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ON THE DYNAMICAL MANIN-MUMFORD CONJECTURE FOR PLANE POLYNOMIAL MAPS

ROMAIN DUJARDIN, CHARLES FAVRE, AND MATTEO RUGGIERO

ABSTRACT. We prove the dynamical Manin-Mumford conjecture for regular polynomial maps of \mathbb{A}^2 and irreducible curves avoiding super-attracting orbits at infinity, over any field of characteristic 0.

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INTRODUCTION

The dynamical Manin-Mumford conjecture for polarized endomorphisms of algebraic varieties, first formulated by S.-W. Zhang in two influential papers [32, Conjecture 2.5] and [33, Conjecture 1.2.1], has been a driving force for the development of the field of arithmetic dynamics. It was realized by Ghioca, Tucker and Zhang [19] and Pazuki [26] that the original formulation of the conjecture was too optimistic, and a modified conjecture was proposed in [19] and more recently in [18]. It can be stated as follows: *let $f : X \rightarrow X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and $Z \subset X$ be a subvariety containing a Zariski dense set of periodic points. Then either Z is preperiodic or Z is special*, in the sense that it is contained in some subvariety Y that is both f^n - and ψ -invariant, for some $n \geq 1$, where ψ is another polarized endomorphism commuting with f^n . Recall that an endomorphism is said to be polarized if there is an ample line bundle $L \rightarrow X$ such that $f^*L \simeq L^d$ for some $d \geq 2$. A basic example is that of non-invertible endomorphisms of \mathbb{P}^k , for which we can take $L = \mathcal{O}(1)$ and d is the degree of f .

Despite its importance, very few cases of the conjecture have been settled so far. One first case is of course the original Manin-Mumford conjecture, which was solved by Raynaud [27, 28]. Viewed as a dynamical statement, it deals with endomorphisms of

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Abelian varieties, and was generalized to related settings, such as commutative algebraic groups (see Hindry [22], and also Lang's classical paper [25]). Uniform versions involving height bounds were subsequently obtained by S.-W. Zhang, David and Philippon, Chambert-Loir and others. We refer to the recent work of Kühne [24] in the semi-abelian case for a latest update, and more references. Closer to algebraic dynamics is the case of polarized endomorphisms of $(\mathbb{P}^1)^k$, which was solved by Ghioca, Nguyen and Ye [17, 16] (see also [18]). Note that such mappings are of product type, that is of the form $f(x_1, \dots, x_k) = (f_1(x_1), \dots, f_k(x_k))$, so their dynamical complexity reduces to dimension 1, which is a key step in the proof.

In this paper we will establish the dynamical Manin-Mumford conjecture for a wide class of 2-dimensional examples, whose dynamical behavior is truly higher dimensional.

Note that besides polarized endomorphisms, a partial answer to the conjecture was given in [11] for plane polynomial automorphisms.

Let \mathbb{k} be any field of characteristic zero. If required we fix an algebraic closure \mathbb{k}^{alg} of \mathbb{k} . Let $f: \mathbb{A}_{\mathbb{k}}^2 \rightarrow \mathbb{A}_{\mathbb{k}}^2$ be a polynomial self-map of the affine plane of degree $d \geq 2$, which in coordinates is written as

$$f(z, w) = (P(z, w), Q(z, w)),$$

with $P, Q \in \mathbb{k}[x, y]$. We say that f is *regular* if it extends to an endomorphism of $\mathbb{P}_{\mathbb{k}}^2$ of degree $d \geq 2$; this means that $P = P_d + \text{l.o.t.}$ and $Q = Q_d + \text{l.o.t.}$, where P_d and Q_d are homogeneous polynomials of degree d without common factors. In particular f induces a rational map $f_{\infty} := [P_d : Q_d]$ on the line at infinity, which is fixed. Note that any endomorphism of $\mathbb{P}_{\mathbb{k}}^2$ with a totally invariant line is conjugate to a regular polynomial map, and that a generic polynomial map of \mathbb{A}^2 whose components are polynomials of degree $\leq d$ is regular.

For regular polynomial maps of \mathbb{A}^2 , it seems that none of the known obstructions to the dynamical Manin-Mumford conjecture can arise. As said above, according to [18], one obstruction would be the existence of a periodic curve C for some endomorphism ψ commuting with f . Such pairs (f, ψ) were classified in [8] over \mathbb{C} (see also [9, 23]), and after a ramified cover they are all induced from a product map or a monomial map on the multiplicative 2-torus. Thanks to [17] it appears that in any such case, C must be also f -preperiodic. Thus we expect that the dynamical Manin-Mumford conjecture holds unconditionally in this case. Our main result confirms this expectation in the vast majority of cases.

Theorem A. *Let f be a regular polynomial endomorphism of \mathbb{A}^2 of degree $d \geq 2$, defined over an arbitrary field \mathbb{k} of characteristic 0.*

Let $C \subset \mathbb{A}^2$ be an irreducible algebraic curve containing infinitely many preperiodic points of f , and suppose that the closure of C in \mathbb{P}^2 contains a point $p \in L_{\infty}$ which is not eventually superattracting. Then C is preperiodic under f .

Corollary B. *Under the assumptions of the theorem, if f has no super-attracting points on the line at infinity, then any irreducible algebraic curve $C \subset \mathbb{P}^2$ containing infinitely many periodic points is preperiodic, that is, the dynamical Manin-Mumford conjecture holds for f .*

Our strategy relies on techniques from Arakelov geometry, in particular on the notion of canonical height, which is now classical in arithmetic dynamics, together with a variety of techniques from holomorphic and non-Archimedean dynamics.

We first assume that \mathbb{k} is a number field, in which case we prove a stronger statement (Theorem 3.1). We first recall in Section 1, that there exists a canonical height h_f on $\mathbb{A}^2(\mathbb{k}^{\text{alg}})$, for which preperiodic points are exactly points of zero height. By a theorem of Zhang, all points of C lying at infinity are preperiodic, hence, by replacing C by some iterate we may assume that they are periodic, and, thanks to our assumptions, one of these periodic points is not superattracting for f_∞ . Fix such a point $p \in \overline{C} \cap L_\infty$. A second consequence of Zhang's theorem is that for each place v of \mathbb{k} , the dynamical Green function $g_{f,v}(z, w) := \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|f^n(z, w)\|$ satisfies

$$(1) \quad \int_C g_{f,v} dd^c g_{f,v} = 0.$$

To make sense of this equality at a finite place, we consider the Berkovich analytification of C and $dd^c g_{f,v}$ is the Laplacian of the subharmonic function $g_{f,v}|_C$ in the sense of Thuillier, see [31].

For an appropriate choice of v , we may suppose that the periodic point p is either repelling or parabolic for f_∞ . In both cases, we construct in Section 2 a local (super-)stable manifold $W_{\text{loc}}^{\text{ss}}(p)$. When p is repelling and v is Archimedean this is classical. We extend this construction to the non-Archimedean setting, and allow for non-repelling p (see Theorem 2.1). Then, in both (repelling or parabolic) cases we establish graph transform type estimates which will be useful for the local analysis of the Green function at p . When p is parabolic this borrows from the work of Hakim [20].

In Section 3 we combine these estimates with condition (1) to show that, near p , C must locally coincide with $W_{\text{loc}}^{\text{ss}}(p)$, which finally implies that C is periodic. Note that a similar argument appears in the recent work [15].

A key point in this argument is that at the chosen place v , p belongs to the Julia set of f_∞ . This is no longer true when f_∞ is superattracting at p , and we are not able to conclude in this case. Still, this situation leads to interesting dynamical considerations and objects, and we plan to come back to this issue in a later work.

Finally, we develop a specialization argument in Section 4 to reduce Theorem A for an arbitrary \mathbb{k} to the case where \mathbb{k} is a number field. So here we rather deal with an algebraic family of endomorphisms of \mathbb{P}^2 parameterized by some algebraic variety S . The main issue is to ensure that for such a family, an infinite set of preperiodic points cannot shrink to a finite set too often on S . To do so, one needs to control collisions of periodic points (using the Shub-Sullivan theorem [29] in the spirit of [11]); and the splitting of local preimages of a super-attracting cycle, a phenomenon that was studied in particular by Chio and Roeder [7].

1. DYNAMICAL HEIGHTS

In this section, we recall some basic facts on canonical heights attached to endomorphisms of the projective plane defined over a number field. Our purpose is to establish Proposition 1.7, which is the key arithmetic geometry input in our main theorem.

Throughout this section, we assume that \mathbb{k} is a number field.

