \left\{\nabla_\lambda \{\bar{\Lambda}_{i/p}(\lambda)\}, \nabla_{\{\bar{\Lambda}_i\}'_{/p}}\right\}, \left\{\nabla_\lambda \{\bar{\Lambda}_{i,p}(\lambda)\}, \nabla_{\{\bar{\Lambda}_i\}_p}\right\}\right)
\end{aligned}$$

The first term describes the main part of the system acting as a background, the second one, the subsystem partly isolated from the background, while the third one represents the interaction between them. Note that the entire system determines the covariant derivatives.

21.3.3 Field formulation of the transition

In the previous paragraphs we have considered a "first quantized" formalism, where the number of states is preserved. More generally we can consider the transitions to depend on the field from which states derive:

$$\Psi_J \left(U^{(j)}/\{\hat{\Lambda}_i\}, \left[U^{(j)}(\{\Lambda_i\}) \right], \lambda \right)$$

A derivation similar to that presented in part 1, shows that the transitions are generated by:

$$\left\langle \lambda, \{\bar{\Lambda}_i\}'_{/p} \middle|_{U^{(j)}} \left\langle h(\{\bar{\Lambda}_{i,p}\}') \middle| \exp \left(i \int_{\lambda((U^{(j)})_0)}^{\lambda(U^{(j)})} L(\Psi_J) d\lambda \right) \middle| \lambda_0, \{\bar{\Lambda}_i\}_{/p} \right\rangle | h_0(\{\bar{\Lambda}_{i,p}\}) \rangle_{U_0^{(j)}}$$

where:

$$\begin{aligned} & L(\Psi_J) \\ &= S \left(\left(\nabla_{\lambda}, \nabla_{\{\bar{\Lambda}_i(\lambda)\}} \right) \Psi_J \left(U^{(j)}/\{\hat{\Lambda}_i\}, \left[U^{(j)}(\{\Lambda_i\}) \right], \lambda \right), \Psi_J \left(U^{(j)}/\{\hat{\Lambda}_i\}, \left[U^{(j)}(\{\Lambda_i\}) \right], \lambda \right) \right) \end{aligned}$$

21.4 Transitions for the operators:

21.4.1 Operatorial form of transition

Under the approximation of global identification the identification $\{\bar{\Lambda}_i\}$, the spaces $\mathcal{H}_{(\lambda_0, \{\{\bar{\Lambda}_i\}(U_{\lambda_0}^{(j)}), U_{\lambda_0}^{(j)})}$ and $\mathcal{H}_{(\lambda, \{\{\bar{\Lambda}_i\}(U_{\lambda}^{(j)}), U_{\lambda}^{(j)})}$ can be identified. so that the transitions can be seen as an operator of the global parameters $\bar{\Lambda}_i, \Pi_{\{\bar{\Lambda}_i\}}$:

$$T_{\lambda_0\lambda} = \exp \left(i(\lambda - \lambda_0) V \left(\{\hat{\Lambda}_i\}, \Pi_{\{\hat{\Lambda}_i\}} \right) \right) \quad (130)$$

such that the transitions:

$$\left\langle \lambda, \{\bar{\Lambda}_i\}' \middle|_{U^{(j)}} \left\langle h(\{\bar{\Lambda}_i\}') \middle| \exp \left(i(\lambda - \lambda_0) V \left(\{\hat{\Lambda}_i\}, \Pi_{\{\hat{\Lambda}_i\}} \right) \right) \middle| \lambda_0, \{\bar{\Lambda}_i\} \right\rangle | h_0(\{\bar{\Lambda}_i\}) \rangle_{U_0^{(j)}} \quad (131)$$

are equal to (123). This is achieved by the usual change of basis: at the infinitesimal level, the matrix elements of a term $\exp \left(i\delta\lambda F \left(\Pi_{\{\hat{\Lambda}_i\}} \right) \right)$ is computed as:

$$\int \exp \left(i\delta\lambda \left(\{\bar{\Lambda}_i\}' - \{\bar{\Lambda}_i\} \right) \pi_{\{\bar{\Lambda}_i\}} \right) \exp \left(i\delta\lambda F \left(\pi_{\{\bar{\Lambda}_i\}} \right) \right) d\pi_{\{\bar{\Lambda}_i\}}$$

where $\pi_{\{\bar{\Lambda}_i\}}$ are eigenstates of $\Pi_{\{\bar{\Lambda}_i(\lambda)\}}$. These integral lead to replace $\pi_{\{\bar{\Lambda}_i\}} \rightarrow \frac{\partial \{\bar{\Lambda}_i(\lambda)\}}{\partial \lambda}$ and transitions of the form (123).

21.4.2 Transition for operators in the eignstates basis

We can use (130) to rewrite operators as a function from of their value at a given λ_0 .

Using the form (119) for states, formula (118) for operators:

$$\left[\{\Lambda'_i\}, \lambda \middle| h(\{\bar{\Lambda}'_i\}) \right]_{U^{(j)'}} \Psi \left[\{\Lambda_i\}, \lambda \middle| U_j / [U_j], [U_j] \right] = \left[\{\Lambda'_i\}, \lambda \middle| U_{j'} / [U_{j'}], [U_{j'}] \right] \Psi \left[\{\Lambda_i\}, \lambda \middle| h(\{\bar{\Lambda}_i\}) \right]_{U^{(j)}}$$

along with (131), the transition for operators is:

$$\begin{aligned}
& [\{\mathbf{\Lambda}'_i\}, \lambda | U_{j'}/[U_{j'}], [U_{j'}]] \Psi [\{\mathbf{\Lambda}_i\}, \lambda | U_j/[U_j], [U_j]] \\
= & \exp\left(-i(\lambda - \lambda_0) V\left(\{\hat{\mathbf{\Lambda}}_i\}, \Pi_{\{\hat{\mathbf{\Lambda}}_i\}}\right)\right) \\
& \times \left[\{\mathbf{\Lambda}'_i\}, \lambda_0 | h\left(\{\bar{\mathbf{\Lambda}}'_i\}\right)_{U^{(j')}}\right] \Psi \left[\{\mathbf{\Lambda}_i\}, \lambda_0 | h\left(\{\bar{\mathbf{\Lambda}}_i\}\right)_{U^{(j)}}\right] \\
& \times \exp\left(i(\lambda - \lambda_0) V\left(\{\hat{\mathbf{\Lambda}}_i\}, \Pi_{\{\hat{\mathbf{\Lambda}}_i\}}\right)\right)
\end{aligned}$$

If we use the decomposition (120), the projection of this operator on the states:

$$\left| \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p} \right\rangle \left| h_{/p}\left(\{\bar{\mathbf{\Lambda}}_i\}_{/p}\right) \right\rangle_{U^{(j)}} \left| \{\bar{\mathbf{\Lambda}}_{i,p}\} \right\rangle \left| h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) \right\rangle_{U^{(j)}}$$

we find an operator:

$$\left[\lambda, \{\bar{\mathbf{\Lambda}}'_i\}_{/p}, \{\bar{\mathbf{\Lambda}}'_{i,p}\}, h_{/p}\left(\{\bar{\mathbf{\Lambda}}'_i\}_{/p}\right), h_p\left(\{\bar{\mathbf{\Lambda}}'_{i,p}\}\right) \right] \Psi \left[\lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_{/p}\left(\{\bar{\mathbf{\Lambda}}_i\}_{/p}\right), h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) \right]$$

If we consider an operator depending only on $h\left(\{\bar{\mathbf{\Lambda}}'_{i,p}\}\right)$, and which is identity over the other compnents, we obtain a decomposition:

$$\left| \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_{/p}\left(\{\bar{\mathbf{\Lambda}}'_i\}_{/p}\right) \right\rangle \left| \{\bar{\mathbf{\Lambda}}_{i,p}\} \right\rangle \left[h_p\left(\{\bar{\mathbf{\Lambda}}'_{i,p}\}, \left(U^{(i)'/p}\right)\right) \right] \Psi \left[h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}, \left(U^{(i)/p}\right)\right) \right] \langle \{\bar{\mathbf{\Lambda}}_{i,p}\} | \langle \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) |$$

Moreover, if the transition operator respects the decomposition:

$$\begin{aligned}
& \exp\left(i(\lambda - \lambda_0) V\left(\{\hat{\mathbf{\Lambda}}_i\}, \Pi_{\{\hat{\mathbf{\Lambda}}_i\}}\right)\right) \\
= & \exp\left(i(\lambda - \lambda_0) \left(V_1\left(\{\bar{\mathbf{\Lambda}}_i\}_{/p}, \Pi_{\{\bar{\mathbf{\Lambda}}_i\}_{/p}}\right) + V_2\left(\{\bar{\mathbf{\Lambda}}_i\}_p, \Pi_{\{\bar{\mathbf{\Lambda}}_i\}_p}\right) \right)\right)
\end{aligned}$$

and that globally $\left| \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_{/p}\left(\{\bar{\mathbf{\Lambda}}'_i\}_{/p}\right) \right\rangle \left| \{\bar{\mathbf{\Lambda}}_{i,p}\} \right\rangle$ is invariant by transition, we are left with:

$$\begin{aligned}
& \left| \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_{/p}\left(\{\bar{\mathbf{\Lambda}}'_i\}_{/p}\right) \right\rangle \left| \{\bar{\mathbf{\Lambda}}_{i,p}\} \right\rangle \left[h_p\left(\{\bar{\mathbf{\Lambda}}'_{i,p}\}\right) \right] \Psi \left[h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) \right] \langle \{\bar{\mathbf{\Lambda}}_{i,p}\} | \langle \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) | \\
= & \left| \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_{/p}\left(\{\bar{\mathbf{\Lambda}}'_i\}_{/p}\right) \right\rangle \left| \{\bar{\mathbf{\Lambda}}_{i,p}\} \right\rangle \exp\left(-i(\lambda - \lambda_0) V_2\left(\{\bar{\mathbf{\Lambda}}_i\}_p, \Pi_{\{\bar{\mathbf{\Lambda}}_i\}_p}\right)\right) \\
& \times \left[h_p\left(\{\bar{\mathbf{\Lambda}}'_{i,p}\}\right) \right] \Psi \left[h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) \right] \\
& \times \exp\left(i(\lambda - \lambda_0) V_2\left(\{\bar{\mathbf{\Lambda}}_i\}_p, \Pi_{\{\bar{\mathbf{\Lambda}}_i\}_p}\right)\right) \langle \{\bar{\mathbf{\Lambda}}_{i,p}\} | \langle \lambda, \{\bar{\mathbf{\Lambda}}_i\}_{/p}, h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) |
\end{aligned}$$

which correspond to the usual definition of fld evolution, with th spc as some inert surrounding in first pprxmtn.

21.4.3 Transformation of operators in the initial basis

Considering only the subsystem wth evolution:

$$\exp\left(-i(\lambda - \lambda_0) V_2\right) \left| \{\bar{\mathbf{\Lambda}}'_{i,p}\} \right\rangle \left[h_p\left(\{\bar{\mathbf{\Lambda}}'_{i,p}\}\right) \right] \Psi \left[h_p\left(\{\bar{\mathbf{\Lambda}}_{i,p}\}\right) \right] \langle \{\bar{\mathbf{\Lambda}}_{i,p}\} | \exp\left(i(\lambda - \lambda_0) V_2\right)$$

and rewrite this operator as a function of the initial basis of operator:

$$\left[\left(\hat{U}^{(i)} \right)' | \left(\hat{U}^{(j)} \right)' \right] \Psi \left[\hat{U}^{(i)} | \hat{U}^{(j)} \right]$$

We can decompose this basis according to our decomposition:

$$U^{(j)} \rightarrow \left(U^{(j)/p}, U^{(j)p} \right)$$

and consider an operator depending only on $U^{(j)p}$, so that it is the identity on other coordinates:

$$\left[\left(U^{(j)p} \right)' \right] \Psi \left[U^{(j)p} \right]$$

The change of basis similar to the inverse transform of (116) writes:

$$\begin{aligned} & [h_p(\{\bar{\Lambda}'_{i,p}\})] \Psi [h_p(\{\bar{\Lambda}_{i,p}\})] \\ = & \int g^\dagger \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}'_{i,p}\}) \right) \left[\left(U^{(j)p} \right)' \right] \Psi \left[U^{(j)p} \right] g \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}_{i,p}\}) \right) d \left(U^{(j)p} \right)' dU^{(j)p} \end{aligned}$$

and using the transitions:

$$\begin{aligned} & |\{\bar{\Lambda}'_{i,p}\}\rangle \int g^\dagger \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}'_{i,p}\}) \right) \left[\lambda, \left(U^{(j)p} \right)' \right] \Psi \left[\lambda, U^{(j)p} \right] g \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}_{i,p}\}) \right) \\ & \times d \left(U^{(j)p} \right)' dU^{(j)p} \langle \{\bar{\Lambda}_{i,p}\} | \\ = & |\{\bar{\Lambda}'_{i,p}\}\rangle \exp(-i(\lambda - \lambda_0) V_2) \\ & \times \int g^\dagger \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}'_{i,p}\}) \right) \left[\left(U^{(j)p} \right)' \right] \Psi \left[U^{(j)p} \right] g \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}_{i,p}\}) \right) d \left(U^{(j)p} \right)' dU^{(j)p} \\ & \times \exp(i(\lambda - \lambda_0) V_2) \langle \{\bar{\Lambda}_{i,p}\} | \end{aligned}$$

where we identified:

$$\left[\left(U^{(j)p} \right)' \right] \Psi \left[U^{(j)p} \right] = \left[\lambda_0, \left(U^{(j)p} \right)' \right] \Psi \left[\lambda_0, U^{(j)p} \right]$$

The usual transformation corresponds to the case where V is diagonal in the $U^{(j)p}$:

$$\begin{aligned} & \exp(-i(\lambda(U^{(j)}) - \lambda_0) V) \left[\left(U^{(j)p} \right)' \right] \Psi_0 \left[U^{(j)p} \right] \exp(i(\lambda(U^{(j)}) - \lambda_0) V) \\ = & \exp(-i(\lambda(U^{(j)}) - \lambda_0) (V \left(\left(U^{(j)p} \right)' \right) - V(U^{(j)p))) \left[\left(U^{(j)p} \right)' \right] \Psi_0 \left[U^{(j)p} \right] \end{aligned}$$

and:

$$\begin{aligned} & |\{\bar{\Lambda}'_{i,p}\}\rangle \int g^\dagger \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}'_{i,p}\}) \right) \left[\lambda, \left(U^{(j)p} \right)' \right] \Psi \left[\lambda, U^{(j)p} \right] \\ & \times g \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}_{i,p}\}) \right) d \left(U^{(j)p} \right)' dU^{(j)p} \langle \{\bar{\Lambda}_{i,p}\} | \\ = & |\{\bar{\Lambda}'_{i,p}\}\rangle \int \exp(-i(\lambda(U^{(j)}) - \lambda_0) (V \left(\left(U^{(j)p} \right)' \right) - V(U^{(j)p))) \left[\left(U^{(j)p} \right)' \right] \\ & \times g^\dagger \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}'_{i,p}\}) \right) \left[\left(U^{(j)p} \right)' \right] \Psi \left[U^{(j)p} \right] g \left(\left(U^{(j)p} \right)', h_p(\{\bar{\Lambda}_{i,p}\}) \right) d \left(U^{(j)p} \right)' dU^{(j)p} \langle \{\bar{\Lambda}_{i,p}\} | \end{aligned}$$

Example For operators:

$$\left[\left(U^{(i)p} \right)' \right] \Psi \left[U^{(i)p} \right] = |U^{(j)p+k}\rangle \langle U^{(j)p}|$$

the conjugate representation writes:

$$|\{\bar{\Lambda}_{i,p+k}\}\rangle \int g^\dagger \left(U^{(j)p+k}, h_{p+k}(\{\bar{\Lambda}_{i,p+k}\}) \right) |U^{(j)p+k}\rangle \langle U^{(j)p} | g \left(U^{(j)p}, h_p(\{\bar{\Lambda}'_{i,p}\}) \right) dU^{(j)p} d \left(U^{(j)p} \right)' \langle \{\bar{\Lambda}_{i,p}\} |$$

In the usual set up, where translation invariance is considered:

$$g^\dagger \left(U^{(j)p+k}, h_{p+k}(\{\bar{\Lambda}_{i,p+k}\}) \right) g \left(U^{(j)p}, h_p(\{\bar{\Lambda}_{i,p}\}) \right) = \exp \left(iU^{(j)p+k/p} \cdot \{\bar{\Lambda}_{i,p+k/p}\} \right)$$

where $\{\bar{\Lambda}_{i,p+k/p}\}$ corresponds to the the degrees of freedom of $\{\bar{\Lambda}_{i,p+k}\}$ except those of $\bar{\Lambda}_{i,p}$.

Consequently, the operator becomes:

$$\left[\lambda^{(i)}, \left(U^{(i)p} \right)' \right] \Psi \left[\lambda, U^{(i)p} \right] = |\{\bar{\Lambda}_{i,p+k}\}\rangle \int \exp \left(iU^{(j)p+k/p} \cdot \{\bar{\Lambda}_{i,p+k/p}\} \right) |U^{(j)p+k}\rangle \langle U^{(j)p} | dU^{(j)p} d \left(U^{(j)p} \right)' \langle \{\bar{\Lambda}_{i,p}\} |$$

similar to a usual field in terms of creation and annihilation operators.

Part IV Further developments

This fourth part serves as a conclusion, presenting various potential developments arising from the present work. We focus on the role of the constraints relating the degeneracy parameters $\{\hat{\Delta}^{[k_i]}\}$ and the state v in the structure of the effective formalism. We posit that this formalism should be locally described by fields on a fibered space. Subsequently, our attention turns to the modifications of the constraints induced by modifications of the apparent state. resulting from changes in the apparent state. From this, we derive linear equations for the modified states similar to some dynamical equations and establish commutation relations between generators of modifications.

