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ON THE DYNAMICAL MANIN-MUMFORD CONJECTURE FOR
PLANE POLYNOMIAL MAPS

ROMAIN DUJARDIN, CHARLES FAVRE, AND MATTEO RUGGIERO

Abstract. We prove the dynamical Manin-Mumford conjecture for regular polynomial maps of $A^2$ and irreducible curves avoiding super-attracting orbits at infinity, over any field of characteristic 0.

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Introduction

The dynamical Manin-Mumford conjecture for polarized endomorphisms of algebraic varieties, first formulated by S.-W. Zhang in two influential papers [32, Conjecture 2.5] and [33, Conjecture 1.2.1], has been a driving force for the development of the field of arithmetic dynamics. It was realized by Ghioca, Tucker and Zhang [19] and Pazuki [26] that the original formulation of the conjecture was too optimistic, and a modified conjecture was proposed in [19] and more recently in [18]. It can be stated as follows: let $f : X \to X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and $Z \subset X$ be a subvariety containing a Zariski dense set of periodic points. Then either $Z$ is preperiodic or $Z$ is special, in the sense that it is contained in some subvariety $Y$ that is both $f^n$- and $\psi$-invariant, for some $n \geq 1$, where $\psi$ is another polarized endomorphism commuting with $f^n$. Recall that an endomorphism is said to be polarized if there is an ample line bundle $L \to X$ such that $f^* L \simeq L^d$ for some $d \geq 2$. A basic example is that of non-invertible endomorphisms of $\mathbb{P}^k$, for which we can take $L = O(1)$ and $d$ is the degree of $f$.

Despite its importance, very few cases of the conjecture have been settled so far. One first case is of course the original Manin-Mumford conjecture, which was solved by Raynaud [27, 28]. Viewed as a dynamical statement, it deals with endomorphisms of...
Abelian varieties, and was generalized to related settings, such as commutative algebraic groups (see Hindry [22], and also Lang’s classical paper [25]). Uniform versions involving height bounds were subsequently obtained by S.-W. Zhang, David and Philippon, Chambert-Loir and others. We refer to the recent work of Kühne [24] in the semi-abelian case for a latest update, and more references. Closer to algebraic dynamics is the case of polarized endomorphisms of \((P^1)^k\), which was solved by Ghioca, Nguyen and Ye [17, 16] (see also [18]). Note that such mappings are of product type, that is of the form \(f(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k))\), so their dynamical complexity reduces to dimension 1, which is a key step in the proof.

In this paper we will establish the dynamical Manin-Mumford conjecture for a wide class of 2-dimensional examples, whose dynamical behavior is truly higher dimensional. Note that besides polarized endomorphisms, a partial answer to the conjecture was given in [11] for plane polynomial automorphisms.

Let \(k\) be any field of characteristic zero. If required we fix an algebraic closure \(k^{alg}\) of \(k\). Let \(f: \mathbb{A}^2_k \to \mathbb{A}^2_k\) be a polynomial self-map of the affine plane of degree \(d \geq 2\), which in coordinates is written as

\[
\begin{align*}
f(z, w) &= (P(z, w), Q(z, w)),
\end{align*}
\]

with \(P, Q \in k[x, y]\). We say that \(f\) is regular if it extends to an endomorphism of \(P^2_k\) of degree \(d \geq 2\); this means that \(P = P_d + \text{l.o.t.}\) and \(Q = Q_d + \text{l.o.t.}\), where \(P_d\) and \(Q_d\) are homogeneous polynomials of degree \(d\) without common factors. In particular \(f\) induces a rational map \(f_\infty := [P_d : Q_d]\) on the line at infinity, which is fixed. Note that any endomorphism of \(P^2_k\) with a totally invariant line is conjugate to a regular polynomial map, and that a generic polynomial map of \(\mathbb{A}^2\) whose components are polynomials of degree \(\leq d\) is regular.

For regular polynomial maps of \(\mathbb{A}^2\), it seems that none of the known obstructions to the dynamical Manin-Mumford conjecture can arise. As said above, according to [18], one obstruction would be the existence of a periodic curve \(C\) for some endomorphism \(\psi\) commuting with \(f\). Such pairs \((f, \psi)\) were classified in [8] over \(\mathbb{C}\) (see also [9, 23]), and after a ramified cover they are all induced from a product map or a monomial map on the multiplicative 2-torus. Thanks to [17] it appears that in any such case, \(C\) must be also \(f\)-preperiodic. Thus we expect that the dynamical Manin-Mumford conjecture holds unconditionally in this case. Our main result confirms this expectation in the vast majority of cases.

**Theorem A.** Let \(f\) be a regular polynomial endomorphism of \(\mathbb{A}^2\) of degree \(d \geq 2\), defined over an arbitrary field \(k\) of characteristic 0.

Let \(C \subset \mathbb{A}^2\) be an irreducible algebraic curve containing infinitely many preperiodic points of \(f\), and suppose that the closure of \(C\) in \(\mathbb{P}^2\) contains a point \(p \in L_\infty\) which is not eventually superattracting. Then \(C\) is preperiodic under \(f\).

**Corollary B.** Under the assumptions of the theorem, if \(f\) has no super-attracting points on the line at infinity, then any irreducible algebraic curve \(C \subset \mathbb{P}^2\) containing infinitely many periodic points is preperiodic, that is, the dynamical Manin-Mumford conjecture holds for \(f\).
Our strategy relies on techniques from Arakelov geometry, in particular on the notion of canonical height, which is now classical in arithmetic dynamics, together with a variety of techniques from holomorphic and non-Archimedean dynamics.

We first assume that $k$ is a number field, in which case we prove a stronger statement (Theorem 3.1). We first recall in Section 1, that there exists a canonical height $h_f$ on $\mathbb{A}^2(k^{alg})$, for which preperiodic points are exactly points of zero height. By a theorem of Zhang, all points of $C$ lying at infinity are preperiodic, hence, by replacing $C$ by some iterate we may assume that they are periodic, and, thanks to our assumptions, one of these periodic points is not superattracting for $f_\infty$. Fix such a point $p \in C \cap L_\infty$. A second consequence of Zhang’s theorem is that or each place $v$ of $k$, the dynamical Green function $g_{f,v}(z,w) := \lim_{n \to \infty} \frac{1}{n} \log^+ \| f^n(z,w) \|$ satisfies

$$\int_C g_{f,v} \, dd^c g_{f,v} = 0. \tag{1}$$

To make sense of this equality at a finite place, we consider the Berkovich analytification of $C$ and $dd^c g_{f,v}$ is the Laplacian of the subharmonic function $g_{f,v}|_C$ in the sense of Thuillier, see [31].

For an appropriate choice of $v$, we may suppose that the periodic point $p$ is either repelling or parabolic for $f_\infty$. In both cases, we construct in Section 2 a local (super-)stable manifold $W^{ss}_{loc}(p)$. When $p$ is repelling and $v$ is Archimedean this is classical. We extend this construction to the non-Archimedean setting, and allow for non-repelling $p$ (see Theorem 2.1). Then, in both (repelling or parabolic) cases we establish graph transform type estimates which will be useful for the local analysis of the Green function at $p$. When $p$ is parabolic this borrows from the work of Hakim [20].

In Section 3 we combine these estimates with condition (1) to show that, near $p$, $C$ must locally coincide with $W^{ss}_{loc}(p)$, which finally implies that $C$ is periodic. Note that a similar argument appears in the recent work [15].

A key point in this argument is that at the chosen place $v$, $p$ belongs to the Julia set of $f_\infty$. This is no longer true when $f_\infty$ is superattracting at $p$, and we are not able to conclude in this case. Still, this situation leads to interesting dynamical considerations and objects, and we plan to come back to this issue in a later work.

Finally, we develop a specialization argument in Section 4 to reduce Theorem A for an arbitrary $k$ to the case where $k$ is a number field. So here we rather deal with an algebraic family of endomorphisms of $\mathbb{P}^2$ parameterized by some algebraic variety $S$. The main issue is to ensure that for such a family, an infinite set of preperiodic points cannot shrink to a finite set too often on $S$. To do so, one needs to control collisions of periodic points (using the Shub-Sullivan theorem [29] in the spirit of [11]); and the splitting of local preimages of a super-attracting cycle, a phenomenon that was studied in particular by Chio and Roeder [7].

1. Dynamical heights

In this section, we recall some basic facts on canonical heights attached to endomorphisms of the projective plane defined over a number field. Our purpose is to establish Proposition 1.7 which is the key arithmetic geometry input in our main theorem.
Throughout this section, we assume that $k$ is a number field.