1.1. Vocabulary of number fields. We denote by $M_{\mathbb{k}} = \{v\}$ the set of places of \mathbb{k} , that is, the set of all multiplicative norms $|\cdot|_v$ on \mathbb{k} that restrict to either the standard euclidean norm $|\cdot|_{\infty}$, or to a p -adic norm $|\cdot|_p$ on \mathbb{Q} for some prime number $p > 1$. We normalize the p -adic norm by $|p|_p = p^{-1}$. We let \mathbb{k}_v be the completion of \mathbb{k} w.r.t. $|\cdot|_v$, and write $n_v := [\mathbb{k}_v : \mathbb{Q}_v]$. The product formula asserts that for any $a \in \mathbb{k}$, we have

$$\prod_{v \in M_{\mathbb{k}}} |a|_v^{n_v} = 1 .$$

The set $M_{\mathbb{k}}$ splits into the finite set $M_{\mathbb{k}}^{\infty}$ of Archimedean places (whose restriction to \mathbb{Q} is $|\cdot|_{\infty}$), and the set of finite (or non-Archimedean) places.

When $v \in M_{\mathbb{k}}^{\infty}$, the algebraic closure \mathbb{C}_v of \mathbb{k}_v is isometric to \mathbb{C} . When v is a finite place extending $|\cdot|_p$ on \mathbb{Q} , then $|\cdot|_v$ extends canonically to $\mathbb{k}_v^{\text{alg}}$, and its completion \mathbb{C}_v is both complete and algebraically closed.

1.2. Regular polynomial maps. Let (z, w) be affine coordinates on the affine plane $\mathbb{A}_{\mathbb{k}}^2$. We also consider homogeneous coordinates $[z_0 : z_1 : z_2]$ on the projective plane $\mathbb{P}_{\mathbb{k}}^2$ and identify the affine plane $\mathbb{A}_{\mathbb{k}}^2$ with the Zariski open set $z_0 \neq 0$ so that $z = z_1/z_0$ and $w = z_2/z_0$. We denote by $L_{\infty} = \{z_0 = 0\}$ the line at infinity.

Let $f: \mathbb{A}_{\mathbb{k}}^2 \rightarrow \mathbb{A}_{\mathbb{k}}^2$ be any polynomial self-map of the affine plane of degree $d \geq 2$ that extends to an endomorphism of $\mathbb{P}_{\mathbb{k}}^2$. In the coordinates z, w , it is given by

$$f(z, w) = (P(z, w), Q(z, w)),$$

where $P, Q \in \mathbb{k}[z, w]$ satisfy $\max\{\deg P, \deg Q\} = d$. The fact that f extends to a regular endomorphism $\bar{f}: \mathbb{P}_{\mathbb{k}}^2 \rightarrow \mathbb{P}_{\mathbb{k}}^2$ is equivalent to say that $P = P_d + \text{l.o.t.}$ and $Q = Q_d + \text{l.o.t.}$, where P_d and Q_d are homogeneous polynomials of degree d without common factors.

For $n \in \mathbb{N}$ we write $f^n(z, w) = (P_n(z, w), Q_n(z, w))$. The restriction of f_{∞} to L_{∞} is an endomorphism of $\mathbb{P}_{\mathbb{k}}^1$ given in homogeneous coordinates by

$$f_{\infty}([z_1 : z_2]) = [P_d(z_1, z_2) : Q_d(z_1, z_2)] .$$

1.3. Green functions. The next proposition follows from the Nullstellensatz (see e.g. [30, Theorem 3.11]).

Proposition 1.1. *For any $v \in M_{\mathbb{k}}$, there exists a constant $C_v \geq 1$ such that*

$$(2) \quad C_v^{-1} \leq \frac{\max\{1, |P(z, w)|_v, |Q(z, w)|_v\}}{\max\{1, |z|_v, |w|_v\}^d} \leq C_v$$

for all $z, w \in \mathbb{C}_v$. Moreover, for all but finitely many $v \in M_{\mathbb{k}}$ we may take $C_v = 1$.

By the previous proposition, the sequence of functions

$$g_{v,n}(z, w) := \frac{1}{d^n} \log \max\{1, |P_n(z, w)|_v, |Q_n(z, w)|_v\}$$

converges uniformly on \mathbb{C}_v^2 to a continuous function g_v , and the next proposition follows.

Proposition 1.2. *For any $v \in M_{\mathbb{k}}$, the function $g_v: \mathbb{C}_v^2 \rightarrow \mathbb{R}_+$ is continuous, it satisfies the invariance equation $g_v \circ f = dg_v$, and we have*

$$|g_v(z, w) - \log \max\{1, |z|_v, |w|_v\}| \leq \frac{\log C_v}{d-1}.$$

The set $\{(z, w) \in \mathbb{C}_v^2, g_v(z, w) = 0\}$ coincides with the set of points having bounded orbits.

We shall also consider the global Green function in \mathbb{C}_v^3 . Write $\tilde{P}(z_0, z_1, z_2) = z_0^d P(\frac{z_1}{z_0}, \frac{z_2}{z_0})$ and $\tilde{Q}(z_0, z_1, z_2) = z_0^d Q(\frac{z_1}{z_0}, \frac{z_2}{z_0})$ so that $F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q})$ is a homogenous map of degree d that lifts f to \mathbb{C}_v^3 . Observe that in homogenous coordinates, (2) can be rewritten as follows:

$$C_v^{-1} \leq \frac{\max\{|z_0|^d, |\tilde{P}(z_0, z_1, z_2)|_v, |\tilde{Q}(z_0, z_1, z_2)|_v\}}{\max\{|z_0|_v, |z_1|_v, |z_2|_v\}^d} \leq C_v$$

so that the next result also holds.

Proposition 1.3. *The function $G_v(z_0, z_1, z_2) = g_v(z_1/z_0, z_2/z_0) + \log |z_0|$ is continuous on $\mathbb{C}_v^3 \setminus \{0\}$, 1-homogeneous (that is, $G_v(\lambda Z) = \log |\lambda| + G_v(Z)$), and satisfies $G_v \circ F = dG_v$. We have*

$$|G_v(z_0, z_1, z_2) - \log \max\{|z_0|_v, |z_1|_v, |z_2|_v\}| \leq \frac{\log C_v}{d-1}.$$

The set $\{(z_0, z_1, z_2) \in \mathbb{C}_v^3, G_v(z_0, z_1, z_2) \leq 0\}$ coincides with the set of points having bounded F -orbits.

1.4. Canonical heights on points. We refer to [6] for generalities on heights. Consider the line bundle on $\mathcal{O}(1) \rightarrow \mathbb{P}_{\mathbb{k}}^2$. The space of sections of this line bundle can be canonically identified with the space of linear forms $a_0 z_0 + a_1 z_1 + a_2 z_2$ with $a_i \in \mathbb{k}$. More precisely, in the trivialization of the bundle over $\{z_i \neq 0\}$, this section is given by $\frac{1}{z_i} a_0 z_0 + a_1 z_1 + a_2 z_2$.

For any $v \in M_{\mathbb{k}}$, we consider the line bundle $\mathcal{O}(1) \rightarrow \mathbb{P}_{\mathbb{C}_v}^2$ and endow it with the metrization $|\cdot|_v$ induced by G_v , in the sense that any section $\sigma = a_0 z_0 + a_1 z_1 + a_2 z_2$ as above

$$(3) \quad |\sigma|_v([z_0 : z_1 : z_2]) = |a_0 z_0 + a_1 z_1 + a_2 z_2| e^{-G_v(z_0, z_1, z_2)}$$

(this expression is well-defined thanks to the homogeneity property of G_v). Let us now explain how this collection of metrizations defines a function on the set of algebraic points in \mathbb{P}^2 as well as on the set of all algebraic curves in $\mathbb{P}_{\mathbb{k}}^2$ defined by an equation with coefficients in \mathbb{k}^{alg} .

For any $p \in (\mathbb{k}^{\text{alg}})^3 \setminus \{0\}$, we set

$$h_f(p) := \frac{1}{N(p)} \sum_{p' \in \text{Gal} \cdot p} \left(\sum_{v \in M_{\mathbb{k}}} n_v G_v(p') \right)$$

where Gal denotes the absolute Galois group of \mathbb{k}^{alg} over \mathbb{k} , and $N(x)$ is the cardinality of the set $\text{Gal} \cdot x \subset (\mathbb{k}^{\text{alg}})^3$.

The product formula entails that $h_f(z_0, z_1, z_2) = h_f(\lambda z_0, \lambda z_1, \lambda z_2)$ for any $\lambda \in \mathbb{k}^{\text{alg}}$ so that we have a well-defined function $h_f: \mathbb{P}^2(\mathbb{k}^{\text{alg}}) \rightarrow \mathbb{R}$.

Proposition 1.4. *The function h_f takes non-negative values, and satisfies $h_f \circ f = dh_f$. The set $\{h_f = 0\}$ coincides with the set of preperiodic points of f . Furthermore, for any $(z, w) \in (\mathbb{k}^{\text{alg}})^2$ we have*

$$h_f(z, w) := \frac{1}{N(z, w)} \sum_{(z', w') \in \text{Gal} \cdot (z, w)} \left(\sum_{v \in M_{\mathbb{k}}} n_v g_v(z', w') \right).$$

As above Gal denotes the absolute Galois group of \mathbb{k}^{alg} over \mathbb{k} , and $N(z, w)$ is the cardinality of the set $\text{Gal} \cdot (z, w) \subset (\mathbb{k}^{\text{alg}})^2$. The proof follows directly from Northcott's theorem, see [4, Corollary 1.1.1].

