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# ON THE DYNAMICAL MANIN-MUMFORD CONJECTURE FOR PLANE POLYNOMIAL MAPS

ROMAIN DUJARDIN, CHARLES FAVRE, AND MATTEO RUGGIERO

ABSTRACT. We prove the dynamical Manin-Mumford conjecture for regular polynomial maps of  $\mathbb{A}^2$  and irreducible curves avoiding super-attracting orbits at infinity, over any field of characteristic 0.

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## INTRODUCTION

The dynamical Manin-Mumford conjecture for polarized endomorphisms of algebraic varieties, first formulated by S.-W. Zhang in two influential papers [32, Conjecture 2.5] and [33, Conjecture 1.2.1], has been a driving force for the development of the field of arithmetic dynamics. It was realized by Ghioca, Tucker and Zhang [19] and Pazuki [26] that the original formulation of the conjecture was too optimistic, and a modified conjecture was proposed in [19] and more recently in [18]. It can be stated as follows: *let  $f : X \rightarrow X$  be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and  $Z \subset X$  be a subvariety containing a Zariski dense set of periodic points. Then either  $Z$  is preperiodic or  $Z$  is special*, in the sense that it is contained in some subvariety  $Y$  that is both  $f^n$ - and  $\psi$ -invariant, for some  $n \geq 1$ , where  $\psi$  is another polarized endomorphism commuting with  $f^n$ . Recall that an endomorphism is said to be polarized if there is an ample line bundle  $L \rightarrow X$  such that  $f^*L \simeq L^d$  for some  $d \geq 2$ . A basic example is that of non-invertible endomorphisms of  $\mathbb{P}^k$ , for which we can take  $L = \mathcal{O}(1)$  and  $d$  is the degree of  $f$ .

Despite its importance, very few cases of the conjecture have been settled so far. One first case is of course the original Manin-Mumford conjecture, which was solved by Raynaud [27, 28]. Viewed as a dynamical statement, it deals with endomorphisms of

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Abelian varieties, and was generalized to related settings, such as commutative algebraic groups (see Hindry [22], and also Lang's classical paper [25]). Uniform versions involving height bounds were subsequently obtained by S.-W. Zhang, David and Philippon, Chambert-Loir and others. We refer to the recent work of Kühne [24] in the semi-abelian case for a latest update, and more references. Closer to algebraic dynamics is the case of polarized endomorphisms of  $(\mathbb{P}^1)^k$ , which was solved by Ghioca, Nguyen and Ye [17, 16] (see also [18]). Note that such mappings are of product type, that is of the form  $f(x_1, \dots, x_k) = (f_1(x_1), \dots, f_k(x_k))$ , so their dynamical complexity reduces to dimension 1, which is a key step in the proof.

In this paper we will establish the dynamical Manin-Mumford conjecture for a wide class of 2-dimensional examples, whose dynamical behavior is truly higher dimensional.

Note that besides polarized endomorphisms, a partial answer to the conjecture was given in [11] for plane polynomial automorphisms.

Let  $\mathbb{k}$  be any field of characteristic zero. If required we fix an algebraic closure  $\mathbb{k}^{\text{alg}}$  of  $\mathbb{k}$ . Let  $f: \mathbb{A}_{\mathbb{k}}^2 \rightarrow \mathbb{A}_{\mathbb{k}}^2$  be a polynomial self-map of the affine plane of degree  $d \geq 2$ , which in coordinates is written as

$$f(z, w) = (P(z, w), Q(z, w)),$$

with  $P, Q \in \mathbb{k}[x, y]$ . We say that  $f$  is *regular* if it extends to an endomorphism of  $\mathbb{P}_{\mathbb{k}}^2$  of degree  $d \geq 2$ ; this means that  $P = P_d + \text{l.o.t.}$  and  $Q = Q_d + \text{l.o.t.}$ , where  $P_d$  and  $Q_d$  are homogeneous polynomials of degree  $d$  without common factors. In particular  $f$  induces a rational map  $f_{\infty} := [P_d : Q_d]$  on the line at infinity, which is fixed. Note that any endomorphism of  $\mathbb{P}_{\mathbb{k}}^2$  with a totally invariant line is conjugate to a regular polynomial map, and that a generic polynomial map of  $\mathbb{A}^2$  whose components are polynomials of degree  $\leq d$  is regular.

For regular polynomial maps of  $\mathbb{A}^2$ , it seems that none of the known obstructions to the dynamical Manin-Mumford conjecture can arise. As said above, according to [18], one obstruction would be the existence of a periodic curve  $C$  for some endomorphism  $\psi$  commuting with  $f$ . Such pairs  $(f, \psi)$  were classified in [8] over  $\mathbb{C}$  (see also [9, 23]), and after a ramified cover they are all induced from a product map or a monomial map on the multiplicative 2-torus. Thanks to [17] it appears that in any such case,  $C$  must be also  $f$ -preperiodic. Thus we expect that the dynamical Manin-Mumford conjecture holds unconditionally in this case. Our main result confirms this expectation in the vast majority of cases.

**Theorem A.** *Let  $f$  be a regular polynomial endomorphism of  $\mathbb{A}^2$  of degree  $d \geq 2$ , defined over an arbitrary field  $\mathbb{k}$  of characteristic 0.*

*Let  $C \subset \mathbb{A}^2$  be an irreducible algebraic curve containing infinitely many preperiodic points of  $f$ , and suppose that the closure of  $C$  in  $\mathbb{P}^2$  contains a point  $p \in L_{\infty}$  which is not eventually superattracting. Then  $C$  is preperiodic under  $f$ .*

**Corollary B.** *Under the assumptions of the theorem, if  $f$  has no super-attracting points on the line at infinity, then any irreducible algebraic curve  $C \subset \mathbb{P}^2$  containing infinitely many periodic points is preperiodic, that is, the dynamical Manin-Mumford conjecture holds for  $f$ .*

Our strategy relies on techniques from Arakelov geometry, in particular on the notion of canonical height, which is now classical in arithmetic dynamics, together with a variety of techniques from holomorphic and non-Archimedean dynamics.

We first assume that  $\mathbb{k}$  is a number field, in which case we prove a stronger statement (Theorem 3.1). We first recall in Section 1, that there exists a canonical height  $h_f$  on  $\mathbb{A}^2(\mathbb{k}^{\text{alg}})$ , for which preperiodic points are exactly points of zero height. By a theorem of Zhang, all points of  $C$  lying at infinity are preperiodic, hence, by replacing  $C$  by some iterate we may assume that they are periodic, and, thanks to our assumptions, one of these periodic points is not superattracting for  $f_\infty$ . Fix such a point  $p \in \overline{C} \cap L_\infty$ . A second consequence of Zhang's theorem is that for each place  $v$  of  $\mathbb{k}$ , the dynamical Green function  $g_{f,v}(z, w) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|f^n(z, w)\|$  satisfies

$$(1) \quad \int_C g_{f,v} dd^c g_{f,v} = 0.$$

To make sense of this equality at a finite place, we consider the Berkovich analytification of  $C$  and  $dd^c g_{f,v}$  is the Laplacian of the subharmonic function  $g_{f,v}|_C$  in the sense of Thuillier, see [31].

For an appropriate choice of  $v$ , we may suppose that the periodic point  $p$  is either repelling or parabolic for  $f_\infty$ . In both cases, we construct in Section 2 a local (super-)stable manifold  $W_{\text{loc}}^{\text{ss}}(p)$ . When  $p$  is repelling and  $v$  is Archimedean this is classical. We extend this construction to the non-Archimedean setting, and allow for non-repelling  $p$  (see Theorem 2.1). Then, in both (repelling or parabolic) cases we establish graph transform type estimates which will be useful for the local analysis of the Green function at  $p$ . When  $p$  is parabolic this borrows from the work of Hakim [20].

In Section 3 we combine these estimates with condition (1) to show that, near  $p$ ,  $C$  must locally coincide with  $W_{\text{loc}}^{\text{ss}}(p)$ , which finally implies that  $C$  is periodic. Note that a similar argument appears in the recent work [15].

A key point in this argument is that at the chosen place  $v$ ,  $p$  belongs to the Julia set of  $f_\infty$ . This is no longer true when  $f_\infty$  is superattracting at  $p$ , and we are not able to conclude in this case. Still, this situation leads to interesting dynamical considerations and objects, and we plan to come back to this issue in a later work.

Finally, we develop a specialization argument in Section 4 to reduce Theorem A for an arbitrary  $\mathbb{k}$  to the case where  $\mathbb{k}$  is a number field. So here we rather deal with an algebraic family of endomorphisms of  $\mathbb{P}^2$  parameterized by some algebraic variety  $S$ . The main issue is to ensure that for such a family, an infinite set of preperiodic points cannot shrink to a finite set too often on  $S$ . To do so, one needs to control collisions of periodic points (using the Shub-Sullivan theorem [29] in the spirit of [11]); and the splitting of local preimages of a super-attracting cycle, a phenomenon that was studied in particular by Chio and Roeder [7].

## 1. DYNAMICAL HEIGHTS

In this section, we recall some basic facts on canonical heights attached to endomorphisms of the projective plane defined over a number field. Our purpose is to establish Proposition 1.7, which is the key arithmetic geometry input in our main theorem.

Throughout this section, we assume that  $\mathbb{k}$  is a number field.

**1.1. Vocabulary of number fields.** We denote by  $M_{\mathbb{k}} = \{v\}$  the set of places of  $\mathbb{k}$ , that is, the set of all multiplicative norms  $|\cdot|_v$  on  $\mathbb{k}$  that restrict to either the standard euclidean norm  $|\cdot|_{\infty}$ , or to a  $p$ -adic norm  $|\cdot|_p$  on  $\mathbb{Q}$  for some prime number  $p > 1$ . We normalize the  $p$ -adic norm by  $|p|_p = p^{-1}$ . We let  $\mathbb{k}_v$  be the completion of  $\mathbb{k}$  w.r.t.  $|\cdot|_v$ , and write  $n_v := [\mathbb{k}_v : \mathbb{Q}_v]$ . The product formula asserts that for any  $a \in \mathbb{k}$ , we have

$$\prod_{v \in M_{\mathbb{k}}} |a|_v^{n_v} = 1 .$$

The set  $M_{\mathbb{k}}$  splits into the finite set  $M_{\mathbb{k}}^{\infty}$  of Archimedean places (whose restriction to  $\mathbb{Q}$  is  $|\cdot|_{\infty}$ ), and the set of finite (or non-Archimedean) places.

