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ON THE DYNAMICAL MANIN-MUMFORD CONJECTURE FOR PLANE POLYNOMIAL MAPS

ROMAIN DUJARDIN, CHARLES FAVRE, AND MATTEO RUGGIERO

Abstract. We prove the dynamical Manin-Mumford conjecture for regular polynomial maps of $\mathbb{A}^2$ and irreducible curves avoiding super-attracting orbits at infinity, over any field of characteristic 0.

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Introduction

The dynamical Manin-Mumford conjecture for polarized endomorphisms of algebraic varieties, first formulated by S.-W. Zhang in two influential papers [32, Conjecture 2.5] and [33, Conjecture 1.2.1], has been a driving force for the development of the field of arithmetic dynamics. It was realized by Ghioca, Tucker and Zhang [19] and Pazuki [26] that the original formulation of the conjecture was too optimistic, and a modified conjecture was proposed in [19] and more recently in [18]. It can be stated as follows: let $f : X \to X$ be a polarized endomorphism of a smooth projective variety over a field of characteristic zero, and $Z \subset X$ be a subvariety containing a Zariski dense set of periodic points. Then either $Z$ is preperiodic or $Z$ is special, in the sense that it is contained in some subvariety $Y$ that is both $f^n$- and $\psi$-invariant, for some $n \geq 1$, where $\psi$ is another polarized endomorphism commuting with $f^n$. Recall that an endomorphism is said to be polarized if there is an ample line bundle $L \to X$ such that $f^*L \simeq L^d$ for some $d \geq 2$. A basic example is that of non-invertible endomorphisms of $\mathbb{P}^k$, for which we can take $L = O(1)$ and $d$ is the degree of $f$.

Despite its importance, very few cases of the conjecture have been settled so far. One first case is of course the original Manin-Mumford conjecture, which was solved by Raynaud [27, 28]. Viewed as a dynamical statement, it deals with endomorphisms of

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Abelian varieties, and was generalized to related settings, such as commutative algebraic groups (see Hindry [22], and also Lang’s classical paper [25]). Uniform versions involving height bounds were subsequently obtained by S.-W. Zhang, David and Philippon, Chambert-Loir and others. We refer to the recent work of Kühne [24] in the semi-abelian case for a latest update, and more references. Closer to algebraic dynamics is the case of polarized endomorphisms of $(\mathbb{P}^1)^k$, which was solved by Ghioca, Nguyen and Ye [17, 16] (see also [18]). Note that such mappings are of product type, that is of the form $f(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k))$, so their dynamical complexity reduces to dimension 1, which is a key step in the proof.

In this paper we will establish the dynamical Manin-Mumford conjecture for a wide class of 2-dimensional examples, whose dynamical behavior is truly higher dimensional. Note that besides polarized endomorphisms, a partial answer to the conjecture was given in [11] for plane polynomial automorphisms.

Let $k$ be any field of characteristic zero. If required we fix an algebraic closure $k_{\text{alg}}$ of $k$. Let $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be a polynomial self-map of the affine plane of degree $d \geq 2$, which in coordinates is written as

$$f(z, w) = (P(z, w), Q(z, w)),$$

with $P, Q \in k[x, y]$. We say that $f$ is regular if it extends to an endomorphism of $\mathbb{P}^2_k$ of degree $d \geq 2$; this means that $P = P_d + \text{l.o.t.}$ and $Q = Q_d + \text{l.o.t.}$, where $P_d$ and $Q_d$ are homogeneous polynomials of degree $d$ without common factors. In particular $f$ induces a rational map $f_{\infty} := [P_d : Q_d]$ on the line at infinity, which is fixed. Note that any endomorphism of $\mathbb{P}^2_k$ with a totally invariant line is conjugate to a regular polynomial map, and that a generic polynomial map of $\mathbb{A}^2$ whose components are polynomials of degree $\leq d$ is regular.

For regular polynomial maps of $\mathbb{A}^2$, it seems that none of the known obstructions to the dynamical Manin-Mumford conjecture can arise. As said above, according to [18], one obstruction would be the existence of a periodic curve $C$ for some endomorphism $\psi$ commuting with $f$. Such pairs $(f, \psi)$ were classified in [8] over $\mathbb{C}$ (see also [9, 23]), and after a ramified cover they are all induced from a product map or a monomial map on the multiplicative 2-torus. Thanks to [17] it appears that in any such case, $C$ must be also $f$-preperiodic. Thus we expect that the dynamical Manin-Mumford conjecture holds unconditionally in this case. Our main result confirms this expectation in the vast majority of cases.