1.5. Analytification of affine curves. Let C be any irreducible algebraic curve in $\mathbb{A}_{\mathbb{k}}^2$ defined by an equation $\{R = 0\}$ with $R \in \mathbb{k}[z, w]$. Denote by \overline{C} the Zariski closure of C in \mathbb{P}^2 .

Fix any place $v \in M_{\mathbb{k}}$. We denote by C_v^{an} the analytification in the sense of Berkovich of C over \mathbb{C}_v . This is a connected, locally connected and locally compact space. When $|\cdot|_v$ is Archimedean, hence \mathbb{C}_v is isometric to \mathbb{C} , C_v^{an} is the complex analytic subspace (possibly with singularities) defined as usual by the vanishing of R in \mathbb{C}_v^2 . When $|\cdot|_v$ is non-Archimedean, then C_v^{an} is defined as the set of multiplicative semi-norms on the ring $\mathbb{C}_v[z, w]/(R)$ whose restriction to the base field equals $|\cdot|_v$. A point is said to be rigid when the semi-norm has non-trivial kernel.

One can also define the analytification of \overline{C} by considering suitable affine coordinates in \mathbb{P}^2 and patching the previous construction in a natural way, see [3, §3.4]. Observe that $\overline{C}_v^{\text{an}} \setminus C_v^{\text{an}}$ consists of rigid points.

Suppose first v is Archimedean. The metrization of $\mathcal{O}(1)$ defined by (3) induces a measure $\mu_{C,v}$ on $\overline{C}_v^{\text{an}}$ which is locally defined by $\mu_{C,v} := \Delta \log |\sigma|_v$ where σ is a local section of $\mathcal{O}(1)$. The plurisubharmonicity of G_v ensures that $\mu_{C,v}$ is a positive measure. The Lelong-Poincaré formula implies that the mass of $\mu_{C,v}$ is equal to $\deg(C)$, and we have $\mu_{C,v} = \Delta(g_v|_{C_v^{\text{an}}})$ on C_v^{an} . Observe that since G_v is continuous, $\mu_{C,v}$ gives no mass to points.

The construction is completely analogous in the non-Archimedean case. We again obtain a positive measure $\mu_{C,v}$ on $\overline{C}_v^{\text{an}}$ of total mass $\deg(C)$ which is given in C_v^{an} by $\mu_{C,v} = \Delta(g_v|_{C_v^{\text{an}}})$ where Δ is the Laplace operator defined by Thuillier [31]. Observe that the continuity of the metrization implies that $\mu_{C,v}$ does not charge any rigid point (but it may still charge some non-rigid point in C_v^{an}). We refer to [6, §1.3] for the details of the constructions.

1.6. Canonical heights on curves. Let C be any irreducible algebraic curve in $\mathbb{A}_{\mathbb{k}}^2$ as in the previous section. We now define the canonical height of the curve \overline{C} following the recipe given in [6, §3.1.2], taking z_0 as a section of $\mathcal{O}(1)$ (note that this section vanishes exactly along the line at infinity). We obtain:

$$(4) \quad h_f(\overline{C}) := \sum_{p \in \overline{C} \cap L_{\infty}} (\overline{C}, L_{\infty})_p \times h_f(p) + \sum_{v \in M_{\mathbb{k}}} \int_{C_v^{\text{an}}} g_v d\mu_v.$$

Note that $h_f(\overline{C}) \geq 0$ because the canonical height is non-negative on closed points, and the Green functions g_v are also non-negative.

Define the essential minimum of h_f by the following formula:

$$\text{essmin}_C(h_f) := \sup_{F \text{ finite} \subset \overline{C}(\mathbb{k}^{\text{alg}})} \inf_{\overline{C}(\mathbb{k}^{\text{alg}}) \setminus F} h_f .$$

Theorem 1.5 (Zhang's inequality [32, Theorem 1.10]). *We have*

$$2 \text{essmin}_C(h_f) \geq \frac{h_f(\overline{C})}{\deg(C)} \geq \text{essmin}_C(h_f) + \inf_{p \in \overline{C}(\mathbb{k}^{\text{alg}})} h_f(p) .$$

Since $h_f(x) = 0$ if and only if x is preperiodic, we obtain:

Corollary 1.6. *An irreducible algebraic curve C containing infinitely many f -preperiodic points satisfies $h_f(\overline{C}) = 0$.*

1.7. A first characterization of special curves.

Proposition 1.7. *Suppose that $C \subset \mathbb{A}_{\mathbb{k}}^2$ is an irreducible algebraic curve containing a sequence of distinct points $p_n \in C(\mathbb{k}^{\text{alg}})$ such that $h_f(p_n) \rightarrow 0$.*

Then all points in $\overline{C} \cap L_{\infty}$ are preperiodic for \overline{f} , and for any $v \in M_{\mathbb{k}}$ the function $g_v|_{C_v^{\text{an}}}$ is harmonic on $\{g_v > 0\}$.

Proof. Note that $h_f(\overline{C}) \geq 0$. By Theorem 1.5, our assumption implies that $\text{essmin}_C(h_f) \leq 0$, therefore $h_f(\overline{C}) = 0$. Then the result follows from (4) and the fact that a point p is f -preperiodic if and only if $h_f(p) = 0$. \square

2. SUPER-STABLE MANIFOLDS AND LOCAL ESTIMATES

2.1. Construction of super-stable manifolds. In this section we work under the following hypothesis: $(\mathbb{k}, |\cdot|)$ is a complete metrized field of characteristic 0 (which may be either Archimedean or non-Archimedean).

Theorem 2.1. *Suppose $f: (\mathbb{A}_{\mathbb{k}}^2, 0) \rightarrow (\mathbb{A}_{\mathbb{k}}^2, 0)$ is a germ of analytic map fixing the origin of the form*

$$(5) \quad f(x, y) = \left(\lambda x + \mu y + g(x, y), y^d(1 + h(x, y)) \right) ,$$

where $d \geq 2$, $\lambda \neq 0$, $\mu \in \mathbb{k}$, $h(0, 0) = 0$, and $g(x, y) = \mathcal{O}(|(x, y)|^2)$. Then there exists a unique smooth analytic curve which is transverse to $\{y = 0\}$ and f -invariant.

We shall call this curve the *local super-stable manifold* of the origin, and denote it by $W_{\text{loc}}^{\text{ss}}(0)$. After a linear change of coordinates of the form $(x, y) \mapsto (x + \frac{\mu}{\lambda}y, y)$, we may and will assume from now on that $\mu = 0$. Expressing the invariant curve as a graph of the form $x = \varphi(y)$ and making a change of coordinates of the form $(x, y) \mapsto (x - \varphi(y), y)$, f takes the form

$$(6) \quad f(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y))) .$$

It follows that f is analytically conjugate to $y \mapsto y^d$ on $W_{\text{loc}}^{\text{ss}}(0)$, hence the terminology.

The result is classical when \mathbb{k} is Archimedean and/or f is locally invertible (see e.g. [21, Appendix]). For convenience we include a proof that works simultaneously in the Archimedean and non-Archimedean settings, and is adapted to $\{y = 0\}$ being superattracting.

Proof. As explained above, we look for an analytic map $y \mapsto \varphi(y)$, with $\varphi(0) = 0$ such that the change of coordinates $\Phi(x, y) = (x + \varphi(y), y)$ satisfies

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y)))$$

with \tilde{g}, \tilde{h} analytic and vanishing at 0. This property is equivalent to the identities:

$$\begin{cases} \lambda\varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + \tilde{h}(0, y))) \\ \tilde{h}(0, y) = h(\varphi(y), y) \end{cases}$$

so that we aim at finding some analytic function φ satisfying

$$\lambda\varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + h(\varphi(y), y))) .$$

For any $r > 0$, let us introduce the Banach space \mathcal{B}_r that consist of those power series $\varphi(y) := \sum_{j \geq 1} a_j y^j$ which are convergent in the disk of radius r , and satisfy $\|\varphi\|_r := \sup_{|y| < r} |\varphi(y)| < \infty$. Note that in the non-Archimedean case, we have $\|\varphi\|_r := \sup_j |a_j| r^j$.

For any $\varphi \in \mathcal{B}_r$, we set

$$T\varphi(y) := \frac{1}{\lambda} \left(\varphi(y^d(1 + h(\varphi(y), y))) - g(\varphi(y), y) \right) .$$

We claim that for $r > 0$ and $\rho > 0$ sufficiently small, T is a well-defined strictly contracting map on $B(0, \rho) \subset \mathcal{B}_r$. Then, applying the Banach fixed point theorem implies the existence of the desired φ .

First, we may suppose that g is analytic in the polydisk of radius r , and since g vanishes up to order 2 at the origin, we have

$$|g(x, y)| \leq C \max\{|x|, |y|\}^2$$

for some $C > 0$ and all $|x|, |y| < r$. Similarly, we may suppose that h is analytic in the polydisk of radius $r > 0$, and that $|h(x, y)| \leq \frac{1}{2}$ for all $|x|, |y| < r$.

