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To cite this version:
Romain Dujardin, Charles Favre, Matteo Ruggiero. On the dynamical Manin-Mumford conjecture for plane polynomial maps. 2023. hal-04361693

HAL Id: hal-04361693
https://hal.science/hal-04361693
Preprint submitted on 22 Dec 2023

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ON THE DYNAMICAL MANIN-MUMFORD CONJECTURE FOR
PLANE POLYNOMIAL MAPS

ROMAIN DUJARDIN, CHARLES FAVRE, AND MATTEO RUGGIERO

Abstract. We prove the dynamical Manin-Mumford conjecture for regular polynomial maps of $A^2$ and irreducible curves avoiding super-attracting orbits at infinity, over any field of characteristic 0.

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Introduction

The dynamical Manin-Mumford conjecture for polarized endomorphisms of algebraic varieties, first formulated by S.-W. Zhang in two influential papers [32, Conjecture 2.5] and [33, Conjecture 1.2.1], has been a driving force for the development of the field of arithmetic dynamics. It was realized by Ghioca, Tucker and Zhang [19] and Pazuki [26] that the original formulation of the conjecture was too optimistic, and a modified conjecture was proposed in [19] and more recently in [18]. It can be stated as follows: let $f : X \to X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and $Z \subset X$ be a subvariety containing a Zariski dense set of periodic points. Then either $Z$ is preperiodic or $Z$ is special, in the sense that it is contained in some subvariety $Y$ that is both $f^n$- and $\psi$-invariant, for some $n \geq 1$, where $\psi$ is another polarized endomorphism commuting with $f^n$. Recall that an endomorphism is said to be polarized if there is an ample line bundle $L \to X$ such that $f^*L \simeq L^d$ for some $d \geq 2$. A basic example is that of non-invertible endomorphisms of $\mathbb{P}^k$, for which we can take $L = O(1)$ and $d$ is the degree of $f$.

Despite its importance, very few cases of the conjecture have been settled so far. One first case is of course the original Manin-Mumford conjecture, which was solved by Raynaud [27, 28]. Viewed as a dynamical statement, it deals with endomorphisms of

Date: December 22, 2023.

M.R. was partially supported by the ANR grant Fatou ANR-17-CE40-0002-01.
Abelian varieties, and was generalized to related settings, such as commutative algebraic groups (see Hindry [22], and also Lang’s classical paper [25]). Uniform versions involving height bounds were subsequently obtained by S.-W. Zhang, David and Philippon, Chambert-Loir and others. We refer to the recent work of Kühne [24] in the semi-abelian case for a latest update, and more references. Closer to algebraic dynamics is the case of polarized endomorphisms of \((\mathbb{P}^1)_k\), which was solved by Ghioca, Nguyen and Ye [17, 16] (see also [18]). Note that such mappings are of product type, that is of the form \(f(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k))\), so their dynamical complexity reduces to dimension 1, which is a key step in the proof.

In this paper we will establish the dynamical Manin-Mumford conjecture for a wide class of 2-dimensional examples, whose dynamical behavior is truly higher dimensional. Note that besides polarized endomorphisms, a partial answer to the conjecture was given in [11] for plane polynomial automorphisms.

Let \(k\) be any field of characteristic zero. If required we fix an algebraic closure \(k^{alg}\) of \(k\). Let \(f: \mathbb{A}^2_k \to \mathbb{A}^2_k\) be a polynomial self-map of the affine plane of degree \(d \geq 2\), which in coordinates is written as

\[
f(z, w) = \left( P(z, w), Q(z, w) \right),
\]

with \(P, Q \in k[x, y]\). We say that \(f\) is regular if it extends to an endomorphism of \(\mathbb{P}^2_k\) of degree \(d \geq 2\); this means that \(P = P_d + l.o.t.\) and \(Q = Q_d + l.o.t.\), where \(P_d\) and \(Q_d\) are homogeneous polynomials of degree \(d\) without common factors. In particular \(f\) induces a rational map \(f_\infty := [P_d : Q_d]\) on the line at infinity, which is fixed. Note that any endomorphism of \(\mathbb{P}^2_k\) with a totally invariant line is conjugate to a regular polynomial map, and that a generic polynomial map of \(\mathbb{A}^2\) whose components are polynomials of degree \(\leq d\) is regular.

For regular polynomial maps of \(\mathbb{A}^2\), it seems that none of the known obstructions to the dynamical Manin-Mumford conjecture can arise. As said above, according to [18], one obstruction would be the existence of a periodic curve \(C\) for some endomorphism \(\psi\) commuting with \(f\). Such pairs \((f, \psi)\) were classified in [8] over \(\mathbb{C}\) (see also [9, 23]), and after a ramified cover they are all induced from a product map or a monomial map on the multiplicative 2-torus. Thanks to [17] it appears that in any such case, \(C\) must be also \(f\)-preperiodic. Thus we expect that the dynamical Manin-Mumford conjecture holds unconditionally in this case. Our main result confirms this expectation in the vast majority of cases.

**Theorem A.** Let \(f\) be a regular polynomial endomorphism of \(\mathbb{A}^2\) of degree \(d \geq 2\), defined over an arbitrary field \(k\) of characteristic 0.

Let \(C \subset \mathbb{A}^2\) be an irreducible algebraic curve containing infinitely many preperiodic points of \(f\), and suppose that the closure of \(C\) in \(\mathbb{P}^2\) contains a point \(p \in L_\infty\) which is not eventually superattracting. Then \(C\) is preperiodic under \(f\).

**Corollary B.** Under the assumptions of the theorem, if \(f\) has no super-attracting points on the line at infinity, then any irreducible algebraic curve \(C \subset \mathbb{P}^2\) containing infinitely many periodic points is preperiodic, that is, the dynamical Manin-Mumford conjecture holds for \(f\).
Our strategy relies on techniques from Arakelov geometry, in particular on the notion of canonical height, which is now classical in arithmetic dynamics, together with a variety of techniques from holomorphic and non-Archimedean dynamics.

We first assume that \( k \) is a number field, in which case we prove a stronger statement (Theorem 3.1). We first recall in Section 1 that there exists a canonical height \( h_f \) on \( \mathbb{A}^2(k^{alg}) \), for which preperiodic points are exactly points of zero height. By a theorem of Zhang, all points of \( C \) lying at infinity are preperiodic, hence, by replacing \( C \) by some iterate we may assume that they are periodic, and, thanks to our assumptions, one of these periodic points is not superattracting for \( f_\infty \). Fix such a point \( p \in \overline{C} \cap L_\infty \). A second consequence of Zhang’s theorem is that for each place \( v \) of \( k \), the dynamical Green function \( g_{f,v}(z,w) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ \| f^n(z,w) \| \) satisfies

\[
\int_C g_{f,v} dd^c g_{f,v} = 0.
\]

To make sense of this equality at a finite place, we consider the Berkovich analytification of \( C \) and \( dd^c g_{f,v} \) is the Laplacian of the subharmonic function \( g_{f,v}|_C \) in the sense of Thuillier, see [31].

For an appropriate choice of \( v \), we may suppose that the periodic point \( p \) is either repelling or parabolic for \( f_\infty \). In both cases, we construct in Section 2 a local (super-)stable manifold \( W^{ss}_{loc}(p) \). When \( p \) is repelling and \( v \) is Archimedean this is classical. We extend this construction to the non-Archimedean setting, and allow for non-repelling \( p \) (see Theorem 2.1). Then, in both (repelling or parabolic) cases we establish graph transform type estimates which will be useful for the local analysis of the Green function at \( p \). When \( p \) is parabolic this borrows from the work of Hakim [20].

In Section 3 we combine these estimates with condition (1) to show that, near \( p \), \( C \) must locally coincide with \( W^{ss}_{loc}(p) \), which finally implies that \( C \) is periodic. Note that a similar argument appears in the recent work [15].

A key point in this argument is that at the chosen place \( v \), \( p \) belongs to the Julia set of \( f_\infty \). This is no longer true when \( f_\infty \) is superattracting at \( p \), and we are not able to conclude in this case. Still, this situation leads to interesting dynamical considerations and objects, and we plan to come back to this issue in a later work.

Finally, we develop a specialization argument in Section 4 to reduce Theorem A for an arbitrary \( k \) to the case where \( k \) is a number field. So here we rather deal with an algebraic family of endomorphisms of \( \mathbb{P}^d \) parameterized by some algebraic variety \( S \). The main issue is to ensure that for such a family, an infinite set of preperiodic points cannot shrink to a finite set too often on \( S \). To do so, one needs to control collisions of periodic points (using the Shub-Sullivan theorem [29] in the spirit of [11]); and the splitting of local preimages of a super-attracting cycle, a phenomenon that was studied in particular by Chio and Roeder [7].

1. Dynamical heights

In this section, we recall some basic facts on canonical heights attached to endomorphisms of the projective plane defined over a number field. Our purpose is to establish Proposition 1.7 which is the key arithmetic geometry input in our main theorem.
Throughout this section, we assume that \( k \) is a number field.

