THREE-TERM ASYMPTOTIC FORMULA FOR LARGE EIGENVALUES OF THE TWO-PHOTON QUANTUM RABI MODEL

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ANNE BOUTET DE MONVEL[1] AND LECH ZIELINSKI[2]

Abstract. We prove that the spectrum of the two-photon quantum Rabi Hamiltonian consists of two eigenvalue sequences $(E_m^+)_{m=0}^\infty$, $(E_m^-)_{m=0}^\infty$ satisfying a three-term asymptotic formula with the remainder estimate $O(m^{-1}\ln m)$ when m tends to infinity. By analogy to the one-photon quantum Rabi model, the leading three terms of this asymptotic formula, describe a generalized rotating-wave approximation for large eigenvalues of the two-photon quantum Rabi model.

Keywords: unbounded self-adjoint operators, discrete spectrum, asymptotic distribution of eigenvalues, quantum Rabi model

Mathematics Subject Classification: 47A75, 81Q10, 47B25, 47B36.

1. General presentation of the paper

1.1. **Introduction.** The simplest interaction between a two-level atom and a classical light field is described by the semi-classical Rabi model [22], [23]. The quantum Rabi model (QRM) couples a two-level system (TLS) with a quantized single-mode radiation and is considered as a particularly important model in quantum electrodynamics: we refer to [8] concerning the historical aspects of the QRM and to [21] for a list of recent research works and experimental realizations of the QRM.

The simplest QRM is defined by the one-photon Hamiltonian $\mathbf{H}_{\rm Rabi}^{(1)}$ given in Definition 1.2(c). The operator $\mathbf{H}_{\rm Rabi}^{(1)}$ is a self-adjoint operator depending on two real parameters: g (the coupling constant) and Δ (the energy separation in the TLS). Its spectrum is discrete and the fundamental question is how to find a good approximation of the corresponding eigenvalues.

The first step in this direction, is the *rotating-wave approximation* (RWA) introduced in the famous paper of Jaynes and Cummings [20]. However, the RWA is a correct approximation only when g is close to 0 and Δ close to 1 for $\mathbf{H}_{\rm Rabi}^{(1)}$ given in Definition 1.2(c). The most popular idea of going beyond the limitations of RWA, bears the name of the *generalized rotating-wave approximation* (GRWA) after Irish [18]. The same idea was considered before by Feranchuk, Komarov, Ulyanenkov [13], under the name of the *zeroth order approximation of the operator method* (see [14]). According to the GRWA, the spectrum of $\mathbf{H}_{\rm Rabi}^{(1)}$ is composed of two eigenvalue sequences: $(E_m^+)_{m=0}^\infty$ and $(E_m^-)_{m=0}^\infty$, satisfying $E_m^\pm \approx E_{m,\rm GRWA}^\pm$ and the formula (25) in [13] claims that

$$E_{m,\rm GRWA}^\pm \approx m - g^2 \pm \mathfrak{r}_m \text{ as } m \to \infty$$

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holds with

$$\mathfrak{r}_m := \frac{\Delta}{2}\frac{\cos\left(4g\sqrt{m}-\frac{\pi}{4}\right)}{\sqrt{2\pi g\sqrt{m}}}. \tag{1.1}$$

In [3]-[4] we proved that if $\mathfrak{r}_m$ is given by (1.1) and $\varepsilon > 0$, then one has

$$E_m^{\pm} = m - g^2 \pm \mathfrak{r}_m + O(m^{-\frac{1}{2}+\varepsilon}) \text{ as } m \to \infty \tag{1.2}$$

and

$$E_{m,\mathrm{GRWA}}^{\pm} = m - g^2 \pm \mathfrak{r}_m + O(m^{-\frac{1}{2}}) \text{ as } m \to \infty \tag{1.3}$$

Thus (1.2)-(1.3) allow one to estimate the quality of GRWA for large eigenvalues of the one-photon QRM. In this paper we will prove a similar result for the two-photon QRM defined by the Hamiltonian $\mathbf{H}_{\mathrm{Rabi}}^{(2)}$ given in Definition 1.2(d).

We remark that the two-photon QRM was used in [15] to describe a two-level atom interacting with squeezed light and we refer to [9] for the overview of works concerning the two-photon quantum Rabi model (see also [21]). In what follows, we always assume that the coupling constant satisfies the condition $0 < g < 1/2$. This hypothesis implies that $\mathbf{H}_{\mathrm{Rabi}}^{(2)}$ is self adjoint and has a discrete spectrum. In [6] we proved that the spectrum of $\mathbf{H}_{\mathrm{Rabi}}^{(2)}$ is composed of two eigenvalue sequences: $(E_m^+)_{m=0}^{\infty}$ and $(E_m^-)_{m=0}^{\infty}$, satisfying

$$E_m^{\pm} = \left(m+\frac{1}{2}\right)\sqrt{1-4g^2} - \frac{1}{2} + O(m^{-1/3}) \text{ as } m \to \infty \tag{1.4}$$

In this paper we will prove the three-term asymptotic formula

$$E_m^{\pm} = \left(m+\frac{1}{2}\right)\sqrt{1-4g^2} - \frac{1}{2} \pm \mathfrak{r}_m + O(m^{-1}\ln m) \text{ as } m \to \infty \tag{1.5}$$

where $\mathfrak{r}_m$ is given by (1.20). By analogy to (1.2)-(1.3), the expression

$$\left(m+\frac{1}{2}\right)\sqrt{1-4g^2} - \frac{1}{2} \pm \mathfrak{r}_m$$

can be viewed as a GRWA for large eigenvalues of the two-photon QRM.

1.2. **Quantum Rabi Model.**

Notation 1.1. (a) In what follows, $\mathbb{Z}$ is the set of integers and $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\}$.

(b) We denote by $\ell^2(\mathbb{N})$ the complex Hilbert space of square-summable sequences $x : \mathbb{N} \to \mathbb{C}$ equipped with the scalar product

$$\langle x, y\rangle_{\ell^2(\mathbb{N})} = \sum_{m=0}^{\infty} \overline{x(m)}y(m) \tag{1.6}$$

and the norm $||x||_{\ell^2(\mathbb{N})} := \langle x, x\rangle_{\ell^2(\mathbb{N})}^{1/2}$. For $s > 0$ we denote

$$\ell^{2,s}(\mathbb{N}) := \{x \in \ell^2(\mathbb{N}) : \sum_{m=0}^{\infty}(1+m^2)^s\,|x(m)|^2 < \infty\} \tag{1.7}$$

(c) The canonical basis of $\ell^2(\mathbb{N})$ is denoted $\{e_n\}_{n\in\mathbb{N}}$ (i.e. $e_n(m) = \delta_{n,m}$ for n, $m \in \mathbb{N}$).

(d) The annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$, are the linear maps $\ell^{2,1/2}(\mathbb{N}) \to \ell^2(\mathbb{N})$ satisfying

$$\hat{a}^\dagger\, e_n = \sqrt{n+1}\, e_{n+1} \text{ for } n \in \mathbb{N} \tag{1.8}$$

$$\hat{a}\, e_0 = 0 \quad \text{and} \quad \hat{a}\, e_n = \sqrt{n}\, e_{n-1} \text{ for } n \in \mathbb{N}\setminus\{0\}. \tag{1.9}$$

(e) Using $(1,0) \in \mathbb{C}^2$ and $(0,1) \in \mathbb{C}^2$ as the canonical basis of the Euclidean space $\mathbb{C}^2$, we denote by σ_x, σ_z, I_2, the linear operators in $\mathbb{C}^2$ defined by the matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.10}$$

Definition 1.2. (a) The two-level system (TLS) Hamiltonian is the linear map in $\mathbb{C}^2$ defined by the matrix

$$H_{\text{TLS}} = \frac{1}{2} \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} = \frac{1}{2} \Delta \sigma_z \tag{1.11}$$

where Δ is a real parameter.

(b) The Hamiltonian of the single-mode radiation is the linear map $H_{\text{rad}} : \ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by the formula

$$H_{\text{rad}} e_n = \hat{a}^\dagger \hat{a}\, e_n = n e_n \text{ for } n \in \mathbb{N}. \tag{1.12}$$

(c) The *one-photon quantum Rabi Hamiltonian* is defined as the linear map $\mathbf{H}^{(1)}_{\text{Rabi}} : \mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \to \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ given by

$$\mathbf{H}^{(1)}_{\text{Rabi}} = I_2 \otimes H_{\text{rad}} + H_{\text{TLS}} \otimes I_{\ell^2(\mathbb{N})} + g\sigma_x \otimes \big(\hat{a} + \hat{a}^\dagger\big), \tag{1.13}$$

where $g > 0$ is the coupling constant.

(d) The *two-photon quantum Rabi Hamiltonian* is defined as the linear map $\mathbf{H}^{(2)}_{\text{Rabi}} : \mathbb{C}^2 \otimes \ell^{2,1}(\mathbb{N}) \to \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$ given by

$$\mathbf{H}^{(2)}_{\text{Rabi}} = I_2 \otimes H_{\text{rad}} + H_{\text{TLS}} \otimes I_{\ell^2(\mathbb{N})} + g\sigma_x \otimes \big(\hat{a}^2 + (\hat{a}^\dagger)^2\big), \tag{1.14}$$

where $g > 0$ is the coupling constant.