1.1. **Vocabulary of number fields.** We denote by $M_k = \{v\}$ the set of places of $k$, that is, the set of all multiplicative norms $|\cdot|_v$ on $k$ that restrict to either the standard euclidean norm $|\cdot|_\infty$, or to a $p$-adic norm $|\cdot|_p$ on $\mathbb{Q}$ for some prime number $p > 1$. We normalize the $p$-adic norm by $|p|_p = p^{-1}$. We let $k_v$ be the completion of $k$ w.r.t. $|\cdot|_v$, and write $n_v := |k_v : \mathbb{Q}_v|$. The product formula asserts that for any $a \in k$, we have

$$\prod_{v \in M_k} |a|_v^{n_v} = 1.$$  

The set $M_k$ splits into the finite set $M^\infty_k$ of Archimedean places (whose restriction to $\mathbb{Q}$ is $|\cdot|_\infty$), and the set of finite (or non-Archimedean) places.

When $v \in M^\infty_k$, the algebraic closure $\mathbb{C}_v$ of $k_v$ is isometric to $\mathbb{C}$. When $v$ is a finite place extending $|\cdot|_p$ on $\mathbb{Q}$, then $|\cdot|_v$ extends canonically to $k_v^{alg}$, and its completion $\mathbb{C}_v$ is both complete and algebraically closed.

1.2. **Regular polynomial maps.** Let $(z, w)$ be affine coordinates on the affine plane $\mathbb{A}_k^2$. We also consider homogeneous coordinates $[z_0 : z_1 : z_2]$ on the projective plane $\mathbb{P}_k^2$ and identify the affine plane $\mathbb{A}_k^2$ with the Zariski open set $z_0 \neq 0$ so that $z = z_1/z_0$ and $w = z_2/z_0$. We denote by $L_\infty = \{z_0 = 0\}$ the line at infinity.

Let $f : \mathbb{A}_k^2 \to \mathbb{A}_k^2$ be any polynomial self-map of the affine plane of degree $d \geq 2$ that extends to an endomorphism of $\mathbb{P}_k^2$. In the coordinates $z, w$, it is given by

$$f(z, w) = (P(z, w), Q(z, w)),$$

where $P, Q \in k[z, w]$ satisfy $\max\{\deg P, \deg Q\} = d$. The fact that $f$ extends to a regular endomorphism $\overline{f} : \mathbb{P}_k^2 \to \mathbb{P}_k^2$ is equivalent to say that $P = P_d + l.o.t.$ and $Q = Q_d + l.o.t.$, where $P_d$ and $Q_d$ are homogeneous polynomials of degree $d$ without common factors.

For $n \in \mathbb{N}$ we write $f^n(z, w) = (P_n(z, w), Q_n(z, w))$. The restriction of $f_\infty$ to $L_\infty$ is an endomorphism of $\mathbb{P}_k^1$ given in homogeneous coordinates by

$$f_\infty([z_1 : z_2]) = [P_d(z_1, z_2) : Q_d(z_1, z_2)].$$

1.3. **Green functions.** The next proposition follows from the Nullstellensatz (see e.g. [30, Theorem 3.11]).

**Proposition 1.1.** For any $v \in M_k$, there exists a constant $C_v \geq 1$ such that

$$C_v^{-1} \leq \frac{\max\{1, |P(z, w)|_v, |Q(z, w)|_v\}}{\max\{1, |z|_v, |w|_v\}^d} \leq C_v$$

for all $z, w \in \mathbb{C}_v$. Moreover, for all but finitely many $v \in M_k$ we may take $C_v = 1$.

By the previous proposition, the sequence of functions

$$g_{v, n}(z, w) := \frac{1}{d^n} \log \max\{1, |P_n(z, w)|_v, |Q_n(z, w)|_v\}$$

converges uniformly on $\mathbb{C}_v^2$ to a continuous function $g_v$, and the next proposition follows.
Proposition 1.2. For any \( v \in M_k \), the function \( g_v : \mathbb{C}^3_v \to \mathbb{R}_+ \) is continuous, it satisfies the invariance equation \( g_v \circ f = dg_v \), and we have

\[
|g_v(z, w) - \log \max\{1, |z|_v, |w|_v\}| \leq \frac{\log C_v}{d - 1}.
\]

The set \( \{(z, w) \in \mathbb{C}^3_v, g_v(z, w) = 0\} \) coincides with the set of points having bounded orbits.

We shall also consider the global Green function in \( \mathbb{C}^3_v \). Write \( \tilde{P}(z_0, z_1, z_2) = z_0^d P\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) \) and \( \tilde{Q}(z_0, z_1, z_2) = z_0^d Q\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) \) so that \( F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q}) \) is a homogenous map of degree \( d \) that lifts \( f \) to \( \mathbb{C}^3_v \). Observe that in homogenous coordinates, \( [2] \) can be rewritten as follows:

\[
C_v^{-1} \leq \frac{\max\{|z_0|^d, |\tilde{P}(z_0, z_1, z_2)|_v, |	ilde{Q}(z_0, z_1, z_2)|_v\}}{\max\{|z_0|_v, |z_1|_v, |z_2|_v\}^d} \leq C_v
\]

so that the next result also holds.

Proposition 1.3. The function \( G_v(z_0, z_1, z_2) = g_v(z_1/z_0, z_2/z_0) + \log |z_0| \) is continuous on \( \mathbb{C}^3_v \setminus \{0\} \), 1-homogeneous (that is, \( G_v(AZ) = \log |A| + G_v(Z) \)), and satisfies \( G_v \circ F = dg_v \). We have

\[
|G_v(z_0, z_1, z_2) - \log \max\{|z_0|_v, |z_1|_v, |z_2|_v\}| \leq \frac{\log C_v}{d - 1}.
\]

The set \( \{(z_0, z_1, z_2) \in \mathbb{C}^3_v, G_v(z_0, z_1, z_2) \leq 0\} \) coincides with the set of points having bounded \( F \)-orbits.

1.4. Canonical heights on points. We refer to [6] for generalities on heights. Consider the line bundle \( \mathcal{O}(1) \to \mathbb{P}^2_k \). The space of sections of this line bundle can be canonically identified with the space of linear forms \( a_0 z_0 + a_1 z_1 + a_2 z_2 \) with \( a_i \in k \). More precisely, in the trivialization of the bundle over \( \{z_i \neq 0\} \), this section is given by \( \frac{1}{z_i} a_0 z_0 + a_1 z_1 + a_2 z_2 \).

For any \( v \in M_k \), we consider the line bundle \( \mathcal{O}(1) \to \mathbb{P}^2_{\mathbb{C}_v} \) and endow it with the metrization \( |\cdot|_v \) induced by \( G_v \), in the sense that any section \( \sigma = a_0 z_0 + a_1 z_1 + a_2 z_2 \) as above

\[
|\sigma|_v([z_0 : z_1 : z_2]) = |a_0 z_0 + a_1 z_1 + a_2 z_2| e^{-G_v(z_0, z_1, z_2)}
\]

(this expression is well-defined thanks to the homogeneity property of \( G_v \)). Let us now explain how this collection of metrizations defines a function on the set of algebraic points in \( \mathbb{P}^2_k \) as well as on the set of all algebraic curves in \( \mathbb{P}^2_k \) defined by an equation with coefficients in \( k_{\text{alg}} \).

For any \( p \in (k_{\text{alg}})^3 \setminus \{0\} \), we set

\[
h_f(p) := \frac{1}{N(p)} \sum_{p' \in \text{Gal} \cdot p} \left( \sum_{v \in M_k} n_v G_v(p') \right)
\]

where \( \text{Gal} \) denotes the absolute Galois group of \( k_{\text{alg}} \) over \( k \), and \( N(x) \) is the cardinality of the set \( \text{Gal} \cdot x \subset (k_{\text{alg}})^3 \).

The product formula entails that \( h_f(z_0, z_1, z_2) = h_f(\lambda z_0, \lambda z_1, \lambda z_2) \) for any \( \lambda \in k_{\text{alg}} \) so that we have a well-defined function \( h_f : \mathbb{P}^2(k_{\text{alg}}) \to \mathbb{R} \).
Proposition 1.4. The function $h_f$ takes non-negative values, and satisfies $h_f \circ f = dh_f$. The set $\{h_f = 0\}$ coincides with the set of preperiodic points of $f$. Furthermore, for any $(z, w) \in (\mathbb{k}^{\text{alg}})^2$ we have

$$h_f(z, w) := \frac{1}{N(z, w)} \sum_{(z', w') \in \text{Gal} \cdot (z, w)} \left( \sum_{v \in M_k} n(v)_v(z', w') \right).$$

As above $\text{Gal}$ denotes the absolute Galois group of $\mathbb{k}^{\text{alg}}$ over $\mathbb{k}$, and $N(z, w)$ is the cardinality of the set $\text{Gal} \cdot (z, w) \subset (\mathbb{k}^{\text{alg}})^2$. The proof follows directly from Northcott’s theorem, see [4 Corollary 1.1.1].

1.5. Analytification of affine curves. Let $C$ be any irreducible algebraic curve in $\mathbb{A}_k^2$ defined by an equation $\{R = 0\}$ with $R \in \mathbb{k}[z, w]$. Denote by $\overline{C}$ the Zariski closure of $C$ in $\mathbb{P}^2$.