1.1. **Vocabulary of number fields.** We denote by \( M_k = \{ v \} \) the set of places of \( k \), that is, the set of all multiplicative norms \(| \cdot |_v\) on \( k \) that restrict to either the standard euclidean norm \(| \cdot |_\infty\), or to a \( p \)-adic norm \(| \cdot |_p\) on \( \mathbb{Q} \) for some prime number \( p > 1 \). We normalize the \( p \)-adic norm by \(| p |_p = p^{-1}\). We let \( k_v \) be the completion of \( k \) w.r.t. \(| \cdot |_v\), and write \( n_v := |k_v : \mathbb{Q}_v|\). The product formula asserts that for any \( a \in k \), we have

\[
\prod_{v \in M_k} |a|_v^{n_v} = 1.
\]

The set \( M_k \) splits into the finite set \( M_k^\infty \) of Archimedean places (whose restriction to \( \mathbb{Q} \) is \(| \cdot |_\infty\)), and the set of finite (or non-Archimedean) places.

When \( v \in M_k^\infty \), the algebraic closure \( \mathbb{C}_v \) of \( k_v \) is isometric to \( \mathbb{C} \). When \( v \) is a finite place extending \(| \cdot |_p\) on \( \mathbb{Q} \), then \(| \cdot |_v\) extends canonically to \( k_v^{alg} \), and its completion \( \mathbb{C}_v \) is both complete and algebraically closed.

1.2. **Regular polynomial maps.** Let \((z, w)\) be affine coordinates on the affine plane \( \mathbb{A}^2_k \). We also consider homogeneous coordinates \([z_0 : z_1 : z_2]\) on the projective plane \( \mathbb{P}^2_k \) and identify the affine plane \( \mathbb{A}^2_k \) with the Zariski open set \( z_0 \neq 0 \) so that \( z = z_1/z_0 \) and \( w = z_2/z_0 \). We denote by \( L_\infty = \{ z_0 = 0 \} \) the line at infinity.

Let \( f : \mathbb{A}^2_k \to \mathbb{A}^2_k \) be any polynomial self-map of the affine plane of degree \( d \geq 2 \) that extends to an endomorphism of \( \mathbb{P}^2_k \). In the coordinates \( z, w \), it is given by

\[
f(z, w) = (P(z, w), Q(z, w)),
\]

where \( P, Q \in k[z, w] \) satisfy \( \max\{\deg P, \deg Q\} = d \). The fact that \( f \) extends to a regular endomorphism \( \overline{f} : \mathbb{P}^2_k \to \mathbb{P}^2_k \) is equivalent to say that \( P = P_d + \text{l.o.t.} \) and \( Q = Q_d + \text{l.o.t.} \), where \( P_d \) and \( Q_d \) are homogeneous polynomials of degree \( d \) without common factors.

For \( n \in \mathbb{N} \) we write \( f^n(z, w) = (P_n(z, w), Q_n(z, w)) \). The restriction of \( f_\infty \) to \( L_\infty \) is an endomorphism of \( \mathbb{P}^1_k \) given in homogeneous coordinates by

\[
f_\infty([z_1 : z_2]) = [P_d(z_1, z_2) : Q_d(z_1, z_2)].
\]

1.3. **Green functions.** The next proposition follows from the Nullstellensatz (see e.g. [30, Theorem 3.11]).

**Proposition 1.1.** For any \( v \in M_k \), there exists a constant \( C_v \geq 1 \) such that

\[
C_v^{-1} \leq \frac{\max\{1, |P(z, w)|_v, |Q(z, w)|_v\}}{\max\{1, |z|_v, |w|_v\}^d} \leq C_v
\]

for all \( z, w \in \mathbb{C}_v \). Moreover, for all but finitely many \( v \in M_k \) we may take \( C_v = 1 \).

By the previous proposition, the sequence of functions

\[
g_{v, n}(z, w) := \frac{1}{d^n} \log \max\{1, |P_n(z, w)|_v, |Q_n(z, w)|_v\}
\]

converges uniformly on \( \mathbb{C}_v^2 \) to a continuous function \( g_v \), and the next proposition follows.
Proposition 1.2. For any \( v \in M_k \), the function \( g_v : \mathbb{C}_v^3 \to \mathbb{R}_+ \) is continuous, it satisfies the invariance equation \( g_v \circ f = dG_v \), and we have
\[
|g_v(z, w) - \log \max\{1, |z|_v, |w|_v\}| \leq \frac{\log C_v}{d-1}.
\]
The set \( \{(z, w) \in \mathbb{C}_v^2, g_v(z, w) = 0\} \) coincides with the set of points having bounded orbits.

We shall also consider the global Green function in \( \mathbb{C}_v^3 \). Write \( \tilde{P}(z_0, z_1, z_2) = z_0^d P\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) \)
and \( \tilde{Q}(z_0, z_1, z_2) = z_0^d Q\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) \) so that \( F(z_0, z_1, z_2) = (z_0^d, \tilde{P}^d, \tilde{Q}) \) is a homogenous map of degree \( d \) that lifts \( f \) to \( \mathbb{C}_v^3 \). Observe that in homogenous coordinates, (2) can be rewritten as follows:
\[
C_v^{-1} \leq \frac{\max\{|z_0|^d, |\tilde{P}(z_0, z_1, z_2)|_v, |\tilde{Q}(z_0, z_1, z_2)|_v\}}{\max\{|z_0|_v, |z_1|_v, |z_2|_v\}^d} \leq C_v
\]
so that the next result also holds.

Proposition 1.3. The function \( G_v(z_0, z_1, z_2) = g_v(z_1/z_0, z_2/z_0) + \log |z_0| \) is continuous on \( \mathbb{C}_v^3 \setminus \{0\} \), 1-homogeneous (that is, \( G_v(AZ) = \log |A| + G_v(Z) \)), and satisfies \( G_v \circ F = dG_v \). We have
\[
|G_v(z_0, z_1, z_2) - \log \max\{|z_0|_v, |z_1|_v, |z_2|_v\}| \leq \frac{\log C_v}{d-1}.
\]
The set \( \{(z_0, z_1, z_2) \in \mathbb{C}_v^3, G_v(z_0, z_1, z_2) \leq 0\} \) coincides with the set of points having bounded \( F \)-orbits.

1.4. Canonical heights on points. We refer to [6] for generalities on heights. Consider the line bundle on \( \mathcal{O}(1) \to \mathbb{P}^2_k \). The space of sections of this line bundle can be canonically identified with the space of linear forms \( a_0z_0 + a_1z_1 + a_2z_2 \) with \( a_i \in k \). More precisely, in the trivialization of the bundle over \( \{z_i \neq 0\} \), this section is given by \( \frac{1}{z_i}a_0z_0 + a_1z_1 + a_2z_2 \).

For any \( v \in M_k \), we consider the line bundle \( \mathcal{O}(1) \to \mathbb{P}^2_{\mathbb{C}_v} \) and endow it with the metrization \( |\cdot|_v \) induced by \( G_v \), in the sense that any section \( \sigma = a_0z_0 + a_1z_1 + a_2z_2 \) as above
\[
|\sigma|_v([z_0 : z_1 : z_2]) = |a_0z_0 + a_1z_1 + a_2z_2|e^{-G_v(z_0, z_1, z_2)}
\]
(this expression is well-defined thanks to the homogeneity property of \( G_v \)). Let us now explain how this collection of metrizations defines a function on the set of algebraic points in \( \mathbb{P}^2_k \) as well as on the set of all algebraic curves in \( \mathbb{P}^2_k \) defined by an equation with coefficients in \( k^{alg} \).

For any \( p \in (k^{alg})^3 \setminus \{0\} \), we set
\[
h_f(p) := \frac{1}{N(p)} \sum_{p' \in \text{Gal} \cdot p} \left( \sum_{v \in M_k} n_vG_v(p') \right)
\]
where \( \text{Gal} \) denotes the absolute Galois group of \( k^{alg} \) over \( k \), and \( N(x) \) is the cardinality of the set \( \text{Gal} \cdot x \subset (k^{alg})^3 \).

The product formula entails that \( h_f(z_0, z_1, z_2) = h_f(\lambda z_0, \lambda z_1, \lambda z_2) \) for any \( \lambda \in k^{alg} \) so that we have a well-defined function \( h_f : \mathbb{P}^2(k^{alg}) \to \mathbb{R} \).
Proposition 1.4. The function \( h_f \) takes non-negative values, and satisfies \( h_f \circ f = dh_f \). The set \( \{ h_f = 0 \} \) coincides with the set of preperiodic points of \( f \). Furthermore, for any \((z, w) \in (k^{\text{alg}})^2\) we have

\[
h_f(z, w) := \frac{1}{N(z, w)} \sum_{(z', w') \in \text{Gal} \cdot (z, w)} \left( \sum_{v \in M_k} n_v g_v(z', w') \right).
\]

As above \( \text{Gal} \) denotes the absolute Galois group of \( k^{\text{alg}} \) over \( k \), and \( N(z, w) \) is the cardinality of the set \( \text{Gal} \cdot (z, w) \subset (k^{\text{alg}})^2 \). The proof follows directly from Northcott’s theorem, see [3 Corollary 1.1.1].

1.5. Analytification of affine curves. Let \( C \) be any irreducible algebraic curve in \( \mathbb{A}^2_k \) defined by an equation \( \{ R = 0 \} \) with \( R \in k[z, w] \). Denote by \( \overline{C} \) the Zariski closure of \( C \) in \( \mathbb{P}^2 \).