**Theorem A.** Let $f$ be a regular polynomial endomorphism of $\mathbb{A}^2$ of degree $d \geq 2$, defined over an arbitrary field $k$ of characteristic 0.

Let $C \subset \mathbb{A}^2$ be an irreducible algebraic curve containing infinitely many preperiodic points of $f$, and suppose that the closure of $C$ in $\mathbb{P}^2$ contains a point $p \in L_\infty$ which is not eventually superattracting. Then $C$ is preperiodic under $f$.

**Corollary B.** Under the assumptions of the theorem, if $f$ has no super-attracting points on the line at infinity, then any irreducible algebraic curve $C \subset \mathbb{P}^2$ containing infinitely many periodic points is preperiodic, that is, the dynamical Manin-Mumford conjecture holds for $f$. 

Our strategy relies on techniques from Arakelov geometry, in particular on the notion of canonical height, which is now classical in arithmetic dynamics, together with a variety of techniques from holomorphic and non-Archimedean dynamics.

We first assume that \( k \) is a number field, in which case we prove a stronger statement (Theorem 3.1). We first recall in Section 1, that there exists a canonical height \( h_f \) on \( \mathbb{A}_k^2(\mathbb{Q}_{\text{alg}}) \), for which preperiodic points are exactly points of zero height. By a theorem of Zhang, all points of \( C \) lying at infinity are preperiodic, hence, by replacing \( C \) by some iterate we may assume that they are periodic, and, thanks to our assumptions, one of these periodic points is not superattracting for \( f_{\infty} \). Fix such a point \( p \in C \cap L_{\infty} \). A second consequence of Zhang’s theorem is that for each place \( v \) of \( k \), the dynamical Green function \( g_{f,v}(z,w) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ \| f^n(z,w) \| \) satisfies

\[
\int_C g_{f,v} dd^c g_{f,v} = 0.
\]

To make sense of this equality at a finite place, we consider the Berkovich analytification of \( C \) and \( dd^c g_{f,v} \) is the Laplacian of the subharmonic function \( g_{f,v}|_C \) in the sense of Thuillier, see [31].

For an appropriate choice of \( v \), we may suppose that the periodic point \( p \) is either repelling or parabolic for \( f_{\infty} \). In both cases, we construct in Section 2 a local (super-)stable manifold \( W_{\text{ss}}(p) \). When \( p \) is repelling and \( v \) is Archimedean this is classical. We extend this construction to the non-Archimedean setting, and allow for non-repelling \( p \) (see Theorem 2.1). Then, in both (repelling or parabolic) cases we establish graph transform type estimates which will be useful for the local analysis of the Green function at \( p \). When \( p \) is parabolic this borrows from the work of Hakim [20].

In Section 3 we combine these estimates with condition (1) to show that, near \( p, C \) must locally coincide with \( W_{\text{ss}}(p) \), which finally implies that \( C \) is periodic. Note that a similar argument appears in the recent work [15].

A key point in this argument is that at the chosen place \( v \), \( p \) belongs to the Julia set of \( f_{\infty} \). This is no longer true when \( f_{\infty} \) is superattracting at \( p \), and we are not able to conclude in this case. Still, this situation leads to interesting dynamical considerations and objects, and we plan to come back to this issue in a later work.

Finally, we develop a specialization argument in Section 4 to reduce Theorem A for an arbitrary \( k \) to the case where \( k \) is a number field. So here we rather deal with an algebraic family of endomorphisms of \( \mathbb{P}^2 \) parameterized by some algebraic variety \( S \). The main issue is to ensure that for such a family, an infinite set of preperiodic points cannot shrink to a finite set too often on \( S \). To do so, one needs to control collisions of periodic points (using the Shub-Sullivan theorem [29] in the spirit of [11]); and the splitting of local preimages of a super-attracting cycle, a phenomenon that was studied in particular by Chio and Roeder [7].

1. Dynamical heights

In this section, we recall some basic facts on canonical heights attached to endomorphisms of the projective plane defined over a number field. Our purpose is to establish Proposition 1.7 which is the key arithmetic geometry input in our main theorem.
Throughout this section, we assume that $k$ is a number field.