Fix any place $v \in M_k$. We denote by $C_v^{an}$ the analytification in the sense of Berkovich of $C$ over $\mathbb{C}_v$. This is a connected, locally connected and locally compact space. When $| \cdot |_v$ is Archimedean, hence $\mathbb{C}_v$ is isometric to $\mathbb{C}$, $C_v^{an}$ is the complex analytic subspace (possibly with singularities) defined as usual by the vanishing of $R$ in $\mathbb{C}_v^2$. When $| \cdot |_v$ is non-Archimedean, then $C_v^{an}$ is defined as the set of multiplicative semi-norms on the ring $\mathbb{C}_v[z, w]/(R)$ whose restriction to the base field equals $| \cdot |_v$. A point is said to be rigid when the semi-norm has non-trivial kernel.

One can also define the analytification of $\overline{C}$ by considering suitable affine coordinates in $\mathbb{P}^2$ and patching the previous construction in a natural way, see [3, §3.4]. Observe that $\overline{C_v^{an}} \setminus C_v^{an}$ consists of rigid points.

Suppose first $v$ is Archimedean. The metrization of $\mathcal{O}(1)$ defined by [3] induces a measure $\mu_{C, v}$ on $\overline{C}_v^{an}$ which is locally defined by $\mu_{C, v} := \Delta \log |\sigma|_v$, where $\sigma$ is a local section of $\mathcal{O}(1)$. The plurisubharmonicity of $G_v$ ensures that $\mu_{C, v}$ is a positive measure. The Lelong-Poincaré formula implies that the mass of $\mu_{C, v}$ is equal to $\deg(C)$, and we have $\mu_{C, v} = \Delta(g_v|_{C_v^{an}})$ on $C_v^{an}$. Observe that since $G_v$ is continuous, $\mu_{C, v}$ gives no mass to points.

The construction is completely analogous in the non-Archimedean case. We again obtain a positive measure $\mu_{C, v}$ on $\overline{C}_v^{an}$ of total mass $\deg(C)$ which is given in $C_v^{an}$ by $\mu_{C, v} = \Delta(g_v|_{C_v^{an}})$ where $\Delta$ is the Laplace operator defined by Thuillier [31]. Observe that the continuity of the metrization implies that $\mu_{C, v}$ does not charge any rigid point (but it may still charge some non-rigid point in $C_v^{an}$). We refer to [3, §1.3] for the details of the constructions.

1.6. Canonical heights on curves. Let $C$ be any irreducible algebraic curve in $\mathbb{A}_k^2$ as in the previous section. We now define the canonical height of the curve $\overline{C}$ following the recipe given in [6, §3.1.2], taking $z_0$ as a section of $\mathcal{O}(1)$ (note that this section vanishes exactly along the line at infinity). We obtain:

$$h_f(\overline{C}) := \sum_{p \in \overline{C} \cap L_{\infty}} (\overline{C}, L_{\infty})_p \times h_f(p) + \sum_{v \in M_k} \int_{C_v^{an}} g_v d\mu_v.$$
Note that \( h_f(C) \geq 0 \) because the canonical height is non-negative on closed points, and the Green functions \( g_v \) are also non-negative.

Define the essential minimum of \( h_f \) by the following formula:
\[
\text{essmin}_C(h_f) := \sup_{F \text{ finite } \subset C(k_{alg})} \inf_{p \in C(k_{alg}) \setminus F} h_f(p).
\]

**Theorem 1.5** (Zhang’s inequality [32, Theorem 1.10]). We have
\[
2 \text{essmin}_C(h_f) \geq h_f(C) \geq \deg(C) \geq \text{essmin}_C(h_f) + \inf_{p \in C(k_{alg})} h_f(p).
\]

Since \( h_f(x) = 0 \) if and only if \( x \) is preperiodic, we obtain:

**Corollary 1.6.** An irreducible algebraic curve \( C \) containing infinitely many \( f \)-preperiodic points satisfies \( h_f(C) = 0 \).

1.7. A first characterization of special curves.

**Proposition 1.7.** Suppose that \( C \subset \mathbb{A}_k^2 \) is an irreducible algebraic curve containing a sequence of distinct points \( p_n \in C(k_{alg}) \) such that \( h_f(p_n) \to 0 \).

Then all points in \( C \cap L_\infty \) are preperiodic for \( f \), and for any \( v \in M_k \) the function \( g_v|_{C_{ss}} \) is harmonic on \( \{g_v > 0\} \).

**Proof.** Note that \( h_f(C) \geq 0 \). By Theorem 1.5 our assumption implies that \( \text{essmin}_C(h_f) \leq 0 \), therefore \( h_f(C) = 0 \). Then the result follows from (4) and the fact that a point \( p \) is \( f \)-preperiodic if and only if \( h_f(p) = 0 \). \( \square \)

2. Super-stable manifolds and local estimates

2.1. Construction of super-stable manifolds. In this section we work under the following hypothesis: \((k, |\cdot|)\) is a complete metrized field of characteristic 0 (which may be either Archimedean or non-Archimedean).

**Theorem 2.1.** Suppose \( f : (\mathbb{A}_k^2,0) \to (\mathbb{A}_k^2,0) \) is a germ of analytic map fixing the origin of the form
\[
f(x,y) = \left( \lambda x + \mu y + g(x,y), y^d(1 + h(x,y)) \right),
\]
where \( d \geq 2, \lambda \neq 0, \mu \in k, h(0,0) = 0, \) and \( g(x,y) = O(|(x,y)|^2) \). Then there exists a unique smooth analytic curve which is transverse to \( \{y = 0\} \) and \( f \)-invariant.

We shall call this curve the local super-stable manifold of the origin, and denote it by \( W^\text{ss}_{\text{loc}}(0) \). After a linear change of coordinates of the form \((x,y) \mapsto (x + \frac{\mu}{\lambda} y, y)\), we may and will assume from now on that \( \mu = 0 \). Expressing the invariant curve as a graph of the form \( x = \varphi(y) \) and making a change of coordinates of the form \((x,y) \mapsto (x - \varphi(y), y)\), \( f \) takes the form
\[
f(x,y) = (\lambda x + x\tilde{g}(x,y), y^d(1 + \tilde{h}(x,y))).
\]
It follows that \( f \) is analytically conjugate to \( y \mapsto y^d \) on \( W^\text{ss}_{\text{loc}}(0) \), hence the terminology.
The result is classical when $k$ is Archimedean and/or $f$ is locally invertible (see e.g. [21, Appendix]). For convenience we include a proof that works simultaneously in the Archimedean and non-Archimedean settings, and is adapted to $\{y = 0\}$ being superattracting.

**Proof.** As explained above, we look for an analytic map $y \mapsto \varphi(y)$, with $\varphi(0) = 0$ such that the change of coordinates $\Phi(x, y) = (x + \varphi(y), y)$ satisfies

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y)))$$

with $\tilde{g}, \tilde{h}$ analytic and vanishing at 0. This property is equivalent to the identities:

$$\begin{align*}
\lambda \varphi(y) + g(\varphi(y), y) &= \varphi(y^d(1 + \tilde{h}(0, y))) \\
\tilde{h}(0, y) &= h(\varphi(y), y)
\end{align*}$$

so that we aim at finding some analytic function $\varphi$ satisfying

$$\lambda \varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + h(\varphi(y), y))).$$

For any $r > 0$, let us introduce the Banach space $B_r$ that consist of those power series $\varphi(y) := \sum_{j \geq 1} a_j y^j$ which are convergent in the disk of radius $r$, and satisfy $||\varphi||_r := \sup_{|y| < r} |\varphi(y)| < \infty$. Note that in the non-Archimedean case, we have $||\varphi||_r := \sup_j |a_j|r^j$.

For any $\varphi \in B_r$, we set

$$T\varphi(y) := \frac{1}{\lambda} \left( \varphi(y^d(1 + h(\varphi(y), y))) - g(\varphi(y), y) \right).$$

We claim that for $r > 0$ and $\rho > 0$ sufficiently small, $T$ is a well-defined strictly contracting map on $B(0, \rho) \subset B_r$. Then, applying the Banach fixed point theorem implies the existence of the desired $\varphi$.

First, we may suppose that $g$ is analytic in the polydisk of radius $r$, and since $g$ vanishes up to order 2 at the origin, we have

$$|g(x, y)| \leq C \max\{|x|, |y|\}^2$$

for some $C > 0$ and all $|x|, |y| < r$. Similarly, we may suppose that $h$ is analytic in the polydisk of radius $r > 0$, and that $|h(x, y)| \leq \frac{1}{2}$ for all $|x|, |y| < r$.

Pick any $\varphi(y) = \sum_j a_j y^j \in B(0, \rho) \subset B_r$. Reduce $r > 0$ if necessary so that $\frac{3}{2} r^d < \rho$. Then $\tilde{\varphi} : y \mapsto \varphi(y^d(1 + h(\varphi(y), y)))$ is well-defined and analytic on the disk of radius $r$.