When  $v \in M_{\mathbb{k}}^{\infty}$ , the algebraic closure  $\mathbb{C}_v$  of  $\mathbb{k}_v$  is isometric to  $\mathbb{C}$ . When  $v$  is a finite place extending  $|\cdot|_p$  on  $\mathbb{Q}$ , then  $|\cdot|_v$  extends canonically to  $\mathbb{k}_v^{\text{alg}}$ , and its completion  $\mathbb{C}_v$  is both complete and algebraically closed.

**1.2. Regular polynomial maps.** Let  $(z, w)$  be affine coordinates on the affine plane  $\mathbb{A}_{\mathbb{k}}^2$ . We also consider homogeneous coordinates  $[z_0 : z_1 : z_2]$  on the projective plane  $\mathbb{P}_{\mathbb{k}}^2$  and identify the affine plane  $\mathbb{A}_{\mathbb{k}}^2$  with the Zariski open set  $z_0 \neq 0$  so that  $z = z_1/z_0$  and  $w = z_2/z_0$ . We denote by  $L_{\infty} = \{z_0 = 0\}$  the line at infinity.

Let  $f: \mathbb{A}_{\mathbb{k}}^2 \rightarrow \mathbb{A}_{\mathbb{k}}^2$  be any polynomial self-map of the affine plane of degree  $d \geq 2$  that extends to an endomorphism of  $\mathbb{P}_{\mathbb{k}}^2$ . In the coordinates  $z, w$ , it is given by

$$f(z, w) = (P(z, w), Q(z, w)),$$

where  $P, Q \in \mathbb{k}[z, w]$  satisfy  $\max\{\deg P, \deg Q\} = d$ . The fact that  $f$  extends to a regular endomorphism  $\bar{f}: \mathbb{P}_{\mathbb{k}}^2 \rightarrow \mathbb{P}_{\mathbb{k}}^2$  is equivalent to say that  $P = P_d + \text{l.o.t.}$  and  $Q = Q_d + \text{l.o.t.}$ , where  $P_d$  and  $Q_d$  are homogeneous polynomials of degree  $d$  without common factors.

For  $n \in \mathbb{N}$  we write  $f^n(z, w) = (P_n(z, w), Q_n(z, w))$ . The restriction of  $f_{\infty}$  to  $L_{\infty}$  is an endomorphism of  $\mathbb{P}_{\mathbb{k}}^1$  given in homogeneous coordinates by

$$f_{\infty}([z_1 : z_2]) = [P_d(z_1, z_2) : Q_d(z_1, z_2)] .$$

**1.3. Green functions.** The next proposition follows from the Nullstellensatz (see e.g. [30, Theorem 3.11]).

**Proposition 1.1.** *For any  $v \in M_{\mathbb{k}}$ , there exists a constant  $C_v \geq 1$  such that*

$$(2) \quad C_v^{-1} \leq \frac{\max\{1, |P(z, w)|_v, |Q(z, w)|_v\}}{\max\{1, |z|_v, |w|_v\}^d} \leq C_v$$

for all  $z, w \in \mathbb{C}_v$ . Moreover, for all but finitely many  $v \in M_{\mathbb{k}}$  we may take  $C_v = 1$ .

By the previous proposition, the sequence of functions

$$g_{v,n}(z, w) := \frac{1}{d^n} \log \max\{1, |P_n(z, w)|_v, |Q_n(z, w)|_v\}$$

converges uniformly on  $\mathbb{C}_v^2$  to a continuous function  $g_v$ , and the next proposition follows.

**Proposition 1.2.** *For any  $v \in M_{\mathbb{k}}$ , the function  $g_v: \mathbb{C}_v^2 \rightarrow \mathbb{R}_+$  is continuous, it satisfies the invariance equation  $g_v \circ f = dg_v$ , and we have*

$$|g_v(z, w) - \log \max\{1, |z|_v, |w|_v\}| \leq \frac{\log C_v}{d-1}.$$

*The set  $\{(z, w) \in \mathbb{C}_v^2, g_v(z, w) = 0\}$  coincides with the set of points having bounded orbits.*

We shall also consider the global Green function in  $\mathbb{C}_v^3$ . Write  $\tilde{P}(z_0, z_1, z_2) = z_0^d P(\frac{z_1}{z_0}, \frac{z_2}{z_0})$  and  $\tilde{Q}(z_0, z_1, z_2) = z_0^d Q(\frac{z_1}{z_0}, \frac{z_2}{z_0})$  so that  $F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q})$  is a homogenous map of degree  $d$  that lifts  $f$  to  $\mathbb{C}_v^3$ . Observe that in homogenous coordinates, (2) can be rewritten as follows:

$$C_v^{-1} \leq \frac{\max\{|z_0|^d, |\tilde{P}(z_0, z_1, z_2)|_v, |\tilde{Q}(z_0, z_1, z_2)|_v\}}{\max\{|z_0|_v, |z_1|_v, |z_2|_v\}^d} \leq C_v$$

so that the next result also holds.

**Proposition 1.3.** *The function  $G_v(z_0, z_1, z_2) = g_v(z_1/z_0, z_2/z_0) + \log |z_0|$  is continuous on  $\mathbb{C}_v^3 \setminus \{0\}$ , 1-homogeneous (that is,  $G_v(\lambda Z) = \log |\lambda| + G_v(Z)$ ), and satisfies  $G_v \circ F = dG_v$ . We have*

$$|G_v(z_0, z_1, z_2) - \log \max\{|z_0|_v, |z_1|_v, |z_2|_v\}| \leq \frac{\log C_v}{d-1}.$$

*The set  $\{(z_0, z_1, z_2) \in \mathbb{C}_v^3, G_v(z_0, z_1, z_2) \leq 0\}$  coincides with the set of points having bounded  $F$ -orbits.*

**1.4. Canonical heights on points.** We refer to [6] for generalities on heights. Consider the line bundle on  $\mathcal{O}(1) \rightarrow \mathbb{P}_{\mathbb{k}}^2$ . The space of sections of this line bundle can be canonically identified with the space of linear forms  $a_0 z_0 + a_1 z_1 + a_2 z_2$  with  $a_i \in \mathbb{k}$ . More precisely, in the trivialization of the bundle over  $\{z_i \neq 0\}$ , this section is given by  $\frac{1}{z_i} a_0 z_0 + a_1 z_1 + a_2 z_2$ .

For any  $v \in M_{\mathbb{k}}$ , we consider the line bundle  $\mathcal{O}(1) \rightarrow \mathbb{P}_{\mathbb{C}_v}^2$  and endow it with the metrization  $|\cdot|_v$  induced by  $G_v$ , in the sense that any section  $\sigma = a_0 z_0 + a_1 z_1 + a_2 z_2$  as above

$$(3) \quad |\sigma|_v([z_0 : z_1 : z_2]) = |a_0 z_0 + a_1 z_1 + a_2 z_2| e^{-G_v(z_0, z_1, z_2)}$$

(this expression is well-defined thanks to the homogeneity property of  $G_v$ ). Let us now explain how this collection of metrizations defines a function on the set of algebraic points in  $\mathbb{P}^2$  as well as on the set of all algebraic curves in  $\mathbb{P}_{\mathbb{k}}^2$  defined by an equation with coefficients in  $\mathbb{k}^{\text{alg}}$ .

For any  $p \in (\mathbb{k}^{\text{alg}})^3 \setminus \{0\}$ , we set

$$h_f(p) := \frac{1}{N(p)} \sum_{p' \in \text{Gal} \cdot p} \left( \sum_{v \in M_{\mathbb{k}}} n_v G_v(p') \right)$$

where  $\text{Gal}$  denotes the absolute Galois group of  $\mathbb{k}^{\text{alg}}$  over  $\mathbb{k}$ , and  $N(x)$  is the cardinality of the set  $\text{Gal} \cdot x \subset (\mathbb{k}^{\text{alg}})^3$ .

The product formula entails that  $h_f(z_0, z_1, z_2) = h_f(\lambda z_0, \lambda z_1, \lambda z_2)$  for any  $\lambda \in \mathbb{k}^{\text{alg}}$  so that we have a well-defined function  $h_f: \mathbb{P}^2(\mathbb{k}^{\text{alg}}) \rightarrow \mathbb{R}$ .

**Proposition 1.4.** *The function  $h_f$  takes non-negative values, and satisfies  $h_f \circ f = dh_f$ . The set  $\{h_f = 0\}$  coincides with the set of preperiodic points of  $f$ . Furthermore, for any  $(z, w) \in (\mathbb{k}^{\text{alg}})^2$  we have*

$$h_f(z, w) := \frac{1}{N(z, w)} \sum_{(z', w') \in \text{Gal} \cdot (z, w)} \left( \sum_{v \in M_{\mathbb{k}}} n_v g_v(z', w') \right).$$

As above  $\text{Gal}$  denotes the absolute Galois group of  $\mathbb{k}^{\text{alg}}$  over  $\mathbb{k}$ , and  $N(z, w)$  is the cardinality of the set  $\text{Gal} \cdot (z, w) \subset (\mathbb{k}^{\text{alg}})^2$ . The proof follows directly from Northcott's theorem, see [4, Corollary 1.1.1].

**1.5. Analytification of affine curves.** Let  $C$  be any irreducible algebraic curve in  $\mathbb{A}_{\mathbb{k}}^2$  defined by an equation  $\{R = 0\}$  with  $R \in \mathbb{k}[z, w]$ . Denote by  $\overline{C}$  the Zariski closure of  $C$  in  $\mathbb{P}^2$ .

Fix any place  $v \in M_{\mathbb{k}}$ . We denote by  $C_v^{\text{an}}$  the analytification in the sense of Berkovich of  $C$  over  $\mathbb{C}_v$ . This is a connected, locally connected and locally compact space. When  $|\cdot|_v$  is Archimedean, hence  $\mathbb{C}_v$  is isometric to  $\mathbb{C}$ ,  $C_v^{\text{an}}$  is the complex analytic subspace (possibly with singularities) defined as usual by the vanishing of  $R$  in  $\mathbb{C}_v^2$ . When  $|\cdot|_v$  is non-Archimedean, then  $C_v^{\text{an}}$  is defined as the set of multiplicative semi-norms on the ring  $\mathbb{C}_v[z, w]/(R)$  whose restriction to the base field equals  $|\cdot|_v$ . A point is said to be rigid when the semi-norm has non-trivial kernel.