1.1. **Vocabulary of number fields.** We denote by $M_k = \{v\}$ the set of places of $k$, that is, the set of all multiplicative norms $|\cdot|_v$ on $k$ that restrict to either the standard euclidean norm $|\cdot|_\infty$, or to a $p$-adic norm $|\cdot|_p$ on $\mathbb{Q}$ for some prime number $p > 1$. We normalize the $p$-adic norm by $|p|_p = p^{-1}$. Let $k_v$ be the completion of $k$ w.r.t. $|\cdot|_v$, and write $n_v := [k_v : \mathbb{Q}_v]$. The product formula asserts that for any $a \in k$, we have

$$\prod_{v \in M_k} |a|_v^{n_v} = 1.$$ 

The set $M_k$ splits into the finite set $M_k^\infty$ of Archimedean places (whose restriction to $\mathbb{Q}$ is $|\cdot|_\infty$), and the set of finite (or non-Archimedean) places.

When $v \in M_k^\infty$, the algebraic closure $\mathbb{C}_v$ of $k_v$ is isometric to $\mathbb{C}$. When $v$ is a finite place extending $|\cdot|_p$ on $\mathbb{Q}$, then $|\cdot|_v$ extends canonically to $k_v^{alg}$, and its completion $\mathbb{C}_v$ is both complete and algebraically closed.

1.2. **Regular polynomial maps.** Let $(z, w)$ be affine coordinates on the affine plane $\mathbb{A}^2_k$. We also consider homogeneous coordinates $[z_0 : z_1 : z_2]$ on the projective plane $\mathbb{P}^2_k$ and identify the affine plane $\mathbb{A}^2_k$ with the Zariski open set $z_0 \neq 0$ so that $z = z_1/z_0$ and $w = z_2/z_0$. We denote by $L_\infty = \{v_0 = 0\}$ the line at infinity.

Let $f : \mathbb{A}^2_k \to \mathbb{A}^2_k$ be any polynomial self-map of the affine plane of degree $d \geq 2$ that extends to an endomorphism of $\mathbb{P}^2_k$. In the coordinates $z, w$, it is given by

$$f(z, w) = (P(z, w), Q(z, w)),$$

where $P, Q \in k[z, w]$ satisfy $\max\{\deg P, \deg Q\} = d$. The fact that $f$ extends to a regular endomorphism $\overline{f} : \mathbb{P}^2_k \to \mathbb{P}^2_k$ is equivalent to say that $P = P_d + l.o.t.$ and $Q = Q_d + l.o.t.$, where $P_d$ and $Q_d$ are homogeneous polynomials of degree $d$ without common factors.

For $n \in \mathbb{N}$ we write $f^n(z, w) = (P_n(z, w), Q_n(z, w))$. The restriction of $f_\infty$ to $L_\infty$ is an endomorphism of $\mathbb{P}^2_k$ given in homogeneous coordinates by

$$f_\infty([z_1 : z_2]) = [P_d(z_1, z_2) : Q_d(z_1, z_2)].$$

1.3. **Green functions.** The next proposition follows from the Nullstellensatz (see e.g. [30, Theorem 3.11]).

**Proposition 1.1.** For any $v \in M_k$, there exists a constant $C_v \geq 1$ such that

$$C_v^{-1} \leq \frac{\max\{1, |P(z, w)|_v, |Q(z, w)|_v\}}{\max\{1, |z|_v, |w|_v\}^d} \leq C_v$$

for all $z, w \in \mathbb{C}_v$. Moreover, for all but finitely many $v \in M_k$ we may take $C_v = 1$.

By the previous proposition, the sequence of functions

$$g_{v,n}(z, w) := \frac{1}{d^n} \log \max\{1, |P_n(z, w)|_v, |Q_n(z, w)|_v\}$$

converges uniformly on $\mathbb{C}_v^2$ to a continuous function $g_v$, and the next proposition follows.
Proposition 1.2. For any $v \in M_k$, the function $g_v : \mathbb{C}^2_v \to \mathbb{R}_+$ is continuous, it satisfies the invariance equation $g_v \circ f = dg_v$, and we have
\[ |g_v(z, w) - \log \max \{1, |z|_v, |w|_v\} | \leq \frac{\log C_v}{d - 1}. \]
The set $\{(z, w) \in \mathbb{C}^2_v, g_v(z, w) = 0\}$ coincides with the set of points having bounded orbits.