In the non-Archimedean case, $|1 + h(\varphi(y), y)| = 1$, so that one has

$$||\tilde{\varphi}(y)||_r = \sup_{j \geq 1} |a_j| r^d \leq r \sup_{j \geq 1} |a_j| r^{d-1} \leq r ||\varphi||_r.$$

In the Archimedean case, the Schwarz lemma yields $|\varphi(y)| \leq \frac{|y|}{\rho} |y|$ for all $|y| < r$, hence $||\tilde{\varphi}||_r \leq \frac{3}{2} r^{d-1} ||\varphi||_r$. Note that we also have:

$$||g(\varphi(y), y)||_r \leq C \max\{||\varphi||_r, r\}^2$$

so for $\varphi \in B(0, \rho)$ we deduce

$$|T\varphi|_r \leq \frac{1}{\lambda} \left( \frac{3}{2} r^{d-1} \rho + C \max\{\rho, r\}^2 \right).$$
By choosing \( \rho = r \) and then \( r \) small enough, this estimate shows that \( T\varphi \) is well-defined on the ball \( B(0, r) \subset B_r \) and \( T(B(0, r)) \subset B(0, r) \).

In order to prove that \( T \) is strictly contracting, observe that we can write
\[
g(x, y) - g(x', y) = (x - x')\tilde{g}(x, x', y)
\]
where \( \tilde{g}(x, x', y) \) is again analytic in the polydisk of radius \( r \), and
\[
|\tilde{g}(x, x', y)| \leq C' \max\{|x|, |x'|, |y|\}
\]
for some constant \( C' > 0 \). For any pair of analytic functions \( \varphi_1, \varphi_2 \in B(0, r) \subset B_r \) we infer:
\[
|g(\varphi_1(y), y) - g(\varphi_2(y), y)| \leq C' \|\varphi_1 - \varphi_2\|_r \max\{|\varphi_1|_r, |\varphi_2|_r, r\}
\]
hence
\[
\|T\varphi_1 - T\varphi_2\|_r \leq \frac{1}{\lambda} (3r^d + C'r) \|\varphi_1 - \varphi_2\|_r.
\]
Again, by choosing \( r \) sufficiently small, we obtain that \( T \) is strictly contracting and we are done. \( \Box \)

### 2.2. The rescaling argument in the repelling case.

We work in \( \mathbb{A}_k^2 \) where \( k \) is an arbitrary complete metrized field of characteristic 0. We start with a preparation lemma.

**Lemma 2.2.** Suppose \( f \) is an analytic map of the form
\[
f(x, y) = \left( \lambda x + xg(x, y), y^d(1 + h(x, y)) \right)
\]
where \( |\lambda| > 1 \), \( d \geq 2 \) and \( g(0) = h(0) = 0 \).

Then there exists an analytic change of coordinates \( \Phi \) such that
\[
\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + xyg(x, y), y^d(1 + xh(x, y)))
\]
for some analytic functions \( \tilde{g}, \tilde{h} \).

Recall that the form (7) is what is obtained from (5) after conjugating to get \( \mu = 0 \) and declaring that the stable manifold of Theorem 2.1 is \( \{ x = 0 \} \).

**Proof.** By Böttcher’s theorem (see [2], Chapter 4) for the non-Archimedean case applied to \( y \mapsto f(0, y) \) we may suppose that \( x \) divides \( h \). Similarly, since \( |\lambda| > 1 \), by we may linearize \( x \mapsto f(x, 0) \), and suppose that \( f \) is of the form \( f(x, y) = (\lambda x + g_1(y)) + x^2y_1(x, y), y^d(1 + \mathcal{O}(x)) \) for some analytic functions \( g_1, h_1 \) with \( g_1(0) = 0 \).

We claim that there exists \( \Phi(x, y) = (x(1 + \varphi(y)), y) \) with \( \varphi(0) = 0 \) such that
\[
\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x^2y_2(x, y), y^d(1 + \mathcal{O}(x)))
\]
for some analytic function \( h_2 \). Indeed, \( \varphi \) is then characterized by the equation
\[
\lambda x + \varphi(y)(1 + g_1(y)) + \mathcal{O}(x^2) = (\lambda x + \mathcal{O}(x^2))(1 + \varphi(y^d))
\]
that is, \( (1 + \varphi(y))(1 + g_1(y)) = (1 + \varphi(y^d)) \), a solution of which is given by the infinite product
\[
1 + \varphi(y) = \prod_{k=0}^{\infty} \left( 1 + g_1(y^{d^k}) \right)^{-1}
\]
and we are done. \( \Box \)
The next result is similar to [11, Lemma 4.2].

**Proposition 2.3.** Suppose $f$ is an analytic map of the form

$$f(x, y) = (\lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y)))$$

where $|\lambda| > 1, d \geq 2$ and $g, h$ are analytic functions.

Then $f^n(\frac{r}{\sqrt[n]{\lambda}}, y) \to (x, 0)$ when $n \to \infty$, uniformly on a polydisk of sufficiently small radius centered at the origin.

**Proof.** Fix $0 < r \leq 1/4$ small enough so that $g, h$ are both analytic on the polydisk of radius $r$ and $|g(x, y)|, |h(x, y)| \leq 1$ for $|x|, |y| < r$. Let us first show that if $|x| \leq \frac{r}{2|\lambda|}$ and $|y| \leq r$, then the $n$ first iterates of $(x, y)$ remain in $\mathbb{D}^2$. We argue by induction. So assume that $(x_0, y_0) \in \mathbb{D}_{r|\lambda|^{-1/2}} \times \mathbb{D}_r$, put $f^1(x_0, y_0) = (x_j, y_j)$, let $k \leq n$ and assume that $(x_j, y_j) \in \mathbb{D}^2$ for $j \leq k - 1$. Observe that for $j \leq k - 1$, $|y_j| \leq 2|y_j|^d$ from which it follows that

$$|y_k| \leq \left(2^{1/(d-1)}|y_0|\right)^{d^k} \leq \left(2^{1/(d-1)r}\right)^{d^k} \leq r.$$

Observe that the first inequality together with $r \leq 1/4$ also yield $|y_j| \leq 2^{-d^j}$. For the first coordinate, we use recursively the relation $x_{j+1} = \lambda x_j(1 + x_jy_jg(x_j, y_j))$ to get

$$|x_k| \leq |\lambda|^k|x_0| \prod_{j=0}^{k-1} \left(1 + |x_j||y_j|\right) \leq \frac{r}{2}|\lambda|^{k-n} \prod_{j=0}^{k-1} \left(1 + 2^{-d^j}\right) \leq r|\lambda|^{k-n},$$

where the last inequality follows from

$$\prod_{j=0}^{k-1} \left(1 + 2^{-d^j}\right) \leq \prod_{j=0}^{k-1} \left(1 + 2^{-2^j}\right) = \frac{2^{2^k} - 1}{2^{2^{k-1}} - 1} < 2,$$

in which the middle equality is easily obtained by induction.

Now take $(x, y) \in \mathbb{D}_{r/2}$, and consider $f^n(\frac{r}{\sqrt[n]{\lambda}}, y)$. Denote as before $(x_0, y_0) = (\frac{x}{\sqrt[n]{\lambda}}, y)$ and $(x_j, y_j) = f^j(x_0, y_0)$. The first part of the proof shows that $(x_j, y_j)$ is well-defined for all $j \leq n$, and that $y_n \to 0$. Now we have

$$x_n = \lambda^n x_0 \prod_{j=0}^{n-1} \left(1 + x_jy_jh(x_j, y_j)\right) = x \prod_{j=0}^{n-1} \left(1 + x_jy_jh(x_j, y_j)\right).$$

The inequality $|\prod (1 + z_j) - 1| \leq \exp (\sum |z_j|) - 1$ shows that to establish the convergence $x_n \to x$ it is enough to show that $\sum_{j=0}^{n-1} |x_jy_jh(x_j, y_j)|$ tends to $0$. But by (9), $|x_j| \leq r|\lambda|^{j-n}$, thus

$$\sum_{j=0}^{n-1} |x_jy_jh(x_j, y_j)| \leq \sum_{j=0}^{n-1} r|\lambda|^{j-n}2^{-d^j} \leq r|\lambda|^{-n} \sum_{j=0}^{\infty} |\lambda|^{j}2^{-d^j},$$

and we are done. \qed
2.3. Graph transform for $\lambda = 1$. In this paragraph we assume that $k = \mathbb{C}$ and $f$ is of the form

$$f(x, y) = \left(x + g(x, y), y^d(1 + h(x, y))\right),$$

with $g(x, y) = O(|x|^2)$, $h(0, 0) = 0$, and $g(x, 0) = cx^{k+1} + O(x^{k+2})$ for some $k \geq 1$ and $c \neq 0$. Observe that $f|_{(y=0)}$ has a parabolic point at the origin with $k$ attracting and $k$ repelling petals (see e.g. [1, §6.5]). An attracting petal is a $f$-invariant (connected and simply-connected) open subset $U$ containing $0$ in its boundary, and such that, for all $z \in U$, $f^n(z) \to 0$ tangentially to some real direction (in our case, to the normalized $k$-th roots of $-c$). A repelling petal is an attracting petal for $f^{-1}$ (they are tangent to the normalized $k$-th roots of $c$). One can chose the $k$ attracting petals and $k$ repelling petals so that their union fills up a punctured neighborhood of the origin.