1.3. **Main result.** In what follows, we assume $0 < g < 1/2$ and introduce

$$\beta := \sqrt{1 - 4g^2}, \tag{1.15}$$

$$\alpha := \arctan\left(\sqrt{\frac{1-2g}{1+2g}}\right). \tag{1.16}$$

Let $\mathbf{H}^{(2)}_{0,\text{Rabi}}$ denote the operator given by (1.14) with $\Delta = 0$, i.e.

$$\mathbf{H}^{(2)}_{0,\text{Rabi}} = I_2 \otimes H_{\text{rad}} + g\sigma_x \otimes \big(\hat{a}^2 + (\hat{a}^\dagger)^2\big). \tag{1.17}$$

If $0 < g < 1/2$, then the spectrum of $\mathbf{H}^{(2)}_{0,\text{Rabi}}$ is explicitly known (see [11], [12]): it is composed of the sequence of eigenvalues

$$E^0_m = m\beta + (\beta - 1)/2, \quad m = 0,\, 1,\, 2, \ldots \tag{1.18}$$

and each eigenvalue E^0_m is of multiplicity 2. Thus $0 < g < 1/2$ ensures the fact that $\mathbf{H}^{(2)}_{0,\text{Rabi}}$ is self-adjoint and has compact resolvent. Since $\mathbf{H}^{(2)}_{\text{Rabi}} - \mathbf{H}^{(2)}_{0,\text{Rabi}}$ is bounded, the operator $\mathbf{H}^{(2)}_{\text{Rabi}}$ is self-adjoint and has compact resolvent if $0 < g < 1/2$. The explicit values of eigenvalues of $\mathbf{H}^{(2)}_{\text{Rabi}}$ are not known when $\Delta \neq 0$, but we can describe their asymptotic behaviour. Our main result is the following

Theorem 1.3. *Assume that $0 < g < 1/2$. Then one can find $\{v_m^+\}_{m\in\mathbb{N}} \cup \{v_m^-\}_{m\in\mathbb{N}}$, an orthonormal basis of $\mathbb{C}^2 \otimes \ell^2(\mathbb{N})$, such that*

$$\mathbf{H}_{\mathrm{Rabi}}^{(2)} v_m^\pm = E_m^\pm v_m^\pm, \quad m = 0,\, 1,\, 2, \dots$$

and the eigenvalue sequences $(E_m^+)_{m\in\mathbb{N}}$, $(E_m^-)_{m\in\mathbb{N}}$, satisfy the large m estimates

$$E_m^\pm = m\beta + (\beta - 1)/2 \pm \mathfrak{r}_m + O(m^{-1}\ln m) \tag{1.19}$$

with $\mathfrak{r}_m$ given by the formula

$$\mathfrak{r}_m = \begin{cases} \dfrac{\Delta}{2}\sqrt{\dfrac{\beta}{\pi g m}}\ \cos((2m+1)\alpha) & \text{if } m \text{ is even} \\ \dfrac{\Delta}{2}\sqrt{\dfrac{\beta}{\pi g m}}\ \sin((2m+1)\alpha) & \text{if } m \text{ is odd,} \end{cases} \tag{1.20}$$

where β is given by (1.15) *and α is given by* (1.16).

Remarks. (a) One has $E_m^\pm - E_m^0 = O(m^{-1/2})$ in spite of the fact that $\mathbf{H}_{\mathrm{Rabi}}^{(2)} - \mathbf{H}_{0,\mathrm{Rabi}}^{(2)}$ is not compact. A similar fact was established in [30] for the one-photon Rabi model.

(b) Following [25], one can prove that the spectrum of $\mathbf{H}_{\mathrm{Rabi}}^{(2)}$ is not discrete if $g \geq \frac{1}{2}$.

1.4. **Overview of earlier results.** The distribution of large eigenvalues of the QRM has been a subject of many numerical investigations. Concerning the earliest physical considerations, we refer to Schmutz [26]. It is well known (see e.g [28]) that the one-photon quantum Rabi Hamiltonian can be expressed as a direct sum of two Jacobi operators, i.e. operators defined by infinite tridiagonal matrices. Similarly, the two-photon quantum Rabi Hamiltonian can be expressed as a direct sum of four Jacobi operators (see [6]). A mathematical study of large eigenvalues of Jacobi matrices was initiated by J. Janas and S. Naboko in the paper [19], which contains fundamental ideas of the method of approximate diagonalizations.

The question of the behaviour of large eigenvalues of $\mathbf{H}_{\mathrm{Rabi}}^{(1)}$, was first posed by E. A. Tur [28]-[29] and it was considered by A. Boutet de Monvel, S. Naboko and L. O. Silva in [1]-[2]. Due to the difficulty of the problem, the authors give a solution for a simpler class of operators ("modified Jaynes-Cummings models"). The two-term asymptotic formula for large eigenvalues of the one-photon quantum Rabi model, was proved by E. A. (Tur) Yanovich in [30] (see also [29]). The three-term asymptotic formula (1.2) was proved in [3]-[4]. We remark that the formula (1.2) allows one to recover the values of parameters g and Δ from the spectrum (see [5]). The two-term asymptotic formula (1.4) for large eigenvalues of the two-photon quantum Rabi model, was proved in [6]. The result of Theorem 1.3 and the main ideas of its proof were described in [7]. The same result was also described in [17].

1.5. **Organization of the paper.** Our approach is based on an analysis of operators acting in $\ell^2(\mathbb{Z})$. These operators are introduced in Section 2.1 and the corresponding three-term asymptotic formula is given in Theorem 2.3. In Theorem 2.3(a) we consider the operators $\tilde{J}_\gamma^0$, which are special case of operators considered by Edward [10]. In Section 2.2 we describe explicit expressions of their eigenvalues and eigenvectors by means of the discrete Fourier transform. In Theorem 2.3(b) we consider the operators

$\tilde{J}^{\delta}_{\gamma}$, which are viewed as perturbations of $\tilde{J}^{0}_{\gamma}$. In Section 3 we show that the assertion of Theorem 2.3(b) follows from a GRWA result given in Proposition 3.1. The proof of Proposition 3.1 begins in Section 4 and is completed in Section 5. In Section 6 we show how to deduce Theorem 1.3 from Theorem 2.3(b). More precisely, we follow the following scheme:

- we prove Proposition 3.1, using Lemma 4.3 and Kato-Temple estimate (Theorem 5.2)
- we prove Theorem 2.3, using Lemma 2.6, Proposition 3.1 and an asymptotic formula from [6]
- we prove Theorem 1.3, using Theorem 2.3, Lemma 6.2 and Lemma 6.3.

2. An auxiliary problem in $\ell^2(\mathbb{Z})$

2.1. Behaviour of large eigenvalues for auxiliary operators $\tilde{J}^{\delta}_{\gamma}$.

Notation 2.1. (a) We denote by $\mathcal{B}(\mathcal{V})$ the algebra of bounded operators on the Banach space $\mathcal{V}$ and $||\cdot||_{\mathcal{B}(\mathcal{V})}$ is the corresponding operator norm.

(b) We denote by $\ell^2(\mathbb{Z})$ the complex Hilbert space of square-summable sequences $x : \mathbb{Z} \to \mathbb{C}$ equipped with the scalar product

$$\langle x, y\rangle_{\ell^2(\mathcal{Z})} = \sum_{k\in\mathbb{Z}} \overline{x(k)}y(k) \tag{2.1}$$

and the norm $||x||_{\ell^2(\mathbb{Z})} := \langle x, x\rangle^{1/2}_{\ell^2(\mathbb{Z})}$. For $s > 0$ we denote

$$\ell^{2,s}(\mathbb{Z}) := \{x \in \ell^2(\mathbb{Z}) :\ ||x||_{\ell^{2,s}(\mathbb{Z})} < \infty\}$$

where

$$||x||_{\ell^{2,s}(\mathbb{Z})} := \Big(\sum_{k\in\mathbb{Z}} (1+k^2)^s\,|x(k)|^2\Big)^{1/2} \tag{2.2}$$

(c) The canonical basis of $\ell^2(\mathbb{Z})$ is denoted $\{\tilde{e}_j\}_{j\in\mathbb{Z}}$ (i.e. $\tilde{e}_j(k) = \delta_{j,k}$ for j, $k \in \mathbb{Z}$) and $\ell^2_{\mathrm{fin}}(\mathbb{Z})$ denotes the set of finite linear combinations of vectors belonging to $\{\tilde{e}_j\}_{j\in\mathbb{Z}}$.

(d) If $(\tilde{d}_j)_{j\in\mathbb{Z}}$ is real valued, then $\tilde{D} := \mathrm{diag}(\tilde{d}_j)_{j\in\mathbb{Z}}$ is the self-adjoint operator in $\ell^2(\mathbb{Z})$ satisfying

$$\tilde{D}\,\tilde{e}_j = \tilde{d}_j\tilde{e}_j \text{ for every } j \in \mathbb{Z}.$$

We denote

$$\Lambda := \mathrm{diag}(j)_{j\in\mathbb{Z}}. \tag{2.3}$$

(e) We denote by S the shift defined in $\ell^2(\mathbb{Z})$ by the formula $(Sx)(j) = x(j-1)$.

Definition 2.2. In what follows, γ, δ and g are fixed real numbers.

(a) We define $\tilde{J}^{0}_{\gamma}$ as the linear map $\ell^{2,1}(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by

$$\tilde{J}^{0}_{\gamma} := \Lambda + g\left(S\big(\Lambda+\gamma\big) + \big(\Lambda+\gamma\big)S^{-1}\right) = \Lambda + g\left(S(\Lambda+\gamma) + \text{ h.c.}\right) \tag{2.4}$$

(b) We define $\tilde{J}^{\delta}_{\gamma}$ as the linear map $\ell^{2,1}(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by the formula

$$\tilde{J}^{\delta}_{\gamma} := \tilde{J}^{0}_{\gamma} + D_{\delta}, \tag{2.5}$$

where

$$D_{\delta} := \mathrm{diag}(\delta(-1)^j)_{j\in\mathbb{Z}}. \tag{2.6}$$

Using the above definition, we find that the action of $\tilde{J}^\delta_\gamma$ can be represented by the following tridiagonal, $\mathbb{Z}\times\mathbb{Z}$ matrix

$$\begin{pmatrix} \ddots & & & & & & \\ & -2+\delta & g(-2+\gamma) & 0 & 0 & 0 & \\ & g(-2+\gamma) & -1-\delta & g(-1+\gamma) & 0 & 0 & \\ & 0 & g(-1+\gamma) & \delta & g\gamma & 0 & \\ & 0 & 0 & g\gamma & 1-\delta & g(1+\gamma) & \\ & 0 & 0 & 0 & g(1+\gamma) & 2+\delta & \\ & & & & & & \ddots \end{pmatrix}$$

whose diagonal entries $\{\tilde{d}_\delta(j)\}_{j\in\mathbb{Z}}$ are given by

$$\tilde{d}_\delta(j) := j + \delta(-1)^j, \tag{2.7}$$

off-diagonal entries $\{\tilde{b}_\gamma(j)\}_{j\in\mathbb{Z}}$ are given by

$$\tilde{b}_\gamma(j) := g(j+\gamma) \tag{2.8}$$

and $\tilde{J}^\delta_\gamma \tilde{e}_j = \tilde{d}_\delta(j)\tilde{e}_j + \tilde{b}_\gamma(j)\tilde{e}_{j+1} + \tilde{b}_\gamma(j-1)\tilde{e}_{j-1}$ for every $j\in\mathbb{Z}$.