We shall also consider the global Green function in $\mathbb{C}^3_v$. Write $P(z_0, z_1, z_2) = z_0^d P(\frac{z_1}{z_0}, \frac{z_2}{z_0})$ and $Q(z_0, z_1, z_2) = z_0^d Q(\frac{z_1}{z_0}, \frac{z_2}{z_0})$ so that $F(z_0, z_1, z_2) = (z_0^d, \tilde{P}, \tilde{Q})$ is a homogenous map of degree $d$ that lifts $f$ to $\mathbb{C}^3_v$. Observe that in homogenous coordinates, $|z_0|_v$ can be rewritten as follows:
\[ C_v^{-1} \leq \max\{|z_0|^d, |\tilde{P}(z_0, z_1, z_2)|_v, |\tilde{Q}(z_0, z_1, z_2)|_v\} \leq C_v \]
so that the next result also holds.

Proposition 1.3. The function $G_v(z_0, z_1, z_2) = g_v(z_1/z_0, z_2/z_0) + \log |z_0|$ is continuous on $\mathbb{C}^3_v \setminus \{0\}$, 1-homogeneous (that is, $G_v(\lambda Z) = \log |\lambda| + G_v(Z)$), and satisfies $G_v \circ F = dG_v$. We have
\[ |G_v(z_0, z_1, z_2) - \log \max\{|z_0|_v, |z_1|_v, |z_2|_v\}| \leq \frac{\log C_v}{d - 1}. \]
The set $\{(z_0, z_1, z_2) \in \mathbb{C}^3_v, G_v(z_0, z_1, z_2) \leq 0\}$ coincides with the set of points having bounded $F$-orbits.

1.4. Canonical heights on points. We refer to [6] for generalities on heights. Consider the line bundle on $\mathcal{O}(1) \to \mathbb{P}^2_k$. The space of sections of this line bundle can be canonically identified with the space of linear forms $a_0z_0 + a_1z_1 + a_2z_2$ with $a_i \in k$. More precisely, in the trivialization of the bundle over $\{z_i \neq 0\}$, this section is given by $\frac{1}{z_i}a_0z_0 + a_1z_1 + a_2z_2$.

For any $v \in M_k$, we consider the line bundle $\mathcal{O}(1) \to \mathbb{P}^2_{\mathbb{C}_v}$ and endow it with the metrization $|.|_v$ induced by $G_v$, in the sense that any section $\sigma = a_0z_0 + a_1z_1 + a_2z_2$ as above
\[ |\sigma|_v([z_0 : z_1 : z_2]) = |a_0z_0 + a_1z_1 + a_2z_2| e^{-G_v(z_0, z_1, z_2)} \]
(this expression is well-defined thanks to the homogeneity property of $G_v$). Let us now explain how this collection of metrizations defines a function on the set of algebraic points in $\mathbb{P}^2_k$ as well as on the set of all algebraic curves in $\mathbb{P}^2_k$ defined by an equation with coefficients in $k_{alg}$.

For any $p \in (k_{alg})^3 \setminus \{0\}$, we set
\[ h_f(p) := \frac{1}{N(p)} \sum_{p' \in \text{Gal} \cdot p} \left( \sum_{v \in M_k} n_v G_v(p') \right) \]
where Gal denotes the absolute Galois group of $k_{alg}$ over $k$, and $N(x)$ is the cardinality of the set $\text{Gal} \cdot x \subset (k_{alg})^3$.

The product formula entails that $h_f(z_0, z_1, z_2) = h_f(\lambda z_0, \lambda z_1, \lambda z_2)$ for any $\lambda \in k_{alg}$ so that we have a well-defined function $h_f : \mathbb{P}^2(k_{alg}) \to \mathbb{R}$. 

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Proposition 1.4. The function $h_f$ takes non-negative values, and satisfies $h_f \circ f = dh_f$. The set $\{ h_f = 0 \}$ coincides with the set of preperiodic points of $f$. Furthermore, for any $(z, w) \in (k^{alg})^2$ we have

$$h_f(z, w) := \frac{1}{N(z, w)} \sum_{(z', w') \in \text{Gal}(z, w)} \left( \sum_{v \in M_k} n_v g_v(z', w') \right) .$$

As above Gal denotes the absolute Galois group of $k^{alg}$ over $k$, and $N(z, w)$ is the cardinality of the set $\text{Gal}(z, w) \subset (k^{alg})^2$. The proof follows directly from Northcott’s theorem, see [4 Corollary 1.1.1].

1.5. Analytification of affine curves. Let $C$ be any irreducible algebraic curve in $\mathbb{A}^2_k$ defined by an equation $\{ R = 0 \}$ with $R \in k[z, w]$. Denote by $\overline{C}$ the Zariski closure of $C$ in $\mathbb{P}^2$.