A vertical graph $V$ in a domain of the form $\Omega \times \mathbb{D}_\rho$ is a submanifold of the form $V := \{(\varphi(y), y, y \in \mathbb{D}_\rho)\}$ for some holomorphic function $\varphi : \mathbb{D}_\rho \to \Omega$. In the next theorem we consider pull backs of such graphs in $\mathbb{D}^2_\rho$ in the graph transform sense, that is, when pulling back some vertical graph under $f$, we keep only the component of $f^{-1}(V) \cap \mathbb{D}^2_\rho$ containing $f^{-1}(V \cap \{y = 0\})$. Abusing notation we simply denote it by $f^{-1}(V)$.

**Theorem 2.4.** Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be of the form (10) with $h(0) = 0$, $g(x, 0) \neq 0$ and $g(x, y) = O(|(x, y)|^2)$. Let $U$ be any repelling petal of $f|_{(y=0)}$ and consider a germ $V$ of analytic curve transverse to $\{y = 0\}$, intersecting $U$.

Then there exists $r > 0$ depending only on $f$, such that for large enough $n$, the analytic sets $f^{-n}(V)$ are vertical graphs in $\mathbb{D}^2_\rho$ converging to $W^s_{loc}(0)$ in the $C^1$ topology.

**Lemma 2.5.** Suppose $f$ is a holomorphic map of the form (10) as in Theorem 2.4. Then there exist an integer $k \geq 1$, and an analytic change of coordinates $\Phi$ such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (x + x^{k+1} + x^{2k+1}\tilde{g}(x, y), y^d(1 + x\tilde{h}(x, y)))$$

for some analytic functions $\tilde{g}, \tilde{h}$.

**Proof.** The proof is essentially contained in [20, Proposition 2.3]. We provide the details for the sake of completeness. By Theorem 2.4 and by applying the Böttcher theorem to $y \mapsto f(0, y)$ we may assume that both $g$ and $h$ are divisible by $x$, so that we may write

$$f(x, y) = \left(x + xg(x, y), y^d(1 + xh(x, y))\right),$$

with $g(0, 0) = 0$.

By a local change of coordinates involving only $x$, we can arrange so that $f(x, 0) = (x + x^{k+1} + O(x^{2k+1}), 0)$. Expand the first coordinate of $f$ in power series of $x$ as follows:

$$x \circ f(x, y) = x(1 + g_0(y)) + x^{k+1}(1 + g_k(y)) + \sum_{j > 0, k} x^{j+1}g_j(y),$$

with $g_j(0) = 0$ for $0 \leq j \leq 2k - 1$.

We claim that for all $n \leq 2k - 1$, we can conjugate $f$ by a germ of invertible holomorphic map such that $g_j \equiv 0$ for every $j \leq n$. Applied to $n = 2k - 1$, this claim implies the proposition.
For $n = 0$ this is done by a change of coordinates of the form $Φ_0(x, y) := (x(1 + \varphi_0(y)), y)$ such that $(1 + g_0)(1 + \varphi_0) = 1 + \varphi_0(y^d)$. This equation can be solved exactly as in Lemma 2.2.

Now assume that $n > 0$, and that the result has been achieved up to $j = n - 1$. Put $Φ_n(x, y) = (x(1 + \varphi_n(y)x^n), y)$, so that $Φ_n^{-1}(x, y) = (x(1 - \varphi_n(y)x^n + O(x^{n+1})), y)$. Depending on the position of $n$ and $k$, we obtain:

$$x \circ (Φ_n^{-1} \circ f \circ Φ_n) = \begin{cases} x + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n < k, \\ x + x^{n+1}(\varphi_k(y) + 1 + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n = k, \\ x + x^{k+1} + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n > k. \end{cases}$$

Therefore, to render the term in $x^{n+1}$ constant, it is enough to solve the equation $-g_n(y) = \varphi_n(y) - \varphi_n(y^d)$ which can be done by setting $\varphi_n(y) = -\sum_{m=0}^{\infty} g_n(y^{de}).$ □

**Proof of Theorem 2.4.** By the previous lemma, we may suppose that

$$f(x, y) = \left( x + x^{k+1} + x^{2k+1} g(x, y), y^d(1 + xh(x, y)) \right)$$

with $g$ and $h$ holomorphic near the origin. Note that $f|_{y=0}$ has a repelling petal along the positive real axis. Fix $r > 0$ such that $f$ is holomorphic and injective on a neighborhood of $D_r$. Reducing and rotating the petal if necessary, we may assume that

$$U := \left\{ x : \text{arg}(x) \in \left( -\frac{\pi}{4k}, \frac{\pi}{4k} \right), \ |x| < r \right\}.$$ 

The holomorphic map $z = (kr^k)^{-1}$ is univalent on $U \times D_r$, and takes its values in

$$Ω_R := \left\{ z : \text{arg}(z) \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right), \ |z| > R \right\},$$

where $R = (kr^k)^{-1}$. The expression of $f$ in the coordinates $(z, y)$ is of the form

$$(12) \quad f(z, y) = \left( z - 1 + \frac{1}{z} a(z, y), y^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right).$$

so that $f$ is now defined in $Ω_R \times D_r$. Fix $M > 0$ such that

$$(13) \quad |a|, \ |b|, \ \left| \frac{∂a}{∂z} \right|, \ \left| \frac{∂a}{∂y} \right|, \ \left| \frac{∂b}{∂z} \right|, \ \left| \frac{∂b}{∂y} \right| \leq M \text{ on } Ω_R \times D_r,$$

and reduce $r$ if necessary so that $r < \frac{1}{10d}$ and $MR^{-1/k} \leq \frac{1}{100}$.

For any $ρ < r$ and $σ > 0$ we let

$$G(ρ, σ) = \{ ϕ : D_ρ \rightarrow Ω_R \text{ holomorphic s.t. } \sup_{D_ρ} |ϕ'| \leq σ \}.$$

**Lemma 2.6.** Suppose that $σρ < \frac{1}{100d}$. For any vertical graph $Γ$ determined by $ϕ \in G(ρ, σ)$, then $f^{-1}Γ$ is a vertical graph determined by a function $ψ \in G(ρ_1, \frac{1}{10})$ where $ρ_1 = \min \left( (ρ/2)^{1/d}, r \right)$ and $\text{Re}(ψ) \geq \text{Re}(ϕ(0)) + 9/10$.

Assuming this result for the moment, let us conclude the proof of the theorem. Let $V$ be any germ of curve intersecting transversally $\{ y = 0 \}$ at $(x_0, 0) \in U$. Then $V$ is a graph of slope $σ$ over some disk $D_ρ$ in the second coordinate, for some $ρ > 0$. Reduce $ρ$ by trimming $V$ if necessary so that $σρ < 1/100d$. Since $r/10 < 1/100d$, the previous
lemma implies that we can define inductively a sequence of vertical graphs \( V = V_0, \ V_n := f^{-1}V_{n-1} \), where \( V_n \) is defined by a holomorphic function \( z = \varphi_n(y) \) of uniformly bounded slope over \( \mathbb{D}_{\rho_n} \), and furthermore \( \rho_n = r \) for large enough \( n \). Moreover, we have \( \text{Re}(\varphi_n) \geq \text{Re}(\varphi(0)) + 9n/10 \), hence, coming back to the \((x, y)\) coordinates, we see that \( V_n = \{ x = (k\varphi_n(y))^{-1/k} \} \) converges in the \( C^1 \)-topology to the curve \( W^\text{ss}_0(0) = \{ x = 0 \} \).