Fix any place \( v \in M_k \). We denote by \( C_v^{\text{an}} \) the analytification in the sense of Berkovich of \( C \) over \( C_v \). This is a connected, locally connected and locally compact space. When \(| \cdot |_v \) is Archimedean, hence \( C_v \) is isometric to \( \mathbb{C} \), \( C_v^{\text{an}} \) is the complex analytic subspace (possibly with singularities) defined as usual by the vanishing of \( R \) in \( \mathbb{C}^2_v \). When \(| \cdot |_v \) is non-Archimedean, then \( C_v^{\text{an}} \) is defined as the set of multiplicative semi-norms on the ring \( \mathbb{C}_v[z, w]/(R) \) whose restriction to the base field equals \(| \cdot |_v \). A point is said to be rigid when the semi-norm has non-trivial kernel.

One can also define the analytification of \( \overline{C} \) by considering suitable affine coordinates in \( \mathbb{P}^2 \) and patching the previous construction in a natural way, see [3 §3.4]. Observe that \( \overline{C}_v^{\text{an}} \setminus C_v^{\text{an}} \) consists of rigid points.

Suppose first \( v \) is Archimedean. The metrization of \( \mathcal{O}(1) \) defined by (3) induces a measure \( \mu_{C,v} \) on \( \overline{C}_v^{\text{an}} \) which is locally defined by \( \mu_{C,v} := \Delta \log |\sigma|_v \), where \( \sigma \) is a local section of \( \mathcal{O}(1) \). The plurisubharmonicity of \( G_v \) ensures that \( \mu_{C,v} \) is a positive measure. The Lelong-Poincaré formula implies that the mass of \( \mu_{C,v} \) is equal to \( \text{deg}(C) \), and we have \( \mu_{C,v} = \Delta(g_v|_{C_v^{\text{an}}}) \) on \( C_v^{\text{an}} \). Observe that since \( G_v \) is continuous, \( \mu_{C,v} \) gives no mass to points.

The construction is completely analogous in the non-Archimedean case. We again obtain a positive measure \( \mu_{C,v} \) on \( \overline{C}_v^{\text{an}} \) of total mass \( \text{deg}(C) \) which is given in \( C_v^{\text{an}} \) by \( \mu_{C,v} = \Delta(g_v|_{C_v^{\text{an}}}) \) where \( \Delta \) is the Laplace operator defined by Thuillier [31]. Observe that the continuity of the metrization implies that \( \mu_{C,v} \) does not charge any rigid point (but it may still charge some non-rigid point in \( C_v^{\text{an}} \)). We refer to [3 §1.3] for the details of the constructions.

1.6. Canonical heights on curves. Let \( C \) be any irreducible algebraic curve in \( \mathbb{A}^2_k \) as in the previous section. We now define the canonical height of the curve \( \overline{C} \) following the recipe given in [3 §3.1.2], taking \( z_0 \) as a section of \( \mathcal{O}(1) \) (note that this section vanishes exactly along the line at infinity). We obtain:

\[
h_f(\overline{C}) := \sum_{p \in \overline{C} \cap L_\infty} (\overline{C}, L_\infty)_p \times h_f(p) + \sum_{v \in M_k} \int_{C_v^{\text{an}}} g_v d\mu_v.
\]
Note that \( h_f(C) \geq 0 \) because the canonical height is non-negative on closed points, and the Green functions \( g_v \) are also non-negative.

Define the essential minimum of \( h_f \) by the following formula:
\[
\text{essmin}_C(h_f) := \sup_{F \text{ finite} \subset \mathbb{C}(\overline{k_{alg}}) \backslash F} \inf_{p \in \mathbb{C}(\overline{k_{alg}}) \backslash F} h_f(p).
\]

**Theorem 1.5 (Zhang’s inequality [32, Theorem 1.10]).** We have
\[
2 \text{ essmin}_C(h_f) \geq \frac{h_f(C)}{\deg(C)} \geq \text{essmin}_C(h_f) + \inf_{p \in \mathbb{C}(\overline{k_{alg}})} h_f(p).
\]

Since \( h_f(x) = 0 \) if and only if \( x \) is preperiodic, we obtain:

**Corollary 1.6.** An irreducible algebraic curve \( C \) containing infinitely many \( f \)-preperiodic points satisfies \( h_f(C) = 0 \).

1.7. A first characterization of special curves.

**Proposition 1.7.** Suppose that \( C \subset \mathbb{A}^2_k \) is an irreducible algebraic curve containing a sequence of distinct points \( p_n \in C(\overline{k_{alg}}) \) such that \( h_f(p_n) \to 0 \).

Then all points in \( C \cap L_\infty \) are preperiodic for \( f \), and for any \( v \in M_k \) the function \( g_v|_{C^*} \) is harmonic on \( \{ g_v > 0 \} \).

**Proof.** Note that \( h_f(C) \geq 0 \). By Theorem 1.5 our assumption implies that \( \text{essmin}_C(h_f) \leq 0 \), therefore \( h_f(C) = 0 \). Then the result follows from (1) and the fact that a point \( p \) is \( f \)-preperiodic if and only if \( h_f(p) = 0 \). \( \square \)

2. **Super-stable manifolds and local estimates**

2.1. **Construction of super-stable manifolds.** In this section we work under the following hypothesis: \((k, |\cdot|)\) is a complete metrized field of characteristic 0 (which may be either Archimedean or non-Archimedean).

**Theorem 2.1.** Suppose \( f : (\mathbb{A}^2_k, 0) \to (\mathbb{A}^2_k, 0) \) is a germ of analytic map fixing the origin of the form
\[
f(x, y) = \left( \lambda x + \mu y + g(x, y), y^d(1 + h(x, y)) \right),
\]
where \( d \geq 2, \lambda \neq 0, \mu \in k, h(0, 0) = 0, \) and \( g(x, y) = O(|(x, y)|^2) \). Then there exists a unique smooth analytic curve which is transverse to \( \{ y = 0 \} \) and \( f \)-invariant.

We shall call this curve the **local super-stable manifold** of the origin, and denote it by \( W_{\text{loc}}^{ss}(0) \). After a linear change of coordinates of the form \( (x, y) \mapsto (x + \frac{\mu}{\lambda} y, y) \), we may and will assume from now on that \( \mu = 0 \). Expressing the invariant curve as a graph of the form \( x = \varphi(y) \) and making a change of coordinates of the form \( (x, y) \mapsto (x - \varphi(y), y) \), \( f \) takes the form
\[
f(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y))).
\]
It follows that \( f \) is analytically conjugate to \( y \mapsto y^d \) on \( W_{\text{loc}}^{ss}(0) \), hence the terminology.
The result is classical when \( k \) is Archimedean and/or \( f \) is locally invertible (see e.g. [21, Appendix]). For convenience we include a proof that works simultaneously in the Archimedean and non-Archimedean settings, and is adapted to \( \{ y = 0 \} \) being superattracting.

**Proof.** As explained above, we look for an analytic map \( y \mapsto \varphi(y) \), with \( \varphi(0) = 0 \) such that the change of coordinates \( \Phi(x, y) = (x + \varphi(y), y) \) satisfies

\[
\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x \tilde{g}(x, y), y^d(1 + \tilde{h}(x, y)))
\]

with \( \tilde{g}, \tilde{h} \) analytic and vanishing at 0. This property is equivalent to the identities:

\[
\begin{cases}
\lambda \varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + \tilde{h}(0, y))) \\
\tilde{h}(0, y) = h(\varphi(y), y)
\end{cases}
\]

so that we aim at finding some analytic function \( \varphi \) satisfying

\[
\lambda \varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + h(\varphi(y), y))) .
\]

For any \( r > 0 \), let us introduce the Banach space \( B_r \) that consist of those power series \( \varphi(y) := \sum_{j \geq 1} a_j y^j \) which are convergent in the disk of radius \( r \), and satisfy \( \| \varphi \| := \sup_{|y| < r} |\varphi(y)| < \infty \). Note that in the non-Archimedean case, we have \( \| \varphi \| := \sup_j |a_j| r^j \).

For any \( \varphi \in B_r \), we set

\[
T \varphi(y) := \frac{1}{\lambda} \left( \varphi(y^d(1 + h(\varphi(y), y))) - g(\varphi(y), y) \right) .
\]

We claim that for \( r > 0 \) and \( \rho > 0 \) sufficiently small, \( T \) is a well-defined strictly contracting map on \( B(0, \rho) \subset B_r \). Then, applying the Banach fixed point theorem implies the existence of the desired \( \varphi \).

First, we may suppose that \( g \) is analytic in the polydisk of radius \( r \), and since \( g \) vanishes up to order 2 at the origin, we have

\[
|g(x, y)| \leq C \max\{|x|, |y|\}^2
\]

for some \( C > 0 \) and all \( |x|, |y| < r \). Similarly, we may suppose that \( h \) is analytic in the polydisk of radius \( r > 0 \), and that \( |h(x, y)| \leq \frac{1}{2} \) for all \( |x|, |y| < r \).