22 Collection of several states and parameters spaces

Given that the space of parameters varies with the state ν , there is no field defined globally on the set:

$$\left\{ \left[\begin{array}{c} \left(p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right) \\ \left[\Psi_J [p_\eta, p_{\eta'}] \right], v \end{array} \right] \right\}$$

In fact, we should rather consider collections of objects:

$$\left\{ \left[\begin{array}{c} \left\{ p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right\}, \left\{ \hat{\Delta}^{[k_i]} \right\} \\ \left\{ p_\eta, p_{\eta'} \right\}, v \left(\left\{ p_\eta, p_{\eta'} \right\} \right) \end{array} \right] \right\}$$

defined by sets of clouds of parameters $\{\hat{\Delta}^{[k_i]}\}$. These parameters describe the eigenstates of some families of operators satisfying the constraints:

$$H \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \mathcal{K}_0 \left(\left\{ p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right\}, \left\{ v^{(\eta)} \left(\left\{ p_\eta, p_{\eta'} \right\} \right) \right\} \right) \right) = 0 \quad (132)$$

where H is a vector with infinite number of components, gathering the components equations of (87). $\{v^{(\eta)}(\{p_\eta, p_{\eta'}\})\}$ is a vector of functionals for several remaining subobjects. The kernel $\mathcal{K}_0 \left(\left\{ p_{l_i}, p_{l'_i l_i}, k_{l'_i} \right\}, \left\{ v^{(\eta)} \left(\left\{ p_\eta, p_{\eta'} \right\} \right) \right\} \right)$ is defined by the background states for some degrees of freedom.

The set of constraints (132) defines one or several manifolds $\{V(v^{(n)})\}$ with sets of points $\{\hat{\mathbf{A}}^{[k_i]}\}$ and depending on states $v^{(n)}$. These manifolds do depend on the symmetries conditioned by the states $|\{v^{(n)}(\{p_\eta, p_{\eta'}\})\}\rangle$, so do their dimensions. $\{V_\alpha(v^{(n)})\}$

When local variations $v^{(n)}$ deform continuously $V(v^{(n)})$, and $V(v^{(n)}) \simeq V(v^{(n)} + \delta v^{(n)})$ we can consider that the $V(v^{(n)})$ form a collection of spaces $\{V_\alpha\}$ fibred on the subspaces of the state space defined by the $v^{(n)}$. Locally the V_α are defined by coordinates $(\{\hat{\mathbf{A}}^{[k_i]}\}, v^{(n)})$ solving (132).

Defining fields over a set V_α corresponds to consider state spaces $H(V_\alpha)$ described locally by the fields $\Psi_{v^{(n)}}(\hat{\mathbf{A}}^{[k_i]})$ or $\Psi(v^{(n)}, \hat{\mathbf{A}}^{[k_i]})$. Maps of inclusion between such spaces:

$$V_\beta \rightarrow V_\alpha$$

corresponds to consider degeneracy subspaces, with inclusion of subsets of degeneracy generators $\{\mathbf{L}\}_\beta \subset \{\mathbf{L}\}_\alpha$. Such maps should translate into maps:

$$H(V_\alpha) \rightarrow H(V_\beta)$$

and products of maps:

$$V_\beta \times_{V_{\beta, \beta'}} V_{\beta'} \rightarrow V_\alpha$$

where $V_{\beta, \beta'}$ represents the constraints between both parameters spaces V_β and $V_{\beta'}$ should translate into the decomposition of fields:

$$\Psi_{v^{(n)}}^{k, \alpha}(\hat{\mathbf{A}}_\alpha^{[k_i]}) \rightarrow \sum \Psi_{v^{(n)}}^{k, \beta}(\hat{\mathbf{A}}_\beta^{[k_i]}) \otimes \Psi_{v^{(n)}}^{k, \beta'}(\hat{\mathbf{A}}_{\beta'}^{[k_i]})$$

where the decomposition is for realizations, as described in the first and second part.

The full system should thus be given by the sets $\{V_\alpha\}$ together with maps between these sets and decomposition of fields corresponding to these maps.

In this context, the state $|\{v^{(n)}(\{p_\eta, p_{\eta'}\})\}\rangle$ conditioning the constraints (132) can be interpreted as a type of global state of the system. The set of parameters changes along these states. However, considering small deviations $|\{\delta v^{(n)}(\{p_\eta, p_{\eta'}\})\}\rangle$ from the state $|\{v^{(n)}(\{p_\eta, p_{\eta'}\})\}\rangle$ may maintain the global state by keeping the constraints invariant. In this case, δv represents a small variation of the state, akin to a microstate. In the next section, we examine the impact of the constraints on this variation.

23 Continuous variation of state and constraint modification.

23.1 First order variation

To express the variation of the constraints resulting from a change in state, we refer to (87), disregarding the realization index and the dependence on $(\Psi_j^{\otimes l})$. Additionally, we implicitly consider the dependencies $\{U_i^k\}_i$ and $(\Psi_j), U_j^l$ to formulate the set of constrained variables in a manner analog to (132):

$$H(\{\mathbf{L}\}, \nu) = 0 \tag{133}$$

or, if we have several types of stts as in part II:

$$H(\{\mathbf{L}\}, \{\nu^{(n)}\}) = 0$$

The constraint has been rewritten in terms of generators of parameters stats.

Modifying states $|\{v^{(n)}\}\rangle$ to $|\{v^{(n)} + \delta v^{(n)}\}\rangle$ modifies the symmetry parameters \mathbf{L} . This may both consist in a modification $\mathbf{L} \rightarrow \mathbf{L} + \mathbf{A}\mathbf{L} + \Delta\mathbf{L}$ where \mathbf{A} is a continuous infinitesimal transformation, and $\Delta\mathbf{L}$ is a discrete deformation changing the number of parameters. For instance, we can suppose a deformation $\Delta\mathbf{L}$ generated by some $\Delta\mathbf{L}^+ + \Delta\mathbf{L}^-$ where $\Delta\mathbf{L}^+$ adds a symmetry generator, and $\Delta\mathbf{L}^-$ removes one generator. The modified set operator writes:

$$\mathbf{L} \rightarrow (\mathbf{L}, \Delta\mathbf{L})$$

The $\Delta\mathbf{L}$ have also to be considered as acting on an additional set of eigenstates. This could correspond to the creation or destruction of a parameter. We write the full deformation of paramtrs $\delta\mathbf{L} = \mathbf{A}\mathbf{L} + (1 + \mathbf{A})\Delta\mathbf{L}$. This modifies the constraint (133) as:

$$H\left(\{\mathbf{L} + \delta\mathbf{L}\}, \{v^{(n)} + \delta v^{(n)}\}\right) = 0 \quad (134)$$

Considering the transformation $(1 + \mathbf{A})$, this becomes, applied on states:

$$H\left(\{\mathbf{L} + \delta\mathbf{L}\}, \{v^{(n)} + \delta v^{(n)}\}\right) \left| \left\{ (1 + \mathbf{A}) \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle = 0 \quad (135)$$

Then, using that $\mathbf{L} + \delta\mathbf{L} = (1 + \mathbf{A})(\mathbf{L} + \Delta\mathbf{L})$ and applying a global transformation $(1 - \mathbf{A})$ yields:

$$\begin{aligned} (1 - \mathbf{A}) H\left(\{\mathbf{L} + \delta\mathbf{L}\}, \{v^{(n)} + \delta v^{(n)}\}\right) (1 + \mathbf{A}) (1 - \mathbf{A}) \left| \left\{ (1 + \mathbf{A}) \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle &= 0 \\ H\left(\{\mathbf{L} + \Delta\mathbf{L}\}, \{v^{(n)} + \delta v^{(n)}\}\right) \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| (1 - \mathbf{A}) \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle &= 0 \\ 0 = (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H\left(\left\{ \left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \hat{\mathbf{A}}^{[\Delta_1]} \right\}, \{v^{(n)} + \delta v^{(n)}\}\right) \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle & \end{aligned}$$

We consider a first order expansion of (134) of the form:

$$\begin{aligned} 0 &= (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)} + \delta v^{(n)}\}\right) \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle \\ &+ (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(n)}}\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)}\}\right) \delta v^{(n)} H\left(\hat{\mathbf{A}}^{[\Delta_1]}\right) \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle \end{aligned} \quad (136)$$

where $H\left(\hat{\mathbf{A}}^{[\Delta_1]}\right)$ encompasses the action of the modified parameters $\hat{\mathbf{A}}^{[\Delta_1]}$. For the sake of simplicity, we have assumed a separable form for this action.

We then write the first order expansion of $H\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)} + \delta v^{(n)}\}\right)$ in $\delta v^{(n)}$ as:

$$H\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)} + \delta v^{(n)}\}\right) = H\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)}\}\right) + H_{\delta v^{(n)}}\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)}\}\right)$$

Using that the following constraint is satisfied at the zeroth order:

$$(1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)}\}\right) \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle = 0$$

formula (136) becomes:

$$\begin{aligned} 0 &= (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H_{\delta v^{(n)}}\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)}\}\right) \delta v^{(n)} \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle \\ &+ (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(n)}}\left(\left\{ \hat{\mathbf{A}}^{[k_i]} \right\}, \{v^{(n)}\}\right) \delta v^{(n)} H\left(\hat{\mathbf{A}}^{[\Delta_1]}\right) \left| \left\{ \hat{\mathbf{A}}^{[k_i]} \right\} \right| \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle \end{aligned} \quad (137)$$

Equation (137) can be written more compactly:

$$H_{\delta v^{(n)}} (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} \delta v^{(n)} \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle + (1 - \mathbf{A})_{\hat{\mathbf{A}}^{[\Delta_1]}} H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(n)}} H\left(\hat{\mathbf{A}}^{[\Delta_1]}\right) \delta v^{(n)} \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle = 0$$

or equivalently as an equation for a state:

$$\left| \delta v^{(\eta)}, \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle = \delta v^{(\eta)} \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle$$

as:

$$\left(H_{\delta v^{(\eta)}} + H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) - \mathbf{A}_{\hat{\mathbf{A}}^{[\Delta_1]}} \left(H_{\delta v^{(\eta)}} + H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \right) \right) \left| \delta v^{(\eta)}, \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle = 0 \quad (138)$$

Equation (138) for state $\left| \delta v^{(\eta)}, \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle$ describes the tangent deformation of the constraint by a state characterized by internal degrees of freedom $\delta v^{(\eta)}$, and apparently a-priori parameters $\hat{\mathbf{A}}^{[\Delta_1]}$. In fact, this parameter keeps track of the projected background, as described in Part II and II.

23.2 Composition of modifications

We can compose two infinitesimal variations preserving the constraints. As before, see (135), we can write for the first modification:

$$H \left(\{ \mathbf{L} + \delta \mathbf{L} \}, \{ v^{(\eta)} + \delta v^{(\eta)} \} \right) \left| \{ (1 + \mathbf{A}_1) \hat{\mathbf{A}}^{[k_i]} \} \right\rangle \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle = 0 \quad (139)$$

Inserting the operator $(1 - \mathbf{A}_1)$, this becomes:

$$0 = (1 - \mathbf{A}_1) H \left(\{ \mathbf{L} + \delta \mathbf{L} \}, \{ v^{(\eta)} + \delta v_1^{(\eta)} \} \right) (1 + \mathbf{A}_1) (1 - \mathbf{A}_1) \left| \{ (1 + \mathbf{A}_1) \hat{\mathbf{A}}^{[k_i]} \} \right\rangle \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle$$

that can be simplified as:

$$0 = H \left(\{ \mathbf{L} + \Delta_1 \mathbf{L} \}, \{ v^{(\eta)} + \delta v_1^{(\eta)} \} \right) \left| \{ \hat{\mathbf{A}}^{[k_i]} \} \right\rangle \left| (1 - \mathbf{A}_1) \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle$$

This transformation can be composed with a second modification $\delta v_2^{(\eta)}$ and we are led to:

$$0 = (1 - \mathbf{A}_{21}) H \left(\{ \mathbf{L} + \Delta_1 \mathbf{L} + \delta_{21} \mathbf{L} \}, \{ v^{(\eta)} + \delta v_1^{(\eta)} + \delta v_2^{(\eta)} \} \right) \left| \{ \hat{\mathbf{A}}^{[k_i]} \} \right\rangle \left| (1 - \mathbf{A}_1) \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle$$

The second transformation \mathbf{A}_{21} models that transformation acts on states already transformed by \mathbf{A}_1 and should a priori depend locally on this transformation. The invariance of the constraints with respect to the composition of transformations becomes:

$$0 = H \left(\{ \mathbf{L} + \Delta_1 \mathbf{L} + \Delta_{21} \mathbf{L} \}, \{ v^{(\eta)} + \delta v_1^{(\eta)} + \delta v_2^{(\eta)} \} \right) \left| \{ \hat{\mathbf{A}}^{[k_i]} \} \right\rangle \left| (1 - \mathbf{A}_{21}) (1 - \mathbf{A}_1) \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle$$

At least, even if the constraints are modified by both transformations, we can derive conditions for these two deformations to commute:

$$\begin{aligned} 0 &= H_{\delta v^{(\eta)}} \left((1 - \mathbf{A}_{21}) (1 - \mathbf{A}_1) - (1 - \mathbf{A}_{12}) (1 - \mathbf{A}_2) \right) \delta v^{(\eta)} \left| \hat{\mathbf{A}}^{[\Delta_1]} \right\rangle \\ &\quad + (1 - \mathbf{A}_{21}) (1 - \mathbf{A}_1) H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{21} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) - (1 - \mathbf{A}_{12}) (1 - \mathbf{A}_2) H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{12} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \delta v^{(\eta)} \end{aligned}$$

That can be factored in the following manner:

$$\begin{aligned} 0 &= \left((1 - \mathbf{A}_{21}) (1 - \mathbf{A}_1) - (1 - \mathbf{A}_{12}) (1 - \mathbf{A}_2) \right) \left(H_{\delta v^{(\eta)}} + \frac{1}{2} \left\{ H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{21} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) + H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{12} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \right\} \right) \delta v^{(\eta)} \\ &\quad + \left((1 - \mathbf{A}_{21}) (1 - \mathbf{A}_1) + (1 - \mathbf{A}_{12}) (1 - \mathbf{A}_2) \right) \frac{1}{2} \left\{ H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{21} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) - H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{12} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \right\} \delta v^{(\eta)} \quad (140) \end{aligned}$$

We may assume that the coordinate transformations write, at least in first approximation:

$$\mathbf{A}_{21} = \mathbf{A}_2 \delta v_2^{(\eta)} + \mathbf{A}' \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right)$$

and symmetrically:

$$\mathbf{A}_{12} = \mathbf{A}_1 \delta v_1^{(\eta)} + \mathbf{A}' \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right)$$

since for $\left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) = 0$, the two modifications are identical and commute.

Similarly, for the modifications of generators we assume:

$$H_{21} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) = H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right)$$

and:

$$H_{12} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) = H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right)$$

We can compute the quantities involved in (140) with these formula:

$$\begin{aligned} & \left((1 - \mathbf{A}_{21}) (1 - \mathbf{A}_1) - (1 - \mathbf{A}_{12}) (1 - \mathbf{A}_2) \right) \\ &= \left((1 - \mathbf{A}_2) (1 - \mathbf{A}_1) - (1 - \mathbf{A}_1) (1 - \mathbf{A}_2) \right) - \mathbf{A}' \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) \left((1 - \mathbf{A}_1) + (1 - \mathbf{A}_2) \right) \end{aligned}$$

and:

$$H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{21} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) - H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{12} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) = H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) \left(H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) + H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \right)$$

and:

$$H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{21} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) + H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} H_{12} \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) = H_{\hat{\mathbf{A}}^{[\Delta_1]} \delta v^{(\eta)}} \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) \left(H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) - H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) \right)$$

The commutation relations should be independent from $\left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right)$, so that any decomposition $\delta v^{(\eta)} = \delta v_2^{(\eta)} + \delta v_1^{(\eta)}$ should lead ultimately to the same result. As a consequence, assuming:

$$(1 - \mathbf{A}_2) (1 - \mathbf{A}_1) - (1 - \mathbf{A}_1) (1 - \mathbf{A}_2) = 0$$

the commutation relations imply $H' \left(\hat{\mathbf{A}}^{[\Delta_1]} \right) = 0$ and $\mathbf{A}' = 0$. That is both "global" part of the transformations commute, and the local modifications cancel. This means that the transformation is global.

This conditions are restrictive, so that we can expect that some holonomy should appear while composing transformations. This is studied in the next section.

23.3 Second order expansion

We consider the second order expansion of a transformation preserving the constraint. This enables to study the consequence of this invariance for a product of states. Define first:

$$\delta v^{(\eta)} = \delta v_a^{(\eta)} \delta v_b^{(\eta)}$$

for two states $\delta v_a^{(\eta)}$ and $\delta v_b^{(\eta)}$, and:

$$\hat{\mathbf{A}}^{[\Delta_1]} = \left| \hat{\mathbf{A}}_a^{[\Delta_1]} \right\rangle \left| \hat{\mathbf{A}}_b^{[\Delta_1]} \right\rangle$$

Assuming that the constraints are invariant at the first order in modifications, the second order variation of the constraints with respect to $\delta v_a^{(\eta)}$ and $\delta v_b^{(\eta)}$ writes with these notations:

$$0 = (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} \delta v_a^{(\eta)} H_{\delta v_a^{(\eta)} \delta v_b^{(\eta)}} \delta v_b^{(\eta)} \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right\rangle + (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} \delta v_a^{(\eta)} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} \delta v_a^{(\eta)} \delta v_b^{(\eta)}} \delta v_b^{(\eta)} H \left(\hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right) \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right\rangle \quad (141)$$

Assume that at least in first approximation the terms $H_{\delta v^{(\eta)} \delta v^{(\eta'')}}$ and $H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} \delta v^{(\eta)} \delta v^{(\eta'')}}$ can be factored:

$$H_{\delta v^{(\eta)} \delta v^{(\eta')}} = H_{a,b} = c H_a H_b$$

and:

$$H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} \delta v^{(\eta)} \delta v^{(\eta')}} = d H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} a} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} b}$$

corresponding to a constraints which is a sum of tensor products of operators.