**Proof of Lemma 2.6.** Let \( \Gamma \) be a vertical graph of equation \( z = \varphi(y) \) in the \((z, y)\) coordinates, with \( \varphi \in \mathcal{G}(\rho, \sigma) \). Then the equation of \( f^{-1}\Gamma \) is given by \( z = \ell(z, y) \), where

$$
\ell(z, y) = \varphi \left( y^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) + 1 - \frac{1}{z} a(z, y).
$$

Fix \( y_0 \in \mathbb{D}_{\rho_1} \). We show that the equation \( z = \ell(z, y_0) \) admits a unique solution \( z \in \Omega_R \).
First, observe that by the estimates \([13]\) on \(|a|\) and \(|b|\), for \( z \in \Omega_R \) we have

$$
|\ell(z, y_0) - (\varphi(0) + 1)| \leq \left| \varphi \left( y_0^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) - \varphi(0) \right| + \frac{M}{R} 
\leq 2\rho_1^d \sigma + \frac{1}{100} \leq \rho \sigma + \frac{1}{10} \leq \frac{1}{10} ,
$$

hence \( \ell(\cdot, y_0) \) maps \( \Omega_R \) to itself. Next, we see that

$$
\left| \frac{\partial \ell}{\partial z}(z, y_0) \right| \leq \left| y_0^d \left( \frac{1}{z^{1/k}} \frac{\partial b}{\partial z} - \frac{1}{kz^{1/k+1} b} \right) \right| \cdot \left| \varphi' \left( y_0^d \left( 1 + \frac{1}{z^{1/k}} b \right) \right) \right| \leq \rho_1^d \frac{2M}{R^{1/k}} \sigma + \frac{2M}{R} \leq \frac{1}{100} \rho \sigma + \frac{1}{10} \leq \frac{1}{10} ,
$$

so \( \ell(\cdot, y_0) \) is a contraction and the equation \( z = \ell(z, y_0) \) has a unique solution. This means that \( f^{-1}\Gamma \) is a vertical graph over \( \mathbb{D}_{\rho_1} \) determined by a holomorphic function \( \psi \) satisfying \( \psi(y) = \ell(\psi(y), y) \). The slope of this graph can be estimated as above:

$$
|\psi'(y)| \leq \frac{|\partial \ell/\partial y|}{1 - |\partial \ell/\partial z|} \leq \frac{10}{9} \left( \rho^d \sigma \left( 1 + \frac{M}{R^{1/k}} \right) + \frac{M}{R^{1/k}} + \frac{M}{R} \right) \leq \frac{1}{10} .
$$

Finally, the estimate \( \text{Re}(\psi) \geq \text{Re}(\varphi(0)) + 9/10 \) follows from \([14]\) and we are done. \( \square \)

### 3. PROOF OF THEOREM A WHEN \( k \) IS A NUMBER FIELD

Here we establish the following more precise form of our main theorem, in the number field case.

**Theorem 3.1.** Let \( k \) be a number field and \( f \) be a regular polynomial map of \( \mathbb{A}^2_k \). Denote by \( h_f \) the induced canonical height.

Suppose that \( C \subset \mathbb{A}^2_k \) is an irreducible algebraic curve containing a sequence of distinct points \( p_n \in C(k^{alb}) \) such that \( h_f(p_n) \to 0 \). If there exists a point of \( C \cap L_\infty \) which is not eventually superattracting, then \( C \) is preperiodic.

Let \( f_\infty \) be the restriction to the line at infinity \( L_\infty \) of the extension of \( f \) to \( \mathbb{P}^2 \). By Proposition \([17]\), all points in \( C \cap L_\infty \) are preperiodic, so we may replace \( f \) by \( f^N \) and \( C \) by \( f^N(C) \), to assume that \( C \cap L_\infty \) contains a fixed point \( p \) which is not super-attracting.
Let \( \lambda = f^\prime_{\infty}(p) \) be the multiplier of \( p\) along \( L_\infty \). Then one of the two following mutually disjoint cases occur:

(a) either \( \lambda \) is a root of unity;
(b) or there is a place \( v \in M_k \) such that \( |\lambda|_v > 1 \).

In the remainder of this section we split the proof of the theorem according to these two cases.

### 3.1. When \( \lambda \) is a root of unity

In this situation we iterate \( f \) further so that \( \lambda = 1 \), and work over the complex numbers. Since \( p \) belongs to the Julia set \( J(f_\infty) \), which is a perfect set, the union of attracting and repelling petals cover a punctured neighborhood of \( p \), and the attracting petals of \( p \) are contained in the Fatou set, we see that there is a repelling periodic point \( q \) of \( f_\infty \) contained in some local repelling petal of \( p \). Then the local (super-)stable manifold of \( q \) is a disk transverse to \( L_\infty \) at \( q \), and by Theorem 2.4 the local truncated pull-backs \( f^{-n}(W_{\text{loc}}^s(q)) \) under \( f^n \) converge to \( W_{\text{loc}}^s(p) \) when \( n \to \infty \).

Assume by way of contradiction that \( \overline{C} \) does not locally coincide with \( W_{\text{loc}}^s(p) \). We claim that \( C \) intersects \( f^{-n}(W_{\text{loc}}^s(q)) \) in \( \mathbb{C}^2 \) close to \( p \) for large \( n \). Indeed, locally near \( p \), \( \overline{C} \cap W_{\text{loc}}^s(p) = \{ p \} \), so by the persistence of proper intersections, \( C \) intersects \( f^{-n}(W_{\text{loc}}^s(q)) \) close to \( p \) for large \( n \). But \( p \notin f^{-n}(W_{\text{loc}}^s(q)) \cap L_\infty \), so these intersection points lie in \( \mathbb{C}^2 \), as claimed. If now \( \Delta \) is a small disk in \( C \) containing one of these intersection points, then by the Inclination Lemma, the derivative of \( f^n \) in the direction of \( \Delta \) tends to infinity, thus \((f^n|_\Delta)\) is not a normal family. On the other hand, by Proposition 1.7 \( g|_\Delta \) is harmonic, which implies that \((f^n|_\Delta)\) is normal (see [13, Prop 5.10]). This contradiction shows that \( \overline{C} \) locally coincides with \( W_{\text{loc}}^s(p) \) near \( p \), so it is fixed under \( f \), and by irreducibility this property propagates to the whole of \( C \). This completes the proof in this case. \( \square \)

**Remark 3.2.** This argument works essentially the same when \( |\lambda|_v > 1 \) at some Archimedean place, and may help understand the non-Archimedean argument below.

### 3.2. When \( |\lambda|_v > 1 \)

We may assume that \( p = [0 : 0 : 1] \), and by Lemma 2.2 find a local analytic isomorphism \( (x, y) \mapsto \psi(x, y) = [z_0(x, y) : z_1(x, y) : 1] \) such that \( \psi(0) = p \), \( z_0(x, 0) = 0 \) so that \( \{ y = 0 \} \) corresponds to the line at infinity \( \mathbb{P}^1 \) and write

\[
\tilde{f} := \psi^{-1} \circ f \circ \psi : (x, y) \mapsto \left( \lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y)) \right).
\]

If \( \overline{C} \) has an analytic branch at \( p \) which coincides with \( \{ x = 0 \} \) then \( C \) is fixed as above, and we are done. Otherwise, we may find a Puiseux parameterization of a branch of \( C \) at \( p \) in the \( (x, y) \) coordinates of the form \( \Gamma(t) = (t^\theta, \gamma(t)) \), where \( \gamma \) is analytic and defined in a small disk \( \mathbb{D}_\delta \), and \( q \in \mathbb{N}^* \). We seek a contradiction.

We lift \( \psi \) to \( \mathbb{A}^3_k \), and set \( \Psi(x, y) = (z_0(x, y), z_1(x, y), 1) \). As in §1.3 we lift \( f \) to a homogeneous polynomial map \( F : \mathbb{A}^3_k \to \mathbb{A}^3_k \), of the form \( F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q}) \). Since \( \Psi \circ \psi^{-1} \) is a local section of the projection \( \mathbb{A}^3 \setminus \{ 0 \} \to \mathbb{P}^2 \), \( \Psi \circ \tilde{f} \) must be a multiple of \( F \circ \Psi \). From the expression of \( F \) we obtain

\[
\Psi \circ \tilde{f} = \frac{F \circ \Psi}{Q \circ \Psi}.
\]

\(^1\)Beware that coordinates are swapped here: \( \{ z_0 = 0 \} \) corresponds to \( \{ y = 0 \} \).
To simplify notation, we write \( F^n(z_0, z_1, z_2) = (\tilde{z}^n, \tilde{P}_n, \tilde{Q}_n) \). We consider the 1-homogeneous Green function \( G_f: \mathbb{A}^n_{\text{an}} \to \mathbb{R} \) of Proposition 1.3; it satisfies \( G_f \circ F = dG_f \) and \( g_f(z_1, z_2) = g_f(1, z_1, z_2) \).

Observe that \( h := G_f \circ \Psi \circ \Gamma \) is a continuous function on \( \mathbb{D}_\delta \). Since \( h(t) = g_f \circ \psi \circ \Gamma(t) \log |z_0 \circ \Gamma(t)| \), by Proposition 1.7, \( h \) is harmonic on \( \{ t \neq 0 \} \). Since it is continuous at 0, it is also harmonic on \( \mathbb{D}_\delta \), see, e.g., [12] Lemma 3.7. Write
\[
d^{\text{eq}} h(t) = G_f \circ F^{\text{eq}} \circ \Psi \circ \Gamma(t)
= G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(t) + \log \left| \tilde{Q}_{nq} \circ \Psi \circ \Gamma(t) \right|.
\]
By Proposition 2.3, \( \tilde{f}[n]\left(\frac{x}{n^q}, y\right) \to (x, 0) \) uniformly in a neighborhood of the origin, hence \( G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(\frac{1}{\lambda(n/q)}) \to G_f(0, z_1(t^q, 0), 1) \) as \( n \to \infty \). On the other hand, since \( d^{\text{eq}} h(\frac{1}{\lambda(n/q)}) - \log \left| \tilde{Q}_{nq} \circ \Psi \circ \Gamma(\frac{1}{\lambda(n/q)}) \right| \) is a sequence of harmonic functions, it follows that \( t \mapsto G_f(0, z_1(t^q, 0), 1) \) is harmonic as well.