Theorem 2.3. *Let $\tilde{J}^0_\gamma$ be given by* (2.4) *and $\tilde{J}^\delta_\gamma$ by* (2.5)-(2.6). *If $0<g<1/2$ then*
(a) the spectrum of $\tilde{J}^0_\gamma$ is composed of a non-decreasing sequence of eigenvalues $\{d_{\gamma,j}\}_{j\in\mathbb{Z}}$ of the form

$$d_{\gamma,j} := \beta j + \Big(\gamma - \frac{1}{2}\Big)(\beta - 1), \tag{2.9}$$

where β is given by (1.15).
(b) the spectrum of $\tilde{J}^\delta_\gamma$ is composed of a non-decreasing sequence of eigenvalues $\{\lambda_j(\tilde{J}^\delta_\gamma)\}_{j\in\mathbb{Z}}$ which can be labeled so that

$$\lambda_j(\tilde{J}^\delta_\gamma) = d_{\gamma,j} + \mathfrak{r}^\delta_\gamma(j) + O(j^{-1}\ln j) \text{ as } j\to\infty \tag{2.10}$$

where $d_{\gamma,j}$ is given by (2.9),

$$\mathfrak{r}^\delta_\gamma(j) = \delta\Big(\frac{\beta}{2\pi g j}\Big)^{1/2}\cos\big(4\alpha j + \hat{\theta}_\gamma\big), \tag{2.11}$$

β is given by (1.15), *α by* (1.16) *and*

$$\hat{\theta}_\gamma = \Big(\gamma - \frac{1}{2}\Big)(4\alpha - \pi) + \frac{\pi}{4}. \tag{2.12}$$

Proof. (a) This result was proved in [10]. We describe a simplified proof in Section 2.2.
(b) See Section 3.2. □

2.2. **Diagonalization of $\tilde{J}^0_\gamma$.** In what follows, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is identified with $]-\pi,\pi]$ and $\mathrm{L}^2(\mathbb{T})$ denotes the Hilbert space of Lebesgue square integrable functions $]-\pi,\pi]\to\mathbb{C}$ equipped with the scalar product

$$\langle f, g\rangle_{\mathrm{L}^2(\mathbb{T})} := \int_{-\pi}^{\pi} \overline{f(\theta)}\, g(\theta)\, \frac{\mathrm{d}\theta}{2\pi} \tag{2.13}$$

and the norm $||f||_{\mathrm{L}^2(\mathbb{T})} = \langle f, f\rangle^{1/2}_{\mathrm{L}^2(\mathbb{T})}$. We let $\mathcal{F}_\mathbb{T}$ denote the isometric isomorphism $\mathrm{L}^2(\mathbb{T})\to\ell^2(\mathbb{Z})$ given by

$$(\mathcal{F}_\mathbb{T} f)(j) = \int_{-\pi}^{\pi} f(\theta)\, \mathrm{e}^{-\mathrm{i}j\theta}\, \frac{\mathrm{d}\theta}{2\pi} \tag{2.14}$$

and consider the operator

$$L^0_\gamma := \mathcal{F}_{\mathbb{T}}^{-1}\tilde{J}^0_\gamma\mathcal{F}_{\mathbb{T}} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta} + g\left(\mathrm{e}^{\mathrm{i}\theta}\Big(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}+\gamma\Big) + \ \text{h.c.}\right)$$

Similarly as in [10] and [6], we observe that the assumption $0 < g < 1/2$ ensures the fact that L^0_γ is the first order linear elliptic differential operator,

$$L^0_\gamma = \frac{1}{2}\left(\big(1+2g\cos(\theta)\big)\Big(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big) \ + \ \text{h.c.}\right) + (2\gamma-1)g\cos(\theta)$$

and we introduce

$$\Phi(\theta) := \int_0^\theta \frac{\beta\,\mathrm{d}\theta'}{1+2g\cos(\theta')}. \tag{2.15}$$

An easy calculation allows one to express the right-hand side of (2.15),

$$\Phi(\theta) = 2\arctan\left(\sqrt{\frac{1-2g}{1+2g}}\tan\Big(\frac{\theta}{2}\Big)\right) \ \text{ if } \ -\pi<\theta<\pi. \tag{2.16}$$

Moreover $\Phi(\pi)=\pi$ and Φ is odd, i.e. $\Phi(-\theta)=-\Phi(\theta)$. Using the change of variable $\eta=\Phi(\theta)$, we can define the unitary operator acting in $\mathrm{L}^2(\mathbb{T})$ according to the formula

$$(U_\Phi f)(\theta) = \Phi'(\theta)^{1/2} f(\Phi(\theta)) \tag{2.17}$$

and the direct computation (see [6]) gives

$$U_\Phi^{-1}\,L^0_\gamma\,U_\Phi = \beta\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\eta} + q_\gamma(\eta)\right) \tag{2.18}$$

with

$$q_\gamma(\eta) := \beta^{-1}(2\gamma-1)g\cos(\Phi^{-1}(\eta)). \tag{2.19}$$

In the following, q_γ is given by (2.19) and

$$\widetilde{q}_\gamma(\eta) := \int_0^\eta q_\gamma(\eta')\,\mathrm{d}\eta'. \tag{2.20}$$

We claim that

$$\tilde{q}_\gamma(\Phi(\theta)) = \Big(\gamma-\frac{1}{2}\Big)(\theta-\beta^{-1}\Phi(\theta)). \tag{2.21}$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\tilde{q}_\gamma(\Phi(\theta))\Big) = q_\gamma(\Phi(\theta))\Phi'(\theta) = \frac{(2\gamma-1)g\cos(\theta)}{\beta}\,\frac{\beta}{1+2g\cos(\theta)} =$$

$$= \frac{2\gamma-1}{2}\left(1-\frac{1}{1+2g\cos(\theta)}\right) = \Big(\gamma-\frac{1}{2}\Big)(1-\beta^{-1}\Phi'(\theta))$$

implies $\tilde{q}_\gamma(\Phi(\theta)) = (\gamma-\frac{1}{2})(\theta-\beta^{-1}\Phi(\theta)) + C_0$ and $C_0=0$ holds due to $\Phi(0)=0$ and $\tilde{q}_\gamma(\Phi(0)) = \tilde{q}_\gamma(0) = 0$.

Using $\Phi(\pm\pi)=\pm\pi$ in (2.21), we compute

$$\langle q_\gamma\rangle := \frac{\tilde{q}_\gamma(\pi)-\tilde{q}_\gamma(-\pi)}{2\pi} = \Big(\gamma-\frac{1}{2}\Big)(1-\beta^{-1}) \tag{2.22}$$

and remark that $\eta\to\langle q_\gamma\rangle\eta - \tilde{q}_\gamma(\eta)$ is a smooth 2π-periodic function. Then we define $(f_{q_\gamma,j})_{j\in\mathbb{Z}}$ to be the orthonormal basis in $\mathrm{L}^2(\mathbb{T})$ given by

$$f_{q_\gamma,j}(\eta) = \mathrm{e}^{\mathrm{i}j\eta}\,\mathrm{e}^{\mathrm{i}(\langle q_\gamma\rangle\eta-\tilde{q}_\gamma(\eta))} \tag{2.23}$$

and (see [7], [10]), for every $j \in \mathbb{Z}$, we get

$$\beta\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\eta}+q_\gamma\right)f_{q_\gamma,j}=d_{\gamma,j}f_{q_\gamma,j}, \tag{2.24}$$

where $d_{\gamma,j}=\beta(j+\langle q_\gamma\rangle)=\beta j+(\gamma-1/2)(\beta-1)$ is given (2.9). Combining (2.24) with (2.18), we obtain

Corollary 2.4. *Let $\{f^0_{\gamma,j}\}_{j\in\mathbb{Z}}$ be the orthonormal basis of $\mathrm{L}^2(\mathbb{T})$ given by*

$$f^0_{\gamma,j}(\theta):=(U_\Phi f_{q_\gamma,j})(\theta)=\Phi'(\theta)^{1/2}\mathrm{e}^{\mathrm{i}j\Phi(\theta)}\,\mathrm{e}^{\mathrm{i}(\langle q_\gamma\rangle\Phi(\theta)-\tilde q_\gamma(\Phi(\theta)))}. \tag{2.25}$$

Then

$$L^0_\gamma f^0_{\gamma,j}=d_{\gamma,j}f^0_{\gamma,j} \tag{2.26}$$

holds with $d_{\gamma,j}$ given (2.9).

It is clear that the assertion of Theorem 2.3(a) follows from Corollary 2.4. Indeed, using (2.26) and $\tilde J^0_\gamma=\mathcal{F}_\mathbb{T}L^0_\gamma\mathcal{F}_\mathbb{T}^{-1}$, we get

$$\tilde J^0_\gamma u^0_{\gamma,j}=d_{\gamma,j}u^0_{\gamma,j}\text{ with }u^0_{\gamma,j}:=\mathcal{F}_\mathbb{T}f^0_{\gamma,j}. \tag{2.27}$$

2.3. **An auxiliary estimate.**

Notation 2.5. (a) For $f\in\mathrm{L}^2(\mathbb{T})$ we write $T_\pi f:=f\circ\tau_\pi$ with $\tau_\pi:\mathbb{T}\to\mathbb{T}$ given by

$$\tau_\pi(\theta+2\pi\mathbb{Z}):=\theta+\pi+2\pi\mathbb{Z} \tag{2.28}$$

(b) For j, $n\in\mathbb{Z}$, we denote

$$V^\delta_\gamma(j,n):=\delta\langle f^0_{\gamma,j},T_\pi f^0_{\gamma,n}\rangle_{\mathrm{L}^2(\mathbb{T})}, \tag{2.29}$$

where $f^0_{\gamma,j}$ is given by (2.25).

Lemma 2.6. *Let $\hat c>0$ be small enough. Then there exists $\hat C>0$ such that*

$$\sup_{\{k\in\mathbb{Z}:\,|k|\le\hat c|n|\}}|V^\delta_\gamma(n+k,n)|\le\hat C|n|^{-1/2} \tag{2.30}$$

holds for every $n\in\mathbb{Z}\setminus\{0\}$.