Pick any $\varphi(y) = \sum_j a_j y^j \in B(0, \rho) \subset \mathcal{B}_r$. Reduce $r > 0$ if necessary so that $\frac{3}{2}r^d < r$. Then $\tilde{\varphi} : y \mapsto \varphi(y^d(1 + h(\varphi(y), y)))$ is well-defined and analytic on the disk of radius r .

In the non-Archimedean case, $|1 + h(\varphi(y), y)| = 1$, so that one has

$$\|\tilde{\varphi}(y)\|_r = \sup_{j \geq 1} |a_j| r^{dj} \leq r \sup_{j \geq 1} |a_j| r^{dj-1} \leq r \|\varphi\|_r .$$

In the Archimedean case, the Schwarz lemma yields $|\varphi(y)| \leq \frac{|\varphi|_r}{r} |y|$ for all $|y| < r$, hence $\|\tilde{\varphi}\|_r \leq \frac{3}{2} r^{d-1} \|\varphi\|_r$. Note that we also have:

$$\|g(\varphi(y), y)\|_r \leq C \max\{\|\varphi\|_r, r\}^2$$

so for $\varphi \in B(0, \rho)$ we deduce

$$|T\varphi|_r \leq \frac{1}{\lambda} \left(\frac{3}{2} r^{d-1} \rho + C \max\{\rho, r\}^2 \right) .$$

By choosing $\rho = r$ and then r small enough, this estimate shows that $T\varphi$ is well-defined on the ball $B(0, r) \subset \mathcal{B}_r$ and $T(B(0, r)) \subset B(0, r)$.

In order to prove that T is strictly contracting, observe that we can write

$$g(x, y) - g(x', y) = (x - x')\hat{g}(x, x', y)$$

where $\hat{g}(x, x', y)$ is again analytic in the polydisk of radius r , and

$$|\hat{g}(x, x', y)| \leq C' \max\{|x|, |x'|, |y|\}$$

for some constant $C' > 0$. For any pair of analytic functions $\varphi_1, \varphi_2 \in B(0, r) \subset \mathcal{B}_r$ we infer:

$$|g(\varphi_1(y), y) - g(\varphi_2(y), y)| \leq C' \|\varphi_1 - \varphi_2\|_r \max\{\|\varphi_1\|_r, \|\varphi_2\|_r, r\}$$

hence

$$\|T\varphi_1 - T\varphi_2\|_r \leq \frac{1}{\lambda}(3r^d + C'r) \|\varphi_1 - \varphi_2\|_r .$$

Again, by choosing r sufficiently small, we obtain that T is strictly contracting and we are done. \square

2.2. The rescaling argument in the repelling case. We work in $\mathbb{A}_{\mathbb{k}}^2$ where \mathbb{k} is an arbitrary complete metrized field of characteristic 0. We start with a preparation lemma.

Lemma 2.2. *Suppose f is an analytic map of the form*

$$(7) \quad f(x, y) = \left(\lambda x + xg(x, y), y^d(1 + h(x, y)) \right)$$

where $|\lambda| > 1$, $d \geq 2$ and $g(0) = h(0) = 0$.

Then there exists an analytic change of coordinates Φ such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x(1 + xy\tilde{g}(x, y)), y^d(1 + x\tilde{h}(x, y)))$$

for some analytic functions \tilde{g}, \tilde{h} .

Recall that the form (7) is what is obtained from (5) after conjugating to get $\mu = 0$ and declaring that the stable manifold of Theorem 2.1 is $\{x = 0\}$.

Proof. By Böttcher's theorem (see [2, Chapter 4] for the non-Archimedean case) applied to $y \mapsto f(0, y)$ we may suppose that x divides h . Similarly, since $|\lambda| > 1$, by we may linearize $x \mapsto f(x, 0)$, and suppose that f is of the form $f(x, y) = (\lambda x(1 + g_1(y)) + x^2 y h_1(x, y), y^d(1 + \mathcal{O}(x)))$ for some analytic functions g_1, h_1 with $g_1(0) = 0$.

We claim that there exists $\Phi(x, y) = (x(1 + \varphi(y)), y)$ with $\varphi(0) = 0$ such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x^2 y h_2(x, y), y^d(1 + \mathcal{O}(x)))$$

for some analytic function h_2 . Indeed, φ is then characterized by the equation

$$\lambda x(1 + \varphi(y))(1 + g_1(y)) + \mathcal{O}(x^2) = (\lambda x + \mathcal{O}(x^2))(1 + \varphi(y^d))$$

that is, $(1 + \varphi(y))(1 + g_1(y)) = (1 + \varphi(y^d))$, a solution of which is given by the infinite product

$$1 + \varphi(y) = \prod_{k=0}^{\infty} \left(1 + g_1 \left(y^{d^k} \right) \right)^{-1}$$

and we are done. \square

The next result is similar to [11, Lemma 4.2].

Proposition 2.3. *Suppose f is an analytic map of the form*

$$(8) \quad f(x, y) = (\lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y)))$$

where $|\lambda| > 1$, $d \geq 2$ and g, h are analytic functions.

Then $f^n(\frac{x}{\lambda^n}, y) \rightarrow (x, 0)$ when $n \rightarrow \infty$, uniformly on a polydisk of sufficiently small radius centered at the origin.

Proof. Fix $0 < r \leq 1/4$ small enough so that g, h are both analytic on the polydisk of radius r and $|g(x, y)|, |h(x, y)| \leq 1$ for $|x|, |y| < r$. Let us first show that if $|x| \leq \frac{r}{2|\lambda|^n}$ and $|y| \leq r$, then the n first iterates of (x, y) remain in \mathbb{D}_r^2 . We argue by induction. So assume that $(x_0, y_0) \in \mathbb{D}_{r|\lambda|^{-n}/2} \times \mathbb{D}_r$, put $f^j(x_0, y_0) = (x_j, y_j)$, let $k \leq n$ and assume that $(x_j, y_j) \in \mathbb{D}_r^2$ for $j \leq k-1$. Observe that for $j \leq k-1$, $|y_{j+1}| \leq 2|y_j|^d$ from which it follows that

$$|y_k| \leq \left(2^{1/(d-1)}|y_0|\right)^{d^k} \leq \left(2^{1/(d-1)}r\right)^{d^k} \leq r.$$

Observe that the first inequality together with $r \leq 1/4$ also yield $|y_j| \leq 2^{-d^j}$. For the first coordinate, we use recursively the relation $x_{j+1} = \lambda x_j(1 + x_j y_j g(x_j, y_j))$ to get

$$(9) \quad |x_k| \leq |\lambda|^k |x_0| \prod_{j=0}^{k-1} (1 + |x_j| |y_j|) \leq \frac{r}{2} |\lambda|^{k-n} \prod_{j=0}^{k-1} (1 + 2^{-d^j}) \leq r |\lambda|^{k-n},$$

where the last inequality follows from

$$\prod_{j=0}^{k-1} (1 + 2^{-d^j}) \leq \prod_{j=0}^{k-1} (1 + 2^{-2^j}) = \frac{2^{2^k} - 1}{2^{2^k - 1}} < 2,$$

in which the middle equality is easily obtained by induction.

Now take $(x, y) \in \mathbb{D}_{r/2}$, and consider $f^n(\frac{x}{\lambda^n}, y)$. Denote as before $(x_0, y_0) = (\frac{x}{\lambda^n}, y)$ and $(x_j, y_j) = f^j(x_0, y_0)$. The first part of the proof shows that (x_j, y_j) is well-defined for all $j \leq n$, and that $y_n \rightarrow 0$. Now we have

$$x_n = \lambda^n x_0 \prod_{j=0}^{n-1} (1 + x_j y_j h(x_j, y_j)) = x \prod_{j=0}^{n-1} (1 + x_j y_j h(x_j, y_j)).$$

The inequality $|\prod(1+z_j) - 1| \leq \exp(\sum|z_j|) - 1$ shows that to establish the convergence $x_n \rightarrow x$ it is enough to show that $\sum_{j=0}^{n-1} |x_j y_j h(x_j, y_j)|$ tends to 0. But by (9), $|x_j| \leq r |\lambda|^{j-n}$, thus

$$\sum_{j=0}^{n-1} |x_j y_j h(x_j, y_j)| \leq \sum_{j=0}^{n-1} r |\lambda|^{j-n} 2^{-d^j} \leq r |\lambda|^{-n} \sum_{j=0}^{\infty} |\lambda|^j 2^{-d^j},$$

and we are done. \square

2.3. Graph transform for $\lambda = 1$. In this paragraph we assume that $\mathbb{k} = \mathbb{C}$ and f is of the form

$$(10) \quad f(x, y) = \left(x + g(x, y), y^d(1 + h(x, y)) \right),$$

with $g(x, y) = \mathcal{O}(|x, y|^2)$, $h(0, 0) = 0$, and $g(x, 0) = cx^{k+1} + \mathcal{O}(x^{k+2})$ for some $k \geq 1$ and $c \neq 0$. Observe that $f|_{\{y=0\}}$ has a parabolic point at the origin with k attracting and k repelling petals (see e.g. [1, §6.5]). An *attracting petal* is a f -invariant (connected and simply-connected) open subset U containing 0 in its boundary, and such that, for all $z \in U$, $f^n(z) \rightarrow 0$ tangentially to some real direction (in our case, to the normalized k -th roots of $-c$). A *repelling petal* is an attracting petal for f^{-1} (they are tangent to the normalized k -th roots of c). One can choose the k attracting petals and k repelling petals so that their union fills up a punctured neighborhood of the origin.