One can also define the analytification of  $\overline{C}$  by considering suitable affine coordinates in  $\mathbb{P}^2$  and patching the previous construction in a natural way, see [3, §3.4]. Observe that  $\overline{C}_v^{\text{an}} \setminus C_v^{\text{an}}$  consists of rigid points.

Suppose first  $v$  is Archimedean. The metrization of  $\mathcal{O}(1)$  defined by (3) induces a measure  $\mu_{C,v}$  on  $\overline{C}_v^{\text{an}}$  which is locally defined by  $\mu_{C,v} := \Delta \log |\sigma|_v$  where  $\sigma$  is a local section of  $\mathcal{O}(1)$ . The plurisubharmonicity of  $G_v$  ensures that  $\mu_{C,v}$  is a positive measure. The Lelong-Poincaré formula implies that the mass of  $\mu_{C,v}$  is equal to  $\deg(C)$ , and we have  $\mu_{C,v} = \Delta(g_v|_{C_v^{\text{an}}})$  on  $C_v^{\text{an}}$ . Observe that since  $G_v$  is continuous,  $\mu_{C,v}$  gives no mass to points.

The construction is completely analogous in the non-Archimedean case. We again obtain a positive measure  $\mu_{C,v}$  on  $\overline{C}_v^{\text{an}}$  of total mass  $\deg(C)$  which is given in  $C_v^{\text{an}}$  by  $\mu_{C,v} = \Delta(g_v|_{C_v^{\text{an}}})$  where  $\Delta$  is the Laplace operator defined by Thuillier [31]. Observe that the continuity of the metrization implies that  $\mu_{C,v}$  does not charge any rigid point (but it may still charge some non-rigid point in  $C_v^{\text{an}}$ ). We refer to [6, §1.3] for the details of the constructions.

**1.6. Canonical heights on curves.** Let  $C$  be any irreducible algebraic curve in  $\mathbb{A}_{\mathbb{k}}^2$  as in the previous section. We now define the canonical height of the curve  $\overline{C}$  following the recipe given in [6, §3.1.2], taking  $z_0$  as a section of  $\mathcal{O}(1)$  (note that this section vanishes exactly along the line at infinity). We obtain:

$$(4) \quad h_f(\overline{C}) := \sum_{p \in \overline{C} \cap L_{\infty}} (\overline{C}, L_{\infty})_p \times h_f(p) + \sum_{v \in M_{\mathbb{k}}} \int_{C_v^{\text{an}}} g_v d\mu_v.$$

Note that  $h_f(\overline{C}) \geq 0$  because the canonical height is non-negative on closed points, and the Green functions  $g_v$  are also non-negative.

Define the essential minimum of  $h_f$  by the following formula:

$$\text{essmin}_C(h_f) := \sup_{F \text{ finite} \subset \overline{C}(\mathbb{k}^{\text{alg}})} \inf_{\overline{C}(\mathbb{k}^{\text{alg}}) \setminus F} h_f .$$

**Theorem 1.5** (Zhang's inequality [32, Theorem 1.10]). *We have*

$$2 \text{essmin}_C(h_f) \geq \frac{h_f(\overline{C})}{\deg(C)} \geq \text{essmin}_C(h_f) + \inf_{p \in \overline{C}(\mathbb{k}^{\text{alg}})} h_f(p) .$$

Since  $h_f(x) = 0$  if and only if  $x$  is preperiodic, we obtain:

**Corollary 1.6.** *An irreducible algebraic curve  $C$  containing infinitely many  $f$ -preperiodic points satisfies  $h_f(\overline{C}) = 0$ .*

### 1.7. A first characterization of special curves.

**Proposition 1.7.** *Suppose that  $C \subset \mathbb{A}_{\mathbb{k}}^2$  is an irreducible algebraic curve containing a sequence of distinct points  $p_n \in C(\mathbb{k}^{\text{alg}})$  such that  $h_f(p_n) \rightarrow 0$ .*

*Then all points in  $\overline{C} \cap L_{\infty}$  are preperiodic for  $\overline{f}$ , and for any  $v \in M_{\mathbb{k}}$  the function  $g_v|_{C_v^{\text{an}}}$  is harmonic on  $\{g_v > 0\}$ .*

*Proof.* Note that  $h_f(\overline{C}) \geq 0$ . By Theorem 1.5, our assumption implies that  $\text{essmin}_C(h_f) \leq 0$ , therefore  $h_f(\overline{C}) = 0$ . Then the result follows from (4) and the fact that a point  $p$  is  $f$ -preperiodic if and only if  $h_f(p) = 0$ .  $\square$

## 2. SUPER-STABLE MANIFOLDS AND LOCAL ESTIMATES

**2.1. Construction of super-stable manifolds.** In this section we work under the following hypothesis:  $(\mathbb{k}, |\cdot|)$  is a complete metrized field of characteristic 0 (which may be either Archimedean or non-Archimedean).

**Theorem 2.1.** *Suppose  $f: (\mathbb{A}_{\mathbb{k}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{k}}^2, 0)$  is a germ of analytic map fixing the origin of the form*

$$(5) \quad f(x, y) = \left( \lambda x + \mu y + g(x, y), y^d(1 + h(x, y)) \right) ,$$

*where  $d \geq 2$ ,  $\lambda \neq 0$ ,  $\mu \in \mathbb{k}$ ,  $h(0, 0) = 0$ , and  $g(x, y) = \mathcal{O}(|(x, y)|^2)$ . Then there exists a unique smooth analytic curve which is transverse to  $\{y = 0\}$  and  $f$ -invariant.*

We shall call this curve the *local super-stable manifold* of the origin, and denote it by  $W_{\text{loc}}^{\text{ss}}(0)$ . After a linear change of coordinates of the form  $(x, y) \mapsto (x + \frac{\mu}{\lambda}y, y)$ , we may and will assume from now on that  $\mu = 0$ . Expressing the invariant curve as a graph of the form  $x = \varphi(y)$  and making a change of coordinates of the form  $(x, y) \mapsto (x - \varphi(y), y)$ ,  $f$  takes the form

$$(6) \quad f(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y))) .$$

It follows that  $f$  is analytically conjugate to  $y \mapsto y^d$  on  $W_{\text{loc}}^{\text{ss}}(0)$ , hence the terminology.



The result is classical when  $\mathbb{k}$  is Archimedean and/or  $f$  is locally invertible (see e.g. [21, Appendix]). For convenience we include a proof that works simultaneously in the Archimedean and non-Archimedean settings, and is adapted to  $\{y = 0\}$  being superattracting.

*Proof.* As explained above, we look for an analytic map  $y \mapsto \varphi(y)$ , with  $\varphi(0) = 0$  such that the change of coordinates  $\Phi(x, y) = (x + \varphi(y), y)$  satisfies

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y)))$$

with  $\tilde{g}, \tilde{h}$  analytic and vanishing at 0. This property is equivalent to the identities:

$$\begin{cases} \lambda\varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + \tilde{h}(0, y))) \\ \tilde{h}(0, y) = h(\varphi(y), y) \end{cases}$$

so that we aim at finding some analytic function  $\varphi$  satisfying

$$\lambda\varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + h(\varphi(y), y))) .$$

For any  $r > 0$ , let us introduce the Banach space  $\mathcal{B}_r$  that consist of those power series  $\varphi(y) := \sum_{j \geq 1} a_j y^j$  which are convergent in the disk of radius  $r$ , and satisfy  $\|\varphi\|_r := \sup_{|y| < r} |\varphi(y)| < \infty$ . Note that in the non-Archimedean case, we have  $\|\varphi\|_r := \sup_j |a_j| r^j$ .

For any  $\varphi \in \mathcal{B}_r$ , we set

$$T\varphi(y) := \frac{1}{\lambda} \left( \varphi(y^d(1 + h(\varphi(y), y))) - g(\varphi(y), y) \right) .$$

We claim that for  $r > 0$  and  $\rho > 0$  sufficiently small,  $T$  is a well-defined strictly contracting map on  $B(0, \rho) \subset \mathcal{B}_r$ . Then, applying the Banach fixed point theorem implies the existence of the desired  $\varphi$ .

First, we may suppose that  $g$  is analytic in the polydisk of radius  $r$ , and since  $g$  vanishes up to order 2 at the origin, we have

$$|g(x, y)| \leq C \max\{|x|, |y|\}^2$$

for some  $C > 0$  and all  $|x|, |y| < r$ . Similarly, we may suppose that  $h$  is analytic in the polydisk of radius  $r > 0$ , and that  $|h(x, y)| \leq \frac{1}{2}$  for all  $|x|, |y| < r$ .

Pick any  $\varphi(y) = \sum_j a_j y^j \in B(0, \rho) \subset \mathcal{B}_r$ . Reduce  $r > 0$  if necessary so that  $\frac{3}{2}r^d < r$ . Then  $\tilde{\varphi} : y \mapsto \varphi(y^d(1 + h(\varphi(y), y)))$  is well-defined and analytic on the disk of radius  $r$ .

In the non-Archimedean case,  $|1 + h(\varphi(y), y)| = 1$ , so that one has

$$\|\tilde{\varphi}(y)\|_r = \sup_{j \geq 1} |a_j| r^{dj} \leq r \sup_{j \geq 1} |a_j| r^{dj-1} \leq r \|\varphi\|_r .$$

In the Archimedean case, the Schwarz lemma yields  $|\varphi(y)| \leq \frac{|\varphi|_r}{r} |y|$  for all  $|y| < r$ , hence  $\|\tilde{\varphi}\|_r \leq \frac{3}{2} r^{d-1} \|\varphi\|_r$ . Note that we also have:

$$\|g(\varphi(y), y)\|_r \leq C \max\{\|\varphi\|_r, r\}^2$$

so for  $\varphi \in B(0, \rho)$  we deduce

$$|T\varphi|_r \leq \frac{1}{\lambda} \left( \frac{3}{2} r^{d-1} \rho + C \max\{\rho, r\}^2 \right) .$$

By choosing  $\rho = r$  and then  $r$  small enough, this estimate shows that  $T\varphi$  is well-defined on the ball  $B(0, r) \subset \mathcal{B}_r$  and  $T(B(0, r)) \subset B(0, r)$ .