Proof. Let us assume that $n\neq0$. By definition (2.25), $f^0_{\gamma,n}=\mathrm{e}^{\mathrm{i}n\Phi}p_\gamma$ holds with

$$p_\gamma(\theta):=\mathrm{e}^{\mathrm{i}\langle q_\gamma\rangle\Phi(\theta)-\mathrm{i}\widetilde q_\gamma(\Phi(\theta))}\beta^{1/2}(1+2g\cos\theta)^{-1/2} \tag{2.31}$$

and we can express

$$V^\delta_\gamma(n+k,n)=\delta\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}n\Psi_{n,k}(\theta)}\,p_\gamma(\theta)\,\overline{(T_\pi p_\gamma)(\theta)}\,\frac{\mathrm{d}\theta}{2\pi} \tag{2.32}$$

with

$$\Psi_{n,k}=\Phi-T_\pi\Phi+\frac{k}{n}\Phi. \tag{2.33}$$

Denote $\Psi:=\Phi-T_\pi\Phi$. Then

$$\Psi'(\theta)=\frac{\beta}{1+2g\cos\theta}-\frac{\beta}{1-2g\cos\theta}=0\ \Leftrightarrow\ \cos\theta=0, \tag{2.34}$$

i.e. Ψ has two critical points: $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since

$$\Psi''(\theta)=\left(\frac{2g\beta}{(1+2g\cos\theta)^2}+\frac{2g\beta}{(1-2g\cos\theta)^2}\right)\sin\theta, \tag{2.35}$$

we obtain $\Psi''(\pm\frac{\pi}{2}) = \pm 4g\beta \neq 0$ and choosing $c_0 > 0$ small enough we can ensure

$$|\Psi''(\theta)| \geq c_0 \text{ for } \theta \in \mathcal{I}_{c_0} := [-\tfrac{\pi}{2} - c_0, -\tfrac{\pi}{2} + c_0] \cup [\tfrac{\pi}{2} - c_0, \tfrac{\pi}{2} + c_0].$$

However $\Psi_{n,k} = \Psi + \frac{k}{n}\Phi$ and denoting $C_0 = \sup|\Phi''|$ we can estimate

$$|\Psi''_{n,k}(\theta)| \geq |\Psi''(\theta)| - C_0 \left|\frac{k}{n}\right|$$

and

$$\left|\frac{k}{n}\right| \leq \frac{c_0}{2C_0} \Longrightarrow |\Psi''_{n,k}(\theta)| \geq \frac{c_0}{2} \text{ for } \theta \in \mathcal{I}_{c_0}. \tag{2.36}$$

Due to van der Corput Lemma (see Lemma 7.1(b)), there exists $C > 0$ such that

$$\left| \int_{\mathcal{I}_{c_0}} \mathrm{e}^{\mathrm{i}n\Phi_{n,k}(\theta)}\, p_\gamma(\theta)\, \overline{(T_\pi p_\gamma)(\theta)}\, \frac{\mathrm{d}\theta}{2\pi} \right| \leq C|n|^{-1/2}. \tag{2.37}$$

Moreover

$$c_1 := \inf\{|\Psi'(\theta)| : \theta \in [-\pi, \pi] \setminus \mathcal{I}_{c_0}\} > 0$$

and denoting $C_1 = \sup|\Phi'|$ we can estimate

$$|\Psi'_{n,k}(\theta)| \geq |\Psi'(\theta)| - C_1 \left|\frac{k}{n}\right|,$$

hence

$$\left|\frac{k}{n}\right| \leq \frac{c_1}{2C_1} \Longrightarrow |\Psi'_{n,k}(\theta)| \geq \frac{c_1}{2} \text{ for } \theta \in [-\pi, \pi] \setminus \mathcal{I}_{c_0}. \tag{2.38}$$

Due to the non-stationary phase estimate (see Lemma 7.1(a)), there exists a constant $C' > 0$ such that

$$\left| \int_{[-\pi,\pi]\setminus\mathcal{I}_{c_0}} \mathrm{e}^{\mathrm{i}n\Phi_{n,k}(\theta)}\, p_\gamma(\theta)\, \overline{(T_\pi p_\gamma)(\theta)}\, \frac{\mathrm{d}\theta}{2\pi} \right| \leq C'|n|^{-1}. \tag{2.39}$$

Combining (2.37), (2.39) and (2.32), we complete the proof. □

3. The GRWA approch

3.1. **Statement of the result.** The idea of GRWA consists in using the diagonal entries of a perturbation as the first correction for eigenvalues of a perturbed diagonal matrix (see [18] and [14]). Our proof of Theorem 2.3 is based on the following variant of the GRWA :

Proposition 3.1. *Let us fix $\beta_0 \in \mathbb{R}$, $\beta > 0$, $\hat{C} > 0$ and $\hat{c} > 0$. Let $J : \ell^{2,1}(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by the formula*

$$J = \beta\Lambda + \beta_0 + V, \tag{3.1}$$

where $\Lambda = \mathrm{diag}(j)_{j\in\mathbb{Z}}$ and V is a bounded self-adjoint operator in $\ell^2(\mathbb{Z})$ satisfying

$$\sup_{\{k\in\mathbb{Z}:\, |k|\leq\hat{c}|n|\}} |\langle \tilde{e}_{n+k}, V\tilde{e}_k\rangle_{\ell^2(\mathbb{Z})}| \leq \hat{C}|n|^{-1/2} \tag{3.2}$$

for every $n \in \mathbb{Z}\setminus\{0\}$. Assume that the spectrum of J is composed of a non-decreasing sequence of eigenvalues $\{\lambda_j(J)\}_{j\in\mathbb{Z}}$ such that

$$\lambda_j(J) = j\beta + \beta_0 + o(1) \text{ as } j \to \infty. \tag{3.3}$$

Then one has

$$\lambda_j(J) = j\beta + \beta_0 + \langle \tilde{e}_j, V\tilde{e}_j\rangle_{\ell^2(\mathbb{Z})} + O(j^{-1}\ln j) \text{ as } j \to \infty \tag{3.4}$$

Proof. See Section 5. □

3.2. **Proof of Theorem 2.3(b).** In what follows, we describe how to deduce the assertion of Theorem 2.3(b) from Proposition 3.1. We begin by introducing $\tilde{U}_\gamma$, the unitary operator in $\ell^2(\mathbb{Z})$ satisfying $\tilde{U}_\gamma \tilde{e}_j = u^0_{\gamma,j}$, where $\{\tilde{e}_j\}_{j\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$ and $\{u^0_{\gamma,j}\}_{j\in\mathbb{Z}}$ is the basis introduced in (2.27). We claim that the hypotheses of Proposition 3.1 are satisfied if

$$J = \tilde{U}_\gamma^{-1} \tilde{J}^\delta_\gamma \tilde{U}_\gamma, \tag{3.5}$$

where J^δ_γ is as in Definition 2.2. Indeed, if β is given by (1.15) and

$$\beta_0 = \Big(\gamma - \frac{1}{2}\Big)(\beta - 1), \tag{3.6}$$

then $\tilde{U}_\gamma^{-1} \tilde{J}^\delta_\gamma \tilde{U}_\gamma = \beta\Lambda + \beta_0 + V^\delta_\gamma$ holds with

$$V^\delta_\gamma := \tilde{U}_\gamma^{-1} D_\delta \, \tilde{U}_\gamma, \tag{3.7}$$

where $D_\delta = \mathrm{diag}(\delta(-1)^j)_{j\in\mathbb{Z}}$ (see (2.6)). Therefore

$$\langle \tilde{e}_j, V^\delta_\gamma \tilde{e}_k\rangle_{\ell^2(\mathbb{Z})} = \langle u^0_{j,\gamma}, D_\delta \, u^0_{j,\gamma}\rangle_{\ell^2(\mathbb{Z})} = \langle f^0_{j,\gamma}, \mathcal{F}_\mathbb{T}^{-1} D_\delta \, \mathcal{F}_\mathbb{T} f^0_{j,\gamma}\rangle_{\mathrm{L}^2(\mathbb{T})}, \tag{3.8}$$

where we used $u^0_{j,\gamma} = \mathcal{F}_\mathbb{T} f^0_{j,\gamma}$ and the isometry $\mathcal{F}_\mathbb{T}$. Since $\mathcal{F}_\mathbb{T}^{-1} D_\delta \mathcal{F}_\mathbb{T} = \delta T_\pi$ holds with T_π as in Notation 2.5, (3.8) gives

$$\langle \tilde{e}_j, V^\delta_\gamma \tilde{e}_k\rangle_{\ell^2(\mathbb{Z})} = V^\delta_\gamma(j,k) \tag{3.9}$$

with $V^\delta_\gamma(\cdot,\cdot)$ expressed by (2.29) and due to Lemma 2.6, the estimate (3.2) holds if $V = V^\delta_\gamma$ is given by (3.7). Moreover, the result of [6, Proposition 5.1] ensures the fact that the spectrum of $\tilde{J}^\delta_\gamma$ is composed of the non-decreasing sequence of eigenvalues $\{\lambda_j(\tilde{J}^\delta_\gamma)\}_{j\in\mathbb{Z}}$, which can be labeled so that one has the estimate

$$\lambda_j(\tilde{J}^\delta_\gamma) = j\beta + \beta_0 + O(j^{-1/3}) \text{ as } j \to \infty, \tag{3.10}$$

i.e. (3.3) holds if J is given by (3.5). Thus all the hypotheses of Proposition 3.1 hold and choosing $\{\lambda_j(\tilde{J}^\delta_\gamma)\}_{j\in\mathbb{Z}}$ to be the sequence of eigenvalues of $\tilde{J}^\delta_\gamma$ such that $\lambda_j(\tilde{J}^\delta_\gamma) = \lambda_j(J)$ for every $j \in \mathbb{Z}$, we obtain

$$\lambda_j(\tilde{J}^\delta_\gamma) = j\beta + \beta_0 + \langle \tilde{e}_j, V^\delta_\gamma \tilde{e}_j\rangle_{\ell^2(\mathbb{Z})} + O(j^{-1}\ln j) \text{ when } j \to \infty \tag{3.11}$$

due to (3.4). The assertion of Theorem 2.3(b) follows from (3.11) and

$$\langle \tilde{e}_j, V^\delta_\gamma \tilde{e}_j\rangle_{\ell^2(\mathbb{Z})} = \mathfrak{r}^\delta_\gamma(j) + O(j^{-1}) \text{ when } j \to \infty. \tag{3.12}$$

where $\mathfrak{r}^\delta_\gamma$ is given by (2.11)-(2.12). The proof of (3.12) is given in Section 3.3.