A vertical graph V in a domain of the form $\Omega \times \mathbb{D}_\rho$ is a submanifold of the form $V := \{(\varphi(y), y), y \in \mathbb{D}_\rho\}$ for some holomorphic function $\varphi : \mathbb{D}_\rho \rightarrow \Omega$. In the next theorem we consider pull backs of such graphs in \mathbb{D}_r^2 in the graph transform sense, that is, when pulling back some vertical graph under f , we keep only the component of $f^{-1}(V) \cap \mathbb{D}_r^2$ containing $f^{-1}(V \cap \{y = 0\})$. Abusing notation we simply denote it by $f^{-1}(V)$.

Theorem 2.4. *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be of the form (10) with $h(0) = 0$, $g(x, 0) \neq 0$ and $g(x, y) = \mathcal{O}(|(x, y)|^2)$. Let U be any repelling petal of $f|_{\{y=0\}}$ and consider a germ V of analytic curve transverse to $\{y = 0\}$, intersecting U .*

Then there exists $r > 0$ depending only on f , such that for large enough n , the analytic sets $f^{-n}(V)$ are vertical graphs in \mathbb{D}_r^2 converging to $W_{\text{loc}}^{\text{ss}}(0)$ in the C^1 topology.

Lemma 2.5. *Suppose f is a holomorphic map of the form (10) as in Theorem 2.4. Then there exist an integer $k \geq 1$, and an analytic change of coordinates Φ such that*

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (x + x^{k+1} + x^{2k+1}\tilde{g}(x, y), y^d(1 + x\tilde{h}(x, y)))$$

for some analytic functions \tilde{g}, \tilde{h} .

Proof. The proof is essentially contained in [20, Proposition 2.3]. We provide the details for the sake of completeness. By Theorem 2.1 and by applying the Böttcher theorem to $y \mapsto f(0, y)$ we may assume that both g and h are divisible by x , so that we may write

$$(11) \quad f(x, y) = \left(x + xg(x, y), y^d(1 + xh(x, y)) \right),$$

with $g(0, 0) = 0$.

By a local change of coordinates involving only x , we can arrange so that $f(x, 0) = (x + x^{k+1} + \mathcal{O}(x^{2k+1}), 0)$. Expand the first coordinate of f in power series of x as follows:

$$x \circ f(x, y) = x(1 + g_0(y)) + x^{k+1}(1 + g_k(y)) + \sum_{j \neq 0, k}^{\infty} x^{j+1}g_j(y),$$

with $g_j(0) = 0$ for $0 \leq j \leq 2k - 1$.

We claim that for all $n \leq 2k - 1$, we can conjugate f by a germ of invertible holomorphic map such that $g_j \equiv 0$ for every $j \leq n$. Applied to $n = 2k - 1$, this claim implies the proposition.

For $n = 0$ this is done by a change of coordinates of the form $\Phi_0(x, y) := (x(1 + \varphi_0(y)), y)$ such that $(1 + g_0)(1 + \varphi_0) = 1 + \varphi_0(y^d)$. This equation can be solved exactly as in Lemma 2.2.

Now assume that $n > 0$, and that the result has been achieved up to $j = n - 1$. Put $\Phi_n(x, y) = (x(1 + \varphi_n(y)x^n), y)$, so that $\Phi_n^{-1}(x, y) = (x(1 - \varphi_n(y)x^n + \mathcal{O}(x^{n+1})), y)$. Depending on the position of n and k , we obtain:

$$x \circ (\Phi_n^{-1} \circ f \circ \Phi_n) = \begin{cases} x + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + \mathcal{O}(x^{n+2}) & \text{if } n < k, \\ x + x^{n+1}(\varphi_k(y) + 1 + g_n(y) - \varphi_n(y^d)) + \mathcal{O}(x^{n+2}) & \text{if } n = k, \\ x + x^{k+1} + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + \mathcal{O}(x^{n+2}) & \text{if } n > k. \end{cases}$$

Therefore, to render the term in x^{n+1} constant, it is enough to solve the equation $-g_n(y) = \varphi_n(y) - \varphi_n(y^d)$ which can be done by setting $\varphi_n(y) = -\sum_{m=0}^{\infty} g_n(y^{d^m})$. \square

Proof of Theorem 2.4. By the previous lemma, we may suppose that

$$f(x, y) = \left(x + x^{k+1} + x^{2k+1}g(x, y), y^d(1 + xh(x, y)) \right)$$

with g and h holomorphic near the origin. Note that $f|_{y=0}$ has a repelling petal along the positive real axis. Fix $r > 0$ such that f is holomorphic and injective on a neighborhood of $\overline{\mathbb{D}}_r^2$. Reducing and rotating the petal if necessary, we may assume that

$$U := \left\{ x : \arg(x) \in \left(-\frac{\pi}{4k}, \frac{\pi}{4k} \right), |x| < r \right\}.$$

The holomorphic map $z = (kx^k)^{-1}$ is univalent on $U \times \mathbb{D}_r$, and takes its values in

$$\Omega_R := \left\{ z : \arg(z) \in \left(-\frac{\pi}{4}, \frac{\pi}{4} \right), |z| > R \right\},$$

where $R = (kr^k)^{-1}$. The expression of f in the coordinates (z, y) is of the form

$$(12) \quad f(z, y) = \left(z - 1 + \frac{1}{z}a(z, y), y^d \left(1 + \frac{1}{z^{1/k}}b(z, y) \right) \right).$$

so that f is now defined in $\Omega_R \times \mathbb{D}_r$. Fix $M > 0$ such that

$$(13) \quad |a|, |b|, \left| \frac{\partial a}{\partial z} \right|, \left| \frac{\partial a}{\partial y} \right|, \left| \frac{\partial b}{\partial z} \right|, \left| \frac{\partial b}{\partial y} \right| \leq M \text{ on } \Omega_R \times \mathbb{D}_r,$$

and reduce r if necessary so that $r < \frac{1}{10d}$ and $MR^{-1/k} \leq \frac{1}{100}$.

For any $\rho < r$ and $\sigma > 0$ we let

$$\mathcal{G}(\rho, \sigma) = \left\{ \varphi : \mathbb{D}_\rho \rightarrow \Omega_R \text{ holomorphic s.t. } \sup_{\mathbb{D}_\rho} |\varphi'| \leq \sigma \right\}.$$

Lemma 2.6. *Suppose that $\sigma\rho < \frac{1}{100d}$. For any vertical graph Γ determined by $\varphi \in \mathcal{G}(\rho, \sigma)$, then $f^{-1}\Gamma$ is a vertical graph determined by a function $\psi \in \mathcal{G}(\rho_1, \frac{1}{10})$ where $\rho_1 = \min((\rho/2)^{1/d}, r)$ and $\text{Re}(\psi) \geq \text{Re}(\varphi(0)) + 9/10$.*

Assuming this result for the moment, let us conclude the proof of the theorem. Let V be any germ of curve intersecting transversally $\{y = 0\}$ at $(x_0, 0) \in U$. Then V is a graph of slope σ over some disk \mathbb{D}_ρ in the second coordinate, for some $\rho > 0$. Reduce ρ by trimming V if necessary so that $\sigma\rho < 1/100d$. Since $r/10 < 1/100d$, the previous

lemma implies that we can define inductively a sequence of vertical graphs $V = V_0$, $V_n := f^{-1}V_{n-1}$, where V_n is defined by a holomorphic function $z = \varphi_n(y)$ of uniformly bounded slope over \mathbb{D}_{ρ_n} , and furthermore $\rho_n = r$ for large enough n . Moreover, we have $\operatorname{Re}(\varphi_n) \geq \operatorname{Re}(\varphi(0)) + 9n/10$, hence, coming back to the (x, y) coordinates, we see that $V_n = \{x = (k\varphi_n(y))^{-1/k}\}$ converges in the C^1 -topology to the curve $W_{\text{loc}}^{\text{ss}}(0) = \{x = 0\}$. \square

Proof of Lemma 2.6. Let Γ be a vertical graph of equation $z = \varphi(y)$ in the (z, y) coordinates, with $\varphi \in \mathcal{G}(\rho, \sigma)$. Then the equation of $f^{-1}\Gamma$ is given by $z = \ell(z, y)$, where

$$\ell(z, y) = \varphi \left(y^d \left(1 + \frac{1}{z^{1/k}} b(z, y) \right) \right) + 1 - \frac{1}{z} a(z, y).$$