In order to prove that  $T$  is strictly contracting, observe that we can write

$$g(x, y) - g(x', y) = (x - x')\hat{g}(x, x', y)$$

where  $\hat{g}(x, x', y)$  is again analytic in the polydisk of radius  $r$ , and

$$|\hat{g}(x, x', y)| \leq C' \max\{|x|, |x'|, |y|\}$$

for some constant  $C' > 0$ . For any pair of analytic functions  $\varphi_1, \varphi_2 \in B(0, r) \subset \mathcal{B}_r$  we infer:

$$|g(\varphi_1(y), y) - g(\varphi_2(y), y)| \leq C' \|\varphi_1 - \varphi_2\|_r \max\{\|\varphi_1\|_r, \|\varphi_2\|_r, r\}$$

hence

$$\|T\varphi_1 - T\varphi_2\|_r \leq \frac{1}{\lambda}(3r^d + C'r) \|\varphi_1 - \varphi_2\|_r .$$

Again, by choosing  $r$  sufficiently small, we obtain that  $T$  is strictly contracting and we are done.  $\square$

**2.2. The rescaling argument in the repelling case.** We work in  $\mathbb{A}_{\mathbb{k}}^2$  where  $\mathbb{k}$  is an arbitrary complete metrized field of characteristic 0. We start with a preparation lemma.

**Lemma 2.2.** *Suppose  $f$  is an analytic map of the form*

$$(7) \quad f(x, y) = \left( \lambda x + xg(x, y), y^d(1 + h(x, y)) \right)$$

where  $|\lambda| > 1$ ,  $d \geq 2$  and  $g(0) = h(0) = 0$ .

*Then there exists an analytic change of coordinates  $\Phi$  such that*

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x(1 + xy\tilde{g}(x, y)), y^d(1 + x\tilde{h}(x, y)))$$

*for some analytic functions  $\tilde{g}, \tilde{h}$ .*

Recall that the form (7) is what is obtained from (5) after conjugating to get  $\mu = 0$  and declaring that the stable manifold of Theorem 2.1 is  $\{x = 0\}$ .

*Proof.* By Böttcher's theorem (see [2, Chapter 4] for the non-Archimedean case) applied to  $y \mapsto f(0, y)$  we may suppose that  $x$  divides  $h$ . Similarly, since  $|\lambda| > 1$ , by we may linearize  $x \mapsto f(x, 0)$ , and suppose that  $f$  is of the form  $f(x, y) = (\lambda x(1 + g_1(y)) + x^2 y h_1(x, y), y^d(1 + \mathcal{O}(x)))$  for some analytic functions  $g_1, h_1$  with  $g_1(0) = 0$ .

We claim that there exists  $\Phi(x, y) = (x(1 + \varphi(y)), y)$  with  $\varphi(0) = 0$  such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x^2 y h_2(x, y), y^d(1 + \mathcal{O}(x)))$$

for some analytic function  $h_2$ . Indeed,  $\varphi$  is then characterized by the equation

$$\lambda x(1 + \varphi(y))(1 + g_1(y)) + \mathcal{O}(x^2) = (\lambda x + \mathcal{O}(x^2))(1 + \varphi(y^d))$$

that is,  $(1 + \varphi(y))(1 + g_1(y)) = (1 + \varphi(y^d))$ , a solution of which is given by the infinite product

$$1 + \varphi(y) = \prod_{k=0}^{\infty} \left( 1 + g_1 \left( y^{d^k} \right) \right)^{-1}$$

and we are done.  $\square$

The next result is similar to [11, Lemma 4.2].

**Proposition 2.3.** *Suppose  $f$  is an analytic map of the form*

$$(8) \quad f(x, y) = (\lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y)))$$

where  $|\lambda| > 1$ ,  $d \geq 2$  and  $g, h$  are analytic functions.

Then  $f^n(\frac{x}{\lambda^n}, y) \rightarrow (x, 0)$  when  $n \rightarrow \infty$ , uniformly on a polydisk of sufficiently small radius centered at the origin.

*Proof.* Fix  $0 < r \leq 1/4$  small enough so that  $g, h$  are both analytic on the polydisk of radius  $r$  and  $|g(x, y)|, |h(x, y)| \leq 1$  for  $|x|, |y| < r$ . Let us first show that if  $|x| \leq \frac{r}{2|\lambda|^n}$  and  $|y| \leq r$ , then the  $n$  first iterates of  $(x, y)$  remain in  $\mathbb{D}_r^2$ . We argue by induction. So assume that  $(x_0, y_0) \in \mathbb{D}_{r|\lambda|^{-n}/2} \times \mathbb{D}_r$ , put  $f^j(x_0, y_0) = (x_j, y_j)$ , let  $k \leq n$  and assume that  $(x_j, y_j) \in \mathbb{D}_r^2$  for  $j \leq k-1$ . Observe that for  $j \leq k-1$ ,  $|y_{j+1}| \leq 2|y_j|^d$  from which it follows that

$$|y_k| \leq \left(2^{1/(d-1)}|y_0|\right)^{d^k} \leq \left(2^{1/(d-1)}r\right)^{d^k} \leq r.$$

Observe that the first inequality together with  $r \leq 1/4$  also yield  $|y_j| \leq 2^{-d^j}$ . For the first coordinate, we use recursively the relation  $x_{j+1} = \lambda x_j(1 + x_j y_j g(x_j, y_j))$  to get

$$(9) \quad |x_k| \leq |\lambda|^k |x_0| \prod_{j=0}^{k-1} (1 + |x_j| |y_j|) \leq \frac{r}{2} |\lambda|^{k-n} \prod_{j=0}^{k-1} (1 + 2^{-d^j}) \leq r |\lambda|^{k-n},$$

where the last inequality follows from

$$\prod_{j=0}^{k-1} (1 + 2^{-d^j}) \leq \prod_{j=0}^{k-1} (1 + 2^{-2^j}) = \frac{2^{2^k} - 1}{2^{2^k - 1}} < 2,$$

in which the middle equality is easily obtained by induction.

Now take  $(x, y) \in \mathbb{D}_{r/2}$ , and consider  $f^n(\frac{x}{\lambda^n}, y)$ . Denote as before  $(x_0, y_0) = (\frac{x}{\lambda^n}, y)$  and  $(x_j, y_j) = f^j(x_0, y_0)$ . The first part of the proof shows that  $(x_j, y_j)$  is well-defined for all  $j \leq n$ , and that  $y_n \rightarrow 0$ . Now we have

$$x_n = \lambda^n x_0 \prod_{j=0}^{n-1} (1 + x_j y_j h(x_j, y_j)) = x \prod_{j=0}^{n-1} (1 + x_j y_j h(x_j, y_j)).$$

The inequality  $|\prod(1+z_j) - 1| \leq \exp(\sum|z_j|) - 1$  shows that to establish the convergence  $x_n \rightarrow x$  it is enough to show that  $\sum_{j=0}^{n-1} |x_j y_j h(x_j, y_j)|$  tends to 0. But by (9),  $|x_j| \leq r |\lambda|^{j-n}$ , thus

$$\sum_{j=0}^{n-1} |x_j y_j h(x_j, y_j)| \leq \sum_{j=0}^{n-1} r |\lambda|^{j-n} 2^{-d^j} \leq r |\lambda|^{-n} \sum_{j=0}^{\infty} |\lambda|^j 2^{-d^j},$$

and we are done.  $\square$

**2.3. Graph transform for  $\lambda = 1$ .** In this paragraph we assume that  $\mathbb{k} = \mathbb{C}$  and  $f$  is of the form

$$(10) \quad f(x, y) = \left( x + g(x, y), y^d(1 + h(x, y)) \right),$$

with  $g(x, y) = \mathcal{O}(|x, y|^2)$ ,  $h(0, 0) = 0$ , and  $g(x, 0) = cx^{k+1} + \mathcal{O}(x^{k+2})$  for some  $k \geq 1$  and  $c \neq 0$ . Observe that  $f|_{\{y=0\}}$  has a parabolic point at the origin with  $k$  attracting and  $k$  repelling petals (see e.g. [1, §6.5]). An *attracting petal* is a  $f$ -invariant (connected and simply-connected) open subset  $U$  containing 0 in its boundary, and such that, for all  $z \in U$ ,  $f^n(z) \rightarrow 0$  tangentially to some real direction (in our case, to the normalized  $k$ -th roots of  $-c$ ). A *repelling petal* is an attracting petal for  $f^{-1}$  (they are tangent to the normalized  $k$ -th roots of  $c$ ). One can choose the  $k$  attracting petals and  $k$  repelling petals so that their union fills up a punctured neighborhood of the origin.

A vertical graph  $V$  in a domain of the form  $\Omega \times \mathbb{D}_\rho$  is a submanifold of the form  $V := \{(\varphi(y), y), y \in \mathbb{D}_\rho\}$  for some holomorphic function  $\varphi : \mathbb{D}_\rho \rightarrow \Omega$ . In the next theorem we consider pull backs of such graphs in  $\mathbb{D}_r^2$  in the graph transform sense, that is, when pulling back some vertical graph under  $f$ , we keep only the component of  $f^{-1}(V) \cap \mathbb{D}_r^2$  containing  $f^{-1}(V \cap \{y = 0\})$ . Abusing notation we simply denote it by  $f^{-1}(V)$ .

**Theorem 2.4.** *Let  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  be of the form (10) with  $h(0) = 0$ ,  $g(x, 0) \neq 0$  and  $g(x, y) = \mathcal{O}(|(x, y)|^2)$ . Let  $U$  be any repelling petal of  $f|_{\{y=0\}}$  and consider a germ  $V$  of analytic curve transverse to  $\{y = 0\}$ , intersecting  $U$ .*

*Then there exists  $r > 0$  depending only on  $f$ , such that for large enough  $n$ , the analytic sets  $f^{-n}(V)$  are vertical graphs in  $\mathbb{D}_r^2$  converging to  $W_{\text{loc}}^{\text{ss}}(0)$  in the  $C^1$  topology.*

**Lemma 2.5.** *Suppose  $f$  is a holomorphic map of the form (10) as in Theorem 2.4. Then there exist an integer  $k \geq 1$ , and an analytic change of coordinates  $\Phi$  such that*

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (x + x^{k+1} + x^{2k+1}\tilde{g}(x, y), y^d(1 + x\tilde{h}(x, y)))$$

for some analytic functions  $\tilde{g}, \tilde{h}$ .

*Proof.* The proof is essentially contained in [20, Proposition 2.3]. We provide the details for the sake of completeness. By Theorem 2.1 and by applying the Böttcher theorem to  $y \mapsto f(0, y)$  we may assume that both  $g$  and  $h$  are divisible by  $x$ , so that we may write

$$(11) \quad f(x, y) = \left( x + xg(x, y), y^d(1 + xh(x, y)) \right),$$

with  $g(0, 0) = 0$ .