In this case, averaging equation (141) over $\left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right\rangle \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right\rangle$:

$$0 = \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right\rangle \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right\rangle \times \left\{ \delta v_a^{(\eta')} \delta v_b^{(\eta')} H_a \delta v_a^{(\eta)} H_b \delta v_b^{(\eta)} + \delta v_a^{(\eta')} \delta v_b^{(\eta')} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} a} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} b} \delta v_a^{(\eta)} H_b \delta v_b^{(\eta)} \right\}$$

can be expanded as:

$$0 = c \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \left| \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} H_a \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \left| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} H_b \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right\rangle + c' \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \left| \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} a} \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle \times \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \left| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} b} \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right\rangle$$

In the case $\left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right\rangle = \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle$ that is, when two identical modifications are performed, we have:

$$0 = c \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \left| \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} H_a \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle^2 + c' \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \left| \left\langle \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_b^{(\eta')} \right| (1 - \mathbf{A})_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]}} H_{\hat{\underline{\mathbf{A}}}^{[\Delta_1]} a} \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle^2$$

and depending on the value of c and c' , states $\left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle \left| \hat{\underline{\mathbf{A}}}^{[\Delta_1]} \right| \delta v_a^{(\eta)} \right\rangle$ may exist or not. This constraint on the number of states should arise in the context of commutations relations, which is the topic of the next section.

24 Discrete transformations

Up until now, our focus has been on continuous deformations of the constraints, and we have explored the implications of state modifications that preserve these constraints. We now delve into the possibilities of discrete deformations that entail a change in the number of parameters.

We will consider two transformations of state $v^{(\eta)}$ that do not preserve the constraints. In other words, these transformations modify the space $V(v^{(\eta)})$. We do not assume that these transformations induce the same modification of the constraints but rather impose some conditions on the

difference between these modifications. These conditions translates into commutation relations between the generators of transformations. To illustrate these ideas we will only consider a simplified example.

Suppose the states space to which the $v^{(n)}$ belong, is composed of sums of tensor products, contingent on the parameters $\{\hat{\underline{\Delta}}^{[k_i]}\}$ through the constraints. We consider in this example that modifying $v^{(n)}$ by tensoring or destroying one state results in transforming the set $\{\hat{\underline{\Delta}}^{[k_i]}\}$ by adding, or removing one or several parameters, with multiplicity. In part I and II this transformation arose from the projection onto some background state (see section 4.3). The degeneracy operators were dependent on the considered state. Tensoring a given state $v^{(n)}$ by some $\delta v^{(n)}$ involves more realizations of the non-projected field in the series expansion of the background state. This increase in the number of realizations induces more symmetries that mix the realizations of the fields.

When considering two successive modifications creating and destroying a state:

$$v^{(n)} \rightarrow v^{(n)} + A_{(\delta v^{(n)})}^- A_{\delta v^{(n)}}^+ v^{(n)}$$

or destroying and creating a state:

$$v^{(n)} \rightarrow v^{(n)} + A_{\delta v^{(n)}}^+ A_{(\delta v^{(n)})}^- v^{(n)}$$

the first modification increases the number of parameters from $\{\hat{\underline{\Delta}}^{[k_i]}\}$ to $\{\{\hat{\underline{\Delta}}^{[k_i]}\}, \hat{\underline{\Delta}}^{[\Delta_1]}\}$ and then reduces this set to $\{\{\hat{\underline{\Delta}}^{[k_i]}\}, \hat{\underline{\Delta}}^{[\Delta_1]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}$ while the second one modifies the set $\{\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}, \hat{\underline{\Delta}}^{[\Delta_1]}\}$. Here, we define $\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}$ as:

$$\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]} = \underset{S \cup \hat{\underline{\Delta}}^{[\Delta_2]} = \{\hat{\underline{\Delta}}^{[k_i]}\}}{\overset{\circ}{\cup}} S$$

where $\overset{\circ}{\cup}$ is the disjoint union. Note that in the definition of $\{\hat{\underline{\Delta}}^{[k_i]}\}$, the value of any parameter $\hat{\underline{\Delta}}^{[k_i]}$, including $\hat{\underline{\Delta}}^{[\Delta_2]}$, can arise with multiplicity. Removing $\hat{\underline{\Delta}}^{[\Delta_2]}$ corresponds to reduce this multiplicity by 1.

The two sets $\{\{\hat{\underline{\Delta}}^{[k_i]}\}, \hat{\underline{\Delta}}^{[\Delta_1]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}$ and $\{\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}, \hat{\underline{\Delta}}^{[\Delta_1]}\}$ are identical if $\hat{\underline{\Delta}}^{[\Delta_2]} \neq \hat{\underline{\Delta}}^{[\Delta_1]}$ whereas:

$$\{\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}, \hat{\underline{\Delta}}^{[\Delta_1]}\} = \{\{\hat{\underline{\Delta}}^{[k_i]}\}, \hat{\underline{\Delta}}^{[\Delta_1]}\} / \hat{\underline{\Delta}}^{[\Delta_1]} \overset{\circ}{\cup} \hat{\underline{\Delta}}^{[\Delta_1]}$$

In terms of modification operators, we can write:

$$\{\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}, \hat{\underline{\Delta}}^{[\Delta_1]}\} = A_{\hat{\underline{\Delta}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Delta}}^{[\Delta_2]}}^- \{\hat{\underline{\Delta}}^{[k_i]}\}$$

and:

$$\{\{\hat{\underline{\Delta}}^{[k_i]}\} / \hat{\underline{\Delta}}^{[\Delta_2]}, \hat{\underline{\Delta}}^{[\Delta_1]}\} = A_{\hat{\underline{\Delta}}^{[\Delta_2]}}^- A_{\hat{\underline{\Delta}}^{[\Delta_1]}}^+ \{\hat{\underline{\Delta}}^{[k_i]}\}$$

where $A_{\hat{\underline{\Delta}}^{[\Delta_1]}}^+$ increases the multiplicity of a value $\hat{\underline{\Delta}}^{[\Delta_1]}$ of the parameter by 1, while $A_{\hat{\underline{\Delta}}^{[\Delta_2]}}^-$ reduces it by 1. These sets gather eigenstates of the set of operators $\{\mathbf{L}\} \cup \left\{ A_{\hat{\underline{\Delta}}^{[\Delta_2]}}^- A_{\hat{\underline{\Delta}}^{[\Delta_1]}}^+, \mathbf{L} \right\}$ and $\{\mathbf{L}\} \cup \left\{ A_{\hat{\underline{\Delta}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Delta}}^{[\Delta_2]}}^-, \mathbf{L} \right\}$ respectively.

In the sequel, we will neglect the continuous transformations $(1 - \mathbf{A}_i)$ on the parameters. As a consequence, given our assumptions, the first modification of the constraints writes:

$$H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right)$$

and its expansion in terms of operators has the form:

$$\begin{aligned} & H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) \\ &= H_{-+}^{(\nu)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ \\ &+ H_{-+}^{(\Lambda)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\hat{\Delta}[\Delta_2]}^- A_{\hat{\Delta}[\Delta_1]}^+ + H_{-+}^{(\nu, \Lambda)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_2^{(\eta)}}^- A_{\hat{\Delta}[\Delta_2]}^- A_{\delta v_1^{(\eta)}}^+ A_{\hat{\Delta}[\Delta_1]}^+ \end{aligned} \quad (142)$$

while the second modification is:

$$\begin{aligned} & H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) \\ &= H_{+-}^{(\nu)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_1^{(\eta)}}^+ A_{\delta v_2^{(\eta)}}^- \\ &+ H_{+-}^{(\Lambda)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- + H_{+-}^{(\nu, \Lambda)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_1^{(\eta)}}^+ A_{\hat{\Delta}[\Delta_1]}^+ A_{\delta v_2^{(\eta)}}^- A_{\hat{\Delta}[\Delta_2]}^- \end{aligned} \quad (143)$$

If $\delta v_1^{(\eta)} \neq \delta v_2^{(\eta)}$, both modified states are identical and the modified constraints have to be identical. Moreover, since:

$$\left\{ \left\{ \hat{\Delta}^{[k_i]} \right\}, \hat{\Delta}^{[\Delta_1]} \right\} / \hat{\Delta}^{[\Delta_2]} = \left\{ \left\{ \hat{\Delta}^{[k_i]} \right\} / \hat{\Delta}^{[\Delta_2]}, \hat{\Delta}^{[\Delta_1]} \right\}$$

for $\hat{\Delta}^{[\Delta_1]} \neq \hat{\Delta}^{[\Delta_2]}$, we can deduce that the difference of modifications:

$$H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) - H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right)$$

should include a factor:

$$\delta \left(\hat{\Delta}^{[\Delta_2]} - \hat{\Delta}^{[\Delta_2]} \right) \delta \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right)$$

We thus impose:

$$\begin{aligned} & H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) - H \left(A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- \left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) \\ &= \delta \left(\hat{\Delta}^{[\Delta_2]} - \hat{\Delta}^{[\Delta_2]} \right) \delta \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right) H_C \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) \end{aligned} \quad (144)$$

with some vector function $H_C \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right)$.

Note that the presence of $\delta \left(\hat{\Delta}^{[\Delta_2]} - \hat{\Delta}^{[\Delta_2]} \right)$ can be checked directly in the term:

$$H_{+-}^{(\nu)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_1^{(\eta)}}^+ A_{\delta v_2^{(\eta)}}^- - H_{-+}^{(\nu)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+$$

and:

$$H_{+-}^{(\Lambda)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\hat{\Delta}[\Delta_1]}^+ A_{\hat{\Delta}[\Delta_2]}^- - H_{-+}^{(\Lambda)} \left(\left\{ \hat{\Delta}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\hat{\Delta}[\Delta_2]}^- A_{\hat{\Delta}[\Delta_1]}^+$$

arising in (144)². Actually, these operators are evaluated between $\langle \hat{\underline{\Lambda}}^{[\Delta_1]}, \delta v_1^{(\eta)} | \langle \hat{\underline{\Lambda}}^{[\Delta_2]}, \delta v_2^{(\eta)} |$ and $| \hat{\underline{\Lambda}}^{[\Delta_1]}, \delta v_1^{(\eta)} \rangle | \hat{\underline{\Lambda}}^{[\Delta_2]}, \delta v_2^{(\eta)} \rangle$

Writing:

$$H_{+-/-+}^{(t)} \left(\left\{ \hat{\underline{\Lambda}}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) = H_{+-/-+}^{(t)}$$

for any index t leads to express the left hand side of (144) as:

$$\begin{aligned} & H \left(A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- \left\{ \hat{\underline{\Lambda}}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) - H \left(A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- \left\{ \hat{\underline{\Lambda}}^{[k_i]} \right\}, \left\{ v^{(\eta)} + A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ v^{(\eta)} \right\} \right) \\ &= \delta \left(\hat{\underline{\Lambda}}^{[\Delta_2]} - \hat{\underline{\Lambda}}^{[\Delta_2]} \right) \left(H_{+-}^{(\nu)} A_{\delta v_1^{(\eta)}}^+ A_{\delta v_2^{(\eta)}}^- - H_{-+}^{(\nu)} A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ \right) \\ &+ \delta \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right) \left(H_{+-}^{(\Lambda)} A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- - H_{-+}^{(\Lambda)} A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ \right) \\ &+ H_{+-}^{(\nu, \Lambda)} A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- - H_{-+}^{(\nu, \Lambda)} \left(\left\{ \hat{\underline{\Lambda}}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ \end{aligned}$$

where:

$$\begin{aligned} A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- &= A_{\delta v_2^{(\eta)}}^- A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- \\ A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ &= A_{\delta v_1^{(\eta)}}^+ A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ \end{aligned}$$

The identification of (145) with the right hand side of (144) leads to:

$$\begin{aligned} & \delta \left(\hat{\underline{\Lambda}}^{[\Delta_2]} - \hat{\underline{\Lambda}}^{[\Delta_2]} \right) \delta \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right) H_C \\ &= \delta \left(\hat{\underline{\Lambda}}^{[\Delta_2]} - \hat{\underline{\Lambda}}^{[\Delta_2]} \right) \left(H_{+-}^{(\nu)} A_{\delta v_1^{(\eta)}}^+ A_{\delta v_2^{(\eta)}}^- - H_{-+}^{(\nu)} A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ \right) \\ &+ \delta \left(\delta v_1^{(\eta)} - \delta v_2^{(\eta)} \right) \left(H_{+-}^{(\Lambda)} A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- - H_{-+}^{(\Lambda)} A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ \right) \\ &+ H_{+-}^{(\nu, \Lambda)} A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- - H_{-+}^{(\nu, \Lambda)} \left(\left\{ \hat{\underline{\Lambda}}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right) A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ \end{aligned}$$

with:

$$H_C = H_C \left(\left\{ \hat{\underline{\Lambda}}^{[k_i]} \right\}, \left\{ v^{(\eta)} \right\} \right)$$

Since the modification are discretized, nothing guarantees the equality $H_{+-}^{(t)} = H_{-+}^{(t)}$. Assume to simplify that we can normalize $|H_{+-}^{(t)}| = |H_{-+}^{(t)}| = 1 = |H_C|$, this implies that:

$$\begin{aligned} 0 &= A_{\delta v_1^{(\eta)}}^+ A_{\delta v_2^{(\eta)}}^- \pm A_{\delta v_2^{(\eta)}}^- A_{\delta v_1^{(\eta)}}^+ + \delta \left(\hat{\underline{\Lambda}}^{[\Delta_2]} - \hat{\underline{\Lambda}}^{[\Delta_2]} \right) \delta \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) \\ 0 &= A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- \pm A_{\hat{\underline{\Lambda}}^{[\Delta_2]}}^- A_{\hat{\underline{\Lambda}}^{[\Delta_1]}}^+ + \delta \left(\hat{\underline{\Lambda}}^{[\Delta_2]} - \hat{\underline{\Lambda}}^{[\Delta_2]} \right) \delta \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) \\ 0 &= A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- \pm A_{\delta v_2^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_2]}}^- A_{\delta v_1^{(\eta)}, \hat{\underline{\Lambda}}^{[\Delta_1]}}^+ + \delta \left(\hat{\underline{\Lambda}}^{[\Delta_2]} - \hat{\underline{\Lambda}}^{[\Delta_2]} \right) \delta \left(\delta v_2^{(\eta)} - \delta v_1^{(\eta)} \right) \end{aligned}$$

leading to some commutation or anticommutation relations between operators involved in the modifications of the constraints.

²These terms are derived from (142), (143) and (144).

Appendix 1: solutions of the saddle points equations and effective theory

We solve formally the saddle point equation for S . The results allow to describe the effective field theory by replacing the results in the field functionals.

Solution of the saddle point

We study the minimization of the action functional for $\Psi_{I,\alpha}^{\otimes k}$.

$$\exp\left(-S\left(\left\{\Psi_{J,\alpha}^{\otimes l}(U_j^l)\right\}_{l,k} \otimes \Psi_{I,\alpha}^{\otimes k}(U_i^k)\right)_\alpha, \left\{v_{\{U_i^k\}}\left(\Psi_{J,\alpha}^{\otimes k}\right) \otimes \Psi_{I,\alpha}^{\otimes k}\right\}_{\{U_i^k\}}\right)$$

The tensor products:

$$\Psi_J^{\otimes l}(U_j^l) \otimes \Psi_I^{\otimes k}(U_i^k)$$

and:

$$v_{\{U_i^k\}}\left(\left\{\Psi_J^{\otimes l}\right\}\right) \otimes \Psi_I^{\otimes k}(U_i^k)$$

will stand implicitly for:

$$\Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes \Psi_{I,\alpha}^{\otimes k}(U_i^k)$$

and:

$$v_{U_i^k}\left\{\Psi_{J,\alpha}^{\otimes k}\right\} \otimes \Psi_{I,\alpha}^{\otimes k}(U_i^k)$$

The indices α will be reintroduced later.