Now, observe that the restriction of \( f \) to the line at infinity is \( f_\infty[z_1 : z_2] = [\tilde{P}(0, z_1, z_2) : \tilde{Q}(0, z_1, z_2)] \), so that \( G_f(0, z_1, z_2) \) is the global Green function of \( f_\infty \). The equilibrium measure of \( f_\infty \) is the probability measure on the analytification of \( L_\infty \) defined by \( \mu_{f_\infty} := \Delta G_f(0, z_1, 1) \) in the chart \( z_2 \neq 0 \). Its support is the Julia set of \( f_\infty \) (see [2] Theorem 13.39), and it contains all repelling (rigid) fixed points, see [2] Theorem 8.7. Therefore, \( z_1 \mapsto G_f(0, z_1, 1) \) cannot be harmonic near 0, hence the function \( t \mapsto G_f(0, z_1(t^q, 0), 1) \) cannot be harmonic either. This contradiction concludes the proof. \( \square \)

Remark 3.3. Under the assumptions of Theorem 3.1, the proof shows that the preperiod \( k \) of \( F \) and the period of \( fh^k(C) \) are exactly the same as that of any of its non-superattracting points at infinity.

4. Proof of Theorem [A] for arbitrary \( \kappa \)

In this section we use a specialization argument to deal with maps defined over arbitrary fields. It shares some arguments with [11] §5 (see also [5] §7). Nevertheless, new ideas are needed to deal with preperiodic points instead of periodic ones.

We are in the setting of Theorem A, so we assume that \( f \) is a regular polynomial map of \( \mathbb{A}^2 \) of degree \( d \geq 2 \) defined over a field \( \kappa \) of characteristic 0, and \( C \) is a curve containing an infinite set \( P = \{ p_n, n \geq 0 \} \) of preperiodic points and whose closure \( \overline{C} \cap L_\infty \) contains at least one point which is not eventually super-attracting.

By enlarging \( \kappa \) if necessary we may assume that it contains the algebraic closure \( \mathbb{Q}_{\text{alg}} \) of its prime field. Let \( R \) be the sub-\( \mathbb{Q}_{\text{alg}} \)-algebra of \( \kappa \) generated by all coefficients defining \( f \) and \( C \). Its fraction field \( K \) is finitely generated over \( \mathbb{Q}_{\text{alg}} \). Let \( S = \text{Spec} \ R \). This is an affine variety defined over \( \mathbb{Q}_{\text{alg}} \), and elements of \( R \) can be seen as regular functions on \( S \).

Inverting some elements of \( R \) if necessary, we may suppose that \( C \) is flat over \( S \), and \( f \) extends as a morphism \( f: \mathbb{P}^2_S \to \mathbb{P}^2_S \). We let \( \pi: \mathbb{P}^2_S \to S \) be the canonical projection, and write \( \mathbb{P}^2_S = \pi^{-1}(s) \). We also let \( \mathcal{A}^2_S := \mathcal{A}^2_S \cap \pi^{-1}(s) \).
For each (scheme theoretic) point $s \in S$, we write $C_s = C \cap \mathbb{A}^2_s$ and let $\overline{C}_s$ be the closure of $C_s$ in $\mathbb{P}^2_s$. The flatness of the morphism $C \to S$ implies $\overline{C}_s$ (hence $C_s$) to be a curve. Similarly, we let $f_s : \mathbb{P}^2_s \to \mathbb{P}^2_s$ be the induced map on the fibers: this is an endomorphism of degree $d$.

We also denote by $p_{n,s} \in \mathbb{A}^2_s$ the specialization of $p_n$. Note that $p_n$ is defined over some finite extension of $K$ which depends on $n$.

The first result does not use the assumption that our infinite set of preperiodic points lies on a curve.

**Proposition 4.1.** Let as above $f : \mathbb{P}^2_S \to \mathbb{P}^2_S$ be a family of endomorphisms over an affine variety defined over $\mathbb{Q}_{\text{alg}}$, and let $\mathcal{P} = \{p_n, \ n \geq 0\}$ be an infinite family of preperiodic points. Then there exists a non-empty Zariski open and dense subset $U \subset S$ such that for any $s \in U \cap S(\mathbb{Q}_{\text{alg}})$, $\mathcal{P}_s = \{p_{n,s}, \ n \geq 0\} \subset \mathbb{A}^2_s$ is infinite.

Before starting the proof, let us fix some additional notation. For each $n \geq 0$, we denote by $k_n$ the preperiod of $p_n$, so that $q_n := f^{k_n}(p_n)$ is the first periodic point in the orbit of $p_n$. We let $\ell_n$ be the (primitive) period of $q_n$.

**Proof.** We may suppose that there exists a parameter $s_0 \in S(\mathbb{Q}_{\text{alg}})$ such that $\mathcal{P}_{s_0}$ is finite (otherwise we take $U = S$ and the proof is complete).

**Lemma 4.2.** The family of periodic points $(q_n)$ is finite.

**Proof.** We follow the arguments of [11, §5]. Set $Q = \{q_n, \ n \geq 0\}$. For each $\ell \geq 1$, we consider the subvariety $\text{Per}_\ell$ of $\mathbb{P}^2_S$ defined by the equation $f^\ell(z) = z$. Since $\mathbb{P}^2_S \to S$ is proper, the structure map $\text{Per}_\ell \to S$ is also proper.

Let $Q_\ell$ be the union of the irreducible components of $\text{Per}_\ell$ containing a point of $\mathcal{P}$. Its underlying set is the Zariski closure of $\mathcal{P} \cap \text{Per}_\ell$, hence $Q_\ell \to S$ is proper. Observe that for $x \in Q_{\ell,s}$, the multiplicity of $x$ as a point of $Q_{\ell,s}$ equals its multiplicity as a fixed point of $f^\ell_s$. By Nakayama’s lemma and the properness of $Q_\ell$ over $S$, the function

$$s \mapsto \sum_{x \in Q_{\ell,s}} \text{mult}_x(Q_{\ell,s})$$

is upper semi-continuous for the Zariski topology, hence

$$\sum_{q \in Q_\ell} \text{mult}_q(Q_\ell) \leq \sum_{x \in Q_{\ell,s_0}} \text{mult}_x(Q_{\ell,s_0}),$$

where the left hand side is the value of (15) at the generic point. By assumption $\mathcal{P}_{s_0}$ is a finite set, hence so does $Q_{s_0} = \{q_1, s_0, \ldots, q_r, s_0\}$ and by the Shub-Sullivan theorem [29], there exists a uniform bound $C > 0$ such that for every $j$, and for any $\ell$, we have

$$\text{mult}_{q_{j,s_0}}(Q_{\ell,s_0}) \leq \text{mult}_{q_{j,s_0}}(\text{Per}_{\ell,s_0}) \leq C.$$

It then follows from (16) that

$$\# Q_\ell \leq \sum_{q \in Q_\ell} \text{mult}_q(Q_\ell) \leq rC$$

hence $\bigcup_{\ell} Q_\ell$ is finite, as was to be shown. \qed
By the previous lemma, replacing $f$ by some iterate $f^N$ we may assume that all periodic points $q_0$ are fixed. Since $\mathcal{P}$ is infinite, one of these fixed points, say $q_1$, admits infinitely many preimages in $\mathcal{P}$. We may denote $q = q_1$ and suppose $\mathcal{P}$ is made of an infinite set of preimages of $q$, that is, (after possible reordering of $\mathcal{P}$) for any $p_n \in \mathcal{P}$ there is a minimal $k_n \geq 0$ such that $f^{k_n}(p_n) = q$, and that $k_{n+1} > k_n$. We may adjoin to $R$ the coordinates of $q$ so that $q \in \mathbb{A}^2(R)$, i.e., for any $s \in S$, $q_s$ is a single point (to say it differently, we replace $S$ by a branched cover of a Zariski open dense subset of $S$).

Let $d(s)$ be the local degree of $f_s$ at $q_s$, which is upper semicontinuous for the Zariski topology. Since $f_s$ is a finite map of degree $d^2$, $d(s) \leq d^2$ for every $s$. Thus there is an analytic hypersurface $H$ such that $d(s) = d_{\text{min}}$ is constant for $s \in S \setminus H$.

We claim that $\mathcal{P}_{s_1}$ is infinite for any $s_1 \in S \setminus H$. We argue again in the complex analytic category fixing an embedding $\mathbb{Q}^{\text{alg}}$ into $\mathbb{C}$. Observe that for any point $s \in S(\mathbb{C})$, $p_{n,s}$ is a finite set included in the fiber $\mathbb{A}^{2,\text{an}}_s \simeq \mathbb{C}^2$ (not necessarily reduced to a single point since $p_n$ lies in a finite extension of $R$).