Pick any \( \varphi(y) = \sum_j a_j y^j \in B(0, \rho) \subset B_r \). Reduce \( r > 0 \) if necessary so that \( \frac{3}{2} r^d < r \). Then \( \tilde{\varphi} : y \mapsto \varphi(y^d(1 + h(\varphi(y), y))) \) is well-defined and analytic on the disk of radius \( r \).

In the non-Archimedean case, \( |1 + h(\varphi(y), y)| = 1 \), so that one has

\[
\| \tilde{\varphi}(y) \|_r = \sup_{j \geq 1} |a_j| r^j \leq r \sup_{j \geq 1} |a_j| r^{j-1} \leq r |\varphi|_r .
\]

In the Archimedean case, the Schwarz lemma yields \( |\varphi(y)| \leq \frac{|y|}{r} |y| \) for all \( |y| < r \), hence \( \| \tilde{\varphi} \|_r \leq \frac{3}{2} r^{d-1} |\varphi|_r \). Note that we also have:

\[
\| g(\varphi(y), y) \|_r \leq C \max\{\| \varphi \|_r, r\}^2
\]

so for \( \varphi \in B(0, \rho) \) we deduce

\[
|T \varphi|_r \leq \frac{1}{\lambda} \left( \frac{3}{2} r^{d-1} \rho + C \max\{\rho, r\}^2 \right) ,
\]
By choosing ρ = r and then r small enough, this estimate shows that T φ is well-defined on the ball B(0, r) ⊂ B r, and T(B(0, r)) ⊂ B(0, r).

In order to prove that T is strictly contracting, observe that we can write
\[ g(x, y) - g(x', y) = (x - x') \hat{g}(x, x', y) \]
where \( \hat{g}(x, x', y) \) is again analytic in the polydisk of radius r, and
\[ |\hat{g}(x, x', y)| \leq C' \max\{|x|, |x'|, |y|\} \]
for some constant \( C' > 0 \). For any pair of analytic functions \( \varphi_1, \varphi_2 \in B(0, r) \subset B_r \) we infer:
\[ |g(\varphi_1(y), y) - g(\varphi_2(y), y)| \leq C' \|\varphi_1 - \varphi_2\| \max\{\|\varphi_1\|, \|\varphi_2\|, r\} \]

hence
\[ \|T\varphi_1 - T\varphi_2\| \leq \frac{1}{\lambda} (3r^d + C' r) \|\varphi_1 - \varphi_2\|_r. \]
Again, by choosing r sufficiently small, we obtain that T is strictly contracting and we are done. \( \square \)

2.2. The rescaling argument in the repelling case. We work in \( \mathbb{A}_k^2 \) where \( k \) is an arbitrary complete metrized field of characteristic 0. We start with a preparation lemma.

Lemma 2.2. Suppose \( f \) is an analytic map of the form
\[ f(x, y) = \left( \lambda x + xg(x, y), y^d(1 + h(x, y)) \right) \]
where \( |\lambda| > 1 \), \( d \geq 2 \) and \( g(0) = h(0) = 0 \).

Then there exists an analytic change of coordinates \( \Phi \) such that
\[ \Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x(1 + xy \hat{g}(x, y)), y^d(1 + x \hat{h}(x, y))) \]
for some analytic functions \( \hat{g}, \hat{h} \).

Recall that the form (7) is what is obtained from (5) after conjugating to get \( \mu = 0 \) and declaring that the stable manifold of Theorem 2.1 is \( \{x = 0\} \).

Proof. By Böttcher’s theorem (see [2, Chapter 4] for the non-Archimedean case) applied to \( y \mapsto f(0, y) \) we may suppose that \( x \) divides \( h \). Similarly, since \( |\lambda| > 1 \), by we may linearize \( x \mapsto f(x, 0) \), and suppose that \( f \) is of the form \( f(x, y) = (\lambda x(1 + g_1(y)) + x^2 y h_1(x, y), y^d(1 + O(x))) \) for some analytic functions \( g_1, h_1 \) with \( g_1(0) = 0 \).

We claim that there exists \( \Phi(x, y) = (x(1 + \varphi(y)), y) \) with \( \varphi(0) = 0 \) such that
\[ \Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x^2 y h_2(x, y), y^d(1 + O(x))) \]
for some analytic function \( h_2 \). Indeed, \( \varphi \) is then characterized by the equation
\[ \lambda x(1 + \varphi(y))(1 + g_1(y)) + O(x^2) = (\lambda x + O(x^2))(1 + \varphi(y^d)) \]
that is, \( (1 + \varphi(y))(1 + g_1(y)) = (1 + \varphi(y^d)) \), a solution of which is given by the infinite product
\[ 1 + \varphi(y) = \prod_{k=0}^{\infty} \left( 1 + g_1 \left( y^{d^k} \right) \right)^{-1} \]
and we are done. \( \square \)
The next result is similar to \[\text{[11, Lemma 4.2]}.\]

**Proposition 2.3.** Suppose \( f \) is an analytic map of the form

\[
(8) \quad f(x, y) = (\lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y)))
\]

where \(|\lambda| > 1\), \(d \geq 2\) and \(g, h\) are analytic functions.

Then \( f^n(\frac{x}{\lambda}, y) \to (x, 0) \) when \( n \to \infty \), uniformly on a polydisk of sufficiently small radius centered at the origin.

**Proof.** Fix \( 0 < r \leq 1/4 \) small enough so that \( g, h \) are both analytic on the polydisk of radius \( r \) and \( |g(x, y)|, |h(x, y)| \leq 1 \) for \(|x|, |y| < r\). Let us first show that if \(|x| \leq \frac{r}{2|\lambda|^n}\) and \(|y| \leq r\), then the \( n \) first iterates of \((x, y)\) remain in \( \mathbb{D}^2_n \). We argue by induction. So assume that \((x_0, y_0) \in \mathbb{D}_{r|\lambda|^{-n/2}} \times \mathbb{D}_r\), put \( f^1(x_0, y_0) = (x_j, y_j)\), let \( k \leq n \) and assume that \((x_j, y_j) \in \mathbb{D}^2_n\) for \( j \leq k - 1 \). Observe that for \( j \leq k - 1 \), \(|y_j| \leq 2|y_j|^d\) from which it follows that

\[
|y_k| \leq \left(2^{1/(d-1)}|y_0|\right)^d \leq \left(2^{1/(d-1)r}\right)^d \leq r.
\]

Observe that the first inequality together with \( r \leq 1/4 \) also yield \(|y_j| \leq 2^{-d/4}\). For the first coordinate, we use recursively the relation \( x_{j+1} = \lambda x_j(1 + x_jy_jg(x_j, y_j))\) to get

\[
(9) \quad |x_k| \leq |\lambda|^k |x_0| \prod_{j=0}^{k-1} (1 + |x_j||y_j|) \leq \frac{r}{2} |\lambda|^{k-n} \prod_{j=0}^{k-1} \left(1 + 2^{-d}\right) \leq r|\lambda|^{k-n},
\]

where the last inequality follows from

\[
\prod_{j=0}^{k-1} \left(1 + 2^{-d}\right) \leq \prod_{j=0}^{k-1} \left(1 + 2^{-2^j}\right) = \frac{2^{2^k} - 1}{2^{2^k} - 1} < 2,
\]

in which the middle equality is easily obtained by induction.

Now take \((x, y) \in \mathbb{D}_{r/2}\), and consider \( f^n(\frac{x}{\lambda^n}, y) \). Denote as before \((x_0, y_0) = (\frac{x}{\lambda^n}, y)\) and \((x_j, y_j) = f^j(x_0, y_0)\). The first part of the proof shows that \((x_j, y_j)\) is well-defined for all \( j \leq n \), and that \( y_n \to 0 \). Now we have

\[
x_n = \lambda^n x_0 \prod_{j=0}^{n-1} (1 + x_jy_jh(x_j, y_j)) = x \prod_{j=0}^{n-1} (1 + x_jy_jh(x_j, y_j)).
\]

The inequality \(|\prod (1 + z_j) - 1| \leq \exp(\sum |z_j|) - 1\) shows that to establish the convergence \( x_n \to x \) it is enough to show that \( \sum_{j=0}^{n-1} |x_jy_jh(x_j, y_j)| \) tends to \( 0 \). But by \((9)\), \(|x_j| \leq r|\lambda|^{j-n}\), thus

\[
\sum_{j=0}^{n-1} |x_jy_jh(x_j, y_j)| \leq \sum_{j=0}^{n-1} r|\lambda|^{j-n}2^{-d} \leq r|\lambda|^{-n} \sum_{j=0}^{\infty} |\lambda|^{j-n} \sum_{k=0}^{\infty} |\lambda|^{j-k-n} \sum_{j=0}^{\infty} |\lambda|^{n-j} 2^{-d},
\]

and we are done. \(\square\)
2.3. Graph transform for $\lambda = 1$. In this paragraph we assume that $k = \mathbb{C}$ and $f$ is of the form

$$f(x, y) = \left( x + g(x, y), y^d(1 + h(x, y)) \right),$$

with $g(x, y) = \mathcal{O}(|x, y|^2)$, $h(0, 0) = 0$, and $g(x, 0) = cx^{k+1} + \mathcal{O}(x^{k+2})$ for some $k \geq 1$ and $c \neq 0$. Observe that $f|_{y=0}$ has a parabolic point at the origin with $k$ attracting and $k$ repelling petals (see e.g. [4, §6.5]). An attracting petal is a $f$-invariant (connected and simply-connected) open subset $U$ containing $0$ in its boundary, and such that, for all $z \in U$, $f^n(z) \to 0$ tangentially to some real direction (in our case, to the normalized $k$-th roots of $-c$). A repelling petal is an attracting petal for $f^{-1}$ (they are tangent to the normalized $k$-th roots of $c$). One can chose the $k$ attracting petals and $k$ repelling petals so that their union fills up a punctured neighborhood of the origin.