We will write the solutions of the equations:

$$\left\{\Psi_J^{\otimes l}(U_j^l) \oplus \Psi_{I,0}^{\otimes k}(U_i^k)\right\}_{\substack{l \leq m \\ l' \leq m'}} \equiv \left\{\Psi_{I,0}^{\otimes k}(U_i^k)\right\}_{\Psi_J^{\otimes l}(U_j^l)}$$

Writing the saddle point equation as:

$$\frac{\delta}{\delta \Psi_{I,\alpha}^{\otimes k}(U_i^k)} S\left(\left\{\Psi_{J,\alpha}^{\otimes l}(U_j^l)\right\}_{l,k} \otimes \Psi_{I,\alpha}^{\otimes k}(U_i^k)\right)_\alpha, \left\{v_{\{U_i^k\}}\left(\Psi_{J,\alpha}^{\otimes k}\right) \otimes \Psi_{I,\alpha}^{\otimes k}\right\}_{\{U_i^k\}} = 0$$

We assume that S is a series in the variables:

$$\left\{\Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes \Psi_{I,\alpha}^{\otimes k}(U_i^k)\right\}_\alpha$$

and the functionals:

$$\left\{v_{\{U_i^k\}}\left(\Psi_{J,\alpha}^{\otimes k}\right) \otimes \Psi_{I,\alpha}^{\otimes k}\right\}_{\{U_i^k\}}$$

so that the saddle point equation for $\Psi_{I,\alpha}^{\otimes k}$ has the form:

$$\begin{aligned}
0 = & \sum_{\substack{s,\beta_1,\dots,\beta_s \\ t,\beta'_1,\dots,\beta'_t}} \prod_i d(U_j^{l_i}/f_{k_i l_i}) d(U_i^{k_i}) \prod_{i'} d(U_{i'}^{k_{i'}}) \\
& \times \mathcal{G}_0 \left(U_i^k, \{U_{j,\beta_r}^l/f_{kl}, U_{i,\beta_r}^k\}, \{U_{i,\beta'_r}^k\}_{i'}, \left\{ \Psi_{J,\alpha'}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha'}^{\otimes k}(U_i^k) \right\}_{\alpha' \neq \alpha}, \left\{ v_{\{U_i^k\}} \left(\Psi_{J,\alpha'}^{\otimes k} \right) \otimes \Psi_{I,\alpha'}^{\otimes k} \right\}_{\{U_i^k\}, \alpha' \neq \alpha} \right) \\
& \times \left(\Psi_{J,\alpha}^{\otimes l}(U_{j,\beta_1}^l/f_{kl}, U_i^k) \prod_{r=2}^s \Psi_{J,\alpha}^{\otimes l}(U_{j,\beta_r}^l/f_{kl}, U_{i,\beta_r}^k) \Psi_{I,\alpha}^{\otimes k}(U_{i,\beta_r}^k) \prod_{r'=1}^t v_{\left\{ U_{i,\beta'_{r'}}^k \right\}} \left(\left\{ \Psi_{J,\alpha}^{\otimes l} \right\} \right) \Psi_{I,\alpha}^{\otimes k}(U_{i,\beta'_{r'}}^k) \right. \\
& \left. + v_{\{U_i^k\}} \left(\left\{ \Psi_{J,\alpha}^{\otimes l} \right\} \right) \prod_{r=1}^s \Psi_{J,\alpha}^{\otimes l}(U_{j,\beta_r}^l/f_{kl}, U_{i,\beta_r}^k) \Psi_{I,\alpha}^{\otimes k}(U_{i,\beta_r}^k) \prod_{r'=2}^t v_{\left\{ U_{i,\beta'_{r'}}^k \right\}} \left(\left\{ \Psi_{J,\alpha}^{\otimes l} \right\} \right) \Psi_{I,\alpha}^{\otimes k}(U_{i,\beta'_{r'}}^k) \right)
\end{aligned} \tag{146}$$

where \mathcal{G}_0 is a kernel depending on $\left\{ \Psi_{J,\alpha'}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha'}^{\otimes k}(U_i^k) \right\}_{\alpha' \neq \alpha}$. We assume that this kernel includes some delta functions:

$$\delta(U_i^k - U_{i,\beta_r}^k)$$

and:

$$\delta(U_i^k - U_{i,\beta'_{r'}}^k)$$

to account for the possibility of some usual quadratic, or higher order, terms in the action:

$$\int \left(\Psi_{J,\alpha}^{\otimes l}(U_j^l) \otimes_{l,k} \Psi_{I,\alpha}^{\otimes k}(U_i^k) \right)^n dU_j^l dU_i^k / f_{kl}$$

This equation (146) can be solved recursively for each series expansion:

$$\begin{aligned}
& \Psi_{I,0,\alpha}^{\otimes k}(U_i^k, \{\Psi_J^{\otimes l}\}) \\
= & \sum_{\substack{s,l_1,\dots,l_s \\ \alpha_i,\dots,\alpha_s}} \prod_i d(U_j^{l_i}/f_{k_i l_i}) d(U_i^{k_i}) \prod_{i'} d(U_{i'}^{k_{i'}}) \\
& \times \mathcal{K}_0 \left(U_i^k, \{U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\}_i, \{U_{i'}^{k_{i'}}\}_{i'}, \{\alpha_i\} \right) \prod \Psi_{J,\alpha_i}^{\otimes l_i}(U_j^{l_i}/f_{k_i l_i}, U_i^{k_i}) \prod v_{\{U_{i'}^{k_{i'}}\}} \left(\left\{ \Psi_{J,\alpha_k}^{\otimes k} \right\} \right)
\end{aligned}$$

We assume that \mathcal{K}_0 is independent from the set $\{\alpha_i\}$, that is:

$$\mathcal{K}_0 \left(U_i^k, \{U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\}_i, \{U_{i'}^{k_{i'}}\}_{i'}, \{\alpha_i\} \right) = \mathcal{K}_0 \left(U_i^k, \{U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\}_i, \{U_{i'}^{k_{i'}}\}_{i'} \right)$$

In the sequel, the realizations are understood. We also write:

$$\begin{aligned}
& \Psi_{I,0}^{\otimes k}(U_i^k, \{\Psi_J^{\otimes l}\}) \\
= & \sum_{s,l_1,\dots,l_s} \int \prod_i d(U_j^{l_i}/f_{k_i l_i}) d(U_i^{k_i}) \prod_{i'} d(U_{i'}^{k_{i'}}) \\
& \times \mathcal{K}_0^v \left(U_i^k, \{U_i^{k_i}, U_j^{l_i}/f_{k_i l_i}\}_i, \{U_{i'}^{k_{i'}}, U_j^{l_{i'}}/f_{k_{i'} l_{i'}}\}_{i'} \right) \prod_i \Psi_J^{\otimes l_i}(U_j^{l_i}/f_{k_i l_i}, U_i^{k_i}) \prod_{i'} \Psi_J^{\otimes k}(U_j^{l_{i'}}/f_{k_{i'} l_{i'}}, U_{i'}^{k_{i'}})
\end{aligned} \tag{147}$$

where:

$$\mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i, \left\{ U_i^{k_{i'}}, U_j^{l_{i'}} / f_{k_{i'} l_{i'}} \right\}_{i'} \right) = \mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i \right) \prod_{i'} v \left(U_i^{k_{i'}}, U_j^{l_{i'}} / f_{k_{i'} l_{i'}} \right) \quad (148)$$

Changing the notation where i, i' is replaced by i , formula (147) writes:

$$\begin{aligned} & \Psi_{I,0}^{\otimes k} (U_i^k, \{\Psi_J^{\otimes l}\}) \\ &= \sum_{s, l_i, \dots, l_s} \prod_i d \left(U_j^{l_i} / f_{k_i l_i} \right) d \left(U_i^{k_i} \right) \mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i \right) \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} / f_{k_i l_i}, U_i^{k_i} \right) \end{aligned} \quad (149)$$

By changing variables in the functionals:

$$\begin{aligned} & \Psi_{I,0}^{\otimes k} (U_i^k, \{\Psi_J^{\otimes l}\}) \\ &= \sum_{s, l_i, \dots, l_s} d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right) d \left(\left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{k_i, l_i} \right) \prod_i \Psi_J^{\otimes l_i} (U_j^{l_i}) \mathcal{K}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_i \right) \end{aligned}$$

Reintroducing the realization indices this becomes:

$$\begin{aligned} & \Psi_{I,0,\alpha}^{\otimes k} (U_i^k, \{\Psi_J^{\otimes l}\}) \\ &= \sum_{s, l_i, \dots, l_s} \int d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right) d \left(\left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{k_i, l_i} \right) \mathcal{K}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_i \right) \left(\sum_{\{\alpha'_i\}_{i \leq s}} \prod_i \Psi_{J,\alpha'_i}^{\otimes l_i} (U_j^{l_i}) \right) \end{aligned} \quad (150)$$

where:

$$\Psi_{J,\alpha}^{\otimes \sum_{i \leq n} l_i} (U_j^{l_i}) = \sum_{\{\alpha'_i\}_{i \leq s}} \prod_i \Psi_{J,\alpha'_i}^{\otimes l_i} (U_j^{l_i})$$

The dependency in α is justified by the fact that the action S is such that an element of $\{\alpha'_i\}_{i \leq n}$ is α or arises in the action in products involving α .

Projected functional

Introducing (150) into the functionals of the form:

$$\int g(v, U_i^k) v_{\{U_i^k\}} (\{\Psi_J^{\otimes l}\}) \Psi_I^{\otimes k} (U_i^k) dv dU_i^k$$

is performed first by evaluating these functionals on states $ev_{(U_i^{l_i})} (\{\Psi_J^{\otimes l_i}\})$ that replaces $\Psi_J^{\otimes l_i}$ by its value $\Psi_J^{\otimes l_i} (U_i^{l_i})$. Then multiply by $v (U_i^{k_i}, U_j^{l_i} / f_{k_i l_i})$. and integrate over $d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right)$ to recover $v_{\{U_i^k\}} (\{\Psi_J^{\otimes l}\})$.

Practically this amounts to remove the integrals over $d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right)$ in (150) and consider:

$$\mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i \right) \prod_{i'} v \left(U_i^{k_{i'}}, U_j^{l_{i'}} / f_{k_{i'} l_{i'}} \right)$$

then, we replace the kernel in (148) by:

$$\begin{aligned}
& \mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i, \left\{ U_i^{k_{i'}}, U_j^{l_{i'}} / f_{k_{i'} l_{i'}} \right\}_{i'} \right) \\
\rightarrow & \mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i, \left\{ U_i^{k_{i'}}, U_j^{l_{i'}} / f_{k_{i'} l_{i'}} \right\}_{i'} \right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \\
= & \mathcal{K}_0^v \left(U_i^k, \left\{ U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right\}_i \right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \prod_{i'} v \left(U_i^{k_{i'}}, U_j^{l_{i'}} / f_{k_{i'} l_{i'}} \right)
\end{aligned} \tag{151}$$

and then resatblish integrals over $d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right)$.

Then the result is inserted in:

$$\int g(v, U_i^k) v_{\{U_i^k\}} \left(\{\Psi_J^{\otimes l}\} \right) \Psi_I^{\otimes k} (U_i^k) dv dU_i^k$$

To write the result, we change the variables arising in the fcnlns. We write:

$$U_i^k = (\{U_j^l\}, U^k / f_{kl})$$

where $\{U_j^l\}$ are coordinates in $U_i^k / (U_i^k / f_{kl})$. Then we replace:

$$U_i^k, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_i = \left(\{U_j^l\}, U_i^{k_i} / f_{k_i l_i} \right), \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_i \rightarrow \{U_j^l\}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_i \tag{152}$$

i.e. we include the variable U^k / f_{kl} in the set indexed by i (by including a label 0 for this variable). Moreover, using (151), we rewrite:

$$\mathcal{K}_0^v \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n} \right) = \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n} \right) \prod v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right)$$

and the functional becomes ultimately:

$$\begin{aligned}
& \int g(v, U_i^k) \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n} \right) \prod v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \\
& \times \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} \left(U_j^{l_i} \right) \prod_i d \left\{ U_j^{l_i}, U_i^{k_i} / f_{k_i l_i}, \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu] \right\}
\end{aligned} \tag{153}$$

Appendix 2: degeneracy

To write the dependency of the saddle point solution we start with the transformatuin parameters. We write:

$$\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{[\{k_i \leq n\}]} [\Psi_J, \nu] \right\}$$

for the set of parameters of the group G_{k_i, \dots, k_n} with varying $k_i \leq n$. The full set of parameters is:

$$\left(\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{[\{k_i \leq n\}]} [\Psi_J, \nu] \right\} \right)_n$$

fr n running from 1 to ∞ .

We also note:

$$\left(\hat{\Lambda}_{\{\alpha_i \leq n\}}^{[\{k_i \leq n\}]} [\Psi_J, \nu] \right)_n$$

the parameters associated to an infinite given sequence $\{k_i\}$. We also assume that the eigenvalues of symmetry parameters satisfy functional relations:

$$h_{k_n} \left(\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} [\Psi_J, U_j^l, \nu] \right\}, h_p((\Psi_J), U_j^l, \nu) \right) = 0$$

We can write the vector kernels of transformations:

$$\begin{aligned} \left[\mathbf{K}_{\{\alpha_i\}_{i \leq n}} (\Psi_J^{\otimes l}) \right]_{(U_i^{k_i})'}^{U_i^{k_i}} &= \left[\mathbf{K}_{i \leq n} (\Psi_J^{\otimes l}) \right]_{(U_i^{k_i})'}^{U_i^{k_i}} \\ &= \sum_{l_i, \{\alpha'_i\}_{i \leq n}} \mathbf{k} \left(\{U_j^{l_i}\}_i, U_i^{k_i}, (U_i^{k_i})' \right) \prod \Psi_{J, \alpha'_i}^{\otimes l_i} (U_j^{l_i}) \\ &= \sum_{l_i} \mathbf{k} \left(\{U_j^{l_i}\}_i, \{U_i^{k_i}\}, \{(U_i^{k_i})'\}_i \right) \sum_{\{\alpha'_i\}_{i \leq n}} \prod \Psi_{J, \alpha'_i}^{\otimes l_i} (U_j^{l_i}) \end{aligned} \quad (154)$$

since the generators are the sum for all realizations arising in the action. The kernel \mathbf{k} has components dual to $\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} [\Psi_J, \nu] \right\}$. We note $\mathbf{L}_{i \leq n} (\Psi_J^{\otimes l})$ the operator with kernel $\mathbf{K}_{i \leq n} (\Psi_J^{\otimes l})$.

The solution is given by the group action:

$$\begin{aligned} &\left\{ \Psi_{I,0,\alpha}^{\otimes k_i} \left(U_i^{k_i}, \left(\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} [\Psi_J, \nu] \right\} \right)_n, \{\Psi_J^{\otimes l}\} \right) \right\} \\ &= \left\{ \exp \left(i \sum \mathbf{L}_{i \leq n} (\Psi_J^{\otimes l}) \cdot \left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}} [\Psi_J, \nu] \right\} \right) \Psi_{I,0,\alpha}^{\otimes k_i} \left(U_i^{k_i}, \{\Psi_J^{\otimes l}\} \right) \right\} \end{aligned} \quad (155)$$

The dependency in $(\Psi_J^{\otimes l})$ kept implicit. To isolate one of the solution:

$$\Psi_{I,0}^{\otimes k} \left(U_i^k, \left(\left\{ \hat{\Lambda}_{\alpha_i}^{[k_i]} [\Psi_J, \nu] \right\}_{i \leq n} \right)_n, \{\Psi_J^{\otimes l}\} \right)$$

we write:

$$\begin{aligned} &\Psi_{I,0,\alpha}^{\otimes k} \left(U_i^k, \left(\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, \{\Psi_J^{\otimes l}\} \right) \\ &= \sum_{k_i} \int d(U_i^{k_i})' \left[\exp \left(i \mathbf{L} (\Psi_J^{\otimes l}) \cdot \left\{ \hat{\Lambda}_{\alpha}^{[k]} [\Psi_J, \nu] \right\} \right) \right]_{(U_i^{k_i})'}^{U_i^k} \Psi_{I,0}^{\otimes k} \left((U_i^{k_i})', \{\Psi_J^{\otimes l}\} \right) \\ &= \prod_n \sum_{k_i \leq n} \int d(U_i^{k_i})' \left[\exp \left(i \mathbf{L}_{i \leq n} (\Psi_J^{\otimes l}) \cdot \left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right) \right]_{(U_i^{k_i})'}^{U_i^k} \Psi_{I,0,\alpha}^{\otimes k} \left((U_i^{k_i})', \{\Psi_J^{\otimes l}\} \right) \end{aligned}$$

and we will use the notation $\left(\left\{ \hat{\Lambda}_{\alpha}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n$ for the set $\left(\left\{ \hat{\Lambda}_{\{\alpha_i \leq n\}}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n$ arising in the realization $\Psi_{I,\alpha,0}^{\otimes k}$. This notation is valid, since the domain of the variables $\hat{\Lambda}_{\alpha}^{[k_i]}$ is the same for all realizations. Consequently, the field writes:

$$\Psi_{I,0,\alpha}^{\otimes k} \left(U_i^k, \left(\left\{ \hat{\Lambda}_{\alpha}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, \{\Psi_J^{\otimes l}\} \right)$$

Particular case

For transformation groups that satisfies:

$$G_{k_i, \dots, k_n} \subset G_{k_i, \dots, k_{n+1}}$$

we can assume that:

$$\hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \subset \hat{\Lambda}_\alpha^{\{k, k_i \leq n\}} [\Psi_J, \nu]$$

so that for given sequence $\{k_i\}$:

$$\left(\hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right)_n$$

is an infinite dimension flag manifold, starting with $\hat{\Lambda}_\alpha^{[k]} [\Psi_J, \nu]$. The entire set of parameters:

$$\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n$$

is thus an infinite number of flag manifolds, all starting at $\hat{\Lambda}_\alpha^{[k]} [\Psi_J, \nu]$.

To simplify we may assume that:

$$\hat{\Lambda}_\alpha^{\{k_i \leq n\}} [\Psi_J, \nu] = \left\{ \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu] \right\}_{k_1, \dots, k_n}$$

and that the groups parameters are constrained such that:

$$\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n = \left\{ \hat{\Lambda}_\alpha^{[k]} [\Psi_J, \nu], \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu] \right\}_i$$

keeping in mind that this represents a distinguished set of points bearing some properties of U_i^k . This cloud of points is a background space. For such parameters, the constraints write:

$$h_{k_n} \left(\left\{ \hat{\Lambda}_{\alpha_i}^{[k_i]} [\Psi_J, U_j^l, \nu] \right\}_{i \leq n}, h_p \left((\Psi_J), U_j^l, \nu \right) \right) = 0$$

Note ultimately that in general:

$$\hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu] = \left(\hat{\Lambda}_\alpha^{k_i} [\Psi_J, \nu] \right)$$

Restriction including constraints

Since we have to consider functionals:

$$\Psi_J^{\otimes l} (U_j^l) \Psi_I^{\otimes k} (U_i^k) \delta (f_{lk} (U_j^l, U_i^k))$$

we have to restrict $\Psi_I^{\otimes k} (U_i^k)$ to $\delta (f_{lk} (U_j^l, U_i^k))$:

$$\Psi_I^{\otimes k} (U_i^k) \rightarrow \Psi_I^{\otimes k} (U_i^k) \delta (f_{lk} (U_j^l, U_i^k))$$

and the same applies to the $U_i^{k_i}$:

$$\Psi_I^{\otimes k_i} (U_i^{k_i}) \rightarrow \Psi_I^{\otimes k_i} (U_i^{k_i}) \delta \left(\left\{ f_{l_i k_i} (U_j^l, U_i^{k_i}) \right\} \right)$$

The notation $\delta\left(\left\{f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right\}\right)$ stands for the product:

$$\prod_{l_i} \delta\left(f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right)$$

that is we take into account the whole constraints $f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)$ for l_i variables.