By a local change of coordinates involving only  $x$ , we can arrange so that  $f(x, 0) = (x + x^{k+1} + \mathcal{O}(x^{2k+1}), 0)$ . Expand the first coordinate of  $f$  in power series of  $x$  as follows:

$$x \circ f(x, y) = x(1 + g_0(y)) + x^{k+1}(1 + g_k(y)) + \sum_{j \neq 0, k}^{\infty} x^{j+1}g_j(y),$$

with  $g_j(0) = 0$  for  $0 \leq j \leq 2k - 1$ .

We claim that for all  $n \leq 2k - 1$ , we can conjugate  $f$  by a germ of invertible holomorphic map such that  $g_j \equiv 0$  for every  $j \leq n$ . Applied to  $n = 2k - 1$ , this claim implies the proposition.

For  $n = 0$  this is done by a change of coordinates of the form  $\Phi_0(x, y) := (x(1 + \varphi_0(y)), y)$  such that  $(1 + g_0)(1 + \varphi_0) = 1 + \varphi_0(y^d)$ . This equation can be solved exactly as in Lemma 2.2.

Now assume that  $n > 0$ , and that the result has been achieved up to  $j = n - 1$ . Put  $\Phi_n(x, y) = (x(1 + \varphi_n(y)x^n), y)$ , so that  $\Phi_n^{-1}(x, y) = (x(1 - \varphi_n(y)x^n + \mathcal{O}(x^{n+1})), y)$ . Depending on the position of  $n$  and  $k$ , we obtain:

$$x \circ (\Phi_n^{-1} \circ f \circ \Phi_n) = \begin{cases} x + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + \mathcal{O}(x^{n+2}) & \text{if } n < k, \\ x + x^{n+1}(\varphi_k(y) + 1 + g_n(y) - \varphi_n(y^d)) + \mathcal{O}(x^{n+2}) & \text{if } n = k, \\ x + x^{k+1} + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + \mathcal{O}(x^{n+2}) & \text{if } n > k. \end{cases}$$

Therefore, to render the term in  $x^{n+1}$  constant, it is enough to solve the equation  $-g_n(y) = \varphi_n(y) - \varphi_n(y^d)$  which can be done by setting  $\varphi_n(y) = -\sum_{m=0}^{\infty} g_n(y^{d^m})$ .  $\square$

*Proof of Theorem 2.4.* By the previous lemma, we may suppose that

$$f(x, y) = \left( x + x^{k+1} + x^{2k+1}g(x, y), y^d(1 + xh(x, y)) \right)$$

with  $g$  and  $h$  holomorphic near the origin. Note that  $f|_{y=0}$  has a repelling petal along the positive real axis. Fix  $r > 0$  such that  $f$  is holomorphic and injective on a neighborhood of  $\overline{\mathbb{D}}_r^2$ . Reducing and rotating the petal if necessary, we may assume that

$$U := \left\{ x : \arg(x) \in \left( -\frac{\pi}{4k}, \frac{\pi}{4k} \right), |x| < r \right\}.$$

The holomorphic map  $z = (kx^k)^{-1}$  is univalent on  $U \times \mathbb{D}_r$ , and takes its values in

$$\Omega_R := \left\{ z : \arg(z) \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right), |z| > R \right\},$$

where  $R = (kr^k)^{-1}$ . The expression of  $f$  in the coordinates  $(z, y)$  is of the form

$$(12) \quad f(z, y) = \left( z - 1 + \frac{1}{z}a(z, y), y^d \left( 1 + \frac{1}{z^{1/k}}b(z, y) \right) \right).$$

so that  $f$  is now defined in  $\Omega_R \times \mathbb{D}_r$ . Fix  $M > 0$  such that

$$(13) \quad |a|, |b|, \left| \frac{\partial a}{\partial z} \right|, \left| \frac{\partial a}{\partial y} \right|, \left| \frac{\partial b}{\partial z} \right|, \left| \frac{\partial b}{\partial y} \right| \leq M \text{ on } \Omega_R \times \mathbb{D}_r,$$

and reduce  $r$  if necessary so that  $r < \frac{1}{10d}$  and  $MR^{-1/k} \leq \frac{1}{100}$ .

For any  $\rho < r$  and  $\sigma > 0$  we let

$$\mathcal{G}(\rho, \sigma) = \left\{ \varphi : \mathbb{D}_\rho \rightarrow \Omega_R \text{ holomorphic s.t. } \sup_{\mathbb{D}_\rho} |\varphi'| \leq \sigma \right\}.$$

**Lemma 2.6.** *Suppose that  $\sigma\rho < \frac{1}{100d}$ . For any vertical graph  $\Gamma$  determined by  $\varphi \in \mathcal{G}(\rho, \sigma)$ , then  $f^{-1}\Gamma$  is a vertical graph determined by a function  $\psi \in \mathcal{G}(\rho_1, \frac{1}{10})$  where  $\rho_1 = \min((\rho/2)^{1/d}, r)$  and  $\text{Re}(\psi) \geq \text{Re}(\varphi(0)) + 9/10$ .*

Assuming this result for the moment, let us conclude the proof of the theorem. Let  $V$  be any germ of curve intersecting transversally  $\{y = 0\}$  at  $(x_0, 0) \in U$ . Then  $V$  is a graph of slope  $\sigma$  over some disk  $\mathbb{D}_\rho$  in the second coordinate, for some  $\rho > 0$ . Reduce  $\rho$  by trimming  $V$  if necessary so that  $\sigma\rho < 1/100d$ . Since  $r/10 < 1/100d$ , the previous

lemma implies that we can define inductively a sequence of vertical graphs  $V = V_0$ ,  $V_n := f^{-1}V_{n-1}$ , where  $V_n$  is defined by a holomorphic function  $z = \varphi_n(y)$  of uniformly bounded slope over  $\mathbb{D}_{\rho_n}$ , and furthermore  $\rho_n = r$  for large enough  $n$ . Moreover, we have  $\operatorname{Re}(\varphi_n) \geq \operatorname{Re}(\varphi(0)) + 9n/10$ , hence, coming back to the  $(x, y)$  coordinates, we see that  $V_n = \{x = (k\varphi_n(y))^{-1/k}\}$  converges in the  $C^1$ -topology to the curve  $W_{\text{loc}}^{\text{ss}}(0) = \{x = 0\}$ .  $\square$

*Proof of Lemma 2.6.* Let  $\Gamma$  be a vertical graph of equation  $z = \varphi(y)$  in the  $(z, y)$  coordinates, with  $\varphi \in \mathcal{G}(\rho, \sigma)$ . Then the equation of  $f^{-1}\Gamma$  is given by  $z = \ell(z, y)$ , where

$$\ell(z, y) = \varphi \left( y^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) + 1 - \frac{1}{z} a(z, y).$$

Fix  $y_0 \in \mathbb{D}_{\rho_1}$ . We show that the equation  $z = \ell(z, y_0)$  admits a unique solution  $z \in \Omega_R$ . First, observe that by the estimates (13) on  $|a|$  and  $|b|$ , for  $z \in \Omega_R$  we have

$$(14) \quad \begin{aligned} |\ell(z, y_0) - (\varphi(0) + 1)| &\leq \left| \varphi \left( y_0^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) - \varphi(0) \right| + \frac{M}{R} \\ &\leq 2\rho_1^d \sigma + \frac{1}{100} \leq \rho\sigma + \frac{1}{100} \leq \frac{1}{10}, \end{aligned}$$

hence  $\ell(\cdot, y_0)$  maps  $\Omega_R$  to itself. Next, we see that

$$\begin{aligned} \left| \frac{\partial \ell}{\partial z}(z, y_0) \right| &\leq \left| y_0^d \left( \frac{1}{z^{1/k}} \frac{\partial b}{\partial z} - \frac{1}{kz^{1/k+1}} b \right) \right| \cdot \left| \varphi' \left( y_0^d \left( 1 + \frac{1}{z^{1/k}} b \right) \right) \right| + \left| \frac{a}{z^2} - \frac{1}{z} \frac{\partial a}{\partial z} \right| \\ &\leq \rho_1^d \frac{2M}{R^{1/k}} \sigma + \frac{2M}{R} \leq \frac{4}{100} \rho\sigma + \frac{2}{100} \leq \frac{1}{10}, \end{aligned}$$

so  $\ell(\cdot, y_0)$  is a contraction and the equation  $z = \ell(z, y_0)$  has a unique solution. This means that  $f^{-1}\Gamma$  is a vertical graph over  $\mathbb{D}_{\rho_1}$  determined by a holomorphic function  $\psi$  satisfying  $\psi(y) = \ell(\psi(y), y)$ . The slope of this graph can be estimated as above:

$$|\psi'(y)| \leq \left| \frac{\partial \ell / \partial y}{1 - \partial \ell / \partial z} \right| \leq \frac{10}{9} \left( \sigma \left( d\rho^{d-1} \left( 1 + \frac{M}{R^{1/k}} \right) + \rho^d \frac{M}{R^{1/k}} \right) + \frac{M}{R} \right) \leq \frac{1}{10}.$$

Finally, the estimate  $\operatorname{Re}(\psi) \geq \operatorname{Re}(\varphi(0)) + 9/10$  follows from (14) and we are done.  $\square$

### 3. PROOF OF THEOREM A WHEN $\mathbb{k}$ IS A NUMBER FIELD

Here we establish the following more precise form of our main theorem, in the number field case.

**Theorem 3.1.** *Let  $\mathbb{k}$  be a number field and  $f$  be a regular polynomial map of  $\mathbb{A}_{\mathbb{k}}^2$ . Denote by  $h_f$  the induced canonical height.*

*Suppose that  $C \subset \mathbb{A}_{\mathbb{k}}^2$  is an irreducible algebraic curve containing a sequence of distinct points  $p_n \in C(\mathbb{k}^{\text{alg}})$  such that  $h_f(p_n) \rightarrow 0$ . If there exists a point of  $\overline{C} \cap L_{\infty}$  which is not eventually superattracting, then  $C$  is preperiodic.*

Let  $f_{\infty}$  be the restriction to the line at infinity  $L_{\infty}$  of the extension of  $f$  to  $\mathbb{P}^2$ . By Proposition 1.7, all points in  $\overline{C} \cap L_{\infty}$  are preperiodic, so we may replace  $f$  by  $f^N$  and  $C$  by  $f^N(C)$ , to assume that  $\overline{C} \cap L_{\infty}$  contains a fixed point  $p$  which is not super-attracting.