A vertical graph $V$ in a domain of the form $\Omega \times \mathbb{D}_\rho$ is a submanifold of the form $V := \{(\varphi(y), y) : \varphi : \mathbb{D}_\rho \to \Omega\}$ for some holomorphic function $\varphi : \mathbb{D}_\rho \to \Omega$. In the next theorem we consider pull backs of such graphs in $\mathbb{D}_\rho^2$ in the graph transform sense, that is, when pulling back some vertical graph under $f$, we keep only the component of $f^{-1}(V) \cap \mathbb{D}_\rho^2$ containing $f^{-1}(V \cap \{y = 0\})$. Abusing notation we simply denote it by $f^{-1}(V)$.

**Theorem 2.4.** Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be of the form (10) with $h(0) = 0$, $g(x, 0) \neq 0$ and $g(x, y) = \mathcal{O}(|x, y|^2)$. Let $U$ be any repelling petal of $f|_{y=0}$ and consider a germ $V$ of analytic curve transverse to $\{y = 0\}$, intersecting $U$.

Then there exists $r > 0$ depending only on $f$, such that for large enough $n$, the analytic sets $f^{-n}(V)$ are vertical graphs in $\mathbb{D}_\rho^2$ converging to $W_{k\rho}(0)$ in the $C^1$ topology.

**Lemma 2.5.** Suppose $f$ is a holomorphic map of the form (10) as in Theorem 2.4. Then there exist an integer $k \geq 1$, and an analytic change of coordinates $\Phi$ such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (x + x^{k+1} + x^{2k+1}\tilde{g}(x, y), y^d(1 + x\tilde{h}(x, y)))$$

for some analytic functions $\tilde{g}, \tilde{h}$.

**Proof.** The proof is essentially contained in [20, Proposition 2.3]. We provide the details for the sake of completeness. By Theorem 2.4 and by applying the Böttcher theorem to $y \to f(0, y)$ we may assume that both $g$ and $h$ are divisible by $x$, so that we may write

$$f(x, y) = \left( x + yg(x, y), y^d(1 + xh(x, y)) \right),$$

with $g(0, 0) = 0$.

By a local change of coordinates involving only $x$, we can arrange so that $f(x, 0) = (x + x^{k+1} + \mathcal{O}(x^{2k+1}), 0)$. Expand the first coordinate of $f$ in power series of $x$ as follows:

$$x \circ f(x, y) = x(1 + g_0(y)) + x^{k+1}(1 + g_k(y)) + \sum_{j \neq 0, k} x^{j+1}g_j(y),$$

with $g_j(0) = 0$ for $0 \leq j \leq 2k - 1$.

We claim that for all $n \leq 2k - 1$, we can conjugate $f$ by a germ of invertible holomorphic map such that $g_j \equiv 0$ for every $j \leq n$. Applied to $n = 2k - 1$, this claim implies the proposition.
For $n = 0$ this is done by a change of coordinates of the form $\Phi_0(x, y) := (x(1 + \varphi_0(y)), y)$ such that $(1 + g_0)(1 + \varphi_0) = 1 + \varphi_0(y^d)$. This equation can be solved exactly as in Lemma 2.2.

Now assume that $n > 0$, and that the result has been achieved up to $j = n - 1$. Put $\Phi_n(x, y) = (x(1 + \varphi_n(y)x^n), y)$, so that $\Phi_n^{-1}(x, y) = (x(1 - \varphi_n(y)x^n + O(x^{n+1})), y)$. Depending on the position of $n$ and $k$, we obtain:

$$x \circ (\Phi_n^{-1} \circ f \circ \Phi_n) = \begin{cases} x + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n < k, \\ x + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n = k, \\ x + x^{k+1} + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n > k. \end{cases}$$

Therefore, to render the term in $x^{n+1}$ constant, it is enough to solve the equation $-g_n(y) = \varphi_n(y) - \varphi_n(y^d)$ which can be done by setting $\varphi_n(y) = -\sum_{m=0}^{\infty} g_n(y^{dm})$.

Proof of Theorem 2.4. By the previous lemma, we may suppose that

$$f(x, y) = \left( x + x^{k+1} + x^{2k+1}g(x, y), y^d(1 + xh(x, y)) \right)$$

with $g$ and $h$ holomorphic near the origin. Note that $f|_{y=0}$ has a repelling petal along the positive real axis. Fix $r > 0$ such that $f$ is holomorphic and injective on a neighborhood of $\mathbb{D}_r$. Reducing and rotating the petal if necessary, we may assume that

$$U := \left\{ x : \arg(x) \in \left(-\frac{\pi}{4k}, \frac{\pi}{4k}\right), |x| < r \right\}.$$ 

The holomorphic map $z = (kx)^{k^{-1}}$ is univalent on $U \times \mathbb{D}_r$, and takes its values in

$$\Omega_R := \left\{ z : \arg(z) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), |z| > R \right\},$$

where $R = (kr)^{k^{-1}}$. The expression of $f$ in the coordinates $(z, y)$ is of the form

$$f(z, y) = \left( z - 1 + \frac{1}{z}a(z, y), y^d \left(1 + \frac{1}{z^{1/k}}b(z, y)\right) \right).$$

so that $f$ is now defined in $\Omega_R \times \mathbb{D}_r$. Fix $M > 0$ such that

$$|a|, |b|, \left| \frac{\partial a}{\partial z} \right|, \left| \frac{\partial b}{\partial z} \right|, \left| \frac{\partial b}{\partial y} \right| \leq M$$

on $\Omega_R \times \mathbb{D}_r$, and reduce $r$ if necessary so that $r < \frac{1}{10M}$ and $M R^{-1/k} \leq \frac{1}{100}$.

For any $\rho < r$ and $\sigma > 0$ we let

$$\mathcal{G}(\rho, \sigma) = \left\{ \varphi : \mathbb{D}_\rho \rightarrow \Omega_R \text{ holomorphic s.t. } \sup_{\mathbb{D}_\rho} |\varphi'| \leq \sigma \right\}.$$

Lemma 2.6. Suppose that $\sigma \rho < \frac{1}{100d}$. For any vertical graph $\Gamma$ determined by $\varphi \in \mathcal{G}(\rho, \sigma)$, then $f^{-1}\Gamma$ is a vertical graph determined by a function $\psi \in \mathcal{G}(\rho_1, \frac{1}{10})$ where $\rho_1 = \min \left( (\rho/2)^{1/d}, r \right)$ and $\text{Re}(\psi) \geq \text{Re}(\varphi(0)) + 9/10$.

Assuming this result for the moment, let us conclude the proof of the theorem. Let $V$ be any germ of curve intersecting transversally $\{ y = 0 \}$ at $x_0 \in U$. Then $V$ is a graph of slope $\sigma$ over some disk $\mathbb{D}_\rho$ in the second coordinate, for some $\rho > 0$. Reduce $\rho$ by trimming $V$ if necessary so that $\sigma \rho < 1/100d$. Since $r/10 < 1/100d$, the previous
lemma implies that we can define inductively a sequence of vertical graphs \( V = V_0, V_n := f^{-1}V_{n-1} \), where \( V_n \) is defined by a holomorphic function \( z = \varphi_n(y) \) of uniformly bounded slope over \( \mathbb{D}_{\rho_n} \), and furthermore \( \rho_n = r \) for large enough \( n \). Moreover, we have \( \text{Re}(\varphi_n) \geq \text{Re}(\varphi(0)) + 9n/10 \), hence, coming back to the \((x, y)\) coordinates, we see that \( V_n = \{x = (k\varphi_n(y))^{-1/k}\} \) converges in the \( C^1\)-topology to the curve \( W_{\text{loc}}^s(0) = \{x = 0\} \).

**Proof of Lemma 2.6.** Let \( \Gamma \) be a vertical graph of equation \( z = \varphi(y) \) in the \((z, y)\) coordinates, with \( \varphi \in \mathcal{G}(\rho, \sigma) \). Then the equation of \( f^{-1}\Gamma \) is given by \( z = \ell(z, y) \), where

\[
\ell(z, y) = \varphi \left( y^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) + 1 - \frac{1}{z} a(z, y).
\]

Fix \( y_0 \in \mathbb{D}_{\rho_1} \). We show that the equation \( z = \ell(z, y_0) \) admits a unique solution \( z \in \Omega_R \).