This means that transformations (155) have to be restricted to projected fields:

$$\begin{aligned} & \left\{ \Psi_{I,0,\alpha}^{\otimes k_i}\left(U_i^{k_i}, \left(\left\{\hat{\mathbf{A}}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}}[\Psi_J, \nu]\right\}\right)_n, \{\Psi_J^{\otimes l}\}\right) \delta\left(\left\{f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right\}\right) \right\} \\ &= \left\{ \exp\left(i \sum \mathbf{L}_{i \leq n}(\Psi_J^{\otimes l}) \cdot \left\{\hat{\mathbf{A}}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}}[\Psi_J, \nu]\right\}\right) \Psi_{I,0,\alpha}^{\otimes k_i}\left(U_i^{k_i}, \{\Psi_J^{\otimes l}\}\right) \delta\left(\left\{f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right\}\right) \right\} \end{aligned} \quad (156)$$

so that we consider transformations that commute with the projections $\delta\left(f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right)$.

These constraints are not independent. Assuming the form:

$$f_{k_i}\left(U_i^{k_i}\right) = f_{l_i}\left(U_j^{l_i}\right) \quad (157)$$

or more generally:

$$f_{k_i}^{(\alpha)}\left(U_i^{k_i}\right) = f_{l_i}\left(\left\{U_j^{l_i}\right\}_\alpha\right) \quad (158)$$

for α running over finite set, such that $\cup_\alpha \left\{U_j^{l_i}\right\}_\alpha = \left\{U_j^{l_i}\right\}$, we can replace:

$$\delta\left(\left\{f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right\}\right) = \prod_\alpha \delta\left(f_{l_i k_i}^{(\alpha)}\left(U_j^{l_i}, U_i^{k_i}\right)\right)$$

and the rest of constraints are compatibility between the $U_j^{l_i}$.

Choosing $\alpha = 1$, for simplicity, we have:

$$\delta\left(\left\{f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right\}\right) = \delta\left(f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right) \quad (159)$$

for some given l_i .

Due to the constraint, the parameters $\left\{\hat{\mathbf{A}}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}}[\Psi_J, \nu]\right\}$ are restricted to subsets:

$$\left\{\hat{\mathbf{A}}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}}[\Psi_J, \nu]\right\}_{\delta\left(f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right)} \subset \left\{\hat{\mathbf{A}}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}}[\Psi_J, \nu]\right\}$$

Practically, when $\Psi_{I,0}^{\otimes k_i}\left(U_i^{k_i}\right)$ is inserted in (153) this subset of parameters is submitted in the s -th term of the sum to s constraints due to the products:

$$\prod_{i=1}^s \Psi_{J, \{\alpha'_i\}}^{\otimes l_i}\left(U_j^{l_i}\right)$$

and this corresponds to consider parameters of transformation that write:

$$\left\{\hat{\mathbf{A}}_{\{\alpha_i \leq n\}}^{\{k_i \leq n\}}\left[\Psi_J, \nu, \left\{U_j^{l_i \leq s}\right\}\right]\right\}$$

where the constraints $\delta\left(f_{l_i k_i}\left(U_j^{l_i}, U_i^{k_i}\right)\right)$ have been solved by introducing the dependency in $\left\{U_j^{l_i \leq s}\right\}$ and by choosing particular l_i in $\left\{U_j^{l_i \leq s}\right\}$ to implement the s constraints (157). In the text, given our assumptions, we will replace:

$$\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq s}\right\}\right]\right\} \rightarrow\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu},\left\{U_j^{l_i}\right\}\right]\right\}$$

or:

$$\left\{\hat{\Lambda}_{\alpha}^{\left[k,\left\{k_i \leq n-1\right\}\right]}\left[\Psi_{J, \nu},\left\{U_j^{l_i}\right\}\right]\right\}$$

if any k is distinguished.

Appendix 3: Projected functional

Disregarding constraint

Without constraints in variables, including some degeneracy in solutions (150) corresponds to consider parameters dependent kernels:

$$\left\{\mathcal{K}_0^v\left(\left\{U_j^{l_i}\right\}_{l_i \leq s},\left\{U_i^{k_i} / f_{k_i l_i}\right\}_{i \leq s},\left(\hat{\Lambda}_{\alpha}^{[k]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq n}\right\}\right],\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu}\right]\right\}_n\right)\right)\right\} \quad (160)$$

Linear combinations of degenerate background amounts to replace:

$$\mathcal{K}_0^v\left(U_i^k,\left\{U_i^{k_i}, U_j^{l_i} / f_{k_i l_i}\right\}_i\right)$$

in (149) or (150) by the following contribution:

$$\begin{aligned} & \Psi_{\alpha}\left(\left(\hat{\Lambda}_{\alpha}^{[k]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq n}\right\}\right],\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu}\right]\right\}_n\right), v\right) \\ & \mathcal{K}_0^v\left(U_i^k,\left\{U_j^{l_i}\right\}_{l_i \leq s},\left\{U_i^{k_i} / f_{k_i l_i}\right\}_{i \leq s},\left(\hat{\Lambda}_{\alpha}^{[k]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq n}\right\}\right],\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu}\right]\right\}_n\right)\right) \end{aligned} \quad (161)$$

We change variables arising in the functionals. We write:

$$U_i^k = \left(\left\{U_j^l\right\}, U^k / f_{kl}\right)$$

where $\left\{U_j^l\right\}$ are coordinates on $U_i^k / \left(U_i^k / f_{kl}\right)$. Then we replace:

$$\begin{aligned} & U_i^k,\left\{\hat{\Lambda}_{\alpha}^{[k]}\left[\Psi_{J, \nu}\right]\right\}_i,\left\{U_i^{k_i} / f_{k_i l_i},\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n-1\right\}\right]}\left[\Psi_{J, \nu}\right]\right\}_i\right\}_i \\ & = \left(\left\{U_j^l\right\}, U_i^{k_i} / f_{k_i l_i}\right),\left\{U_i^{k_i} / f_{k_i l_i},\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu}\right]\right\}_i\right)_i \rightarrow\left\{U_j^l\right\},\left\{U_i^{k_i} / f_{k_i l_i}, \hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n\right\}\right]}\left[\Psi_{J, \nu}\right]\right\}_i \end{aligned} \quad (162)$$

That is we include the variable U^k / f_{kl} in the set indexed by i (this is done by including a label 0 for this variable). Similarly we replace:

$$\hat{\Lambda}_{\alpha}^{[k]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq n}\right\}\right],\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k_i \leq n-1\right\}\right]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq n}\right\}\right]\right\}_i \rightarrow\left\{\hat{\Lambda}_{\alpha}^{\left[\left\{k, k_i \leq n-1\right\}\right]}\left[\Psi_{J, \nu},\left\{U_j^{l_i \leq n}\right\}\right]\right\}_i$$

Moreover, using (151), we rewrite the kernel as:

$$\begin{aligned} & \mathcal{K}_0^v \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \\ &= \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \end{aligned}$$

where S runs over the subset of $\{1, \dots, n\}$.

As a consequence, in a functional of tensor products:

$$\Psi_J^{\otimes l} (U_j^l) \otimes_{l,k} \Psi_I^{\otimes k} (U_i^k)$$

a copy for a single realization:

$$\int g(v, U_i^k) v_{\{U_i^k\}} (\{\Psi_J^{\otimes l}\}) \Psi_I^{\otimes k} (U_i^k) dv dU_i^k$$

is thus replaced by:

$$\begin{aligned} & \int g(v, U_i^k) \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \\ & \times \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} (U_j^{l_i}) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \prod_i d \left\{ U_j^{l_i}, U_i^{k_i} / f_{k_i l_i}, \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu] \right\} \end{aligned} \quad (163)$$

As in the core of the text, we write:

$$v_{\{U_i^k\}} (\{\Psi_{J,\alpha}^{\otimes l}\}) = \int v (\{U_j^l\}, U_i^l / f_{kl}) \Psi_{J,\alpha}^{\otimes l} (U_j^l)$$

and use our previous change of variable (162), to include $\{U_j^l\}, U_i^l / f_{kl}$ in the set indexed by i . Ultimately, functionl (163) rewrites:

$$\begin{aligned} & \int g(v, U_i^k) \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \\ & \times \Psi_{J,\alpha}^{\otimes l} (U_j^l) \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} (U_j^{l_i}) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \\ & \times \prod_i d \left\{ U_j^{l_i}, U_i^{k_i} / f_{k_i l_i} \right\} d \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \end{aligned} \quad (164)$$

including constraint

We include the constraints for parameters. We choose n values of l_i in $\{U_j^{l_i \leq n}\}$ arising in (164) to implement the n constraints (157). We then come back to the derivation of (153) and remove the integrals over $d \left(\left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{l_i} \right)$ in (163) to write the functionals:

$$\begin{aligned} & \int g(v, U_i^k) \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \prod_i v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \\ & \times \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} (U_j^{l_i}) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \prod_i d \left\{ U_j^{l_i}, \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu] \right\} \end{aligned} \quad (165)$$

This has projected the functional over the states defined by the set $\left\{ \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} \left(U_j^{l_i} \right) \right\}$. This implies the constraints (159):

$$\delta \left(f_{l_i k_i} \left(U_j^{l_i}, U_i^{k_i} \right) \right)$$

and the restriction:

$$\Psi_I^{\otimes k_i} \left(U_i^{k_i} \right) \delta \left(\left\{ f_{l_i k_i} \left(U_j^{l_i}, U_i^{k_i} \right) \right\} \right)$$

can be expressed by sing th vrbl $U_j^{l_i}$, s tht w wrt th prmtrs dpndnc:

$$\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i \leq n} \right\} \right] \right\} \right)_n$$

and functional (164) rewrites:

$$\begin{aligned} & \int g \left(v, U_i^k \right) \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i \leq n} \right\} \right] \right\} \right)_n \right) \sum_{S, 0 \in S} \prod_{i \in S} v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) \\ & \times \Psi_{J, \alpha}^{\otimes l} \left(U_j^l \right) \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} \left(U_j^{l_i} \right) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i \leq n} \right\} \right] \right\} \right)_n, v \right) \\ & \times \prod_i d \left\{ U_j^{l_i}, U_i^{k_i} / f_{k_i l_i} \right\} d \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i \leq n} \right\} \right] \right\} \right)_n \end{aligned} \quad (166)$$

We choose the form:

$$v \left(U_i^{k_i}, U_j^{l_i} / f_{k_i l_i} \right) = v \left(U_j^{l_i} \right) \delta \left(f_{k_i l_i} \left(U_i^{k_i}, U_j^{l_i} \right) \right)$$

corresponding to current-current type interactions. As a consequence, defining as before:

$$\Psi_{J, \alpha}^{\otimes \sum_{i \leq n} l_i} \left(U_j^{l_i} \right) = \sum_{\{\alpha'_i\}} \prod_i \Psi_{J, \{\alpha'_i\}}^{\otimes l_i} \left(U_j^{l_i} \right)$$

as well as:

$$\begin{aligned} & \bar{g} \left(v, \left\{ U_j^{l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i \leq n} \right\} \right] \right\} \right)_n \right) \\ & = \int g \left(v, U_i^k \right) \mathcal{K}_0 \left(\left\{ U_j^{l_i} \right\}_{l_i}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu, \left\{ U_j^{l_i \leq n} \right\} \right] \right\} \right)_n \right) dU_i^k / f_{kl} \prod dU_i^{k_i} / f_{k_i l_i} \end{aligned}$$

Formula (166) is given by:

$$\begin{aligned} & \int \bar{g} \left(v, \left\{ U_j^{l_i} \right\}_{i \leq n}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu \right] \right\} \right)_n \right) \tag{167} \\ & \times \sum_{S, 0 \in S} \prod_{i \in S} v \left(U_j^{l_i} \right) \Psi_{J, \alpha}^{\otimes l} \left(U_j^l \right) \Psi_{J, \alpha}^{\otimes \sum_{i \leq n} l_i} \left(U_j^{l_i} \right) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu \right] \right\} \right)_n, v \right) \\ & \equiv \int \bar{g} \left(v, \left\{ U_j^{l_i}, \hat{\Lambda}_\alpha^{[k_i]} \left[\Psi_J, \nu, U_j^l \right] \right\}_{i \leq n} \right) v \left(\left(U_j^{l_i} \right) \right) \left(\Psi_{J, \alpha}^{\otimes (l + \sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \left[\Psi_J, \nu \right] \right\} \right)_n, v \right) \right) \end{aligned}$$

with:

$$\begin{aligned}
& \Psi_{J,\alpha}^{\otimes(l+\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \\
&= \Psi_{J,\alpha}^{\otimes(l+\sum_i l_i)} \left(U_i^l, \left\{ U_j^{l_i} \right\} \right) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \\
&= \Psi_{J,\alpha}^{\otimes l} (U_i^l) \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(U_j^{l_i} \right) \Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right)
\end{aligned}$$

and:

$$v \left(\left(U_j^{l_i} \right) \right) = \sum_{S, 0 \in S} \prod_{i \in S} v \left(U_j^{l_i} \right)$$

As a consequence, functional (167) becomes:

$$\int \bar{g} \left(v, \left\{ U_j^{l_i}, \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu, U_j^{l_i}] \right\}_i \right) v \left(\left(U_j^{l_i} \right) \right) \left(\Psi_{J,\alpha}^{\otimes(l+\sum_i l_i)} \left(U_j^{l_i}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \right)$$

Setting $\bar{g}_v \rightarrow \bar{g}$ since the functionals are arbitrary and the v dependency can be absorbed in the definition of \bar{g} , this yields:

$$\int v \left(\left(U_j^{l_i} \right) \right) \bar{g} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \Psi_{J,\alpha}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right)$$

In the case where:

$$\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n = \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n$$

are independent from α , we can sum over the index and we have:

$$\begin{aligned}
& \int v \left(\left(U_j^{l_i} \right) \right) \bar{g} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \sum_\alpha \Psi_{J,\alpha}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \\
&\equiv \int v \left(\left(U_j^{l_i} \right) \right) \bar{g} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right)
\end{aligned}$$

with:

$$\begin{aligned}
& \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right) \\
&= \sum_\alpha \Psi_{J,\alpha}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right)
\end{aligned}$$

Note that if the dependency in:

$$\Psi_{J,\alpha}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n, v \right)$$

have the form:

$$\int k_v \left(U_j^{l_i} \right) \Psi_{J,\alpha}^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right)$$

The functional becomes:

$$\int v \left(\left(U_j^{l_i} \right) \right) \bar{g} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right) \Psi_J^{\otimes(\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_n \right)$$

with $v \left(\left(U_j^{l_i} \right) \right)$ rescaled:

$$v \left(\left(U_j^{l_i} \right) \right) \rightarrow \prod k_v \left(U_j^{l_i} \right) \sum_{S, 0 \in S} \prod_{i \in S} v \left(U_j^{l_i} \right)$$

Remark

If we choose for the functionals:

$$v_{\left\{ \begin{smallmatrix} U_j^l / f_{lk} \\ U_i^k \end{smallmatrix} \right\}} \{ \Psi_J^{\otimes l} \} = \Psi_J^{\otimes l} (U_j^l)$$

that is a density which is a Dirac measure:

$$v \left(U_j^{l_i} \right) = \delta_v (U_j^{l_i})$$

we find:

$$\begin{aligned} & \int \bar{g} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}[\{k, k_i \leq n-1\}] [\Psi_J, \nu] \right\} \right)_n \right) \Psi_J^{\otimes (\sum_i l_i)} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}[\{k, k_i \leq n-1\}] [\Psi_J, \nu] \right\} \right)_n, v \right) \\ & \equiv F_{f, lin} \left(\Psi_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}[\{k, k_i \leq n-1\}] [\Psi_J, \nu] \right\} \right)_n, v \right) \right) \end{aligned}$$

while if the $\hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu]$ not independent of the copies $\Psi_{J, \alpha}^{\otimes k p_0}$, identification is local.

Remark

In (160) and (37), the kernel can be considered as representation of the $G^{k, \{k_i \leq n-1\}}$ with $n \leq s$, so that the kernel considered are rather:

$$\left\{ \mathcal{K}_0^v \left(\left\{ U_j^{l_i} \right\}_{l_i \leq s}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq s}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_{n \leq s} \right) \right\} \quad (168)$$

which corresponds to include the following contribution in (149):

$$\Psi_\alpha \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_{n \leq s}, v \right) \mathcal{K}_0^v \left(\left\{ U_j^{l_i} \right\}_{l_i \leq s}, \left\{ U_i^{k_i} / f_{k_i l_i} \right\}_{i \leq s}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu] \right\} \right)_{n \leq s} \right) \quad (169)$$

and the field is:

$$\sum_\alpha \Psi_{J, \alpha}^{\otimes (\sum_{i=1}^s l_i)} \left(\left\{ U_j^{l_i} \right\}_{i \leq s}, \left(\left\{ \hat{\Lambda}[\{k, k_i \leq n-1\}] [\Psi_J, \nu] \right\} \right)_{n \leq s}, v \right)$$

Appendix 4: invariance

We start by assuming a projective invariance:

$$\int v \left(\left(U_j^{l_i} \right) \right) \frac{\delta \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l]}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} \nabla_{\hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}}} \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right)$$

is not equal to 0, but satisfies:

$$\begin{aligned} & \int v \left(\left(U_j^{l_i} \right) \right) \frac{\delta \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l]}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left(\left(U_j^{l_i} \right)' \right)} \nabla_{\hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}}} \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\ & = \int v \left(\left(U_j^{l_i} \right) \right) \frac{\delta' V \left(\left\{ \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right) \right)}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left(\left(U_j^{l_i} \right)', \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right)} \\ & \quad \times \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right) \end{aligned} \quad (70)$$

and we show that we can define, at least locally, a field depending on parameters independent from field variations.