Let  $\lambda = f'_\infty(p)$  be the multiplier of  $p$  along  $L_\infty$ . Then one of the two following mutually disjoint cases occur:

- (a) either  $\lambda$  is a root of unity;
- (b) or there is a place  $v \in M_{\mathbb{k}}$  such that  $|\lambda|_v > 1$ .

In the remainder of this section we split the proof of the theorem according to these two cases.

**3.1. When  $\lambda$  is a root of unity.** In this situation we iterate  $f$  further so that  $\lambda = 1$ , and work over the complex numbers. Since  $p$  belongs to the Julia set  $J(f_\infty)$ , which is a perfect set, the union of attracting and repelling petals cover a punctured neighborhood of  $p$ , and the attracting petals of  $p$  are contained in the Fatou set, we see that there is a repelling periodic point  $q$  of  $f_\infty$  contained in some local repelling petal of  $p$ . Then the local (super-)stable manifold of  $q$  is a disk transverse to  $L_\infty$  at  $q$ , and by Theorem 2.4, the local truncated pull-backs  $f^{-n}(W_{\text{loc}}^{\text{ss}}(q))$  under  $f^n$  converge to  $W_{\text{loc}}^{\text{ss}}(p)$  when  $n \rightarrow \infty$ .

Assume by way of contradiction that  $\overline{C}$  does not locally coincide with  $W_{\text{loc}}^{\text{ss}}(p)$ . We claim that  $C$  intersects  $f^{-n}(W_{\text{loc}}^{\text{ss}}(q))$  in  $\mathbb{C}^2$  close to  $p$  for large  $n$ . Indeed, locally near  $p$ ,  $\overline{C} \cap W_{\text{loc}}^{\text{ss}}(p) = \{p\}$ , so by the persistence of proper intersections,  $\overline{C}$  intersects  $f^{-n}(W_{\text{loc}}^{\text{ss}}(q))$  close to  $p$  for large  $n$ . But  $p \notin f^{-n}(W_{\text{loc}}^{\text{ss}}(q)) \cap L_\infty$ , so these intersection points lie in  $\mathbb{C}^2$ , as claimed. If now  $\Delta$  is a small disk in  $C$  containing one of these intersection points, then by the Inclination Lemma, the derivative of  $f^n$  in the direction of  $\Delta$  tends to infinity, thus  $(f^n|_\Delta)$  is not a normal family. On the other hand, by Proposition 1.7  $g|_\Delta$  is harmonic, which implies that  $(f^n|_\Delta)$  is normal (see [13, Prop 5.10]). This contradiction shows that  $\overline{C}$  locally coincides with  $W_{\text{loc}}^{\text{ss}}(p)$  near  $p$ , so it is fixed under  $f$ , and by irreducibility this property propagates to the whole of  $C$ . This completes the proof in this case.  $\square$

**Remark 3.2.** This argument works essentially the same when  $|\lambda|_v > 1$  at some Archimedean place, and may help understand the non-Archimedean argument below.

**3.2. When  $|\lambda|_v > 1$ .** We may assume that  $p = [0 : 0 : 1]$ , and by Lemma 2.2 find a local analytic isomorphism  $(x, y) \mapsto \psi(x, y) = [z_0(x, y) : z_1(x, y) : 1]$  such that  $\psi(0) = p$ ,  $z_0(x, 0) = 0$  so that  $\{y = 0\}$  corresponds to the line at infinity<sup>1</sup>, and write

$$\tilde{f} := \psi^{-1} \circ f \circ \psi : (x, y) \mapsto (\lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y))) .$$

If  $\overline{C}$  has an analytic branch at  $p$  which coincides with  $\{x = 0\}$  then  $C$  is fixed as above, and we are done. Otherwise, we may find a Puiseux parameterization of a branch of  $\overline{C}$  at  $p$  in the  $(x, y)$  coordinates of the form  $\Gamma(t) = (t^q, \gamma(t))$ , where  $\gamma$  is analytic and defined in a small disk  $\mathbb{D}_\delta$ , and  $q \in \mathbb{N}^*$ . We seek a contradiction.

We lift  $\psi$  to  $\mathbb{A}_{\mathbb{k}}^3$ , and set  $\Psi(x, y) = (z_0(x, y), z_1(x, y), 1)$ . As in §1.3, we lift  $f$  to a homogeneous polynomial map  $F: \mathbb{A}_{\mathbb{k}}^3 \rightarrow \mathbb{A}_{\mathbb{k}}^3$ , of the form  $F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q})$ . Since  $\Psi \circ \psi^{-1}$  is a local section of the projection  $\mathbb{A}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ ,  $\Psi \circ \tilde{f}$  must be a multiple of  $F \circ \Psi$ . From the expression of  $F$  we obtain

$$\Psi \circ \tilde{f} = \frac{F \circ \Psi}{\tilde{Q} \circ \Psi} .$$

<sup>1</sup>Beware that coordinates are swapped here :  $\{z_0 = 0\}$  corresponds to  $\{y = 0\}$ .

To simplify notation, we write  $F^n(z_0, z_1, z_2) = (z_0^{d^n}, \tilde{P}_n, \tilde{Q}_n)$ . We consider the 1-homogeneous Green function  $G_f: \mathbb{A}_v^{3,\text{an}} \rightarrow \mathbb{R}$  of Proposition 1.3; it satisfies  $G_f \circ F = dG_f$  and  $g_f(z_1, z_2) = G_f(1, z_1, z_2)$ .

Observe that  $h := G_f \circ \Psi \circ \Gamma$  is a continuous function on  $\mathbb{D}_\delta$ . Since  $h(t) = g_f \circ \psi \circ \Gamma(t) + \log |z_0 \circ \Gamma(t)|$ , by Proposition 1.7,  $h$  is harmonic on  $\{t \neq 0\}$ . Since it is continuous at 0, it is also harmonic on  $\mathbb{D}_\delta$ , see, e.g., [12, Lemma 3.7]. Write

$$\begin{aligned} d^{nq} h(t) &= G_f \circ F^{nq} \circ \Psi \circ \Gamma(t) \\ &= G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(t) + \log \left| \tilde{Q}_{nq} \circ \Psi \circ \Gamma(t) \right|. \end{aligned}$$

By Proposition 2.3,  $\tilde{f}[n](\frac{x}{\lambda^n}, y) \rightarrow (x, 0)$  uniformly in a neighborhood of the origin, hence

$$G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma\left(\frac{t}{\lambda^{n/q}}\right) \rightarrow G_f(0, z_1(t^q, 0), 1)$$

as  $n \rightarrow \infty$ . On the other hand, since  $d^{nq} h\left(\frac{t}{\lambda^{n/q}}\right) - \log \left| \tilde{Q}_{nq} \circ \Psi \circ \Gamma\left(\frac{t}{\lambda^{n/q}}\right) \right|$  is a sequence of harmonic functions, it follows that  $t \mapsto G_f(0, z_1(t^q, 0), 1)$  is harmonic as well.

Now, observe that the restriction of  $f$  to the line at infinity is  $f_\infty[z_1 : z_2] = [\tilde{P}(0, z_1, z_2) : \tilde{Q}(0, z_1, z_2)]$ , so that  $G_f(0, z_1, z_2)$  is the global Green function of  $f_\infty$ . The equilibrium measure of  $f_\infty$  is the probability measure on the analytification of  $L_\infty$  defined by  $\mu_{f_\infty} := \Delta G_f(0, z_1, 1)$  in the chart  $z_2 \neq 0$ . Its support is the Julia set of  $f_\infty$  (see [2, Theorem 13.39]), and it contains all repelling (rigid) fixed points, see [2, Theorem 8.7]. Therefore,  $z_1 \mapsto G_f(0, z_1, 1)$  cannot be harmonic near 0, hence the function  $t \mapsto G_f(0, z_1(t^q, 0), 1)$  cannot be harmonic either. This contradiction concludes the proof.  $\square$

**Remark 3.3.** Under the assumptions of Theorem 3.1, the proof shows that the preperiod  $k$  of  $F$  and the period of  $f^k(C)$  are exactly the same as that of any of its non-superattracting points at infinity.

#### 4. PROOF OF THEOREM A FOR ARBITRARY $\mathbb{k}$

In this section we use a specialization argument to deal with maps defined over arbitrary fields. It shares some arguments with [11, §5] (see also [5, §7]). Nevertheless, new ideas are needed to deal with preperiodic points instead of periodic ones.

We are in the setting of Theorem A, so we assume that  $f$  is a regular polynomial map of  $\mathbb{A}^2$  of degree  $d \geq 2$  defined over a field  $\mathbb{k}$  of characteristic 0, and  $C$  is a curve containing an infinite set  $\mathcal{P} = \{p_n, n \geq 0\}$  of preperiodic points and whose closure  $\overline{C} \cap L_\infty$  contains at least one point which is not eventually super-attracting.

By enlarging  $\mathbb{k}$  if necessary we may assume that it contains the algebraic closure  $\mathbb{Q}^{\text{alg}}$  of its prime field. Let  $R$  be the sub- $\mathbb{Q}^{\text{alg}}$ -algebra of  $\mathbb{k}$  generated by all coefficients defining  $f$  and  $C$ . Its fraction field  $K$  is finitely generated over  $\mathbb{Q}^{\text{alg}}$ . Let  $S = \text{Spec } R$ . This is an affine variety defined over  $\mathbb{Q}^{\text{alg}}$ , and elements of  $R$  can be seen as regular functions on  $S$ .

Inverting some elements of  $R$  if necessary, we may suppose that  $C$  is flat over  $S$ , and  $f$  extends as a morphism  $f: \mathbb{P}_S^2 \rightarrow \mathbb{P}_S^2$ . We let  $\pi: \mathbb{P}_S^2 \rightarrow S$  be the canonical projection, and write  $\mathbb{P}_s^2 = \pi^{-1}(s)$ . We also let  $\mathbb{A}_s^2 := \mathbb{A}_S^2 \cap \pi^{-1}(s)$ .



For each (scheme theoretic) point  $s \in S$ , we write  $C_s = C \cap \mathbb{A}_s^2$  and let  $\overline{C}_s$  be the closure of  $C_s$  in  $\mathbb{P}_s^2$ . The flatness of the morphism  $C \rightarrow S$  implies  $\overline{C}_s$  (hence  $C_s$ ) to be a curve. Similarly, we let  $f_s: \mathbb{P}_s^2 \rightarrow \mathbb{P}_s^2$  be the induced map on the fibers: this is an endomorphism of degree  $d$ .