First, observe that by the estimates \((13)\) on \(|a|\) and \(|b|\), for \( z \in \Omega_R \) we have

\[
|\ell(z, y_0) - (\varphi(0) + 1)| \leq \left| \varphi \left( y_0^d \left( 1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) - \varphi(0) \right| + \frac{M}{R} \leq 2\rho^d \sigma + \frac{1}{100} \leq \rho \sigma + \frac{1}{10} \leq 10,
\]

hence \( \ell(\cdot, y_0) \) maps \( \Omega_R \) to itself. Next, we see that

\[
\left| \frac{\partial \ell}{\partial z}(z, y_0) \right| \leq y_0^d \left( \frac{1}{z^{1/k}} \frac{\partial b}{\partial z} - \frac{1}{kz^{1/k+1} b} \right) \left| \varphi' \right| \left( y_0^d \left( 1 + \frac{1}{z^{1/k}} b \right) \right) + \frac{a}{z^2} - \frac{1}{z} \frac{\partial a}{\partial z} \leq \rho^d \frac{2M}{R^{1/k}} \sigma + \frac{2M}{R^{1/k}} \leq \frac{1}{10},
\]

so \( \ell(\cdot, y_0) \) is a contraction and the equation \( z = \ell(z, y_0) \) has a unique solution. This means that \( f^{-1}\Gamma \) is a vertical graph over \( \mathbb{D}_{\rho_1} \) determined by a holomorphic function \( \psi \) satisfying \( \psi(y) = \ell(\psi(y), y) \). The slope of this graph can be estimated as above:

\[
|\psi'(y)| \leq \frac{\left| \frac{\partial \psi}{\partial y} \right|}{1 - \left| \frac{\partial \psi}{\partial z} \right|} \leq \frac{10}{9} \left( \sigma \left( d^{d-1} \left( 1 + \frac{M}{R^{1/k}} \right) \right) + \rho^d \frac{M}{R^{1/k}} + \frac{M}{R^{1/k}} \right) \leq \frac{1}{10}.
\]

Finally, the estimate \( \text{Re}(\psi) \geq \text{Re}(\varphi(0)) + 9/10 \) follows from \((14)\) and we are done. \( \Box \)

3. Proof of Theorem A when \( k \) is a number field

Here we establish the following more precise form of our main theorem, in the number field case.

**Theorem 3.1.** Let \( k \) be a number field and \( f \) be a regular polynomial map of \( \mathbb{A}^2_k \). Denote by \( h_f \) the induced canonical height.

Suppose that \( C \subset \mathbb{A}^2_k \) is an irreducible algebraic curve containing a sequence of distinct points \( p_n \in C(k^{alb}) \) such that \( h_f(p_n) \to 0 \). If there exists a point of \( C \cap L_\infty \) which is not eventually superattracting, then \( C \) is preperiodic.

Let \( f_\infty \) be the restriction to the line at infinity \( L_\infty \) of the extension of \( f \) to \( \mathbb{P}^2 \). By Proposition 1.7, all points in \( C \cap L_\infty \) are preperiodic, so we may replace \( f \) by \( f^N \) and \( C \) by \( f^N(C) \), to assume that \( C \cap L_\infty \) contains a fixed point \( p \) which is not super-attracting.
Let \( \lambda = f_\infty^\ast(p) \) be the multiplier of \( p \) along \( L_\infty \). Then one of the two following mutually disjoint cases occur:

(a) either \( \lambda \) is a root of unity;
(b) or there is a place \( v \in M_\infty \) such that \(|\lambda|_v > 1\).

In the remainder of this section we split the proof of the theorem according to these two cases.

3.1. When \( \lambda \) is a root of unity. In this situation we iterate \( f \) further so that \( \lambda = 1 \)

and work over the complex numbers. Since \( p \) belongs to the Julia set \( J(f_\infty) \), which is a

perfect set, the union of attracting and repelling petals cover a punctured neighborhood of \( p \), and the attracting petals of \( p \) are contained in the Fatou set, we see that there is a

repelling periodic point \( q \) of \( f_\infty \) contained in some local repelling petal of \( p \). Then the local (super-)stable manifold of \( q \) is a disk transverse to \( L_\infty \) at \( q \), and by Theorem 2.4

the local truncated pull-backs \( f^{-n}(W_{\text{loc}}^{\text{ss}}(q)) \) under \( f^n \) converge to \( W_{\text{loc}}^{\text{ss}}(p) \) when \( n \to \infty \).

Assume by way of contradiction that \( \overline{C} \) does not locally coincide with \( W_{\text{loc}}^{\text{ss}}(p) \). We claim that \( \overline{C} \) intersects \( f^{-n}(W_{\text{loc}}^{\text{ss}}(q)) \) in \( \mathbb{C}^2 \) close to \( p \) for large \( n \). Indeed, locally near \( p \),

\( \overline{C} \cap W_{\text{loc}}^{\text{ss}}(p) = \{ p \} \), so by the persistence of proper intersections, \( \overline{C} \) intersects \( f^{-n}(W_{\text{loc}}^{\text{ss}}(q)) \) close to \( p \) for large \( n \). But \( p \notin f^{-n}(W_{\text{loc}}^{\text{ss}}(q)) \cap L_\infty \), so these intersection points lie in \( \mathbb{C}^2 \), as claimed. If now \( \Delta \) is a small disk in \( C \) containing one of these intersection points, then by

the Inclination Lemma, the derivative of \( f^n \) in the direction of \( \Delta \) tends to infinity, thus

\( (f^n|_\Delta) \) is not a normal family. On the other hand, by Proposition 1.7 \( g|_\Delta \) is harmonic,

which implies that \( (f^n|_\Delta) \) is normal (see [13, Prop 5.10]). This contradiction shows that

\( \overline{C} \) locally coincides with \( W_{\text{loc}}^{\text{ss}}(p) \) near \( p \), so it is fixed under \( f \), and by irreducibility this property propagates to the whole of \( C \). This completes the proof in this case. \( \square \)

**Remark 3.2.** This argument works essentially the same when \(|\lambda|_v > 1\) at some Archime-

dean place, and may help understand the non-Archimedean argument below.

3.2. When \(|\lambda|_v > 1\). We may assume that \( p = [0 : 0 : 1] \), and by Lemma 2.2 find a

local analytic isomorphism \( (x, y) \mapsto \psi(x, y) = [z_0(x, y) : z_1(x, y) : 1] \) such that \( \psi(0) = p \),

\( z_0(x, 0) = 0 \) so that \( \{ y = 0 \} \) corresponds to the line at infinity,\(^1\) and write

\[ \tilde{f} := \psi^{-1} \circ f \circ \psi : (x, y) \mapsto \left( \lambda x(1 + z_2g(x, y)), y^d(1 + xh(x, y)) \right) \].

If \( \overline{C} \) has an analytic branch at \( p \) which coincides with \( \{ x = 0 \} \) then \( C \) is fixed as above,

and we are done. Otherwise, we may find a Puiseux parameterization of a branch of \( C \) at \( p \) in the \( (x, y) \) coordinates of the form

\( \Gamma(t) = (t^\gamma, \gamma(t)) \), where \( \gamma \) is analytic and defined in a small disk \( \mathbb{D}_\delta \), and \( q \in \mathbb{N}^* \). We seek a contradiction.

We lift \( \psi \) to \( \mathbb{A}^3_\mathbb{K} \), and set \( \Psi(x, y) = (z_0(x, y), z_1(x, y), 1) \). As in §1.3 we lift \( f \) to a

homogeneous polynomial map \( F : \mathbb{A}^3_\mathbb{K} \to \mathbb{A}^3_\mathbb{K} \), of the form \( F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q}) \). Since \( \Psi \circ \psi^{-1} \) is a local section of the projection \( \mathbb{A}^3 \setminus \{ 0 \} \to \mathbb{P}^2 \), \( \Psi \circ \tilde{f} \) must be a multiple of \( F \circ \Psi \). From the expression of \( F \) we obtain

\[ \Psi \circ \tilde{f} = \frac{F \circ \Psi}{Q \circ \Psi} \].

\(^1\)Beware that coordinates are swapped here: \( \{ z_0 = 0 \} \) corresponds to \( \{ y = 0 \} \).
To simplify notation, we write $F^n(z_0, z_1, z_2) = (z_0^n, \tilde{P}_n, \tilde{Q}_n)$. We consider the 1-homogeneous Green function $G_f: \mathbb{A}^2_n \to \mathbb{R}$ of Proposition 1.3; it satisfies $G_f \circ F = dG_f$ and $g_f(z_1, z_2) = G_f(1, z_1, z_2)$.

Observe that $h := G_f \circ \Psi \circ \Gamma$ is a continuous function on $\mathbb{D}_\delta$. Since $h(t) = g_f \circ \psi \circ \Gamma(t) + \log |z_0 \circ \Gamma(t)|$, by Proposition 1.7, $h$ is harmonic on $\{t \neq 0\}$. Since it is continuous at 0, it is also harmonic on $\mathbb{D}_\delta$, see, e.g., [12] Lemma 3.7. Write

$$d^\text{eq} h(t) = G_f \circ F^{nq} \circ \Psi \circ \Gamma(t)$$

$$= G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(t) + \log \left| \tilde{Q}[nq] \circ \Psi \circ \Gamma(t) \right|.$$ 

By Proposition 2.3, $\tilde{f}[n](x, y) \to (x, 0)$ uniformly in a neighborhood of the origin, hence

$$G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(\frac{x}{n^q}) \to G_f(0, z_1(1^q, 0), 1)$$

as $n \to \infty$. On the other hand, since $d^\text{eq} h(\frac{x}{n^q}) - \log \left| \tilde{Q}[nq] \circ \Psi \circ \Gamma(\frac{x}{n^q}) \right|$ is a sequence of harmonic functions, it follows that $t \mapsto G_f(0, z_1(t^q, 0), 1)$ is harmonic as well.