Note that the gradients are not included in the right hand side since they can be removed by redefinition of:

$$\Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \right)$$

Functional V is assumed to be defined as a series of functionals of the type:

$$\begin{aligned} & V \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \right\} \right) \\ &= \sum_u \prod_{p_i \leq u} \int \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \Big|_I \\ & \times A \left(\left\{ \left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right\}_I \right) \end{aligned}$$

Computing the variation:

$$\int v \left((U_j^{l_i}) \right) \frac{\delta \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right)}{\delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)}$$

and using assumption (170) yields:

$$\begin{aligned} & \int v \left((U_j^{l_i}) \right) \frac{\delta \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right)}{\delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)} \\ &= \int v \left(U_i^l \right) \frac{\delta' \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n, v \right) \right)}{\delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)} \\ &+ \int v \left(\left\{ (U_j^{l_i}) \right\} \right) \underline{V}_\alpha \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \right\} \right) \\ & \times \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \end{aligned} \tag{171}$$

with:

$$\begin{aligned} & \underline{V}_\alpha \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \right) \right\} \right) \\ &= \frac{\delta' V \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right) \right) \right) \right)}{\delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})', \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\}_n \right) \right)} \end{aligned}$$

and equation (171) writes:

$$\begin{aligned}
& \int v \left((U_j^{l_i}) \right) \frac{\delta \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right)}{\delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)} \\
&= v \left((U_j^{l_i})' \right) \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ (U_j^{l_i})' \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\
&+ \int_{(U_j^{l_i}) / (U_j^{l_i})'} v \left((U_j^{l_i}) \right) \underline{V}_\alpha \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right\} \right) \\
&\times \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right)
\end{aligned}$$

This can be factored as:

$$\begin{aligned}
& \delta \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\
&= \int \left(\delta \left(U_j^{l_i} - (U_j^{l_i})' \right) + \underline{V}_\alpha \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right) \right) \right) \\
&\times \delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right) \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\
&= \delta' \hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right)
\end{aligned}$$

where:

$$\begin{aligned}
& \delta' \hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left((U_j^{l_i}) \right) \\
&= \int \left(\delta \left(U_j^{l_i} - (U_j^{l_i})' \right) + \underline{V}_\alpha \left(\left\{ \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right) \right) \right) \delta \Psi_{J,\alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)
\end{aligned}$$

As a consequence, the initial variation of the field rewrites:

$$\begin{aligned}
& \delta \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\
&= \delta' \hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left((U_j^{l_i}) \right) \Psi_{J,\alpha}^{\otimes \sum_i l_i} \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right)
\end{aligned}$$

which is performed at fixed $\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} [\Psi_J, \nu, U_j^l] \right\} \right)_n$. Thus, even if $\Psi_{J,\alpha}^{\otimes \sum_i l_i}$ is not invariant, there is a related field $\hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i}$ that can be defined locally as a function of an invariant family of parametrers $\hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}}$.

The solution:

$$\hat{\Psi}_{J,\alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}, \left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \right\} \right)_n, v, \alpha \right) \delta \left(f \left(\left(\left\{ \hat{\Lambda}_\alpha^{\{k, k_i \leq n-1\}} \right\} \right)_n, \left(\Psi_{J,0,\alpha}^{\otimes l_i}, v \right) \right) \right)$$

satisfies the equation:

$$\begin{aligned} & \int \hat{v} \left((U_j^{l_i}) \right) \frac{\delta \left\{ \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right\}}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\mathbf{O}}} \hat{\Psi}_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\ & \simeq \int v \left(\left\{ (U_j^{l_i}) \right\} \right) \frac{\delta' \hat{V} \left(\left\{ \hat{\Psi}_{J, \alpha}^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right) \right)}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left(\left\{ (U_j^{l_i})' \right\}, \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right)} \end{aligned}$$

with:

$$\hat{v} \left((U_j^{l_i}) \right) = \int v \left((U_j^{l_i})'' \right) \frac{\delta \Psi_{J, \alpha}^{\otimes \sum_i l_i} \left((U_j^{l_i})'' \right)}{\delta \hat{\Psi}_{J, \alpha}^{\otimes \sum_i l_i} \left(U_j^{l_i} \right)}$$

If the $\frac{\delta \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu, U_j^l]}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)}$ are independent from realization, as before, the sum over realizations verifies the equation:

$$\begin{aligned} & \int \hat{v} \left((U_j^{l_i}) \right) \frac{\delta \hat{\Lambda}_\alpha^{[k_i]} [\Psi_J, \nu, U_j^l]}{\delta \Psi_{J, \alpha}^{\otimes l_i} \left((U_j^{l_i})' \right)} \nabla_{\hat{\mathbf{O}}} \hat{\Psi}_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \\ & \simeq v \left(\left\{ (U_j^{l_i}) \right\} \right) \frac{\delta' \hat{V} \left(\hat{\Psi}_J^{\otimes \sum_i l_i} \left(\left\{ U_j^{l_i} \right\}_i, \left(\left\{ \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right\} \right)_n \right) \right)}{\delta \Psi_J^{\otimes l_i} \left(\left\{ (U_j^{l_i})' \right\}, \hat{\Lambda}_\alpha^{[k, k_i \leq n-1]} [\Psi_J, \nu, U_j^l] \right)} \end{aligned}$$

Appendix 5: projection over eigenvalues of operators

Projected functional for single eigenvalue

Starting with the fct (47):

$$\langle \Psi_I^{\otimes k} (U_i^k) \rangle_{\hat{\Lambda}} = v_I^{\otimes k} \left(U_i^k, \left\{ \Psi_{J, \alpha}^{\otimes l_i} \right\}, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right)$$

computing these averages of the field in the space spanned by states $F_0^{\hat{\Lambda}} \left[\left\{ \Psi_{I, \alpha}^{\otimes k_i} \right\}, \left\{ \Psi_{J, \alpha}^{\otimes l_i} \right\}, v \right]$. needs to consider linear combinations over these states with coefficients $\Psi \left(\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right)$. averages $\langle \Psi_I^{\otimes k} (U_i^k) \rangle_{\hat{\Lambda}}$ are thus combined:

$$\begin{aligned} \langle \Psi_I^{\otimes k} (U_i^k) \rangle & \rightarrow \int \Psi \left(\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) \langle \Psi_I^{\otimes k} (U_i^k) \rangle_{\hat{\Lambda}} d\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \\ & = \int \Psi \left(\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) v_I^{\otimes k} \left(U_i^k, \left\{ \Psi_{J, \alpha}^{\otimes l_i} \right\}, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) d\hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \end{aligned} \quad (172)$$

Use that:

$$v_I^{\otimes k} \left(U_i^k, \left\{ \Psi_{J, \alpha}^{\otimes l_i} \right\}, \hat{\Lambda}_{\infty, \alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right)$$

has an expansion:

$$\begin{aligned} & v_I^{\otimes k} \left(U_i^k, \left\{ \Psi_{J,\alpha}^{\otimes l_i} \right\}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \\ &= \sum_{s,l_1,\dots,l_s} \int d \left(\left\{ U_j^{l_i} \right\}_{l_i} \right) \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \mathcal{V}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \end{aligned}$$

As a consequence, replacing:

$$\Psi_I^{\otimes k} \left(U_i^k \right) \delta \left(f_{lk} \left(U_j^l, U_i^k \right) \right)$$

in the functional (44) by its average (172) and change variables as before:

$$\begin{aligned} U_j^l, \left\{ U_j^{l_i} \right\}_i &\rightarrow \left\{ U_j^{l_i} \right\}_i \\ \Psi_J^{\otimes l} \left(U_j^l \right) \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) &\rightarrow \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \end{aligned}$$

leads to the functional, written for a single realization:

$$\int \bar{g} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) \prod_i \Psi_J^{\otimes l_i} \left(U_j^{l_i} \right) \Psi \left(\hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right)$$

where:

$$\bar{g} \left(\left\{ U_j^{l_i}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k_i]} \left[\Psi_J, \nu, U_j^{l_i} \right] \right\}_i \right) = \int g \left(U_i^k / f_{kl}, U_j^l \right) \mathcal{V}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) dU_i^k / f_{kl}$$

Introducing the eigenvalue explicit:

$$\begin{aligned} \mathcal{V}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right) &\rightarrow \mathcal{V}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right) \right) \\ \Psi \left(\hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v \right) &\rightarrow \Psi \left(\hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v, \lambda \left(\Psi_J^{\otimes k} \right) \right) \end{aligned}$$

As before if the set:

$$\left\{ \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\} = \left\{ \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right] \right\}$$

does not depend on the realiztn, this becomes:

$$\begin{aligned} & F_{f,lin} \left(\left\{ \Psi_J^{\otimes l} \left(U_j^l \right) \right\}_l \right) \\ &= \int \bar{g} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right) \right) \Psi_J^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\underline{\Delta}}_{\infty,\alpha}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right) \end{aligned}$$

where:

$$\Psi_J^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right) = \sum_{\alpha} \Psi_{J,\alpha}^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}_i, \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right), v \right)$$

and:

$$\Psi_{J,\alpha}^{\otimes \sum l_i} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], v, \lambda \left(\Psi_J^{\otimes k} \right) \right) = \prod_i \Psi_{J,\alpha}^{\otimes l_i} \left(U_j^{l_i} \right) \Psi_{\alpha} \left(\hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right) \right)$$

and:

$$\begin{aligned} & \bar{g} \left(\left\{ U_j^{l_i} \right\}, \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right) \right) \\ &= \int g \left(U_i^k / f_{kl}, U_j^l \right) \mathcal{V}_0^v \left(U_i^k, \left\{ U_j^{l_i} \right\}_{l_i}, \hat{\underline{\Delta}}_{\infty}^{[k]} \left[\Psi_J, \nu, \left\{ U_j^{l_i} \right\} \right], \lambda \left(\Psi_J^{\otimes k} \right) \right) dU_i^k / f_{kl} \end{aligned}$$

Average over several eigenspaces

If we consider the average over several eigenspaces, we write:

$$\left(\left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right)$$

the variables:

$$\left(\lambda \left[\Psi_J, \nu, \{U_j^{l_i}\} \right], \hat{\underline{\mathbf{A}}}_\infty^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right)$$

with the eigenvalues of the operator considered. λ can be multi-valued. The eigenstates write:

$$F \left[[\Psi_J^{\otimes l}], \{\Psi^k\}, \left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right]$$

and the average values in this state are:

$$\langle \Psi_I^{\otimes k} (U_i^k) \rangle = v_I^{\otimes k} \left(U_i^k, [\Psi_J^{\otimes l}], \lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right)$$

where:

$$\begin{aligned} & v_I^{\otimes k} \left(U_i^k, [\Psi_J^{\otimes l}], \left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right) \\ &= \int \Psi_I^{\otimes k} (U_i^k) \left| F_0 \left[[\Psi_J^{\otimes l}], \{\Psi_J^{\otimes l}\}, \left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right] \right|^2 \prod \mathcal{D} \{ \Psi_J^{\otimes l} \} \end{aligned}$$

and functionl becomes:

$$\begin{aligned} & F_{f,lin} \left(\left\{ \Psi_{J,s_{p'l'},0}^{\otimes k_{l'}} \left(\left((U_j)^{(p_0,p'l_0)} \right)^{k_{l'}} \right) \right\}_{(p'l_0)} \right) \\ &= \sum_m \sum_{D_j^{p_0,p'l,m}} \sum_{m'} \sum_{D_{j,p_u}^{(p_0,p'l_0,p_l,p'l'),m,m'}} \int g \left(\{[p_0,p'l_0]\}_{l'}, \left\{ \left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) \\ & \times \Psi_{J,s_{p'l'},0}^{\otimes k_{l'}} \left([p_0,p'l_0], \left\{ \left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \right\}_{\substack{l \leq m \\ l' \leq m'}} \right) d \{[p_0,p'l_0]\}_{0,l'} d \left(\lambda, \hat{\underline{\mathbf{A}}}_\infty^{[k]} \right) \left[\Psi_J, \nu, \{U_j^{l_i}\} \right] \end{aligned}$$

Appendix 6: general form for amplitudes

The transport $P_{\lambda\lambda+\delta\lambda}$ is performed through generator of translation operator:

$$\left(\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) - \Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right) \right) \frac{\delta}{\delta \Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right)}$$

Since $\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda + \delta\lambda \right)$ and $\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right)$ do not act on the same space, their difference is not the derivative of an operator $\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right)$ that can be defined for every λ . This derivative has to be corrected to account for the change in spaces induced by the modification of λ .

Given the constraints:

$$\begin{aligned} & \left(\Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda + \delta\lambda \right) - \Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right) \right) \\ &= \delta\lambda \left(\frac{\partial \Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right)}{\partial \lambda} + \left((A_\lambda)_{k'}^{k'} \left(U_j / [U_j] \right) \Psi_J \left((U_j / [U_j])_{k'}, \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right) \right)_k \right) \\ &= \delta\lambda \nabla_\lambda \Psi_J \left(U_j / [U_j], \left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \right\}, \lambda \right) \end{aligned}$$

and the transport writes:

$$P_{\lambda\lambda+\delta\lambda} = \exp \left(\int i\delta\lambda \underline{\nabla}_\lambda \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \frac{\delta}{\delta \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right)} \mathcal{D} \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right)$$

The matrix elements of:

$$\frac{\delta}{\delta \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right)}$$

are computed using dual basis, involving amplitudes of the form:

$$\exp \left(i \left(\Psi'_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) - \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right) \Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right)$$

where $\Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right)$ are elements of dual basis. Using the parallel transport back to $\lambda + \delta\lambda$ we can write:

$$\Psi'_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \rightarrow \Psi'_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda + \delta\lambda \right)$$

so that the matrices element involving $\frac{\delta}{\delta \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right)}$ writes:

$$\exp \left(i\delta\lambda \underline{\nabla}_\lambda \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \left(\Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right) \right)$$

The matrix contribution of:

$$\exp \left(i\delta\lambda \underline{\nabla}_\lambda \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \left(\Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right) \right) \exp \left(iF \left(\Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right) \right)$$

is then obtained by saddle point equation and we find:

$$\Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) = \underline{\nabla}_\lambda \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) + \text{something including field}$$

Matrices elements involving $\Pi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right)$ are then constrained by:

$$h_{k_i} \left(\left\{ \hat{\underline{\mathbf{A}}}^{[k_i]} \left[\Psi_J, U^{(j)} \right] \right\}_i, h_p \left((\Psi_J), U^{(j)} \right) \right) = 0$$

Non local terms write:

$$\begin{aligned} & \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\} + \delta \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) - \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \\ &= \delta \{\hat{\underline{\mathbf{A}}}^{[k_i]}\} \left(\nabla_{\{\hat{\underline{\mathbf{A}}}^{[k_i]}\}} \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) + \left(\left(A_{\{\hat{\underline{\mathbf{A}}}^{[k_i]}\}} \right)_k^{k'} \left(U_j/[U_j] \right) \Psi_J \left((U_j/[U_j])_{k'}, \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right)_k \right) \\ &= \delta \{\hat{\underline{\mathbf{A}}}^{[k_i]}\} \underline{\nabla}_{\{\hat{\underline{\mathbf{A}}}^{[k_i]}\}} \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \end{aligned}$$

and matrix elements:

$$\begin{aligned} & \left\langle \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda + \delta\lambda \right) \middle| \delta T_{\lambda\lambda+\delta\lambda} \middle| \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right\rangle \\ &= \left\langle \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda + \delta\lambda \right) \middle| \exp(i\delta\lambda S(\Psi_J)) \middle| \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right\rangle \end{aligned}$$

infinitesimally, this is generated:

$$\begin{aligned} & S(\Psi_J) \\ &= S \left(\underline{\nabla}_\lambda \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right), \underline{\nabla}_{\{\hat{\underline{\mathbf{A}}}^{[k_i]}\}} \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right), \Psi_J \left(U_j/[U_j], \{\hat{\underline{\mathbf{A}}}^{[k_i]}\}, \lambda \right) \right) \end{aligned}$$

Appendix 7 General formalism. Projected functional and effective field

Saddle point approach

Starting with a generic functional and inserting the solution of the saddle point equation yields:

$$\int g \left([p_l, p_{l'}]^{k_{l'}}, v \right) \times v_{\{[p_l, p_{l'}]^{k_{l'}}\}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \\ \times \Psi_{J, 0, s_{p_{l'}, l}}^{\otimes k_{l'} \otimes k_{l'}} \left([p_l, p_{l'}]^{k_{l'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) dv$$

with $v_{\{[p_l, p_{l'}]^{k_{l'}}\}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right)$ was defined in (71):

$$v_{[p_l, p_{l'}]^{k_{l'}}} \left(\Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \right) \\ = \int v \left(\overline{[p_0, p_{l'}]^{k_{l'}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right) \times \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'}]^{k_{l'}}}, \{[p_l, p_{l'}]^{k_{l'}}\} \right) d \left(\overline{[p_0, p_{l'}]^{k_{l'}}} \right)$$

Expanding the solution leads to:

$$\sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} \int g^{\mathcal{K}} \left(\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'}, 0}}^{\otimes k_{l'}, i} \left(\overline{[p_0, p_{l'}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P} \neq \emptyset} v_{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \left\{ \Psi_J^{\otimes k_{l'}} \right\} \\ \times \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) d \left(\left\{ \overline{[p_0, p_{l'}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) dv$$

where:

$$g^{\mathcal{K}} \left(\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ \overline{[p_0, p_{l'}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ = \sum_l g \left([p_l, p_{l'}]^{k_{l'}}, v \right) \mathcal{K}_0 \left(\left\{ \overline{[p_0, p_{l'}]^{k_{l'}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1, l_1 \neq l \\ l'_1 \leq m'_1}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right)$$

and:

$$\begin{aligned} & \prod_{\substack{l_2 \leq m_2 \\ l'_2 \leq m'_2}} v_{[p_{l_2}, p_{l'_2 l_2}]^{k_{l'_2}}} \left\{ \Psi_J^{\otimes k_{l'_2}} \right\} \\ = & \prod_{\substack{l_2 \leq m_2 \\ l'_2 \leq m'_2}} \int v \left(\overline{[p'_0, p'_{l'_0}]^{k_{l'_2}}}, \left\{ [p_{l_2}, p_{l'_2 l_2}]^{k_{l'_2}} \right\} \right) \times \Psi_{J, s_{p_{l'_0}}, 0}^{\otimes k_{l'_2}} \left(\overline{[p'_0, p'_{l'_0}]^{k_{l'_2}}}, \left\{ [p_{l_2}, p_{l'_2 l_2}]^{k_{l'_2}} \right\} \right) d \left(\overline{[p'_0, p'_{l'_0}]^{k_{l'_2}}} \right) \end{aligned}$$