Fix $y_0 \in \mathbb{D}_{\rho_1}$. We show that the equation $z = \ell(z, y_0)$ admits a unique solution $z \in \Omega_R$. First, observe that by the estimates (13) on $|a|$ and $|b|$, for $z \in \Omega_R$ we have

$$(14) \quad \begin{aligned} |\ell(z, y_0) - (\varphi(0) + 1)| &\leq \left| \varphi \left(y_0^d \left(1 + \frac{1}{z^{1/k}} b(z, y_0) \right) \right) - \varphi(0) \right| + \frac{M}{R} \\ &\leq 2\rho_1^d \sigma + \frac{1}{100} \leq \rho\sigma + \frac{1}{100} \leq \frac{1}{10}, \end{aligned}$$

hence $\ell(\cdot, y_0)$ maps Ω_R to itself. Next, we see that

$$\begin{aligned} \left| \frac{\partial \ell}{\partial z}(z, y_0) \right| &\leq \left| y_0^d \left(\frac{1}{z^{1/k}} \frac{\partial b}{\partial z} - \frac{1}{kz^{1/k+1}} b \right) \right| \cdot \left| \varphi' \left(y_0^d \left(1 + \frac{1}{z^{1/k}} b \right) \right) \right| + \left| \frac{a}{z^2} - \frac{1}{z} \frac{\partial a}{\partial z} \right| \\ &\leq \rho_1^d \frac{2M}{R^{1/k}} \sigma + \frac{2M}{R} \leq \frac{4}{100} \rho\sigma + \frac{2}{100} \leq \frac{1}{10}, \end{aligned}$$

so $\ell(\cdot, y_0)$ is a contraction and the equation $z = \ell(z, y_0)$ has a unique solution. This means that $f^{-1}\Gamma$ is a vertical graph over \mathbb{D}_{ρ_1} determined by a holomorphic function ψ satisfying $\psi(y) = \ell(\psi(y), y)$. The slope of this graph can be estimated as above:

$$|\psi'(y)| \leq \left| \frac{\partial \ell / \partial y}{1 - \partial \ell / \partial z} \right| \leq \frac{10}{9} \left(\sigma \left(d\rho^{d-1} \left(1 + \frac{M}{R^{1/k}} \right) + \rho^d \frac{M}{R^{1/k}} \right) + \frac{M}{R} \right) \leq \frac{1}{10}.$$

Finally, the estimate $\operatorname{Re}(\psi) \geq \operatorname{Re}(\varphi(0)) + 9/10$ follows from (14) and we are done. \square

3. PROOF OF THEOREM A WHEN \mathbb{k} IS A NUMBER FIELD

Here we establish the following more precise form of our main theorem, in the number field case.

Theorem 3.1. *Let \mathbb{k} be a number field and f be a regular polynomial map of $\mathbb{A}_{\mathbb{k}}^2$. Denote by h_f the induced canonical height.*

Suppose that $C \subset \mathbb{A}_{\mathbb{k}}^2$ is an irreducible algebraic curve containing a sequence of distinct points $p_n \in C(\mathbb{k}^{\text{alg}})$ such that $h_f(p_n) \rightarrow 0$. If there exists a point of $\overline{C} \cap L_{\infty}$ which is not eventually superattracting, then C is preperiodic.

Let f_{∞} be the restriction to the line at infinity L_{∞} of the extension of f to \mathbb{P}^2 . By Proposition 1.7, all points in $\overline{C} \cap L_{\infty}$ are preperiodic, so we may replace f by f^N and C by $f^N(C)$, to assume that $\overline{C} \cap L_{\infty}$ contains a fixed point p which is not super-attracting.

Let $\lambda = f'_\infty(p)$ be the multiplier of p along L_∞ . Then one of the two following mutually disjoint cases occur:

- (a) either λ is a root of unity;
- (b) or there is a place $v \in M_{\mathbb{k}}$ such that $|\lambda|_v > 1$.

In the remainder of this section we split the proof of the theorem according to these two cases.

3.1. When λ is a root of unity. In this situation we iterate f further so that $\lambda = 1$, and work over the complex numbers. Since p belongs to the Julia set $J(f_\infty)$, which is a perfect set, the union of attracting and repelling petals cover a punctured neighborhood of p , and the attracting petals of p are contained in the Fatou set, we see that there is a repelling periodic point q of f_∞ contained in some local repelling petal of p . Then the local (super-)stable manifold of q is a disk transverse to L_∞ at q , and by Theorem 2.4, the local truncated pull-backs $f^{-n}(W_{\text{loc}}^{\text{ss}}(q))$ under f^n converge to $W_{\text{loc}}^{\text{ss}}(p)$ when $n \rightarrow \infty$.

Assume by way of contradiction that \bar{C} does not locally coincide with $W_{\text{loc}}^{\text{ss}}(p)$. We claim that C intersects $f^{-n}(W_{\text{loc}}^{\text{ss}}(q))$ in \mathbb{C}^2 close to p for large n . Indeed, locally near p , $\bar{C} \cap W_{\text{loc}}^{\text{ss}}(p) = \{p\}$, so by the persistence of proper intersections, \bar{C} intersects $f^{-n}(W_{\text{loc}}^{\text{ss}}(q))$ close to p for large n . But $p \notin f^{-n}(W_{\text{loc}}^{\text{ss}}(q)) \cap L_\infty$, so these intersection points lie in \mathbb{C}^2 , as claimed. If now Δ is a small disk in C containing one of these intersection points, then by the Inclination Lemma, the derivative of f^n in the direction of Δ tends to infinity, thus $(f^n|_\Delta)$ is not a normal family. On the other hand, by Proposition 1.7 $g|_\Delta$ is harmonic, which implies that $(f^n|_\Delta)$ is normal (see [13, Prop 5.10]). This contradiction shows that \bar{C} locally coincides with $W_{\text{loc}}^{\text{ss}}(p)$ near p , so it is fixed under f , and by irreducibility this property propagates to the whole of C . This completes the proof in this case. \square

Remark 3.2. This argument works essentially the same when $|\lambda|_v > 1$ at some Archimedean place, and may help understand the non-Archimedean argument below.

3.2. When $|\lambda|_v > 1$. We may assume that $p = [0 : 0 : 1]$, and by Lemma 2.2 find a local analytic isomorphism $(x, y) \mapsto \psi(x, y) = [z_0(x, y) : z_1(x, y) : 1]$ such that $\psi(0) = p$, $z_0(x, 0) = 0$ so that $\{y = 0\}$ corresponds to the line at infinity¹, and write

$$\tilde{f} := \psi^{-1} \circ f \circ \psi : (x, y) \mapsto (\lambda x(1 + xyg(x, y)), y^d(1 + xh(x, y))) .$$

If \bar{C} has an analytic branch at p which coincides with $\{x = 0\}$ then C is fixed as above, and we are done. Otherwise, we may find a Puiseux parameterization of a branch of \bar{C} at p in the (x, y) coordinates of the form $\Gamma(t) = (t^q, \gamma(t))$, where γ is analytic and defined in a small disk \mathbb{D}_δ , and $q \in \mathbb{N}^*$. We seek a contradiction.

We lift ψ to $\mathbb{A}_{\mathbb{k}}^3$, and set $\Psi(x, y) = (z_0(x, y), z_1(x, y), 1)$. As in §1.3, we lift f to a homogeneous polynomial map $F: \mathbb{A}_{\mathbb{k}}^3 \rightarrow \mathbb{A}_{\mathbb{k}}^3$, of the form $F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q})$. Since $\Psi \circ \psi^{-1}$ is a local section of the projection $\mathbb{A}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$, $\Psi \circ \tilde{f}$ must be a multiple of $F \circ \Psi$. From the expression of F we obtain

$$\Psi \circ \tilde{f} = \frac{F \circ \Psi}{\tilde{Q} \circ \Psi} .$$

¹Beware that coordinates are swapped here : $\{z_0 = 0\}$ corresponds to $\{y = 0\}$.

To simplify notation, we write $F^n(z_0, z_1, z_2) = (z_0^{d^n}, \tilde{P}_n, \tilde{Q}_n)$. We consider the 1-homogeneous Green function $G_f: \mathbb{A}_v^{3,\text{an}} \rightarrow \mathbb{R}$ of Proposition 1.3; it satisfies $G_f \circ F = dG_f$ and $g_f(z_1, z_2) = G_f(1, z_1, z_2)$.