Now, observe that the restriction of $f$ to the line at infinity is $f_\infty[z_1 : z_2] = [\tilde{P}(0, z_1, z_2) : \tilde{Q}(0, z_1, z_2)]$, so that $G_f(0, z_1, z_2)$ is the global Green function of $f_\infty$. The equilibrium measure of $f_\infty$ is the probability measure on the analytification of $L_\infty$ defined by $\mu_{f_\infty} := \Delta G_f(0, z_1, 1)$ in the chart $z_2 \neq 0$. Its support is the Julia set of $f_\infty$ (see [2] Theorem 13.39), and it contains all repelling (rigid) fixed points, see [2] Theorem 8.7. Therefore, $z_1 \mapsto G_f(0, z_1, 1)$ cannot be harmonic near 0, hence the function $t \mapsto G_f(0, z_1(t^q, 0), 1)$ cannot be harmonic either. This contradiction concludes the proof.

**Remark 3.3.** Under the assumptions of Theorem 3.1, the proof shows that the preperiod $k$ of $F$ and the period of $f^k(C)$ are exactly the same as that of any of its non-superattracting points at infinity.

### 4. Proof of Theorem A for arbitrary $k$

In this section we use a specialization argument to deal with maps defined over arbitrary fields. It shares some arguments with [11] §5 (see also [5] §7). Nevertheless, new ideas are needed to deal with preperiodic points instead of periodic ones.

We are in the setting of Theorem A, so we assume that $f$ is a regular polynomial map of $\mathbb{A}^2$ of degree $d \geq 2$ defined over a field $k$ of characteristic 0, and $C$ is a curve containing an infinite set $\mathbb{P} = \{ p_n, \ n \geq 0 \}$ of preperiodic points and whose closure $\overline{C} \cap L_\infty$ contains at least one point which is not eventually super-attracting.

By enlarging $k$ if necessary we may assume that it contains the algebraic closure $\overline{\mathbb{Q}}$ of its prime field. Let $R$ be the sub-$\overline{\mathbb{Q}}$-algebra of $k$ generated by all coefficients defining $f$ and $C$. Its fraction field $K$ is finitely generated over $\overline{\mathbb{Q}}$. Let $S = \text{Spec} \ R$. This is an affine variety defined over $\overline{\mathbb{Q}}$, and elements of $R$ can be seen as regular functions on $S$.

Inverting some elements of $R$ if necessary, we may suppose that $C$ is flat over $S$, and $f$ extends as a morphism $f: \mathbb{P}^2_S \to \mathbb{P}^2_S$. We let $\pi: \mathbb{P}^2_S \to S$ be the canonical projection, and write $\mathbb{P}^2_s = \pi^{-1}(s)$. We also let $\mathbb{A}^2_s := \mathbb{A}^2_S \cap \pi^{-1}(s)$.
For each (scheme theoretic) point \( s \in S \), we write \( C_s = C \cap \mathbb{A}_s^2 \) and let \( \overline{C}_s \) be the closure of \( C_s \) in \( \mathbb{P}_s^2 \). The flatness of the morphism \( C \to S \) implies \( \overline{C}_s \) (hence \( C_s \)) to be a curve. Similarly, we let \( f_s : \mathbb{P}_s^2 \to \mathbb{P}_s^2 \) be the induced map on the fibers: this is an endomorphism of degree \( d \).

We also denote by \( p_{n,s} \in \mathbb{A}_s^2 \) the specialization of \( p_n \). Note that \( p_n \) is defined over some finite extension of \( K \) which depends on \( n \).

The first result does not use the assumption that our infinite set of preperiodic points lies on a curve.

**Proposition 4.1.** Let as above \( f : \mathbb{P}_S^2 \to \mathbb{P}_S^2 \) be a family of endomorphisms over an affine variety defined over \( \mathbb{Q}^{alg} \), and let \( P = \{ p_n, n \geq 0 \} \) be an infinite family of preperiodic points. Then there exists a non-empty Zariski open and dense subset \( U \subset S \) such that for any \( s \in U \cap S(\mathbb{Q}^{alg}) \), \( P_s = \{ p_{n,s}, n \geq 0 \} \subset \mathbb{A}_s^2 \) is infinite.

Before starting the proof, let us fix some additional notation. For each \( n \geq 0 \), we denote by \( k_n \) the preperiod of \( p_n \), so that \( q_n := f^{k_n}(p_n) \) is the first periodic point in the orbit of \( p_n \). We let \( \ell_n \) be the (primitive) period of \( q_n \).

**Proof.** We may suppose that there exists a parameter \( s_0 \in S(\mathbb{Q}^{alg}) \) such that \( P_{s_0} \) is finite (otherwise we take \( U = S \) and the proof is complete).

**Lemma 4.2.** The family of periodic points \((q_n)\) is finite.

**Proof.** We follow the arguments of [11, §5]. Set \( Q = \{ q_n, n \geq 0 \} \). For each \( \ell \geq 1 \), we consider the subvariety \( \text{Per}_\ell \) of \( \mathbb{P}_S^2 \) defined by the equation \( f^\ell(z) = z \). Since \( \mathbb{P}_S^2 \to S \) is proper, the structure map \( \text{Per}_\ell \to S \) is also proper.

Let \( Q_\ell \) be the union of the irreducible components of \( \text{Per}_\ell \) containing a point of \( P \). Its underlying set is the Zariski closure of \( P \cap \text{Per}_\ell \), hence \( Q_\ell \) is proper. Observe that for \( x \in Q_\ell,s \), the multiplicity of \( x \) as a point of \( Q_\ell,s \) equals its multiplicity as a fixed point of \( f_\ell \). By Nakayama’s lemma and the properness of \( Q_\ell \) over \( S \), the function

\[
(15) \quad s \mapsto \sum_{x \in Q_\ell,s} \text{mult}_x(Q_\ell,s)
\]

is upper semi-continuous for the Zariski topology, hence

\[
(16) \quad \sum_{q \in Q_\ell} \text{mult}_q(Q_\ell) \leq \sum_{x \in Q_\ell,s_0} \text{mult}_x(Q_\ell,s_0),
\]

where the left hand side is the value of \( (15) \) at the generic point. By assumption \( P_{s_0} \) is a finite set, hence so does \( Q_{s_0} = \{ q_{1,s_0}, \ldots, q_{r,s_0} \} \) and by the Shub-Sullivan theorem [29], there exists a uniform bound \( C > 0 \) such that for every \( j \), and for any \( \ell \), we have

\[
\text{mult}_{q_{j,s_0}}(Q_\ell,s_0) \leq \text{mult}_{q_{j,s_0}}(\text{Per}_\ell,s_0) \leq C.
\]

It then follows from \( (16) \) that

\[
\# Q_\ell \leq \sum_{q \in Q_\ell} \text{mult}_q(Q_\ell) \leq rC
\]

hence \( \bigcup_\ell Q_\ell \) is finite, as was to be shown. \( \square \)
By the previous lemma, replacing \( f \) by some iterate \( f^N \) we may assume that all periodic points \( q_s \) are fixed. Since \( \mathcal{P} \) is infinite, one of these fixed points, say \( q_1 \), admits infinitely many preimages in \( \mathcal{P} \). We may denote \( q = q_1 \) and suppose \( \mathcal{P} \) is made of an infinite set of preimages of \( q \), that is, (after possible reordering of \( \mathcal{P} \)) for any \( p_n \in \mathcal{P} \), there is a minimal \( k_n \geq 0 \) such that \( f^{k_n}(p_n) = q \), and that \( k_{n+1} > k_n \). We may adjoin to \( R \) the coordinates of \( q \) so that \( q \in k^2(R) \), i.e., for any \( s \in S \), \( q_s \) is a single point (to say it differently, we replace \( S \) by a branched cover of a Zariski open dense subset of \( S \)).

Let \( d(s) \) be the local degree of \( f_s \) at \( q_s \), which is upper semicontinuous for the Zariski topology. Since \( f_s \) is a finite map of degree \( d^2 \), \( d(s) \leq d^2 \) for every \( s \). Thus there is an analytic hypersurface \( H \) such that \( d(s) = d_{\min} \) is constant for \( s \in S \setminus H \).

We claim that \( \mathcal{P}_{s_1} \) is infinite for any \( s_1 \in S \setminus H \). We argue again in the complex analytic category fixing an embedding \( \mathbb{Q}^{\text{alg}} \rightarrow \mathbb{C} \). Observe that for any point \( s \in S(\mathbb{C}) \), \( p_{n,s} \) is a finite set included in the fiber \( k^2_{s,\text{an}} \simeq \mathbb{C}^2 \) (not necessarily reduced to a single point since \( p_n \) lies in a finite extension of \( R \)).