3.3. **Proof of** (3.12)**.** Due to (2.32),

$$V^\delta_\gamma(n,n) = \delta \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\Psi(\theta)} h_\gamma(\theta) \frac{\mathrm{d}\theta}{2\pi}, \tag{3.13}$$

where $\Psi := \Phi - T_\pi\Phi$ and $h_\gamma(\theta) = p_\gamma(\theta)\,\overline{(T_\pi p_\gamma)(\theta)}$ with p_γ given by (2.31). Since $\pm\frac{\pi}{2}$ are non-degenerated critical points of Ψ (see (2.34), (2.35))), the stationary phase formula gives

$$V^\delta_\gamma(n,n) = \delta \sum_{\nu=\pm 1} \frac{h_\gamma(\nu\pi/2)\mathrm{e}^{\mathrm{i}n\Psi(\nu\pi/2)+\mathrm{i}\nu\pi/4}}{\sqrt{2\pi n|\Psi''(\nu\pi/2)|}} + O(n^{-1}). \tag{3.14}$$

Since Φ is odd, we get

$$\Psi(\pm\pi/2) = \Phi(\pm\pi/2) - \Phi(\mp\pi/2) = 2\Phi(\pm\pi/2) = \pm 2\Phi(\pi/2) = \pm 4\alpha \tag{3.15}$$

where (2.16) was used to write $\Phi(\pi/2) = 2\alpha$ with α given by (1.16). By definition (2.31), one has $p_\gamma(\pi/2) = \sqrt{\beta}\,\mathrm{e}^{\mathrm{i}\tilde{\theta}_\gamma}$ with

$$\tilde{\theta}_\gamma := \langle q_\gamma\rangle\Phi(\pi/2) - \tilde{q}_\gamma(\Phi(\pi/2)) \tag{3.16}$$

and $\tilde{q}_\gamma \circ \Phi$ is odd, hence $p_\gamma(-\pi/2) = \sqrt{\beta}\,\mathrm{e}^{-\mathrm{i}\tilde{\theta}_\gamma}$ and $h_\gamma(\pi/2) = \beta\mathrm{e}^{2\mathrm{i}\tilde{\theta}_\gamma}$. Moreover, one has $h_\gamma(-\pi/2) = \overline{h_\gamma(\pi/2)}$, hence

$$h_\gamma(\pm\pi/2) = \beta\mathrm{e}^{\pm 2\mathrm{i}\tilde{\theta}_\gamma} \tag{3.17}$$

and we get two conjugated terms corresponding to $\nu = \pm 1$ in (3.14). Therefore, using (3.17), (3.15) and $|\Psi''(\pm\pi/2)| = 4g\beta$ in (3.14), we get

$$\begin{aligned} V_\gamma^\delta(n,n) &= \delta\,\frac{2\mathrm{Re}\big(\beta\mathrm{e}^{2\mathrm{i}\tilde{\theta}_\gamma}\mathrm{e}^{\mathrm{i}4n\alpha+\mathrm{i}\pi/4}\big)}{\sqrt{2\pi n\cdot 4g\beta}} + O(n^{-1}) \\ &= \delta\,\frac{\sqrt{\beta}\,\cos(4n\alpha + 2\tilde{\theta}_\gamma + \pi/4)}{\sqrt{2\pi n g}} + O(n^{-1}). \end{aligned}$$

To complete the proof of (3.12), it remains to check that $\hat{\theta}_\gamma := 2\tilde{\theta}_\gamma + \pi/4$ is given by (2.12). However, (2.22) gives

$$\langle q_\gamma\rangle\Phi(\pi/2) = \Big(\gamma - \frac{1}{2}\Big)\big(\Phi(\pi/2) - \beta^{-1}\Phi(\pi/2)\big), \tag{3.18}$$

(2.21) gives

$$-\tilde{q}_\gamma(\Phi(\pi/2)) = \Big(\gamma - \frac{1}{2}\Big)\big(\beta^{-1}\Phi(\pi/2) - \pi/2\big), \tag{3.19}$$

and summing up (3.18), (3.19), we find that (3.16) gives

$$\tilde{\theta}_\gamma = \Big(\gamma - \frac{1}{2}\Big)(\Phi(\pi/2) - \pi/2) = \Big(\gamma - \frac{1}{2}\Big)(2\alpha - \pi/2) \tag{3.20}$$

and $\hat{\theta}_\gamma = 2\tilde{\theta}_\gamma + \frac{\pi}{4} = (\gamma - \frac{1}{2})(4\alpha - \pi) + \frac{\pi}{4}$ as claimed in (2.12).

4. Auxiliary operators Q_n

4.1. **Definition of Q_n.** Our aim is to introduce a sequence of auxiliary self-adjoint operators $\{Q_n\}_{n\in\mathbb{N}^*}$ which will be used in Section 5 to estimate $\lambda_n(J)$ thanks to an analysis of the operator

$$J_n' := \mathrm{e}^{-\mathrm{i}Q_n} J \mathrm{e}^{\mathrm{i}Q_n}. \tag{4.1}$$

Definition 4.1. Let V be a self-adjoint bounded operator in $\ell^2(\mathbb{Z})$ and denote

$$V(j,k) := (V\tilde{e}_k)(j) = \langle \tilde{e}_j, V\tilde{e}_k\rangle_{\ell^2(\mathbb{Z})} \text{ for } (j,k)\in\mathbb{Z}^2.$$

For $n\in\mathbb{N}^*$ define $(Q_n(j,k))_{(j,k)\in\mathbb{Z}^2}$ by

$$Q_n(j,k) = \begin{cases} \mathrm{i}\,\dfrac{V(j,n)}{\beta(j-n)} & \text{when } j\neq n \text{ and } k = n \\ \mathrm{i}\,\dfrac{V(n,k)}{\beta(n-k)} & \text{when } j = n \text{ and } k\neq n \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

Then

$$Q_n(j,k) \neq 0 \Rightarrow \big(\text{ either } (j \neq n \text{ and } k = n) \text{ or } (j = n \text{ and } k \neq n)\big) \tag{4.3}$$

and the relation

$$(Q_n\tilde{e}_k)(j) = \langle \tilde{e}_j, Q_n\tilde{e}_k\rangle_{\ell^2(\mathbb{Z})} = Q_n(j,k) \text{ for } (j,k) \in \mathbb{Z}^2$$

defines in $\ell^2(\mathbb{Z})$ a self-adjoint Hilbert-Schmidt operator Q_n due to

$$\sum_{(j,k)\in\mathbb{Z}^2} |Q_n(j,k)|^2 \leq \frac{2}{\beta^2}\sum_{j\in\mathbb{Z}} |V(j,n)|^2 = \frac{2}{\beta^2}||V\tilde{e}_n||^2_{\ell^2(\mathbb{Z})} \leq \frac{2}{\beta^2}||V||^2_{\mathcal{B}(\ell^2(\mathbb{Z}))}$$

Notation 4.2. If $n \in \mathbb{Z}$ then $\widehat{\Pi}_n \in \mathcal{B}(\ell^2(\mathbb{Z}))$ (respectively $\widetilde{\Pi}_n \in \mathcal{B}(\ell^2(\mathbb{Z}))$) denotes the orthogonal projection on $\mathbb{C}\tilde{e}_n$ (respectively on $(\mathbb{C}\tilde{e}_n)^\perp$), i.e.

$$\widehat{\Pi}_n x := \langle \tilde{e}_n, x\rangle_{\ell^2(\mathbb{Z})}\tilde{e}_n = x(n)\tilde{e}_n \quad \text{and} \quad \widetilde{\Pi}_n := I - \widehat{\Pi}_n.$$

4.2. **Properties of Q_n.** We observe that (4.3) implies

$$\widehat{\Pi}_n Q_n \widehat{\Pi}_n = 0 = \widetilde{\Pi}_n Q_n \widetilde{\Pi}_n \tag{4.4}$$

and

$$Q_n = \widetilde{\Pi}_n Q_n \widehat{\Pi}_n + \widehat{\Pi}_n Q_n \widetilde{\Pi}_n. \tag{4.5}$$

More properties of Q_n are given in

Lemma 4.3. *Assume that V satisfies the assumption of Proposition 3.1, i.e. V is a bounded self-adjoint operator in $\ell^2(\mathbb{Z})$ satisfying*

$$|j-n| \leq \hat{c}|n| \Rightarrow |V(j,n)| \leq \hat{C}|n|^{-1/2}, \tag{4.6}$$

where $\hat{C}$, $\hat{c} > 0$ are independent of n. Let $Q_n \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be as in Definition 4.1. Then

(a) one has large n estimate

$$||Q_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} = O(n^{-1/2}) \tag{4.7}$$

(b) $\ell^{2,1}(\mathbb{Z})$ is an invariant subspace of Q_n and for every $x \in \ell^{2,1}(\mathbb{Z})$,

$$\mathrm{i}\beta[Q_n, \Lambda]x = \mathrm{i}\beta(Q_n\Lambda - \Lambda Q_n)x = V_n x \tag{4.8}$$

holds with

$$V_n := \widetilde{\Pi}_n V \widehat{\Pi}_n + \widehat{\Pi}_n V \widetilde{\Pi}_n. \tag{4.9}$$

(c) $\ell^{2,1}(\mathbb{Z})$ is an invariant subspace of $\mathrm{e}^{\mathrm{i}tQ_n}$ and $t \to \mathrm{e}^{\mathrm{i}tQ_n}x$ is of class $\mathrm{C}^\infty(\mathbb{R}; \ell^{2,1}(\mathbb{Z}))$ if $x \in \ell^{2,1}(\mathbb{Z})$.

Proof. (a) By definition of Q_n,

$$||Q_n\tilde{e}_n||^2_{\ell^2(\mathbb{Z})} = \sum_{j\in\mathbb{Z}\setminus\{n\}} |Q_n(j,n)|^2 = \sum_{j\in\mathbb{Z}\setminus\{n\}} \frac{|V(j,n)|^2}{\beta^2(j-n)^2} \leq \beta^{-2}(\mathcal{M}_n + \mathcal{M}'_n)$$

holds with

$$\mathcal{M}_n := \sum_{\{j\in\mathbb{Z}:\, 0<|j-n|\leq\hat{c}n\}} \frac{|V(j,n)|^2}{(j-n)^2} \leq \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\hat{C}^2 n^{-1}}{m^2} = C_1 n^{-1}$$

due to (4.6) and

$$\mathcal{M}'_n := \sum_{\{j\in\mathbb{Z}:\, |j-n|>\hat{c}n\}} \frac{|V(j,n)|^2}{(j-n)^2} \leq \sum_{j\in\mathbb{Z}} \frac{|V(j,n)|^2}{\hat{c}^2 n^2} = \frac{||V\tilde{e}_n||^2_{\ell^2(\mathbb{Z})}}{\hat{c}^2 n^2} \leq C_2 n^{-2}.$$

This shows that $||Q_n\widehat{\Pi}_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} = O(n^{-1/2})$. Moreover, due to (4.5),

$$||Q_n\widetilde{\Pi}_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} = ||\widehat{\Pi}_n Q_n\widetilde{\Pi}_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq ||\widehat{\Pi}_n Q_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} = ||Q_n\widehat{\Pi}_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))}$$

is $O(n^{-1/2})$, which completes the proof of (4.7).