Gathering p_{l_1} and $p_{l_2} \rightarrow p_{l_1}$, and $[p_0, p_{l'_0}]^{k_{l'_2}}$ nd $\overline{[p'_0, p'_{l'_0}]^{k_{l'_2}}}$, the previous formula becoms:

$$\begin{aligned} & \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} \int \bar{g}^{\mathcal{K}} \left(\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ \overline{[p_0, p_{l'_0}]^{k_{l'_2}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J[l', 0], v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ \times & \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'_0}}, 0}^{\otimes k_{l'_2}, i} \left(\overline{[p_0, p_{l'_0}]^{k_{l'_2}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J[l', 0], v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) \\ \times & d \left(\left\{ \overline{[p_0, p_{l'_0}]^{k_{l'_2}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) dv \end{aligned}$$

where:

$$\begin{aligned} & \bar{g}^{\mathcal{K}} \left(\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ \overline{[p_0, p_{l'_0}]^{k_{l'_2}}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J[l', 0], v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ = & \sum_{\mathcal{P}, \mathcal{P}_1} g^{\mathcal{K}} \left(\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}_1}, \left\{ [p_0, p_{l'_0}]^{k_{l'_2}} \right\}_{\mathcal{P}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\Psi_J[l', 0], v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ \times & \prod_{\mathcal{P}^c} v \left(\left\{ [p_0, p_{l'_0}]^{k_{l'_2}} \right\}_{\mathcal{P}^c}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}_1^c} \right) \end{aligned}$$

along with:

$$\left\{ [p_0, p_{l'_0}]^{k_{l'_2}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} = \left\{ \left\{ [p_0, p_{l'_0}]^{k_{l'_2}} \right\}_{\mathcal{P}}, \left\{ [p_0, p_{l'_0}]^{k_{l'_2}} \right\}_{\mathcal{P}^c} \right\}$$

and:

$$\left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} = \left\{ \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}_1}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}_1^c} \right\}$$

Changing variable leads to rewrite the functiona as:

$$\begin{aligned} & \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} \int \bar{g}^{\mathcal{K}} \left(\left\{ \overline{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, \left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'0}}, 0}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) d \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \overline{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) dv \end{aligned}$$

with:

$$\overline{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} = \overline{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} / \prod_{p_{1,l}, \dots, p_{m',l}} f_{p_{1,l}, \dots, p_{m',l}}$$

and this is equal to:

$$\begin{aligned} & \equiv \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} \int \bar{g}^{\mathcal{K}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, v, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'0}}, 0}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}}, v \right) d \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) dv \end{aligned}$$

Ultimately, if we choose for the functional:

$$v_{\{[p_l, p_{l'l}]\}^{k_{l'}}} = v_{\substack{[p_0, p_{l'0}]^{k_{l'}} \\ [p_l, p_{l'l}]\}^{k_{l'}}, \alpha} \left\{ \Psi_{J, \alpha, s_{p_{l'0}}, 0}^{\otimes k_{l'}} \right\}$$

evaluated at $\Psi_{J, \alpha, s_{p_{l'0}}, 0}^{\otimes k_{l'}}$ or equivalently if we consdr sbbjcts that are chosen as eigenstates of oprtrs

Λ the functional simplifies:

$$\begin{aligned} & \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} \int \bar{g}^{\mathcal{K}} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}}, \left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, [p_0, p_{l'0}]^{k_{l'}}] \end{array} \right] \right\} \right) \\ & \times \prod_{\substack{l \leq s \\ l' \leq s'}} \Psi_{J, s_{p_{l'0}}, 0}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \Psi \left(\left\{ \left[\begin{array}{c} (p_{l_1}, p_{l'_1 l_1}, k_{l'_1}) \\ [\{\Psi_J[l', 0]\}, [p_0, p_{l'0}]^{k_{l'}}] \end{array} \right] \right\}, v \right) d \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{\substack{l \leq s \\ l' \leq s'}} \right) \end{aligned}$$

Projection over eigenstates of operator

By changing variables in (100), it writes:

$$\begin{aligned}
& F_{f,lin} \left(\left\{ \Psi_{J,s_{p'l'},0}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right\}_{(p_{l'0})} \right) \\
&= \sum_m \sum_{m'} \int \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} d \left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{l' \leq s} \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \bar{g} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{l' \leq s} \right. \\
&\quad \left. , \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \\
&\times \prod_{\substack{l' \leq s \\ l' \leq s'}} \Psi_{J,s_{p'l'},0}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \left\{ \Psi_J^{\otimes k_{l'}} \right\} \Psi \left(\left(\left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right) \right) \\
&\quad \bar{g} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{l' \leq s} \right. \\
&\quad \left. , \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \\
&= \int g \left(\left\{ [p_l, p_{l'l}] \right\}_{l,l'} \right) \\
&\quad \prod_{l'} \mathcal{V}_0 \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{l' \leq s} \right. \\
&\quad \left. , \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\substack{l_1 \leq m_1 \\ l'_1 \leq m'_1}} \right. \\
&\quad \left. , \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \overline{d [p_{l_i}, p_{l'_i l_i}]^{k_{l'_i}}}
\end{aligned}$$

Introducing the sum over copies:

$$\begin{aligned}
& F_{f,lin} \left(\left\{ \Psi_{J,s_{p'l'},0}^{\otimes k_{l'}} \left([p_0, p_{l'0}]^{k_{l'}} \right) \right\}_{(p_{l'0})} \right) \\
&= \sum_m \sum_{m'} \int \sum_{\alpha} \sum_{\substack{(s,s',[p,p',k']) \\ (m,m',[p,p',k'])_1}} d \left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{l' \leq s} \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \bar{g} \left(\left\{ [p_0, p_{l'0}]^{k_{l'}} \right\}_{l' \leq s} \right. \\
&\quad \left. , \left\{ \left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \\
&\times \prod_{\substack{l' \leq s \\ l' \leq s'}} \Psi_{J,\alpha,s_{p'l'},0}^{\otimes k_{l'}} \left(\overline{[p_0, p_{l'0}]^{k_{l'}}}, \left\{ [p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}} \right\}_{\mathcal{P}} \right) \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1 l_1}]^{k_{l'_1}}} \left\{ \Psi_J^{\otimes k_{l'}} \right\} \Psi_{\alpha} \left(\left(\left[\begin{array}{c} [p_{l_i}, p_{l'_i l_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right) \right)
\end{aligned}$$

As before if the set $\left\{ \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right\}$ independent of the realization, it becomes:

$$\begin{aligned}
& F_{f,lin} \left(\left\{ \Psi_{J, s_{p_{l'_i}, 0}}^{\otimes k_{l'_i}} \left([p_0, p_{l'_0}]^{k_{l'_i}} \right) \right\}_{(p_{l'_0})} \right) \\
&= \sum_m \sum_{m'} \int \sum_{\substack{(s, s', [p, p', k']) \\ (m, m', [p, p', k'])_1}} d \left\{ [p_0, p_{l'_0}]^{k_{l'_i}} \right\}_{l' \leq s'}^{l' \leq s} d \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \\
&\times \bar{g} \left(\left\{ [p_0, p_{l'_0}]^{k_{l'_i}} \right\}_{l' \leq s'}^{l' \leq s}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \Psi_{J, \prod_{l'} s_{p_{l'_i}, 0}}^{\otimes \sum_{l'} k_{l'_i}} \left(\prod_{l'} [p_0, p_{l'_0}]^{k_{l'_i}, i}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right)
\end{aligned}$$

where:

$$\begin{aligned}
& \Psi_{J, \prod_{l'} s_{p_{l'_i}, 0}}^{\otimes \sum_{l'} k_{l'_i}} \left(\prod_{l'} [p_0, p_{l'_0}]^{k_{l'_i}, i}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \\
&= \sum_{\mathcal{P}, \alpha} \prod_{\substack{l' \leq s \\ l' \leq s'}} \Psi_{J, \alpha, s_{p_{l'_i}, 0}}^{\otimes k_{l'_i}} \left(\overline{[p_0, p_{l'_0}]^{k_{l'_i}}}, \left\{ [p_{l_1}, p_{l'_1}]^{k_{l'_1}} \right\}_{\mathcal{P}}, \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) \\
&\times \prod_{\mathcal{P}^c} v_{[p_{l_1}, p_{l'_1}]^{k_{l'_1}}} \left\{ \Psi_J^{\otimes k_{l'_i}} \right\} \Psi_{\alpha} \left(\left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\} \right) d \left\{ \left[\begin{array}{c} [p_l, p_{l'_i}, k_{l'_i}] \\ [\{\Psi_J[l', 0]\}, v] \end{array} \right] \right\}
\end{aligned}$$

Appendix 8. Example

An example can be detailed by considering a system involving some boson + fermions system, classically described by the hamiltonian (the free fermion part is omitted):

$$H = \int A(k) k^2 A(-k) + \bar{\psi}(k_1) A(k) \gamma \psi(-k_1 - k)$$

or by the alternative form:

$$\int A(k) k^2 A(-k) + \int \bar{\psi}(k_1) A(k) \gamma \psi(-k_2) \delta(k + k_1 - k_2)$$

We consider this system defined by two spaces of states, one for bosons and one for fermions constrained by momentum conservation. We will rewrite the constraint in a form suitable for the present formalism and then project the state space for bosons along eigenspaces depending on the fermion degrees of freedom. This corresponds to the case where one type of states, i.e., the bosons (corresponding above to the states depending on $U^{(i)}$), is projected onto some subspace to produce an effective theory for the fermions (the subspace parameterized by $U^{(j)}$). However, due to constraints, this projection also depends on the states parameterized by $U^{(j)}$.

The application of our procedure follows several steps. First, rewriting the constraints in terms of operators. Then decomposing the operator along which we want to project in terms of eigenvalues of fermion degrees of freedom. Then, given this decomposition, project each partial Hamiltonian in this decomposition onto the lowest boson eigenstate.

Constraint

First, introducing:

$$\bar{\psi}(k_1) A(k) \gamma \psi(-k_2) = A(k) B(k_1 - k_2)$$

whose commutation relation with momentum operators are given by:

$$\begin{aligned} \left[K_a^{(k)}, A(\hat{k}) \right] &= k A(k) \delta(k - \hat{k}) \\ \left[K_b^{(k')}, B(\hat{k}') \right] &= k' B(k') \delta(k' - \hat{k}') \end{aligned}$$

by imposing the constraints:

$$\left[K_a^{(k)}, A(k) \right] = - \left[K_b^{(k')}, B(k') \right]$$

we recover the initial form including $\delta(k + k_1 - k_2)$. In the basis $|\lambda, k\rangle_A$ for $A(k)$, $|\lambda, k'\rangle_B$ for $B(k')$ the states satisfying the constraint are:

$$\begin{aligned} &\prod |\lambda_1, k_1\rangle_A |\lambda_2, -k_1\rangle_A \dots |\lambda_l, k_l\rangle_A |\mu_1, k_1\rangle_B |\mu_2, -k_1\rangle_B \dots |\mu_l, -k_l\rangle_B \\ &\times \prod_{\hat{k} \neq k_i, \hat{k}' \neq -k_i} |0, \hat{k}\rangle_A \dots |0, \hat{k}'\rangle_B \dots \end{aligned}$$

Decomposition of the hamiltonian in fermions eigenstates

To project the boson states on some eigenspaces, we have to decompose the interaction terms in a way that is diagonal in the fermion degrees of freedom. We write $|S^{(j)}\rangle$ the common eigenvectors of:

$$\int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1$$

and rewrite the Hamiltonian for A :

$$\begin{aligned} H_A &= \int k a_k^+ a_k^- + \left[\langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 | S^{(j)} \rangle \right] a_k^+ + \left[\langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 | S^{(j)} \rangle \right] a_k^- \\ &= \int k \left(a_k^+ + \frac{1}{k} \left[\langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 | S^{(j)} \rangle \right] \right) \left(a_k^- + \frac{1}{k} \left[\langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 | S^{(j)} \rangle \right] \right) \\ &\quad - \frac{1}{k} \left| \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 | S^{(j)} \rangle \right|^2 \end{aligned}$$

For a given state $S^{(j)}$, this can be written $H_A(S^{(j)})$ and this corresponds to the operator $\hat{H}_i(U^{(j)})$ in th decomposition (105). This hamiltonian depends on the fermion degrees of freedom.

Boson eigenstates

The eigenstates for $H_A(|S^{(j)}\rangle)$ are, for gvn state $|S^{(j)}\rangle$:

$$\begin{aligned} \prod \left(a_k^+ + \frac{1}{k} \left[\langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 | S^{(j)} \rangle \right] \right) |\hat{0}\rangle &= \prod \hat{a}_k^+ |\hat{0}_k(S^{(j)})\rangle \\ |\hat{0}_k(S^{(j)})\rangle &= \exp\left(-\frac{c_k^2(S^{(j)})}{2}\right) \exp\left(-c_k(S^{(j)}) a_k^+\right) |0\rangle \\ c_k(S^{(j)}) &= \frac{1}{k} \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 | S^{(j)} \rangle \end{aligned}$$

where we define:

$$|\hat{0}_k(S^{(j)})\rangle \equiv |k, S^{(j)}\rangle$$

. Then:

$$|\hat{0}(k, S^{(j)})\rangle |\hat{0}(k_1 + k_2, S^{(j)})\rangle \delta(k_1 + k_2 - k)$$

Decomposition in bosons eigenstates

The interaction part:

$$\bar{\psi}(k_1) (A(k) \gamma) \psi(-k_1 - k)$$

can then be rewritten decomposing the boson states according to the basis $|N_k\rangle \langle N_k|$ with:

$$N_k = \left(a_k^+ + \frac{1}{k} \left\langle S^{(j)} \left| \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 \right| S^{(j)} \right\rangle \right) \left(a_k^- + \frac{1}{k} \left\langle S^{(j)} \left| \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 \right| S^{(j)} \right\rangle \right)$$

We have:

$$\begin{aligned} & \bar{\psi}(k_1) (A(k) \gamma) \psi(-k_1 - k) \\ = & \int \sum_i |S^{(j)}\rangle \sum_{N_k, N'_k} \langle S^{(j)} | |N_k\rangle \langle N_k| A(k) |N'_k\rangle \langle N'_k| \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 |N'_k\rangle |S^{(j)}\rangle \langle S^{(j)}| dk \end{aligned}$$

Projection on lowest boson eigenstate

Now we project:

$$\sum_{N_k, N'_k} |N_k\rangle \langle N_k| A(k) |N'_k\rangle \langle N'_k|$$

on the lowest boson eigenstate:

$$|\hat{0}_k(S^{(j)})\rangle \langle \hat{0}_k(S^{(j)})|$$

so that the field $A(k)$ is replaced by:

$$\begin{aligned} & |N_k\rangle \langle N_k| A(k) |N'_k\rangle \langle N'_k| \\ \rightarrow & |\hat{0}_k(S^{(j)})\rangle \sum_{N_k, N'_k} \langle \hat{0}_k(S^{(j)}) | |N_k\rangle \langle N_k| A(k) |N'_k\rangle \langle N'_k| |\hat{0}_k(S^{(j)})\rangle \langle \hat{0}_k(S^{(j)})| \end{aligned}$$

Gvn that:

$$\langle N_{k'} | A(k)_1 | N_k \rangle$$

is non nul for $N_{k'} = N_k \pm 1$, we can use that;

$$\langle N_{k'} | A(k)_1 | N_k \rangle = \sqrt{n+1} \delta_{k', k}$$

to rewrite:

$$\begin{aligned} & \sum_{N_k, N'_k} \langle \hat{0}_k(S^{(j)}) | |N_k\rangle \langle N_k| A(k) |N'_k\rangle \langle N'_k| |\hat{0}_k(S^{(j)})\rangle \\ = & \sum_n \exp\left(-\frac{c_k^2(S^{(j)})}{2}\right) c_k^{n+1}(S^{(j)}) \frac{1}{\sqrt{(n+1)!}} \sqrt{n+1} \frac{1}{\sqrt{n!}} c_k^n(S^{(j)}) \exp\left(-\frac{c_k^2(S^{(j)})}{2}\right) \\ = & \sum_n \exp\left(-\frac{c_k^2(S^{(j)})}{2}\right) \exp\left(-\frac{c_k^2(S^{(j)})}{2}\right) \exp(c_k^2(S^{(j)})) c_k(S^{(j)}) \\ = & c_k(S^{(j)}) \end{aligned}$$

As a consequence, the projection of the interaction part along the lowest eigenstates for A wrts:

$$\begin{aligned}
& \bar{\psi}(k_1) (A(k) \gamma) \psi(-k_1 - k) \\
= & \int \sum_i |S^{(j)}\rangle \sum_{N_k, N'_k} \langle S^{(j)} | N_k \rangle \langle N_k | A(k) | N'_k \rangle \langle N'_k | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 | N'_k \rangle |S^{(j)}\rangle \langle S^{(j)} | dk \\
\rightarrow & \int \sum_i |S^{(j)}\rangle |\hat{0}_k(S^{(j)})\rangle c_k(S^{(j)}) \langle \hat{0}_k(S^{(j)}) | \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 |S^{(j)}\rangle \langle S^{(j)} | dk \\
= & \int \sum_i |S^{(j)}\rangle |\hat{0}_k(S^{(j)})\rangle c_k(S^{(j)}) \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 |S^{(j)}\rangle \langle \hat{0}_k(S^{(j)}) | \langle S^{(j)} | dk \\
= & \int \sum_i |S^{(j)}\rangle |\hat{0}_k(S^{(j)})\rangle \frac{1}{k} \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 |S^{(j)}\rangle \\
& \times \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 |S^{(j)}\rangle \langle \hat{0}_k(S^{(j)}) | \langle S^{(j)} | dk
\end{aligned}$$

If:

$$\prod_k |\hat{0}_k(S^{(j)})\rangle \simeq |\hat{0}\rangle$$

the projection reduces to:

$$\begin{aligned}
& \bar{\psi}(k_1) (A(k) \gamma) \psi(-k_1 - k) \\
\rightarrow & |\hat{0}\rangle \int \left(\int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 \right) \frac{1}{k} \left(\int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 \right) dk \langle \hat{0} |
\end{aligned}$$

Note that the projection on the $|\hat{0}_k(S^{(j)})\rangle$ leads to effective amplitudes:

$$\int \sum_i |S^{(j)}\rangle \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 \frac{1}{k} \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 |S^{(j)}\rangle \langle S^{(j)} | dk$$

that is, to an effective self interaction term:

$$\int \left(\int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 \frac{1}{k} \int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 \right) dk$$

Remark 1

The results corresponds in term of path integral to integrate the boson degrees of freedom:

$$\begin{aligned}
& \int \exp \left(- \int (A(k) k^2 A(-k) + \bar{\psi}(k_1) (a_k^+ \gamma) \psi(-k_1 - k) + \bar{\psi}(k_1) (a_k^- \gamma) \psi(-k_1 + k)) \right) \mathcal{D}A(k) \\
= & \exp \left(- \int \left(\int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 \right) \frac{1}{k} \left(\int \bar{\psi}(k_1) \gamma \psi(-k_1 - k) dk_1 \right) \right)
\end{aligned}$$

with boundary conditions $A(k) + \bar{\psi}(k_1) \gamma \psi(-k_1 - k) \rightarrow 0$: which projects on the $|\hat{0}_k(S^{(j)})\rangle$.