Observe that $h := G_f \circ \Psi \circ \Gamma$ is a continuous function on \mathbb{D}_δ . Since $h(t) = g_f \circ \psi \circ \Gamma(t) + \log |z_0 \circ \Gamma(t)|$, by Proposition 1.7, h is harmonic on $\{t \neq 0\}$. Since it is continuous at 0, it is also harmonic on \mathbb{D}_δ , see, e.g., [12, Lemma 3.7]. Write

$$\begin{aligned} d^{nq} h(t) &= G_f \circ F^{nq} \circ \Psi \circ \Gamma(t) \\ &= G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(t) + \log \left| \tilde{Q}_{nq} \circ \Psi \circ \Gamma(t) \right|. \end{aligned}$$

By Proposition 2.3, $\tilde{f}[n](\frac{x}{\lambda^n}, y) \rightarrow (x, 0)$ uniformly in a neighborhood of the origin, hence

$$G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma\left(\frac{t}{\lambda^{n/q}}\right) \rightarrow G_f(0, z_1(t^q, 0), 1)$$

as $n \rightarrow \infty$. On the other hand, since $d^{nq} h\left(\frac{t}{\lambda^{n/q}}\right) - \log \left| \tilde{Q}_{nq} \circ \Psi \circ \Gamma\left(\frac{t}{\lambda^{n/q}}\right) \right|$ is a sequence of harmonic functions, it follows that $t \mapsto G_f(0, z_1(t^q, 0), 1)$ is harmonic as well.

Now, observe that the restriction of f to the line at infinity is $f_\infty[z_1 : z_2] = [\tilde{P}(0, z_1, z_2) : \tilde{Q}(0, z_1, z_2)]$, so that $G_f(0, z_1, z_2)$ is the global Green function of f_∞ . The equilibrium measure of f_∞ is the probability measure on the analytification of L_∞ defined by $\mu_{f_\infty} := \Delta G_f(0, z_1, 1)$ in the chart $z_2 \neq 0$. Its support is the Julia set of f_∞ (see [2, Theorem 13.39]), and it contains all repelling (rigid) fixed points, see [2, Theorem 8.7]. Therefore, $z_1 \mapsto G_f(0, z_1, 1)$ cannot be harmonic near 0, hence the function $t \mapsto G_f(0, z_1(t^q, 0), 1)$ cannot be harmonic either. This contradiction concludes the proof. \square

Remark 3.3. Under the assumptions of Theorem 3.1, the proof shows that the preperiod k of F and the period of $f^k(C)$ are exactly the same as that of any of its non-superattracting points at infinity.

4. PROOF OF THEOREM A FOR ARBITRARY \mathbb{k}

In this section we use a specialization argument to deal with maps defined over arbitrary fields. It shares some arguments with [11, §5] (see also [5, §7]). Nevertheless, new ideas are needed to deal with preperiodic points instead of periodic ones.

We are in the setting of Theorem A, so we assume that f is a regular polynomial map of \mathbb{A}^2 of degree $d \geq 2$ defined over a field \mathbb{k} of characteristic 0, and C is a curve containing an infinite set $\mathcal{P} = \{p_n, n \geq 0\}$ of preperiodic points and whose closure $\overline{C} \cap L_\infty$ contains at least one point which is not eventually super-attracting.

By enlarging \mathbb{k} if necessary we may assume that it contains the algebraic closure \mathbb{Q}^{alg} of its prime field. Let R be the sub- \mathbb{Q}^{alg} -algebra of \mathbb{k} generated by all coefficients defining f and C . Its fraction field K is finitely generated over \mathbb{Q}^{alg} . Let $S = \text{Spec } R$. This is an affine variety defined over \mathbb{Q}^{alg} , and elements of R can be seen as regular functions on S .

Inverting some elements of R if necessary, we may suppose that C is flat over S , and f extends as a morphism $f: \mathbb{P}_S^2 \rightarrow \mathbb{P}_S^2$. We let $\pi: \mathbb{P}_S^2 \rightarrow S$ be the canonical projection, and write $\mathbb{P}_s^2 = \pi^{-1}(s)$. We also let $\mathbb{A}_s^2 := \mathbb{A}_S^2 \cap \pi^{-1}(s)$.

For each (scheme theoretic) point $s \in S$, we write $C_s = C \cap \mathbb{A}_s^2$ and let \overline{C}_s be the closure of C_s in \mathbb{P}_s^2 . The flatness of the morphism $C \rightarrow S$ implies \overline{C}_s (hence C_s) to be a curve. Similarly, we let $f_s: \mathbb{P}_s^2 \rightarrow \mathbb{P}_s^2$ be the induced map on the fibers: this is an endomorphism of degree d .

We also denote by $p_{n,s} \in \mathbb{A}_s^2$ the specialization of p_n . Note that p_n is defined over some finite extension of K which depends on n .

The first result does not use the assumption that our infinite set of preperiodic points lies on a curve.

Proposition 4.1. *Let as above $f: \mathbb{P}_S^2 \rightarrow \mathbb{P}_S^2$ be a family of endomorphisms over an affine variety defined over \mathbb{Q}^{alg} , and let $\mathcal{P} = \{p_n, n \geq 0\}$ be an infinite family of preperiodic points. Then there exists a non-empty Zariski open and dense subset $U \subset S$ such that for any $s \in U \cap S(\mathbb{Q}^{\text{alg}})$, $\mathcal{P}_s = \{p_{n,s}, n \geq 0\} \subset \mathbb{A}_s^2$ is infinite.*

Before starting the proof, let us fix some additional notation. For each $n \geq 0$, we denote by k_n the preperiod of p_n , so that $q_n := f^{k_n}(p_n)$ is the first periodic point in the orbit of p_n . We let ℓ_n be the (primitive) period of q_n .

Proof. We may suppose that there exists a parameter $s_0 \in S(\mathbb{Q}^{\text{alg}})$ such that \mathcal{P}_{s_0} is finite (otherwise we take $U = S$ and the proof is complete).

Lemma 4.2. *The family of periodic points (q_n) is finite.*

Proof. We follow the arguments of [11, §5]. Set $\mathcal{Q} = \{q_n, n \geq 0\}$. For each $\ell \geq 1$, we consider the subvariety Per_ℓ of \mathbb{P}_S^2 defined by the equation $f^\ell(z) = z$. Since $\mathbb{P}_S^2 \rightarrow S$ is proper, the structure map $\text{Per}_\ell \rightarrow S$ is also proper.

Let \mathcal{Q}_ℓ be the union of the irreducible components of Per_ℓ containing a point of \mathcal{P} . Its underlying set is the Zariski closure of $\mathcal{P} \cap \text{Per}_\ell$, hence $\mathcal{Q}_\ell \rightarrow S$ is proper. Observe that for $x \in \mathcal{Q}_{\ell,s}$, the multiplicity of x as a point of $\mathcal{Q}_{\ell,s}$ equals its multiplicity as a fixed point of f_s^ℓ . By Nakayama's lemma and the properness of \mathcal{Q}_ℓ over S , the function

$$(15) \quad s \mapsto \sum_{x \in \mathcal{Q}_{\ell,s}} \text{mult}_x(\mathcal{Q}_{\ell,s})$$

is upper semi-continuous for the Zariski topology, hence

$$(16) \quad \sum_{q \in \mathcal{Q}_\ell} \text{mult}_q(\mathcal{Q}_\ell) \leq \sum_{x \in \mathcal{Q}_{\ell,s_0}} \text{mult}_x(\mathcal{Q}_{\ell,s_0}),$$

where the left hand side is the value of (15) at the generic point. By assumption \mathcal{P}_{s_0} is a finite set, hence so does $\mathcal{Q}_{s_0} = \{q_{1,s_0}, \dots, q_{r,s_0}\}$ and by the Shub-Sullivan theorem [29], there exists a uniform bound $C > 0$ such that for every j , and for any ℓ , we have

$$\text{mult}_{q_{j,s_0}}(\mathcal{Q}_{\ell,s_0}) \leq \text{mult}_{q_{j,s_0}}(\text{Per}_{\ell,s_0}) \leq C .$$

It then follows from (16) that

$$\#\mathcal{Q}_\ell \leq \sum_{q \in \mathcal{Q}_\ell} \text{mult}_q(\mathcal{Q}_\ell) \leq rC$$

hence $\bigcup_\ell \mathcal{Q}_\ell$ is finite, as was to be shown. \square

By the previous lemma, replacing f by some iterate f^N we may assume that all periodic points q_n are fixed. Since \mathcal{P} is infinite, one of these fixed points, say q_1 , admits infinitely many preimages in \mathcal{P} . We may denote $q = q_1$ and suppose \mathcal{P} is made of an infinite set of preimages of q , that is, (after possible reordering of \mathcal{P}) for any $p_n \in \mathcal{P}$ there is a minimal $k_n \geq 0$ such that $f^{k_n}(p_n) = q$, and that $k_{n+1} > k_n$. We may adjoin to R the coordinates of q so that $q \in \mathbb{A}^2(R)$, i.e., for any $s \in S$, q_s is a single point (to say it differently, we replace S by a branched cover of a Zariski open dense subset of S).

Let $d(s)$ be the local degree of f_s at q_s , which is upper semicontinuous for the Zariski topology. Since f_s is a finite map of degree d^2 , $d(s) \leq d^2$ for every s . Thus there is an analytic hypersurface H such that $d(s) = d_{\min}$ is constant for $s \in S \setminus H$.