We also denote by  $p_{n,s} \in \mathbb{A}_s^2$  the specialization of  $p_n$ . Note that  $p_n$  is defined over some finite extension of  $K$  which depends on  $n$ .

The first result does not use the assumption that our infinite set of preperiodic points lies on a curve.

**Proposition 4.1.** *Let as above  $f: \mathbb{P}_S^2 \rightarrow \mathbb{P}_S^2$  be a family of endomorphisms over an affine variety defined over  $\mathbb{Q}^{\text{alg}}$ , and let  $\mathcal{P} = \{p_n, n \geq 0\}$  be an infinite family of preperiodic points. Then there exists a non-empty Zariski open and dense subset  $U \subset S$  such that for any  $s \in U \cap S(\mathbb{Q}^{\text{alg}})$ ,  $\mathcal{P}_s = \{p_{n,s}, n \geq 0\} \subset \mathbb{A}_s^2$  is infinite.*

Before starting the proof, let us fix some additional notation. For each  $n \geq 0$ , we denote by  $k_n$  the preperiod of  $p_n$ , so that  $q_n := f^{k_n}(p_n)$  is the first periodic point in the orbit of  $p_n$ . We let  $\ell_n$  be the (primitive) period of  $q_n$ .

*Proof.* We may suppose that there exists a parameter  $s_0 \in S(\mathbb{Q}^{\text{alg}})$  such that  $\mathcal{P}_{s_0}$  is finite (otherwise we take  $U = S$  and the proof is complete).

**Lemma 4.2.** *The family of periodic points  $(q_n)$  is finite.*

*Proof.* We follow the arguments of [11, §5]. Set  $\mathcal{Q} = \{q_n, n \geq 0\}$ . For each  $\ell \geq 1$ , we consider the subvariety  $\text{Per}_\ell$  of  $\mathbb{P}_S^2$  defined by the equation  $f^\ell(z) = z$ . Since  $\mathbb{P}_S^2 \rightarrow S$  is proper, the structure map  $\text{Per}_\ell \rightarrow S$  is also proper.

Let  $\mathcal{Q}_\ell$  be the union of the irreducible components of  $\text{Per}_\ell$  containing a point of  $\mathcal{P}$ . Its underlying set is the Zariski closure of  $\mathcal{P} \cap \text{Per}_\ell$ , hence  $\mathcal{Q}_\ell \rightarrow S$  is proper. Observe that for  $x \in \mathcal{Q}_{\ell,s}$ , the multiplicity of  $x$  as a point of  $\mathcal{Q}_{\ell,s}$  equals its multiplicity as a fixed point of  $f_s^\ell$ . By Nakayama's lemma and the properness of  $\mathcal{Q}_\ell$  over  $S$ , the function

$$(15) \quad s \mapsto \sum_{x \in \mathcal{Q}_{\ell,s}} \text{mult}_x(\mathcal{Q}_{\ell,s})$$

is upper semi-continuous for the Zariski topology, hence

$$(16) \quad \sum_{q \in \mathcal{Q}_\ell} \text{mult}_q(\mathcal{Q}_\ell) \leq \sum_{x \in \mathcal{Q}_{\ell,s_0}} \text{mult}_x(\mathcal{Q}_{\ell,s_0}),$$

where the left hand side is the value of (15) at the generic point. By assumption  $\mathcal{P}_{s_0}$  is a finite set, hence so does  $\mathcal{Q}_{s_0} = \{q_{1,s_0}, \dots, q_{r,s_0}\}$  and by the Shub-Sullivan theorem [29], there exists a uniform bound  $C > 0$  such that for every  $j$ , and for any  $\ell$ , we have

$$\text{mult}_{q_{j,s_0}}(\mathcal{Q}_{\ell,s_0}) \leq \text{mult}_{q_{j,s_0}}(\text{Per}_{\ell,s_0}) \leq C .$$

It then follows from (16) that

$$\#\mathcal{Q}_\ell \leq \sum_{q \in \mathcal{Q}_\ell} \text{mult}_q(\mathcal{Q}_\ell) \leq rC$$

hence  $\bigcup_\ell \mathcal{Q}_\ell$  is finite, as was to be shown.  $\square$

By the previous lemma, replacing  $f$  by some iterate  $f^N$  we may assume that all periodic points  $q_n$  are fixed. Since  $\mathcal{P}$  is infinite, one of these fixed points, say  $q_1$ , admits infinitely many preimages in  $\mathcal{P}$ . We may denote  $q = q_1$  and suppose  $\mathcal{P}$  is made of an infinite set of preimages of  $q$ , that is, (after possible reordering of  $\mathcal{P}$ ) for any  $p_n \in \mathcal{P}$  there is a minimal  $k_n \geq 0$  such that  $f^{k_n}(p_n) = q$ , and that  $k_{n+1} > k_n$ . We may adjoin to  $R$  the coordinates of  $q$  so that  $q \in \mathbb{A}^2(R)$ , i.e., for any  $s \in S$ ,  $q_s$  is a single point (to say it differently, we replace  $S$  by a branched cover of a Zariski open dense subset of  $S$ ).

Let  $d(s)$  be the local degree of  $f_s$  at  $q_s$ , which is upper semicontinuous for the Zariski topology. Since  $f_s$  is a finite map of degree  $d^2$ ,  $d(s) \leq d^2$  for every  $s$ . Thus there is an analytic hypersurface  $H$  such that  $d(s) = d_{\min}$  is constant for  $s \in S \setminus H$ .

We claim that  $\mathcal{P}_{s_1}$  is infinite for any  $s_1 \in S \setminus H$ . We argue again in the complex analytic category fixing an embedding  $\mathbb{Q}^{\text{alg}}$  into  $\mathbb{C}$ . Observe that for any point  $s \in S(\mathbb{C})$ ,  $p_{n,s}$  is a finite set included in the fiber  $\mathbb{A}_s^{2,\text{an}} \simeq \mathbb{C}^2$  (not necessarily reduced to a single point since  $p_n$  lies in a finite extension of  $R$ ).

Fix an analytic neighborhood  $V$  of  $q_{s_1}$  in  $\mathbb{P}_{s_1}^{2,\text{an}}(\mathbb{C})$  such that  $f_{s_1}^{-1}(q_{s_1}) \cap \bar{V} = \{q_{s_1}\}$ . Since  $d(s)$  is locally constant near  $s_1$ , there is an analytic neighborhood  $W$  of  $s_1$  in  $S^{\text{an}}(\mathbb{C})$  such that for  $s \in W$ ,

$$(17) \quad f_s^{-1}(q_s) \cap V = \{q_s\}$$

Choose any  $n > m$ , and suppose by contradiction that  $p_{m,s_1} = p_{n,s_1}$ . Since  $k_n - 1 \geq k_m$ , we have

$$f_{s_1}^{k_n-1}(p_{n,s_1}) = f_{s_1}^{k_n-1}(p_{m,s_1}) = q_{s_1}.$$

Thus, for  $s$  close to  $s_1$ , the finite set  $f_{s_1}^{k_n-1}(p_{n,s_1})$  is contained in  $V$ , hence by (17),  $f_s^{k_n-1}(p_{n,s}) = q_s$ , and by analytic continuation this property holds throughout  $S$ , which contradicts the definition of  $k_n$ .

This shows that the  $p_{n,s}$  are all distinct for all  $s \in S \setminus H$ , and concludes the proof of Proposition 4.1.  $\square$

**Proposition 4.3.** *Let  $f: \mathbb{A}_{\mathbb{k}}^2 \rightarrow \mathbb{A}_{\mathbb{k}}^2$  be a regular polynomial map and  $C \subset \mathbb{A}_{\mathbb{k}}^2$  be an algebraic curve containing infinitely many preperiodic points. Then every point of  $\bar{C} \cap L_{\infty}$  is preperiodic under  $f|_{L_{\infty}}$ .*

*Proof.* We keep the same formalism and notation as above, so that  $f$  is viewed as a family over  $S$ . Write  $\bar{C} \cap L_{\infty} = \{c_1, \dots, c_r\}$  and without loss of generality enlarge  $R$  so that the points at infinity  $c_i$  have their coordinates in  $R$ . Fix  $i \in \{1, \dots, r\}$  for the remainder of the proof and consider  $c = c_i$ . By Proposition 4.1, there is a Zariski open subset  $U$  such that for any  $s \in U \cap S(\mathbb{Q}^{\text{alg}})$ ,  $f_s$  admits infinitely many preperiodic points on  $C_s$ . Therefore, by Proposition 1.7, for every such  $s$ ,  $c_s$  is preperiodic. Fix  $s_0 \in U \cap S(\mathbb{Q}^{\text{alg}})$ , then  $c_{s_0}$  eventually falls on a periodic point  $q_{s_0}$ . Replacing  $f$  by  $f^N$  and  $C$  by  $f^N(C)$  for some  $N$ , we may assume that  $q_{s_0}$  is fixed and  $c_{s_0} = q_{s_0}$ . Enlarging  $R$  again if necessary we may assume that  $q_{s_0}$  is the specialization at  $s = s_0$  of a fixed point  $q \in \mathbb{P}^2(R)$  of  $f$ .

Our purpose is to show that  $c = q$ . To simplify notation we write  $\hat{f} = f|_{L_{\infty}}$ . Note that the multiplier  $\mu := \hat{f}'_{s_0}(q_{s_0})$  belongs to  $\mathbb{Q}^{\text{alg}}$ . It follows from Kronecker's theorem that either  $\mu$  is a root of unity or there is a place  $v$  on  $\mathbb{Q}^{\text{alg}}$  such that  $|\mu|_v < 1$ .

*Case 1.*  $\mu$  is not a root of unity.

Fix a place  $v$  on  $\mathbb{Q}^{\text{alg}}$  such that  $|\mu|_v < 1$ , and consider the completion  $\mathbb{C}_v$  of  $(\mathbb{Q}^{\text{alg}}, |\cdot|_v)$ . We then argue in the analytic topology in the Berkovich analytification of  $\mathbb{P}_{\mathbb{C}_v}^2$  and  $S_{\mathbb{C}_v}$ .