Remark 2:

We can rewrite the constraints in terms of conditions on parameter spaces. In this example, states:

$$\prod_k |\hat{0}_k(S^{(j)})\rangle |S^{(j)}\rangle$$

can be parametrized as:

$$\prod_k |\hat{0}_k(\lambda_k)\rangle |S^{(j)}\rangle \delta\left(\lambda_k - \langle S^{(j)} | \int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1 |S^{(j)}\rangle\right)$$

where states $|\hat{0}_k(\lambda_k)\rangle$ are defined by:

$$|\hat{0}_k(\lambda_k)\rangle = \exp\left(-\frac{\lambda_k^2}{2k^2}\right) \exp\left(-\frac{\lambda_k}{k} a_k^+\right) |0\rangle$$

The $\int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1$ commute and the $|S^{(j)}\rangle$ are thus parametrized as:

$$|\{\mu_k\}_{k \in \mathbb{R}}\rangle$$

so that states become:

$$\prod_k |\hat{0}_k(\lambda_k)\rangle |\{\mu_k\}_{k \in \mathbb{R}}\rangle \delta(\lambda_k - \mu_k) = \prod_k |\hat{0}_k(\lambda_k)\rangle |\{\lambda_k\}_{k \in \mathbb{R}}\rangle$$

with the condition that $|\{\lambda_k\}_{k \in \mathbb{R}}\rangle = 0$ for non admissible values of the $\{\lambda_k\}$. The eigenstates of $\int \bar{\psi}(k_1) \gamma \psi(-k_1 + k) dk_1$ and $A(k)$ are constrained by δ condition.

Appendix 9. Formulation in terms of constrained prtrs and description of States:

This section describes more precisely the constraints in the initial parameters spaces as constraints between operators. We also study the consequences of the operator constraints on the states. We provide an example in the end.

Projection operator and Constraints in terms of operators

Starting with operators $(U_j) H_i(U_j)$ acting on the states spaces over the parameter space U_i and depending on the U_j parameters, we assume they can be written as combinations of some operator $\hat{\mathbf{A}}_i(U_j)$ similar to some band hamiltonin:

$$(U_j) H_i(U_j) = H_i(U_j, \hat{\mathbf{A}}_i(U_j)) \quad (173)$$

where $\hat{\mathbf{A}}_i(U^{(j)})$ is a combination of the operators $\mathbf{U}^{(i)}$ of multiplication by sum of tensor products of U_i and $\Pi^{(i)}(U^{(i)})$ is conjugate to the $\mathbf{U}^{(i)}$. The combination depends on $(U^{(j)})$. Acting on the U_i states, prtrs $\hat{\mathbf{A}}_i(U_j)$ belong the the space of operators on the larger states space spanned by the $|\hat{U}^{(i)} | \hat{U}^{(j)}\rangle$. We aim at writing the constraint between $\hat{U}^{(i)}$ and $\hat{U}^{(j)}$ degrees of freedom as a relation between the $\hat{\mathbf{A}}_i(U_j)$ and some operators acting on the state space corresponding to the $\hat{U}^{(j)}$.

We start by considering by some functions, possibly multi-valued, $A^i(U^{(j)})$ satisfying:

$$F(A^i, U^{(j)}) = 0$$

with F a vector value function of $(A^i, U^{(j)})$.

This can be written as an equation operators with eigenvalues $A^i, U^{(j)}$ when operators commute:

$$F(\mathbf{A}^i, \mathbf{U}^{(j)}) = 0$$

The commutation condition is satisfied if we assume that \mathbf{A}^i is a function of the unconstrained generators $\mathbf{\Lambda}_i$ and conjugate $\Pi_{\mathbf{\Lambda}_i}^i$. This is an identity for the matrices elements. The constraints become in terms of operators:

$$F\left(\mathbf{A}^i(\{\mathbf{\Lambda}_i\}, \{\Pi_{\mathbf{\Lambda}_i}^i\}), \mathbf{U}^{(j)}\right) \equiv F\left(\{\mathbf{\Lambda}_i\}, \{\Pi_{\mathbf{\Lambda}_i}^i\}, \mathbf{U}^{(j)}\right) = 0$$

with Π^i conjugate to the $\mathbf{\Lambda}_i$ and $\mathbf{U}^{(j)}$ defined as multiplication over the states $\{|U_j\rangle\}_\lambda$. This allows to rewrite some combination A^j of the eigenvalues of $\mathbf{U}^{(j)}$ as functions of the eigenvals of the $(\mathbf{\Lambda}_i)$. This expression is not local since $\{\mathbf{\Lambda}_i\}, \{\Pi_{\mathbf{\Lambda}_i}^i\}$ do not commute.

The states for gvn λ can then be written:

$$\left\{ |\Sigma(\lambda, \mathbf{\Lambda}_i)\rangle_{\times F(\mathbf{\Lambda}_i, \Pi^i, U_j)} |U_j\rangle \right\}_\lambda$$

As an example, the states $\mathbf{A}^i(\mathbf{\Lambda}_i, \Pi^i) |0\rangle$ can be assumed to be series $\sum_{\alpha, \beta} (\mathbf{\Lambda}_i)^\alpha (\Pi^i)^\beta$ acting on some vacuum $|0\rangle$ with $\mathbf{\Lambda}_i = (\mathbf{\Lambda}_i)^+ + (\mathbf{\Lambda}_i)^-$, $\Pi^i = (\mathbf{\Lambda}_i)^+ - (\mathbf{\Lambda}_i)^-$, and we have:

$$\mathbf{A}^i(\mathbf{\Lambda}_i, \Pi^i) |0\rangle = \sum_m a_m \left((\mathbf{\Lambda}_i)^+ \right)^m |0\rangle$$

and the eigenvalues of $A_i(\mathbf{\Lambda}_i, \Pi^i)$ are sums of tensor powers of eigenvalues $(\mathbf{\Lambda}_i)^{\otimes m}$.

The constrt written in terms of eigenvals:

$$F\left(A^i, U^{(j)}\right) = 0$$

translate in writting p coordinates of states $|U^{(j)}\rangle$ by functionals h of series $\sum_m b_m^j (\mathbf{\Lambda}_i)^{\otimes m}$ and $|U^{(j)}\rangle = \left| h\left(\sum_m b_m^j (\mathbf{\Lambda}_i)^{\otimes m}\right), U^{(j/p)} \right\rangle$ where $U^{(j/p)}$ describes the remaining degrees of freedom.

The states can then be written as:

$$|\lambda(U_j), \{\mathbf{\Lambda}_i\}\rangle \left| h(\{\{\mathbf{\Lambda}_i\}\}), U^{(j/p)} \right\rangle \quad (174)$$

an infinite number of coordinates Λ_i of $\mathbf{\Lambda}_i$ are involvd in the series expansion of this state. More precisely, an infinite sequence consistng of an increasing number of points arises in the expansion. This is similar to the flag manifold described in the first part. Note that we also recover the form of states divided into parameter dependent states and remaining degrees of freedom.

Similar to some covariant formulation, we can also consider the states:

$$|\Sigma(\lambda(U_j), \{\mathbf{\Lambda}_i\})\rangle \left| h(\{\{\mathbf{\Lambda}_i\}\}), U^{(j/p)} \right\rangle$$

In the sequel, we will assume that the constraint is global so that we can write the $(U_j) H_i (U_j)$ as:

$$H_i \left[U^{(j)}, \{\mathbf{\Lambda}_i\} \right] \delta \left(f^{(p_j)} \left(\{\mathbf{\Lambda}_i\}, A^{(j)} \right) \right) \quad (175)$$

where the $\{\mathbf{\Lambda}_i\}$ is a set of unconstrained operators. The constraint has to be understood for the matrices elements of the $f^{(p_j)}(\{\mathbf{\Lambda}_i\}, A^{(j)})$ in a tensor basis of eigenvectors of $\{\mathbf{\Lambda}_i\}$ and $A^{(j)}$. The expansion of $f^{(p_j)}(\{\mathbf{\Lambda}_i\}, A^{(j)})$ is non local and involves infinite number of eigenvlues $\{\mathbf{\Lambda}_i\}$ through integrals.

The eigenstates write:

$$|\lambda(U_j), \{\mathbf{\Lambda}_i\}\rangle \left| h(\{\{\mathbf{\Lambda}_i\}\}), U^{(j/p)} \right\rangle$$

and the $\{\mathbf{\Lambda}_i\}$ rrepresent degeneracies. Or covariantly:

$$|\Sigma(\lambda(U_j), \{\mathbf{\Lambda}_i\})\rangle \left| h(\{\{\mathbf{\Lambda}_i\}\}), U^{(j/p)} \right\rangle$$

Remark: For given $U^{(j)}$ the $\Sigma(\lambda(U_j), \{\mathbf{\Lambda}_i\})$ are eigenstates of commuting set of operators: $\{\mathbf{\Lambda}_i\}(U^{(j)})$:

$$\left(\mathbf{\Lambda}_i^{(1)} \mathbf{\Lambda}_i^{(2)} - \mathbf{\Lambda}_i^{(2)} \mathbf{\Lambda}_i^{(1)} \right) |\Sigma(\lambda(U_j), \{\mathbf{\Lambda}_i\})\rangle = 0$$

Description of the constraint

Using (175):

$$H_i \left[U^{(j)}, \mathbf{A}^{(i)} \right] \delta \left(f^{(p_j)} \left(U^{(j)}, \mathbf{A}^{(i)} \right) \right)$$

there are p_j operators $\mathbf{A}^{(i)}$ combination of some components of $\mathbf{\Lambda}_i$ and conjugate $\Pi_{\mathbf{\Lambda}_i}^i$. Function f with p_j components constrains the $A^{(i)}$ and $U^{(j)}$. When no confusion arises, p_j is written p . The index corresponds to the number of contact point $U^{(j)}$, $U^{(i)}$. There are p_j eigenvalues of $\mathbf{A}^{(i)}$ that can be expressed as functionals of the $\{\mathbf{\Lambda}_i\}$, eigenvalues of $\{\hat{\mathbf{\Lambda}}_i\}$ (see (174)). Introducing the corresponding p_j eigenstates $|a^{(i)}\rangle$ of the operators $\mathbf{A}^{(i)}$:

$$|U^{(j)}\rangle \simeq \text{Vect} \left\{ |a^{(i)}\rangle \right\} |U^{(j)/p}\rangle$$

The state $|U^{(j)/p}\rangle$ denotes the remaining independent degrees of freedom. The $|a^{(i)}\rangle$ generate a subspace defined by parameters $U^{(j)p}$.

$$|U^{(j)}\rangle \simeq |U^{(j)p}\rangle |U^{(j)/p}\rangle$$

After projection on some eigenstates for $\lambda^{(i)}(U^{(j)})$, $\{\mathbf{\Lambda}_i\}$:

$$|\lambda^{(i)}(U^{(j)}), \{\mathbf{\Lambda}_i\}\rangle \delta \left(f \left(\{\mathbf{\Lambda}_i\}, a^{(j)} \right) \right) |U^{(j)}\rangle \simeq |\lambda^{(i)}(U^{(j)}), \{\mathbf{\Lambda}_i\}\rangle |h(\{\{\mathbf{\Lambda}_i\}\}, U^{(j)/p})\rangle |U^{(j)/p}\rangle \quad (176)$$

The function $h(\{\mathbf{\Lambda}_i\}, U^{(j)/p})$ with p components. The dependency in $U^{(j)/p}$ will be considered implicit and we write $h(\{\mathbf{\Lambda}_i\})$.

The states $|\lambda^{(i)}(U^{(j)}), \{\mathbf{\Lambda}_i\}\rangle |h(\{\mathbf{\Lambda}_i\})\rangle$ combine with coefficient $H(h(\{\mathbf{\Lambda}_i\}))$ to produce also states:

$$\int H(h(\{\mathbf{\Lambda}_i\})) |\lambda^{(i)}, \{\mathbf{\Lambda}_i\}\rangle |h(\{\mathbf{\Lambda}_i\})\rangle$$

that correspond to "wave functions" $|H(\{\mathbf{\Lambda}_i\})\rangle$.

We can assume that the eigenvals $\{\mathbf{\Lambda}_i\}$ can be divided into $\{\mathbf{\Lambda}_i\}_p$ and $\{\mathbf{\Lambda}_i\}_{/p}$ so that:

$$|\lambda^{(i)}(U^{(j)}), \{\mathbf{\Lambda}_i\}\rangle |h(\{\mathbf{\Lambda}_i\}, U^{(j)/p})\rangle |U^{(j)/p}\rangle = |\lambda^{(i)}(U^{(j)}), \{\mathbf{\Lambda}_i\}_{/p}\rangle |h(\{\mathbf{\Lambda}_i\}_p, U^{(j)/p})\rangle |U^{(j)/p}\rangle$$

Example

Assume the eigenstates of $\mathbf{A}^{(i)}$

$$|A^{(i)}\rangle = \left| \left(a^{(i)}, n_{a^{(i)}}, \mu_{a^{(i)}} \right) \right\rangle$$

Considering the particular form of (175):

$$\begin{aligned} & H_i \left[U^{(j)}, O^{(i)} \right] \delta \left(f^{(p_j)} \left(U^{(j)}, A^{(i)} \right) \right) \\ &= \sum \left(\left(\left\{ \mathbf{\Lambda}_i \left(U^{(j)} \right) \right\} \right)^2 - \alpha^i \left(U^{(j)} \right) \left\{ \mathbf{\Lambda}_i \left(U^{(j)} \right) \right\} \right)^2 \\ & \quad \times \delta \left(f^{(p_j)} \left(U^{(j)}, A^{(i)} \right) \right) + \alpha \left(U^{(j)} \right) \end{aligned}$$

where:

$$\begin{aligned} \alpha^i \left(U^{(j)} \right) &= \left\langle A^{(i)} \right| \delta \left(N^i - N^j \right) \left| A^{(i)} \right\rangle \delta \left(f^{(p_j)} \left(U^{(j)}, A^{(i)} \right) \right) \\ \alpha \left(U^{(j)} \right) &= \left\langle A^{(i)} \right| N^j \left| A^{(j)} \right\rangle \end{aligned}$$

and $h(\{\mathbf{\Lambda}_i(U^{(j)})\}) = h_j(\mathbf{\Lambda}_i)$

$$\begin{aligned}
|h(\{\{\mathbf{\Lambda}_i(U^{(j)})\})\})\rangle &= |h(\{\mathbf{\Lambda}_i(U^{(j)})\}, n_{a^{(j)}}, \mu_{a^{(j)}})\rangle \\
&\equiv |\{\mathbf{\Lambda}_i\}, n_{\bar{\sigma}^{(i)}}, \mu_{\bar{\sigma}^{(i)}}\rangle \\
&\equiv (A_{\mu_{\bar{\sigma}^{(i)}}}^+)^{n_{\bar{\sigma}^{(i)}}}(\{\mathbf{\Lambda}_i\})|0\rangle
\end{aligned}$$

and for several $\{\mathbf{\Lambda}_i(U^{(j)})\}$

$$\begin{aligned}
|h(\{\mathbf{\Lambda}_i\})\rangle &= |h(\{\mathbf{\Lambda}_i\}, n_{a^{(j)}}, \mu_{a^{(j)}})\rangle \\
&\equiv |\{\mathbf{\Lambda}_i\}, n_{\bar{\sigma}^{(i)}}, \mu_{\bar{\sigma}^{(i)}}\rangle \\
&\equiv (A_{\mu_{\{\mathbf{\Lambda}_i\}}}^+)^{n_{\{\mathbf{\Lambda}_i\}}}(\{\mathbf{\Lambda}_i\})|0\rangle
\end{aligned}$$