We claim that \mathcal{P}_{s_1} is infinite for any $s_1 \in S \setminus H$. We argue again in the complex analytic category fixing an embedding \mathbb{Q}^{alg} into \mathbb{C} . Observe that for any point $s \in S(\mathbb{C})$, $p_{n,s}$ is a finite set included in the fiber $\mathbb{A}_s^{2,\text{an}} \simeq \mathbb{C}^2$ (not necessarily reduced to a single point since p_n lies in a finite extension of R).

Fix an analytic neighborhood V of q_{s_1} in $\mathbb{P}_{s_1}^{2,\text{an}}(\mathbb{C})$ such that $f_{s_1}^{-1}(q_{s_1}) \cap \bar{V} = \{q_{s_1}\}$. Since $d(s)$ is locally constant near s_1 , there is an analytic neighborhood W of s_1 in $S^{\text{an}}(\mathbb{C})$ such that for $s \in W$,

$$(17) \quad f_s^{-1}(q_s) \cap V = \{q_s\}$$

Choose any $n > m$, and suppose by contradiction that $p_{m,s_1} = p_{n,s_1}$. Since $k_n - 1 \geq k_m$, we have

$$f_{s_1}^{k_n-1}(p_{n,s_1}) = f_{s_1}^{k_n-1}(p_{m,s_1}) = q_{s_1}.$$

Thus, for s close to s_1 , the finite set $f_{s_1}^{k_n-1}(p_{n,s_1})$ is contained in V , hence by (17), $f_s^{k_n-1}(p_{n,s}) = q_s$, and by analytic continuation this property holds throughout S , which contradicts the definition of k_n .

This shows that the $p_{n,s}$ are all distinct for all $s \in S \setminus H$, and concludes the proof of Proposition 4.1. \square

Proposition 4.3. *Let $f: \mathbb{A}_{\mathbb{k}}^2 \rightarrow \mathbb{A}_{\mathbb{k}}^2$ be a regular polynomial map and $C \subset \mathbb{A}_{\mathbb{k}}^2$ be an algebraic curve containing infinitely many preperiodic points. Then every point of $\overline{C} \cap L_{\infty}$ is preperiodic under $f|_{L_{\infty}}$.*

Proof. We keep the same formalism and notation as above, so that f is viewed as a family over S . Write $\overline{C} \cap L_{\infty} = \{c_1, \dots, c_r\}$ and without loss of generality enlarge R so that the points at infinity c_i have their coordinates in R . Fix $i \in \{1, \dots, r\}$ for the remainder of the proof and consider $c = c_i$. By Proposition 4.1, there is a Zariski open subset U such that for any $s \in U \cap S(\mathbb{Q}^{\text{alg}})$, f_s admits infinitely many preperiodic points on C_s . Therefore, by Proposition 1.7, for every such s , c_s is preperiodic. Fix $s_0 \in U \cap S(\mathbb{Q}^{\text{alg}})$, then c_{s_0} eventually falls on a periodic point q_{s_0} . Replacing f by f^N and C by $f^N(C)$ for some N , we may assume that q_{s_0} is fixed and $c_{s_0} = q_{s_0}$. Enlarging R again if necessary we may assume that q_{s_0} is the specialization at $s = s_0$ of a fixed point $q \in \mathbb{P}^2(R)$ of f .

Our purpose is to show that $c = q$. To simplify notation we write $\hat{f} = f|_{L_{\infty}}$. Note that the multiplier $\mu := \hat{f}'_{s_0}(q_{s_0})$ belongs to \mathbb{Q}^{alg} . It follows from Kronecker's theorem that either μ is a root of unity or there is a place v on \mathbb{Q}^{alg} such that $|\mu|_v < 1$.

Case 1. μ is not a root of unity.

Fix a place v on \mathbb{Q}^{alg} such that $|\mu|_v < 1$, and consider the completion \mathbb{C}_v of $(\mathbb{Q}^{\text{alg}}, |\cdot|_v)$. We then argue in the analytic topology in the Berkovich analytification of $\mathbb{P}_{\mathbb{C}_v}^2$ and $S_{\mathbb{C}_v}$.

Fix a neighborhood W of s_0 in $S_{\mathbb{C}_v}^{\text{an}}$ such that for $s \in W$, q_s is attracting, and a neighborhood V of q_s in L_∞ independent of $s \in W$ such that $\hat{f}_s(V) \subset V$ and for any $z \in V$, $f_s^n(z)$ converges to q_s as $n \rightarrow \infty$. Reducing W if necessary we may assume that for any $s \in W$, c_s belongs to V . For $s \in W \cap S(\mathbb{Q}^{\text{alg}})$, c_s is preperiodic and converges to q_s , so it is preperiodic to q_s , that is, there exists a minimal $k = k(s)$ such that $f_s^{k(s)}(c_s) = q_s$. Now we use an argument similar to that of Proposition 4.1: let $H \subset S$ be a hypersurface such that the local degree of f_s at q_s is locally minimal outside H and fix $s_1 \in W \setminus H$. Then, there is a neighborhood W_1 of s_1 in $W \setminus H$ and a neighborhood $V_1 \subset V$ of q_1 such that for any $s \in W_1$, $f_s(V_1) \subset V_1$ and $f_s^{-1}(q_s) \cap V_1 = \{q_s\}$. From this it follows that the only point eventually falling onto q_s in V_1 is q_s itself. Therefore if $s \in W_1$ is so close to s_1 that $c_s \in V_1$, we infer that $c_s = q_s$, and finally $c = q$ by analytic continuation.

Case 2. μ is a root of unity.

To deal with this case we embed \mathbb{Q}^{alg} into \mathbb{C} , and work at the complex place. Recall that a holomorphic family $(g_\lambda)_{\lambda \in \Lambda}$ of rational maps on $\mathbb{P}^1(\mathbb{C})$, parameterized by a connected complex manifold is trivial if any two members are conjugate by a Möbius transformation, depending holomorphically on Λ . If c is persistently preperiodic we are done, so assume that c is not persistently preperiodic.

Under our assumptions, there is a dense set $S(\mathbb{Q}^{\text{alg}})$ of parameters such that c_s is preperiodic, but c is not persistently preperiodic. Thus by Chio-Roeder [7, Theorem 2.7] (see also [10, Theorem 4]) every such parameter belongs to the bifurcation locus of the marked family (f_∞, c) (note that c is *not* a critical point here). As a consequence, the bifurcation locus of the family is equal to $S_{\mathbb{C}}^{\text{an}}$. (Note that it is enough to work in a neighborhood of s_0 , away from possible singularities of $S_{\mathbb{C}}^{\text{an}}$.)

A first possibility is that the family $(\hat{f}_s)_{s \in S_{\mathbb{C}}^{\text{an}}}$ is non-trivial. Then Gauthier [14, Theorem A] implies that $J(\hat{f}_s) = L_\infty$ for all s . But since \hat{f}_{s_0} has a rationally indifferent fixed point, it admits an attracting petal and $J(\hat{f}_{s_0}) \neq L_\infty$. This contradiction shows that the family $(\hat{f}_s)_{s \in S_{\mathbb{C}}^{\text{an}}}$ is trivial.

Now the situation is that there is a holomorphic family φ_s of Möbius transformations such that $\varphi_s \hat{f}_s \varphi_s^{-1} = g$ is a fixed rational map g on \mathbb{P}^1 with a rationally indifferent fixed point at 0. After this conjugacy, the marked family (\hat{f}, c) becomes $(g, \varphi(c))$. Since c coincides with q at s_0 and by assumption c is not persistently preperiodic, there is an open set Ω of parameters such that for $s \in \Omega$, $\varphi_s(c_s)$ belongs to some attracting petal associated to 0 for g . This contradicts the fact that $\varphi_s(c_s)$ must be preperiodic for a dense set of parameters, and the proof is complete. \square

Conclusion of the proof of Theorem A. By Proposition 4.3, any point at infinity of C is preperiodic, and, by assumption, one of these points, say c , is preperiodic to a non-superattracting periodic point p . Replace f by f^N and C by $f^N(C)$ for some N , so that $c = p$ is fixed. By Proposition 4.1, there is a non-trivial Zariski open subset $U \subset S$

such that for every $s \in U \cap S(\mathbb{Q}^{\text{alg}})$, C_s contains infinitely many preperiodic points. Then, since p_s is fixed and not superattracting for f_s , Theorem 3.1 asserts that C_s is preperiodic, and more precisely fixed, under f_s (see Remark 3.3). The density of $U \cap S(\mathbb{Q}^{\text{alg}})$ in S (for the Zariski or analytic topology) then implies that $f(C) = C$, and the proof is complete. \square

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SORBONNE UNIVERSITÉ, LABORATOIRE DE PROBABILITÉS, STATISTIQUE ET MODÉLISATION, UMR 8001, 4 PLACE JUSSIEU, 75005 PARIS, FRANCE

Email address: `romain.dujardin@sorbonne-universite.fr`

CNRS - CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ, ÉCOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE

Email address: `charles.favre@polytechnique.edu`

UNIVERSITÉ DE PARIS AND SORBONNE UNIVERSITÉ, CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, F-75006 PARIS, FRANCE

Email address: `matteo.ruggiero@imj-prg.fr`