Fix a neighborhood  $W$  of  $s_0$  in  $S_{\mathbb{C}_v}^{\text{an}}$  such that for  $s \in W$ ,  $q_s$  is attracting, and a neighborhood  $V$  of  $q_s$  in  $L_\infty$  independent of  $s \in W$  such that  $\hat{f}_s(V) \subset V$  and for any  $z \in V$ ,  $f_s^n(z)$  converges to  $q_s$  as  $n \rightarrow \infty$ . Reducing  $W$  if necessary we may assume that for any  $s \in W$ ,  $c_s$  belongs to  $V$ . For  $s \in W \cap S(\mathbb{Q}^{\text{alg}})$ ,  $c_s$  is preperiodic and converges to  $q_s$ , so it is preperiodic to  $q_s$ , that is, there exists a minimal  $k = k(s)$  such that  $f_s^{k(s)}(c_s) = q_s$ . Now we use an argument similar to that of Proposition 4.1: let  $H \subset S$  be a hypersurface such that the local degree of  $f_s$  at  $q_s$  is locally minimal outside  $H$  and fix  $s_1 \in W \setminus H$ . Then, there is a neighborhood  $W_1$  of  $s_1$  in  $W \setminus H$  and a neighborhood  $V_1 \subset V$  of  $q_1$  such that for any  $s \in W_1$ ,  $f_s(V_1) \subset V_1$  and  $f_s^{-1}(q_s) \cap V_1 = \{q_s\}$ . From this it follows that the only point eventually falling onto  $q_s$  in  $V_1$  is  $q_s$  itself. Therefore if  $s \in W_1$  is so close to  $s_1$  that  $c_s \in V_1$ , we infer that  $c_s = q_s$ , and finally  $c = q$  by analytic continuation.

*Case 2.*  $\mu$  is a root of unity.

To deal with this case we embed  $\mathbb{Q}^{\text{alg}}$  into  $\mathbb{C}$ , and work at the complex place. Recall that a holomorphic family  $(g_\lambda)_{\lambda \in \Lambda}$  of rational maps on  $\mathbb{P}^1(\mathbb{C})$ , parameterized by a connected complex manifold is trivial if any two members are conjugate by a Möbius transformation, depending holomorphically on  $\Lambda$ . If  $c$  is persistently preperiodic we are done, so assume that  $c$  is not persistently preperiodic.

Under our assumptions, there is a dense set  $S(\mathbb{Q}^{\text{alg}})$  of parameters such that  $c_s$  is preperiodic, but  $c$  is not persistently preperiodic. Thus by Chio-Roeder [7, Theorem 2.7] (see also [10, Theorem 4]) every such parameter belongs to the bifurcation locus of the marked family  $(f_\infty, c)$  (note that  $c$  is *not* a critical point here). As a consequence, the bifurcation locus of the family is equal to  $S_{\mathbb{C}}^{\text{an}}$ . (Note that it is enough to work in a neighborhood of  $s_0$ , away from possible singularities of  $S_{\mathbb{C}}^{\text{an}}$ .)

A first possibility is that the family  $(\hat{f}_s)_{s \in S_{\mathbb{C}}^{\text{an}}}$  is non-trivial. Then Gauthier [14, Theorem A] implies that  $J(\hat{f}_s) = L_\infty$  for all  $s$ . But since  $\hat{f}_{s_0}$  has a rationally indifferent fixed point, it admits an attracting petal and  $J(\hat{f}_{s_0}) \neq L_\infty$ . This contradiction shows that the family  $(\hat{f}_s)_{s \in S_{\mathbb{C}}^{\text{an}}}$  is trivial.

Now the situation is that there is a holomorphic family  $\varphi_s$  of Möbius transformations such that  $\varphi_s \hat{f}_s \varphi_s^{-1} = g$  is a fixed rational map  $g$  on  $\mathbb{P}^1$  with a rationally indifferent fixed point at 0. After this conjugacy, the marked family  $(\hat{f}, c)$  becomes  $(g, \varphi(c))$ . Since  $c$  coincides with  $q$  at  $s_0$  and by assumption  $c$  is not persistently preperiodic, there is an open set  $\Omega$  of parameters such that for  $s \in \Omega$ ,  $\varphi_s(c_s)$  belongs to some attracting petal associated to 0 for  $g$ . This contradicts the fact that  $\varphi_s(c_s)$  must be preperiodic for a dense set of parameters, and the proof is complete.  $\square$

*Conclusion of the proof of Theorem A.* By Proposition 4.3, any point at infinity of  $C$  is preperiodic, and, by assumption, one of these points, say  $c$ , is preperiodic to a non-superattracting periodic point  $p$ . Replace  $f$  by  $f^N$  and  $C$  by  $f^N(C)$  for some  $N$ , so that  $c = p$  is fixed. By Proposition 4.1, there is a non-trivial Zariski open subset  $U \subset S$

such that for every  $s \in U \cap S(\mathbb{Q}^{\text{alg}})$ ,  $C_s$  contains infinitely many preperiodic points. Then, since  $p_s$  is fixed and not superattracting for  $f_s$ , Theorem 3.1 asserts that  $C_s$  is preperiodic, and more precisely fixed, under  $f_s$  (see Remark 3.3). The density of  $U \cap S(\mathbb{Q}^{\text{alg}})$  in  $S$  (for the Zariski or analytic topology) then implies that  $f(C) = C$ , and the proof is complete.  $\square$

## REFERENCES

- [1] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Complex analytic dynamical systems. 11
- [2] Robert L. Benedetto. *Dynamics in one non-archimedean variable*, volume 198 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019. 9, 15
- [3] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990. 6
- [4] Gregory S. Call and Joseph H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993. 6
- [5] Serge Cantat and Romain Dujardin. Finite orbits for large groups of automorphisms of projective surfaces. *Compositio Math.*, 160:120–175, 2024. 15
- [6] Antoine Chambert-Loir. Heights and measures on analytic spaces. A survey of recent results, and some remarks. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume II*, volume 384 of *London Math. Soc. Lecture Note Ser.*, pages 1–50. Cambridge Univ. Press, Cambridge, 2011. 5, 6
- [7] Ivan Chio and Roland K. W. Roeder. Chromatic zeros on hierarchical lattices and equidistribution on parameter space. *Ann. Inst. Henri Poincaré D*, 8(4):491–536, 2021. 3, 18
- [8] Tien-Cuong Dinh. Sur les endomorphismes polynomiaux permutables de  $\mathbb{C}^2$ . *Ann. Inst. Fourier (Grenoble)*, 51(2):431–459, 2001. 2
- [9] Tien-Cuong Dinh and Nessim Sibony. Sur les endomorphismes holomorphes permutables de  $\mathbb{P}^k$ . *Math. Ann.*, 324(1):33–70, 2002. 2
- [10] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.*, 130(4):979–1032, 2008. 18
- [11] Romain Dujardin and Charles Favre. The dynamical Manin-Mumford problem for plane polynomial automorphisms. *J. Eur. Math. Soc. (JEMS)*, 19(11):3421–3465, 2017. 2, 3, 10, 15, 16
- [12] Charles Favre and Thomas Gauthier. Classification of special curves in the space of cubic polynomials. *Int. Math. Res. Not. IMRN*, (2):362–411, 2018. 15
- [13] John Erik Fornæss and Nessim Sibony. Complex dynamics in higher dimension. II. In *Modern methods in complex analysis (Princeton, NJ, 1992)*, volume 137 of *Ann. of Math. Stud.*, pages 135–182. Princeton Univ. Press, Princeton, NJ, 1995. 14
- [14] Thomas Gauthier. Dynamical pairs with an absolutely continuous bifurcation measure. *Ann. Fac. Sci. Toulouse Math. (6)*, 32(2):203–230, 2023. 18
- [15] Thomas Gauthier, Johan Taflin, and Gabriel Vigny. Sparsity of postcritically finite maps of  $\mathbb{P}^k$  and beyond: A complex analytic approach, 2023. 3
- [16] Dragos Ghioca, Khoa D. Nguyen, and Hexi Ye. The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for endomorphisms of  $(\mathbb{P}^1)^n$ . *Compos. Math.*, 154(7):1441–1472, 2018. 2
- [17] Dragos Ghioca, Khoa D. Nguyen, and Hexi Ye. The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for split rational maps. *J. Eur. Math. Soc. (JEMS)*, 21(5):1571–1594, 2019. 2
- [18] Dragos Ghioca and Thomas J. Tucker. A reformulation of the dynamical Manin-Mumford conjecture. *Bull. Aust. Math. Soc.*, 103(1):154–161, 2021. 1, 2
- [19] Dragos Ghioca, Thomas J. Tucker, and Shouwu Zhang. Towards a dynamical Manin-Mumford conjecture. *Int. Math. Res. Not. IMRN*, (22):5109–5122, 2011. 1

- [20] Monique Hakim. Attracting domains for semi-attractive transformations of  $\mathbb{C}^p$ . *Publ. Mat.*, 38(2):479–499, 1994. 3, 11
- [21] Michael Herman and Jean-Christophe Yoccoz. Generalizations of some theorems of small divisors to non-Archimedean fields. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 408–447. Springer, Berlin, 1983. 8
- [22] Marc Hindry. Autour d’une conjecture de Serge Lang. *Invent. Math.*, 94(3):575–603, 1988. 2
- [23] Lucas Kaufmann. Commuting pairs of endomorphisms of  $\mathbb{P}^2$ . *Ergodic Theory Dynam. Systems*, 38(3):1025–1047, 2018. 2
- [24] Lars Kühne. Points of small height on semiabelian varieties. *J. Eur. Math. Soc. (JEMS)*, 24(6):2077–2131, 2022. 2
- [25] Serge Lang. Division points on curves. *Ann. Mat. Pura Appl. (4)*, 70:229–234, 1965. 2
- [26] Fabien Pazuki. Zhang’s conjecture and squares of abelian surfaces. *C. R. Math. Acad. Sci. Paris*, 348(9-10):483–486, 2010. 1
- [27] Michel Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1):207–233, 1983. 1
- [28] Michel Raynaud. Sous-variétés d’une variété abélienne et points de torsion. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 327–352. Birkhäuser Boston, Boston, MA, 1983. 1
- [29] Michael Shub and Dennis Sullivan. A remark on the Lefschetz fixed point formula for differentiable maps. *Topology*, 13:189–191, 1974. 3, 16
- [30] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007. 4
- [31] Amaury Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d’Arakelov*. Theses, Université Rennes 1, October 2005. 3, 6
- [32] Shou-Wu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995. 1, 7
- [33] Shou-Wu Zhang. Distributions in algebraic dynamics. In *Surveys in differential geometry. Vol. X*, volume 10 of *Surv. Differ. Geom.*, pages 381–430. Int. Press, Somerville, MA, 2006. 1

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