(b) Let V_n be given by (4.9). Then one has

$$V_n(j,k) := (V_n\tilde{e}_k)(j) = \langle \tilde{e}_j, V_n\tilde{e}_k\rangle_{\ell^2(\mathbb{Z})} = \begin{cases} V(j,n) \text{ when } j \neq n \text{ and } k = n \\ V(n,k) \text{ when } j = n \text{ and } k \neq n \\ 0 \quad \text{otherwise} \end{cases} \tag{4.10}$$

and combining (4.10) with (4.2) we obtain

$$\mathrm{i}\beta(j-k)Q_n(j,k) = -V_n(j,k) \text{ for every } j,k \in \mathbb{Z}. \tag{4.11}$$

However, (4.11) implies

$$\mathrm{i}\beta(j+\tfrac{1}{2})(Q_n\tilde{e}_k)(j) = \mathrm{i}\beta\big(Q_n(k+\tfrac{1}{2})\tilde{e}_k\big)(j) - (V_n\tilde{e}_k)(j) \tag{4.12}$$

and ensures $Q_n\tilde{e}_k \in \ell^{2,1}(\mathbb{Z})$. Moreover, (4.12) ensures the equality

$$\mathrm{i}\beta(\Lambda+\tfrac{1}{2})Q_n x = \mathrm{i}\beta Q_n(\Lambda+\tfrac{1}{2})x - V_n x \tag{4.13}$$

for $x \in \ell^2_{\mathrm{fin}}(\mathbb{Z})$. If $y \in \ell^2_{\mathrm{fin}}(\mathbb{Z})$, then using (4.13) with $x = (\Lambda+\frac{1}{2})^{-1}y$, we obtain

$$\mathrm{i}\beta(\Lambda+\tfrac{1}{2})Q_n(\Lambda+\tfrac{1}{2})^{-1}y = \mathrm{i}\beta Q_n' y \tag{4.14}$$

with $Q_n' \in \mathcal{B}(\ell^2(\mathbb{Z}))$ given by

$$Q_n' := Q_n - (\mathrm{i}\beta)^{-1}V_n(\Lambda+\tfrac{1}{2})^{-1}. \tag{4.15}$$

Due to (4.14), for every $y \in \ell^2_{\mathrm{fin}}(\mathbb{Z})$ one has

$$\mathrm{i}\beta Q_n(\Lambda+\tfrac{1}{2})^{-1}y = \mathrm{i}\beta(\Lambda+\tfrac{1}{2})^{-1}Q_n' y \tag{4.16}$$

and by continuity, (4.16) holds for every $y \in \ell^2(\mathbb{Z})$. Consider now $x \in \ell^{2,1}(\mathbb{Z})$ and take $y = (\Lambda+\frac{1}{2})x$ in (4.16). This gives

$$\mathrm{i}\beta Q_n x = \mathrm{i}\beta(\Lambda+\tfrac{1}{2})^{-1}Q_n'(\Lambda+\tfrac{1}{2})x \in \ell^{2,1}(\mathbb{Z}) \tag{4.17}$$

and

$$\mathrm{i}\beta(\Lambda+\tfrac{1}{2})Q_n x = \mathrm{i}\beta Q_n'(\Lambda+\tfrac{1}{2})x = \mathrm{i}\beta Q_n(\Lambda+\tfrac{1}{2})x - V_n x, \tag{4.18}$$

implying (4.8) for every $x \in \ell^{2,1}(\mathbb{Z})$.

(c) We first observe that (4.16) holds for every $y \in \ell^2(\mathbb{Z})$ and implies (4.14) for every $y \in \ell^2(\mathbb{Z})$. Since (4.14) implies $(Q_n')^m y = (\Lambda+\frac{1}{2})(Q_n)^m(\Lambda+\frac{1}{2})^{-1}y$ for every $m \in \mathbb{N}$ and $y \in \ell^2(\mathbb{Z})$, we have

$$\mathrm{e}^{\mathrm{i}tQ_n}(\Lambda+\tfrac{1}{2})^{-1}y = \lim_{N\to\infty}\sum_{m=0}^{N}\frac{(\mathrm{i}t)^m}{m!}(Q_n)^m(\Lambda+\tfrac{1}{2})^{-1}y =$$

$$= \lim_{N\to\infty}\sum_{m=0}^{N}\frac{(\mathrm{i}t)^m}{m!}(\Lambda+\tfrac{1}{2})^{-1}(Q_n')^m y = (\Lambda+\tfrac{1}{2})^{-1}\mathrm{e}^{\mathrm{i}tQ_n'}y \in \ell^{2,1}(\mathbb{Z})$$

if $y \in \ell^2(\mathbb{Z})$ and setting $x = (\Lambda+\frac{1}{2})^{-1}y$, we find that $t \to (\Lambda+\frac{1}{2})\mathrm{e}^{\mathrm{i}tQ_n}x = \mathrm{e}^{\mathrm{i}tQ_n'}y$ is $\mathrm{C}^\infty(\mathbb{R};\ell^2(\mathbb{Z}))$ for every $x \in \ell^{2,1}(\mathbb{Z})$. □

5. Proof of Proposition 3.1

5.1. **Taylor's expansion formula.** We will investigate $J'_n = \mathrm{e}^{-\mathrm{i}Q_n} J \mathrm{e}^{\mathrm{i}Q_n}$, using a special version of the Taylor's expansion formula. Assume that B and $Q \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and denote

$$F_{tQ}(B) := \mathrm{e}^{-\mathrm{i}tQ}\, B\, \mathrm{e}^{\mathrm{i}tQ} \text{ for } t \in \mathbb{R}, \tag{5.1}$$

$$\mathrm{ad}^1_{\mathrm{i}Q}(B) := [B, \mathrm{i}Q] = \mathrm{i}(BQ - QB)$$

and

$$\mathrm{ad}^{m+1}_{\mathrm{i}Q}(B) := [\mathrm{ad}^m_{\mathrm{i}Q}(B), \mathrm{i}Q] \text{ for } m \in \mathbb{N}^*.$$

Then

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} F_{tQ}(B) = \mathrm{e}^{-\mathrm{i}tQ}\, \mathrm{ad}^m_{\mathrm{i}Q}(B)\, \mathrm{e}^{\mathrm{i}tQ} = F_{tQ}(\mathrm{ad}^m_{\mathrm{i}Q}(B)) \tag{5.2}$$

and the Taylor formula gives

$$F_{tQ}(B) = B + \sum_{m=1}^{N-1} \frac{t^m}{m!} \mathrm{ad}^m_{\mathrm{i}Q}(B) + \mathcal{R}^{t,N}_Q(B) \tag{5.3}$$

with

$$\mathcal{R}^{t,N}_Q(B) := \frac{t^N}{(N-1)!} \int_0^1 F_{stQ}(\mathrm{ad}^N_{\mathrm{i}Q}(B))\, (1-s)^{N-1}\, \mathrm{d}s. \tag{5.4}$$

We can also consider the case when B is an unbounded symmetric operator in $\ell^2(\mathbb{Z})$, defined on a dense domain $\mathcal{D}(B)$. Suppose that $\mathcal{D}(B)$ is an invariant subspace for Q and $\mathrm{e}^{\mathrm{i}tQ}$ for every $t \in \mathbb{R}$. If $t \to B\mathrm{e}^{\mathrm{i}tQ}x$ is $C^1(\mathbb{R}, \ell^2(\mathbb{Z}))$ for every $x \in \mathcal{D}(B)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{e}^{\mathrm{i}tQ}x, B\mathrm{e}^{\mathrm{i}tQ}y\rangle_{\ell^2(\mathbb{Z})} = \langle B\mathrm{e}^{\mathrm{i}tQ}x, \mathrm{i}Q\mathrm{e}^{\mathrm{i}tQ}y\rangle_{\ell^2(\mathbb{Z})} + \langle \mathrm{i}Q\mathrm{e}^{\mathrm{i}tQ}x, B\mathrm{e}^{\mathrm{i}tQ}y\rangle_{\ell^2(\mathbb{Z})}$$

holds for every x, $y \in \mathcal{D}(B)$. If the form $(x, y) \to \langle Bx, \mathrm{i}Qy\rangle_{\ell^2(\mathbb{Z})} + \langle \mathrm{i}Qx, By\rangle_{\ell^2(\mathbb{Z})}$ can be extended from $\mathcal{D}(B) \times \mathcal{D}(B)$ to a bounded form on $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$, then we can introduce $[B, \mathrm{i}Q] \in \mathcal{B}(\ell^2(\mathbb{Z}))$ defined by this form and we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} F_{tQ}(B) = \mathrm{e}^{-\mathrm{i}tQ}\, [B, \mathrm{i}Q]\, \mathrm{e}^{\mathrm{i}tQ} = F_{tQ}([B, \mathrm{i}Q]),$$

hence (5.3)-(5.4) still hold for every $N \in \mathbb{N}^*$.

5.2. **Step 1 of the proof of Proposition 3.1.**

Notation 5.1. If L_n, L'_n are linear maps defined on a dense subspace of the Banach space $\mathcal{V}$ and $L_n - L'_n$ can be extended to a bounded operator on $\mathcal{V}$, then $||L_n - L'_n||_{\mathcal{B}(\mathcal{V})}$ denotes the norm of this extension and the notation $L_n = L'_n + O(\eta_n)$ means that choosing $n_0 \in \mathbb{N}$ and $C > 0$ large enough, we ensure $||L_n - L'_n||_{\mathcal{B}(\mathcal{V})} \leq C\eta_n$ for all $n \geq n_0$. If x_n, $x'_n \in \mathcal{V}$, then we write $x_n = y_n + O(\eta_n)$ if and only if we can find $C > 0$, $n_0 \in \mathbb{N}$, such that $||x_n - x'_n||_{\mathcal{V}} \leq C\eta_n$ holds for all $n \geq n_0$.