Fix an analytic neighborhood \( V \) of \( q_{s_1} \) in \( \mathbb{P}^2_{s,\text{an}}(\mathbb{C}) \) such that \( f_{s_1}^{-1}(q_{s_1}) \cap V = \{ q_{s_1} \} \). Since \( d(s) \) is locally constant near \( s_1 \), there is an analytic neighborhood \( W \) of \( s_1 \) in \( S^\text{an}(\mathbb{C}) \) such that for \( s \in W \),

\[
(17) \quad f_s^{-1}(q_s) \cap V = \{ q_s \}
\]

Choose any \( n > m \), and suppose by contradiction that \( p_{m,s_1} = p_{n,s_1} \). Since \( k_n - 1 \geq k_m \), we have

\[
f_{s_1}^{k_n-1}(p_{n,s_1}) = f_{s_1}^{k_n-1}(p_{m,s_1}) = q_{s_1}.
\]

Thus, for \( s \) close to \( s_1 \), the finite set \( f_{s_1}^{k_n-1}(p_{n,s_1}) \) is contained in \( V \), hence by (17), \( f_{s_1}^{k_n-1}(p_{n,s}) = q_s \), and by analytic continuation this property holds throughout \( S \), which contradicts the definition of \( k_n \).

This shows that the \( p_{n,s} \) are all distinct for all \( s \in S \setminus H \), and concludes the proof of Proposition 4.1.

**Proposition 4.3.** Let \( f : \mathbb{A}^2_{\mathbb{C}} \rightarrow \mathbb{A}^2_{\mathbb{C}} \) be a regular polynomial map and \( C \subset \mathbb{A}^2_{\mathbb{C}} \) be an algebraic curve containing infinitely many preperiodic points. Then every point of \( \overline{C} \cap L_\infty \) is preperiodic under \( f|_{L_\infty} \).

**Proof.** We keep the same formalism and notation as above, so that \( f \) is viewed as a family over \( S \). Write \( \overline{C} \cap L_\infty = \{ c_1, \ldots, c_r \} \) and without loss of generality enlarge \( R \) so that the points at infinity \( c_i \) have their coordinates in \( R \). Fix \( i \in \{ 1, \ldots, r \} \) for the remainder of the proof and consider \( c = c_i \). By Proposition 4.1 there is a Zariski open subset \( U \) such that for any \( s \in U \cap S(\mathbb{Q}^{\text{alg}}) \), \( f_s \) admits infinitely many preperiodic points on \( C_s \). Therefore, by Proposition 4.1, for every such \( s \), \( c_s \) is preperiodic. Fix \( s_0 \in U \cap S(\mathbb{Q}^{\text{alg}}) \), then \( c_{s_0} \) eventually falls on a periodic point \( q_{s_0} \). Replacing \( f \) by \( f^N \) and \( C \) by \( f^N(C) \) for some \( N \), we may assume that \( q_{s_0} \) is fixed and \( c_{s_0} = q_{s_0} \). Enlarging \( R \) again if necessary we may assume that \( q_{s_0} \) is the specialization at \( s = s_0 \) of a fixed point \( q \in \mathbb{P}^2(R) \) of \( f \).

Our purpose is to show that \( c = q \). To simplify notation we write \( \hat{f} = f|_{L_\infty} \). Note that the multiplier \( \mu := \hat{f}'_{s_0}(q_{s_0}) \) belongs to \( \mathbb{Q}^{\text{alg}} \). It follows from Kronecker’s theorem that either \( \mu \) is a root of unity or there is a place \( v \) on \( \mathbb{Q}^{\text{alg}} \) such that \( |\mu|_v < 1 \).
Case 1. \( \mu \) is not a root of unity.

Fix a place \( v \) on \( \mathbb{Q}^{\text{alg}} \) such that \(|\mu|_v < 1\), and consider the completion \( \mathbb{C}_v \) of \((\mathbb{Q}^{\text{alg}}, |\cdot|_v)\). We then argue in the analytic topology in the Berkovich analytification of \( \mathbb{P}^2_{\mathbb{C}_v} \) and \( S_{\mathbb{C}_v} \).

Fix a neighborhood \( W \) of \( s_0 \) in \( S_{\mathbb{C}_v}^{\text{an}} \) such that for \( s \in W \), \( q_s \) is attracting, and a neighborhood \( V \) of \( q_s \) in \( L_\infty \) independent of \( s \in W \) such that \( f_s(V) \subset V \) and for any \( z \in V \), \( f^n_s(z) \) converges to \( q_s \) as \( n \to \infty \). Reducing \( W \) if necessary we may assume that for any \( s \in W \), \( c_s \) belongs to \( V \). For \( s \in W \cap S(\mathbb{Q}^{\text{alg}}) \), \( c_s \) is preperiodic and converges to \( q_s \), so it is preperiodic to \( q_s \), that is, there exists a minimal \( k = k(s) \) such that \( f^{k(s)}_s(c_s) = q_s \). Now we use an argument similar to that of Proposition 4.1 let \( H \subset S \) be a hypersurface such that the local degree of \( f_s \) at \( q_s \) is locally minimal outside \( H \) and fix \( s_1 \in W \setminus H \). Then, there is a neighborhood \( W_1 \) of \( s_1 \) in \( W \setminus H \) and a neighborhood \( V_1 \subset V \) of \( q_1 \) such that for any \( s \in W_1 \), \( f_s(V_1) \subset V_1 \) and \( f^{-1}_s(q_s) \cap V_1 = \{ q_s \} \). From this it follows that the only point eventually falling onto \( q_s \) in \( V_1 \) is \( q_s \) itself. Therefore if \( s \in W_1 \) is so close to \( s_1 \) that \( c_s \in V_1 \), we infer that \( c_s = q_s \), and finally \( c = q \) by analytic continuation.

Case 2. \( \mu \) is a root of unity.

To deal with this case we embed \( \mathbb{Q}^{\text{alg}} \) into \( \mathbb{C} \), and work at the complex place. Recall that a holomorphic family \( (g_\lambda)_{\lambda \in \Lambda} \) of rational maps on \( \mathbb{P}^1(\mathbb{C}) \), parameterized by a connected complex manifold is trivial if any two members are conjugate by a Möbius transformation, depending holomorphically on \( \Lambda \). If \( c \) is persistently preperiodic we are done, so assume that \( c \) is not persistently preperiodic.

Under our assumptions, there is a dense set \( S(\mathbb{Q}^{\text{alg}}) \) of parameters such that \( c_s \) is preperiodic, but \( c \) is not persistently preperiodic. Thus by Chio-Roeder [7, Theorem 2.7] (see also [10, Theorem 4]) every such parameter belongs to the bifurcation locus of the marked family \( (f_\infty, c) \) (note that \( c \) is not a critical point here). As a consequence, the bifurcation locus of the family is equal to \( S_{\mathbb{C}_v}^{\text{an}} \). (Note that it is enough to work in a neighborhood of \( s_0 \), away from possible singularities of \( S_{\mathbb{C}_v}^{\text{an}} \)).

A first possibility is that the family \( (\hat{f}_s)_{s \in S_{\mathbb{C}_v}^{\text{an}}} \) is non-trivial. Then Gauthier [14, Theorem A] implies that \( J(\hat{f}_s) = L_\infty \) for all \( s \). But since \( \hat{f}_{s_0} \) has a rationally indifferent fixed point, it admits an attracting petal and \( J(\hat{f}_{s_0}) \neq L_\infty \). This contradiction shows that the family \( (\hat{f}_s)_{s \in S_{\mathbb{C}_v}^{\text{an}}} \) is trivial.

Now the situation is that there is a holomorphic family \( \varphi_s \) of Möbius transformations such that \( \varphi_s \hat{f}_s \varphi_s^{-1} = g \) is a fixed rational map \( g \) on \( \mathbb{P}^1 \) with a rationally indifferent fixed point at \( 0 \). After this conjugacy, the marked family \( (\hat{f}, c) \) becomes \((g, \varphi(c))\). Since \( c \) coincides with \( q \) at \( s_0 \) and by assumption \( c \) is not persistently preperiodic, there is an open set \( \Omega \) of parameters such that for \( s \in \Omega \), \( \varphi_s(c_s) \) belongs to some attracting petal associated to \( 0 \) for \( g \). This contradicts the fact that \( \varphi_s(c_s) \) must be preperiodic for a dense set of parameters, and the proof is complete. \( \square \)

Conclusion of the proof of Theorem A. By Proposition 4.3, any point at infinity of \( C \) is preperiodic, and, by assumption, one of these points, say \( c \), is preperiodic to a non-superattracting periodic point \( p \). Replace \( f \) by \( f^N \) and \( C \) by \( f^N(C) \) for some \( N \), so that \( c = p \) is fixed. By Proposition 4.1 there is a non-trivial Zariski open subset \( U \subset S \)
such that for every $s \in U \cap S(Q_{\text{alg}})$, $C_s$ contains infinitely many preperiodic points. Then, since $p_s$ is fixed and not superattracting for $f_s$, Theorem 3.1 asserts that $C_s$ is preperiodic, and more precisely fixed, under $f_s$ (see Remark 3.3). The density of $U \cap S(Q_{\text{alg}})$ in $S$ (for the Zariski or analytic topology) then implies that $f(C) = C$, and the proof is complete. \hfill \Box

References


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