Fix an analytic neighborhood $V$ of $q_{s_1}$ in $\mathbb{P}^{2,\text{an}}_{s_1}(\mathbb{C})$ such that $f^{-1}_{s_1}(q_{s_1}) \cap V = \{q_{s_1}\}$. Since $d(s)$ is locally constant near $s_1$, there is an analytic neighborhood $W$ of $s_1$ in $S^{\text{an}}(\mathbb{C})$ such that for $s \in W$,

\begin{equation}
    f^{-1}_{s}(q_s) \cap V = \{q_s\}
\end{equation}

Choose any $n > m$, and suppose by contradiction that $p_{m,s_1} = p_{n,s_1}$. Since $k_n - 1 \geq k_m$, we have

\[ f^{k_n}_{s_1}(p_{n,s_1}) = f^{k_n}_{s_1}(p_{m,s_1}) = q_{s_1}. \]

Thus, for $s$ close to $s_1$, the finite set $f^{k_n-1}_{s_1}(p_{n,s_1})$ is contained in $V$, hence by (17), $f^{k_n-1}_{s_1}(p_{n,s}) = q_s$, and by analytic continuity this property holds throughout $S$, which contradicts the definition of $k_n$.

This shows that the $p_{n,s}$ are all distinct for all $s \in S \setminus H$, and concludes the proof of Proposition 4.1.

**Proposition 4.3.** Let $f: \mathbb{A}^2_k \to \mathbb{A}^2_k$ be a regular polynomial map and $C \subset \mathbb{A}^2_k$ be an algebraic curve containing infinitely many preperiodic points. Then every point of $\overline{C} \cap L_\infty$ is preperiodic under $f|_{L_\infty}$.

**Proof.** We keep the same formalism and notation as above, so that $f$ is viewed as a family over $S$. Write $\overline{C} \cap L_\infty = \{c_1, \ldots, c_r\}$ and without loss of generality enlarge $R$ so that the points at infinity $c_i$ have their coordinates in $R$. Fix $i \in \{1, \ldots, r\}$ for the remainder of the proof and consider $c = c_i$. By Proposition 4.1 there is a Zariski open subset $U$ such that for any $s \in U \cap S(\mathbb{Q}^{\text{alg}})$, $f_s$ admits infinitely many preperiodic points on $C_s$.

Therefore, by Proposition 4.1, for every such $s$, $c_s$ is preperiodic. Fix $s_0 \in U \cap S(\mathbb{Q}^{\text{alg}})$, then $c_{s_0}$ eventually falls on a periodic point $q_{s_0}$. Replacing $f$ by $f^N$ and $C$ by $f^N(C)$ for some $N$, we may assume that $q_{s_0}$ is fixed and $c_{s_0} = q_{s_0}$. Enlarging $R$ again if necessary we may assume that $q_{s_0}$ is the specializes at $s = s_0$ of a fixed point $q \in \mathbb{P}^2(R)$ of $f$.

Our purpose is to show that $c = q$. To simplify notation we write $\hat{f} = f|_{L_\infty}$. Note that the multiplier $\mu := \hat{f}_{s_0}(q_{s_0})$ belongs to $\mathbb{Q}^{\text{alg}}$. It follows from Kronecker’s theorem that either $\mu$ is a root of unity or there is a place $v$ on $\mathbb{Q}^{\text{alg}}$ such that $|\mu|_v < 1$. The proof is completed by Remark 3.1.\[\square\]
that

**Conclusion of the proof of Theorem A.** By Proposition 4.3, any point at infinity of $C$ is preperiodic, and, by assumption, one of these points, say $c$, is preperiodic to a non-superattracting periodic point $p$. Replace $f$ by $f^N$ and $C$ by $f^N(C)$ for some $N$, so that $c = p$ is fixed. By Proposition 4.1, there is a non-trivial Zariski open subset $U \subset S$

Case 1. $\mu$ is not a root of unity.

Fix a place $v$ on $\mathbb{Q}^{\text{alg}}$ such that $|\mu|_v < 1$, and consider the completion $\mathbb{C}_v$ of $(\mathbb{Q}^{\text{alg}}, |\cdot|_v)$. We then argue in the analytic topology in the Berkovich analytification of $\mathbb{P}^2_{\mathbb{C}_v}$ and $S_{\mathbb{C}_v}$.

Fix a neighborhood $W$ of $s_0$ in $S_{\mathbb{C}_v}$ such that for $s \in W$, $q_s$ is attracting, and a neighborhood $V$ of $q_s$ in $L_\infty$ independent of $s \in W$ such that $\hat{f}_s(V) \subset V$ and for any $z \in V$, $f^n_s(z)$ converges to $q_s$ as $n \to \infty$. Reducing $W$ if necessary we may assume that for any $s \in W$, $c_s$ belongs to $V$. For $s \in W \cap S(\mathbb{Q}^{\text{alg}})$, $c_s$ is preperiodic and converges to $q_s$, so it is preperiodic to $q_s$, that is, there exists a minimal $k = k(s)$ such that $f^{k(s)}_s(c_s) = q_s$. Now we use an argument similar to that of Proposition 4.1. Let $H \subset S$ be a hypersurface such that the local degree of $f_s$ at $q_s$ is locally minimal outside $H$ and fix $s_1 \in W \setminus H$. Then, there is a neighborhood $W_1$ of $s_1$ in $W \setminus H$ and a neighborhood $V_1 \subset V$ of $q_1$ such that for any $s \in W_1$, $f_s(V_1) \subset V_1$ and $f_s^{-1}(q_s) \cap V_1 = \{q_s\}$. From this it follows that the only point eventually falling onto $q_s$ in $V_1$ is $q_s$ itself. Therefore if $s \in W_1$ is so close to $s_1$ that $c_s \in V_1$, we infer that $c_s = q_s$, and finally $c = q$ by analytic continuation.

Case 2. $\mu$ is a root of unity.

To deal with this case we embed $\mathbb{Q}^{\text{alg}}$ into $\mathbb{C}$, and work at the complex place. Recall that a holomorphic family $(g_\lambda)_{\lambda \in \Lambda}$ of rational maps on $\mathbb{P}^1(\mathbb{C})$, parameterized by a connected complex manifold is trivial if any two members are conjugate by a Möbius transformation, depending holomorphically on $\Lambda$. If $c$ is persistently preperiodic we are done, so assume that $c$ is not persistently preperiodic.

Under our assumptions, there is a dense set $S(\mathbb{Q}^{\text{alg}})$ of parameters such that $c_s$ is preperiodic, but $c$ is not persistently preperiodic. Thus by Chio-Roeder [7, Theorem 2.7] (see also [10, Theorem 4]) every such parameter belongs to the bifurcation locus of the marked family $(f_{\infty}, c)$ (note that $c$ is not a critical point here). As a consequence, the bifurcation locus of the family is equal to $S_{\mathbb{C}_v}$. (Note that it is enough to work in a neighborhood of $s_0$, away from possible singularities of $S_{\mathbb{C}_v}$.)

A first possibility is that the family $(\hat{f}_s)_{s \in S_{\mathbb{C}_v}}$ is non-trivial. Then Gauthier [13, Theorem A] implies that $J(\hat{f}_s) = L_\infty$ for all $s$. But since $\hat{f}_{s_0}$ has a rationally indifferent fixed point, it admits an attracting petal and $J(\hat{f}_{s_0}) \neq L_\infty$. This contradiction shows that the family $(\hat{f}_s)_{s \in S_{\mathbb{C}_v}}$ is trivial.

Now the situation is that there is a holomorphic family $\varphi_s$ of Möbius transformations such that $\varphi_s f_s \varphi_s^{-1} = g$ is a fixed rational map $g$ on $\mathbb{P}^1$ with a rationally indifferent fixed point at 0. After this conjugacy, the marked family $(\hat{f}, c)$ becomes $(g, \varphi(c))$. Since $c$ coincides with $q$ at $s_0$ and by assumption $c$ is not persistently preperiodic, there is an open set $\Omega$ of parameters such that for $s \in \Omega$, $\varphi_s(c_s)$ belongs to some attracting petal associated to 0 for $g$. This contradicts the fact that $\varphi_s(c_s)$ must be preperiodic for a dense set of parameters, and the proof is complete.


such that for every \( s \in U \cap S(Q_{\text{alg}}) \), \( C_s \) contains infinitely many preperiodic points. Then, since \( p_s \) is fixed and not superattracting for \( f_s \), Theorem 3.1 asserts that \( C_s \) is preperiodic, and more precisely fixed, under \( f_s \) (see Remark 3.3). The density of \( U \cap S(Q_{\text{alg}}) \) in \( S \) (for the Zariski or analytic topology) then implies that \( f(C) = C \), and the proof is complete. \( \square \)

References


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