In what follows, V satisfies the hypotheses of Proposition 3.1 and without any loss of generality we assume $\beta_0 = 0$, i.e. $J = \beta\Lambda + V$. We consider

$$J'_n = \mathrm{e}^{-\mathrm{i}Q_n}(\beta\Lambda + V)\mathrm{e}^{\mathrm{i}Q_n} \tag{5.5}$$

and claim the large n estimate

$$J'_n = \beta\Lambda + V - V_n + [V - \tfrac{1}{2}V_n, \mathrm{i}Q_n] + O(n^{-1}), \tag{5.6}$$

where $V_n = \widetilde{\Pi}_n V \widehat{\Pi}_n + \widehat{\Pi}_n V \widetilde{\Pi}_n$ is given by (4.9).
We first use (5.3)-(5.4) with $N = 2$ and write

$$\mathrm{e}^{-\mathrm{i}Q_n} V \mathrm{e}^{\mathrm{i}Q_n} = V + [V, \mathrm{i}Q_n] + \mathcal{R}^{1,2}_{Q_n}(V), \tag{5.7}$$

$$||\mathcal{R}^{1,2}_{Q_n}(V)||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq ||\mathrm{ad}^2_{\mathrm{i}Q_n}(V)||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq C||Q_n||^2_{\mathcal{B}(\ell^2(\mathbb{Z}))}. \tag{5.8}$$

We also use (5.3)-(5.4) with $N = 3$ and write

$$\mathrm{e}^{-\mathrm{i}Q_n} \beta\Lambda \mathrm{e}^{\mathrm{i}Q_n} = \beta\Lambda + [\beta\Lambda, \mathrm{i}Q_n] + \tfrac{1}{2}[[\beta\Lambda, \mathrm{i}Q_n], \mathrm{i}Q_n] + \mathcal{R}^{1,3}_{Q_n}(\beta\Lambda), \tag{5.9}$$

$$||\mathcal{R}^{1,3}_{Q_n}(\beta\Lambda)||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq ||\mathrm{ad}^3_{\mathrm{i}Q_n}(\beta\Lambda)||_{\mathcal{B}(\ell^2(\mathbb{Z}))}. \tag{5.10}$$

Using $[\beta\Lambda, \mathrm{i}Q_n] = -V_n$ (see Lemma 4.3b) in (5.9)-(5.10), we obtain

$$\mathrm{e}^{-\mathrm{i}Q_n} \beta\Lambda \mathrm{e}^{\mathrm{i}Q_n} = \beta\Lambda - V_n - [\tfrac{1}{2}V_n, \mathrm{i}Q_n] + \mathcal{R}^{1,3}_{Q_n}(\beta\Lambda), \tag{5.11}$$

$$||\mathcal{R}^{1,3}_{Q_n}(\beta\Lambda)||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq ||\mathrm{ad}^2_{\mathrm{i}Q_n}(V_n)||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq C||Q_n||^2_{\mathcal{B}(\ell^2(\mathbb{Z}))} \tag{5.12}$$

and summing up (5.11), (5.7), we obtain

$$\mathrm{e}^{-\mathrm{i}Q_n} J \mathrm{e}^{\mathrm{i}Q_n} = \beta\Lambda + V - V_n + [V - \tfrac{1}{2}V_n, \mathrm{i}Q_n] + \mathcal{R}^{1,2}_{Q_n}(V) + \mathcal{R}^{1,3}_{Q_n}(\beta\Lambda), \tag{5.13}$$

$$||\mathcal{R}^{1,2}_{Q_n}(V) + \mathcal{R}^{1,3}_{Q_n}(\beta\Lambda)||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq C||Q_n||^2_{\mathcal{B}(\ell^2(\mathbb{Z}))}. \tag{5.14}$$

To complete the proof, we observe that $||Q_n||^2_{\mathcal{B}(\ell^2(\mathbb{Z}))} = O(n^{-1})$ follows from (4.7).

5.3. **Step 2 of the proof of Proposition 3.1.** We denote

$$W_n := [V - \tfrac{1}{2}V_n, \mathrm{i}Q_n] \tag{5.15}$$

and we claim that the following estimate

$$W_n(n,n) := (W_n \tilde{e}_n)(n) = \langle \tilde{e}_n, W_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})} = O(n^{-1} \ln n) \tag{5.16}$$

holds when $n \to \infty$.

Indeed, $W_n = [V_n', \mathrm{i}Q_n]$ holds with $V_n' := V - \frac{1}{2}V_n$ and

$$|V_n'(j,k)| \leq 2|V(j,k)| \text{ for all } j,k \in \mathbb{Z}, \tag{5.17}$$

hence we can estimate

$$|(V_n' Q_n)(n,n)| \leq \sum_{j \in \mathbb{Z}} 2|V(n,j)Q_n(j,n)| \leq$$

$$\leq \sum_{j \in \mathbb{Z} \setminus \{n\}} \frac{2|V(j,n)|^2}{\beta|j-n|} \leq 2\beta^{-1}(\mathcal{M}_n + \mathcal{M}_n')$$

with

$$\mathcal{M}_n := \sum_{\{j \in \mathbb{Z}:\, 0 < |j-n| \leq \hat{c}n\}} \frac{|V(j,n)|^2}{|j-n|} \leq \sum_{\{m \in \mathbb{Z}:\, 0 < |m| \leq \hat{c}n\}} \frac{\hat{C}^2 n^{-1}}{|m|} \leq C_1 n^{-1} \ln n$$

due to (4.6) and

$$\mathcal{M}_n' := \sum_{\{j \in \mathbb{Z}:\, |j-n| > \hat{c}n\}} \frac{|V(j,n)|^2}{|j-n|} \leq \sum_{j \in \mathbb{Z}} \frac{|V(j,n)|^2}{\hat{c}n} = \frac{||V\tilde{e}_n||^2_{\ell^2(\mathbb{Z})}}{\hat{c}n} \leq C_2 n^{-1}.$$

5.4. **Step 3 of the proof of Proposition 3.1.** Let J'_n be as above and denote

$$d'_n := \beta n + V(n,n). \tag{5.18}$$

We claim that the following estimates

$$\langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})} = d'_n + O(n^{-1}\ln n) \tag{5.19}$$

$$(J'_n - \langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})})\tilde{e}_n = O(n^{-1/2}) \tag{5.20}$$

hold when $n \to \infty$.

Indeed, writing

$$\beta\Lambda + V - V_n = \beta\Lambda + V - \widetilde{\Pi}_n V \widehat{\Pi}_n - \widehat{\Pi}_n V \widetilde{\Pi}_n$$
$$= \beta\Lambda + \widehat{\Pi}_n V \widehat{\Pi}_n + \widetilde{\Pi}_n V \widetilde{\Pi}_n = \beta\Lambda + V(n,n)\widehat{\Pi}_n + \widetilde{\Pi}_n V \widetilde{\Pi}_n$$

in (5.6), we find

$$J'_n \tilde{e}_n = \big(\beta\Lambda + V(n,n)\widehat{\Pi}_n + \widetilde{\Pi}_n V \widetilde{\Pi}_n + W_n\big)\tilde{e}_n + O(n^{-1}), \tag{5.21}$$

where $W_n = [V - \frac{1}{2}V_n, \mathrm{i}Q_n]$ is given by (5.15). However (5.21) implies

$$J'_n \tilde{e}_n = d'_n \tilde{e}_n + W_n \tilde{e}_n + O(n^{-1}), \tag{5.22}$$

hence

$$\langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})} = d'_n + \langle \tilde{e}_n, W_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})} + O(n^{-1}) \tag{5.23}$$

and (5.19) follows from (5.16).

In order to prove (5.20), we observe that (5.19) implies

$$(J'_n - \langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})})\tilde{e}_n = (J'_n - d'_n)\tilde{e}_n + O(n^{-1}\ln n) \tag{5.24}$$

and (5.22) ensures

$$(J'_n - d'_n)\tilde{e}_n = W_n \tilde{e}_n + O(n^{-1}). \tag{5.25}$$

Thus (5.20) follows from (5.24), (5.25) and

$$||W_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} \leq C||Q_n||_{\mathcal{B}(\ell^2(\mathbb{Z}))} = O(n^{-1/2}).$$

5.5. **End of the proof of Proposition 3.1.** In what follows, $\sigma(L)$ denotes the spectrum of the operator L. We assume that the hypotheses of Proposition 3.1 hold and $\beta_0 = 0$. Then (3.3) ensures $\lambda_n(J) = \beta n + o(1)$ as $n \to \infty$ and

$$\sigma(J) \cap [\beta(n - \tfrac{1}{2}), \beta(n + \tfrac{1}{2})] = \{\lambda_n(J)\} \text{ for } n \geq n_0 \tag{5.26}$$

if $n_0 \in \mathbb{N}$ is chosen large enough. Let $\{\lambda_j(J'_n)\}_{j\in\mathbb{Z}}$ be the sequence of eigenvalues of J'_n labeled so that $\lambda_j(J'_n) = \lambda_j(J)$ for every $j \in \mathbb{Z}$. Then (5.26) gives

$$\sigma(J'_n) \cap [\beta(n - \tfrac{1}{2}), \beta(n + \tfrac{1}{2})] = \{\lambda_n(J'_n)\} \text{ for } n \geq n_0 \tag{5.27}$$

and we will use $L = \tilde{J}'_n$, $x = e_n$, $d' = \beta(n - \frac{1}{2})$ and $d'' = \beta(n + \frac{1}{2})$ with $n \geq n_0$ in

Theorem 5.2. (Kato-Temple) *Assume that the operator L is self-adjoint in the Hilbert space $\mathcal{H}$ and has exactly one eigenvalue λ in the interval $[d', d'']$. If x is an element of the domain of L such that $\eta := \langle x, Lx\rangle_{\mathcal{H}}$ belongs to $]d', d''[$, then*

$$\eta - \frac{\|(L-\eta)x\|^2}{\eta - d'} \leq \lambda \leq \eta + \frac{\|(L-\eta)x\|^2}{d'' - \eta}. \tag{5.28}$$

Proof. See [16]. □

We claim that the assumptions of Theorem 5.2 are satisfied for $n \geq n_0$ if n_0 is chosen large enough. Indeed, (5.27) holds and

$$\eta := \langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})} \in [\beta(n - \tfrac{1}{4}), \beta(n + \tfrac{1}{4})] \text{ for } n \geq n_0 \tag{5.29}$$

holds due to (5.19) and $V(n,n) = O(n^{-1/2})$ as $n \to \infty$. Since (5.29) implies the inequality $\min\{d'' - \eta, \eta - d'\} \geq \frac{1}{4}\beta$, the estimate (5.28) gives

$$|\lambda_n(J'_n) - \langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})}| \leq 4\beta^{-1} ||(J'_n \tilde{e}_n - \langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})})\tilde{e}_n||^2 = O(n^{-1}), \tag{5.30}$$

where the last estimate is due to (5.20). The estimate (5.30) implies

$$\lambda_n(J'_n) = \langle \tilde{e}_n, J'_n \tilde{e}_n \rangle_{\ell^2(\mathbb{Z})} + O(n^{-1}) \tag{5.31}$$

and using (5.19) in the right hand side of (5.31), we obtain

$$\lambda_n(J) = \lambda_n(J'_n) = d'_n + O(n^{-1} \ln n),$$

completing the proof of Proposition 3.1.

6. Proof of Theorem 1.3

6.1. **Step 1 of the proof of Theorem 1.3.** As before $0 < g < \frac{1}{2}$ and $\{e_m\}_{m \in \mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$.

Definition 6.1. For $\nu = \pm 1$, $\mu = 0, 1$, we define $\hat{J}_{\nu,\mu}$ as the linear map $\hat{J}_{\nu,\mu} : \ell^{2,1}(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by

$$\begin{cases} \hat{J}_{\nu,\mu} e_m = \hat{d}_{\nu,\mu}(m) e_m + \hat{b}_\mu(m) e_{m+1} + \hat{b}_\mu(m-1) e_{m-1} \text{ for } m \in \mathbb{N}^*, \\ \hat{J}_{\nu,\mu} e_0 = \hat{d}_{\nu,\mu}(0) e_0 + \hat{b}_\mu(0) e_1 \end{cases} \tag{6.1}$$

where

$$\begin{cases} \hat{d}_{\nu,\mu}(m) := 2m + \mu + (-1)^m \nu \Delta/2 \\ \hat{b}_\mu(m) := g\sqrt{(2m+1+\mu)(2m+2+\mu)}. \end{cases} \tag{6.2}$$

Lemma 6.2. *The operator* $\mathbf{H}^{(2)}_{\mathrm{Rabi}}$ *is similar to the direct sum*

$$\hat{J}_{1,0} \oplus \hat{J}_{-1,0} \oplus \hat{J}_{1,1} \oplus \hat{J}_{-1,1} \tag{6.3}$$

Proof. See Section 2.2 in [6]. □

6.2. **Step 2 of the proof of Theorem 1.3.** Let $\tilde{J}^\delta_\gamma$, $\tilde{d}_\delta$, $\tilde{b}_\gamma$ be as in Definition 2.2. If $\hat{d}_{\nu,\mu}$ and $\hat{b}_\mu$ are given by (6.2), then it is easy to check the following relations

$$\begin{cases} \hat{d}_{\pm 1,0}(m) = 2\tilde{d}_{\pm\Delta/4}(m) \\ \hat{b}_0(m) = 2\tilde{b}_{3/4}(m) + O(m^{-1}) \text{ when } m \to \infty \end{cases} \tag{6.4}$$

$$\begin{cases} \hat{d}_{\pm 1,1}(m) = 2\tilde{d}_{\pm\Delta/4}(m) + 1 \\ \hat{b}_1(m) = 2\tilde{b}_{5/4}(m) + O(m^{-1}) \text{ when } m \to \infty \end{cases} \tag{6.5}$$

The above properties allow us to prove

Lemma 6.3. *Let $0 < g < \frac{1}{2}$ and let $\{\lambda_n(\hat{J}_{\nu,\mu})\}_{n\in\mathbb{N}}$ denote the non-decreasing sequence of eigenvalues of $\hat{J}_{\nu,\mu}$. If $\{\lambda_j(\tilde{J}^\delta_\gamma)\}_{j\in\mathbb{Z}}$ is the sequence of eigenvalues of $\tilde{J}^\delta_\gamma$ satisfying* (2.10)*, then one has*

$$\lambda_n(\hat{J}_{\pm 1,0}) = 2\lambda_n(\tilde{J}^{\pm\Delta/4}_{3/4}) + O(n^{-1}) \text{ as } n \to \infty \tag{6.6}$$

and

$$\lambda_n(\hat{J}_{\pm 1,1}) = 2\lambda_n(\tilde{J}^{\pm\Delta/4}_{5/4}) + 1 + O(n^{-1}) \text{ as } n \to \infty. \tag{6.7}$$

Proof. We identify $\ell^2(\mathbb{N})$ with $\{x \in \ell^2(\mathbb{Z}) : x(k) = 0 \text{ for } k \in \mathbb{Z} \setminus \mathbb{N}\}$. Let $\Pi_{\geq 0}$ be the orthogonal projection $\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ and let $\hat{J}^\delta_\gamma := \Pi_{\geq 0}\tilde{J}^\delta_\gamma|_{\ell^2(\mathbb{N})}$. Identifying $\tilde{e}_m$ with e_m when $m \in \mathbb{N}$, we get

$$\begin{cases} \hat{J}^\delta_\gamma e_m = \hat{d}_\delta(m)e_m + \hat{b}_\gamma(m)e_{m+1} + \hat{b}_\gamma(m-1)e_{m-1} \text{ for } m \in \mathbb{N}^*, \\ \hat{J}^\delta_\gamma e_0 = \hat{d}_\delta(0)e_0 + \hat{b}_\gamma(0)e_1 \end{cases} \tag{6.8}$$

We observe that (6.4)-(6.5) imply

$$2\hat{J}^{\pm\Delta/4}_{3/4} - \hat{J}_{\pm 1,0} \in \mathcal{B}(\ell^2(\mathbb{N}), \ell^{2,1}(\mathbb{N})), \tag{6.9}$$

$$2\hat{J}^{\pm\Delta/4}_{5/4} + 1 - \hat{J}_{\pm 1,1} \in \mathcal{B}(\ell^2(\mathbb{N}), \ell^{2,1}(\mathbb{N})) \tag{6.10}$$

and using the result of Rozenblum [24, Theorem 1.1] as indicated in the proof of (3.6) in [6, Section 3.2], we find that (6.9)-(6.10) ensure

$$2\lambda_n(\hat{J}^{\pm\Delta/4}_{3/4}) = \lambda_n(2\hat{J}^{\pm\Delta/4}_{3/4}) = \lambda_n(\hat{J}_{\pm 1,0}) + O(n^{-1}) \text{ as } n \to \infty, \tag{6.11}$$

$$2\lambda_n(\hat{J}^{\pm\Delta/4}_{5/4}) + 1 = \lambda_n(2\hat{J}^{\pm\Delta/4}_{5/4} + 1) = \lambda_n(\hat{J}_{\pm 1,1}) + O(n^{-1}) \text{ as } n \to \infty, \tag{6.12}$$

where $\{\lambda_n(\hat{J}^\delta_\gamma)\}_{n\in\mathbb{N}}$ is the non-decreasing sequence of eigenvalues of $\hat{J}^\delta_\gamma$.

Let $\{\lambda_j(\tilde{J}^\delta_\gamma)\}_{j\in\mathbb{Z}}$ be the sequence of eigenvalues of $\tilde{J}^\delta_\gamma$ satisfying (2.10). We complete the proof, combining (6.11)-(6.12) and the estimate

$$\lambda_n(\hat{J}^\delta_\gamma) - \lambda_n(\tilde{J}^\delta_\gamma) = O(n^{-N}) \text{ as } n \to \infty, \tag{6.13}$$

which holds for every $N \geq 1$. The estimate (6.13) is proved in Section 6.1 of [6]. □

6.3. **End of the proof of Theorem 1.3.** Using the asymptotic estimates (2.10) in the right hand side of (6.6) and (6.7), we obtain

$$\lambda_n(\hat{J}_{\pm 1,0}) = 2\left(n\beta + \frac{1}{4}(\beta - 1) + \mathfrak{r}^{\pm\Delta/4}_{3/4}(n)\right) + O(n^{-1}\ln n), \tag{6.14}$$

$$\lambda_n(\hat{J}_{\pm 1,1}) = 2\left(n\beta + \frac{3}{4}(\beta - 1) + \mathfrak{r}^{\pm\Delta/4}_{5/4}(n)\right) + 1 + O(n^{-1}\ln n). \tag{6.15}$$

Due to (2.12),

$$\hat{\theta}_{3/4} = \frac{1}{4}(4\alpha - \pi) + \frac{\pi}{4} = \alpha,$$

$$\hat{\theta}_{5/4} = \frac{3}{4}(4\alpha - \pi) + \frac{\pi}{4} = 3\alpha - \frac{\pi}{2}$$

and (6.15)-(6.15) give

$$\lambda_n(\hat{J}_{\pm 1,0}) = 2n\beta + \frac{1}{2}(\beta - 1) \pm \frac{\Delta}{2}\left(\frac{\beta}{2\pi g n}\right)^{1/2}\cos\big((4n+1)\alpha\big) + O(n^{-1}\ln n),$$

$$\lambda_n(\hat{J}_{\pm 1,1}) = (2n+1)\beta + \frac{1}{2}(\beta - 1) \pm \frac{\Delta}{2}\left(\frac{\beta}{2\pi g n}\right)^{1/2}\sin\big((4n+3)\alpha\big) + O(n^{-1}\ln n)$$

Due to Lemma 6.2, $\sigma(\mathbf{H}^{(2)}_{\mathrm{Rabi}}) = \sigma(\hat{J}_{1,0}) \cup \sigma(\hat{J}_{-1,0}) \cup \sigma(\hat{J}_{1,1}) \cup \sigma(\hat{J}_{-1,1})$ and we complete the proof, taking

$$\begin{cases} E^{\pm}_{2n} = \lambda_n(\hat{J}_{\pm 1,0}) \\ E^{\pm}_{2n+1} = \lambda_n(\hat{J}_{\pm 1,1}) \end{cases} \tag{6.16}$$

for $n \in \mathbb{N}$.

7. Appendix: estimates of oscillatory integrals

Lemma 7.1. *Let us fix real numbers θ_0, θ_1 and $c_0 > 0$. Assume that $\{\Psi_\tau\}_{\tau\in\mathcal{T}}$ is a family of functions $[\theta_0, \theta_1] \to \mathbb{R}$, bounded in $C^2([\theta_0, \theta_1])$, i.e. $\sup_{\theta_0\le\theta\le\theta_1} |\Psi^{(k)}_\tau(\theta)| < \infty$ for $k = 0, 1, 2$. For $\lambda \in \mathbb{R}$ and $h \in C^2([\theta_0, \theta_1])$ we denote*

$$M_\tau(\lambda) := \int_{\theta_0}^{\theta_1} \mathrm{e}^{\mathrm{i}\lambda\Psi_\tau(\theta)} h(\theta)\,\mathrm{d}\theta. \tag{7.1}$$

(a) If $|\Psi'_\tau(\theta)| \ge c_0 > 0$ holds for every $\theta \in [\theta_0, \theta_1]$ and $\tau \in \mathcal{T}$, then there is a constant C such that the estimate

$$|M_\tau(\lambda)| \le C|\lambda|^{-1} \tag{7.2}$$

holds for all $\tau \in \mathcal{T}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

(b) If $|\Psi''_\tau(\theta)| \ge c_0 > 0$ holds for every $\theta \in [\theta_0, \theta_1]$ and $\tau \in \mathcal{T}$, then there is a constant C such that the estimate

$$|M_\tau(\lambda)| \le C|\lambda|^{-1/2} \tag{7.3}$$

holds for all $\tau \in \mathcal{T}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. (a) The non-stationary phase estimate (7.2) follows by using the standard integration by parts.
(b) The estimate (7.3) follows from van der Corput Lemma (see [27, Section VIII.1.2, p. 354]). □

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