Fix any place $v \in M_k$. We denote by $C^\text{an}_v$ the analytification in the sense of Berkovich of $C$ over $\mathbb{C}_v$. This is a connected, locally connected and locally compact space. When $| \cdot |_v$ is Archimedean, hence $\mathbb{C}_v$ is isometric to $\mathbb{C}$, $C^\text{an}_v$ is the complex analytic subspace (possibly with singularities) defined as usual by the vanishing of $R$ in $\mathbb{C}_v^2$. When $| \cdot |_v$ is non-Archimedean, then $C^\text{an}_v$ is defined as the set of multiplicative semi-norms on the ring $\mathbb{C}_v[z, w]/(R)$ whose restriction to the base field equals $| \cdot |_v$. A point is said to be rigid when the semi-norm has non-trivial kernel.

One can also define the analytification of $\overline{C}$ by considering suitable affine coordinates in $\mathbb{P}^2$ and patching the previous construction in a natural way, see [3 §3.4]. Observe that $\overline{C}^\text{an}_v \setminus C^\text{an}_v$ consists of rigid points.

Suppose first $v$ is Archimedean. The metrization of $\mathcal{O}(1)$ defined by [3] induces a measure $\mu_{C, v}$ on $\overline{C}^\text{an}_v$ which is locally defined by $\mu_{C, v} := \Delta \log |\sigma|_v$, where $\sigma$ is a local section of $\mathcal{O}(1)$. The plurisubharmonicity of $G_v$ ensures that $\mu_{C, v}$ is a positive measure. The Lelong-Poincaré formula implies that the mass of $\mu_{C, v}$ is equal to $\deg(C)$, and we have $\mu_{C, v} = \Delta (g_v|_{C^\text{an}_v})$ on $C^\text{an}_v$. Observe that since $G_v$ is continuous, $\mu_{C, v}$ gives no mass to points.

The construction is completely analogous in the non-Archimedean case. We again obtain a positive measure $\mu_{C, v}$ on $\overline{C}^\text{an}_v$ of total mass $\deg(C)$ which is given in $C^\text{an}_v$ by $\mu_{C, v} = \Delta (g_v|_{C^\text{an}_v})$ where $\Delta$ is the Laplace operator defined by Thuillier [31]. Observe that the continuity of the metrization implies that $\mu_{C, v}$ does not charge any rigid point (but it may still charge some non-rigid point in $C^\text{an}_v$). We refer to [3 §1.3] for the details of the constructions.

1.6. Canonical heights on curves. Let $C$ be any irreducible algebraic curve in $\mathbb{A}^2_k$ as in the previous section. We now define the canonical height of the curve $\overline{C}$ following the recipe given in [3 §3.1.2], taking $z_0$ as a section of $\mathcal{O}(1)$ (note that this section vanishes exactly along the line at infinity). We obtain:

$$(4) \quad h_f(\overline{C}) := \sum_{p \in \overline{C} \cap L_\infty} (\overline{C}, L_\infty)_p \times h_f(p) + \sum_{v \in M_k} \int_{\overline{C}^\text{an}_v} g_v \mu_v .$$
Note that $h_f(C) \geq 0$ because the canonical height is non-negative on closed points, and the Green functions $g_v$ are also non-negative.

Define the essential minimum of $h_f$ by the following formula:

$$\text{essmin}_C(h_f) := \sup_{F \text{ finite} \subset C(k_{\text{alg}})} \inf_{C(k_{\text{alg}}) \setminus F} h_f.$$ 

**Theorem 1.5** (Zhang’s inequality [32, Theorem 1.10]). We have

$$2 \text{essmin}_C(h_f) \geq \frac{h_f(C)}{\deg(C)} \geq \text{essmin}_C(h_f) + \inf_{p \in C(k_{\text{alg}})} h_f(p).$$

Since $h_f(x) = 0$ if and only if $x$ is preperiodic, we obtain:

**Corollary 1.6.** An irreducible algebraic curve $C$ containing infinitely many $f$-preperiodic points satisfies $h_f(C) = 0$.

### 1.7. A first characterization of special curves.

**Proposition 1.7.** Suppose that $C \subset \mathbb{A}^2_k$ is an irreducible algebraic curve containing a sequence of distinct points $p_n \in C(k_{\text{alg}})$ such that $h_f(p_n) \to 0$.

Then all points in $\overline{C} \cap L_\infty$ are preperiodic for $f$, and for any $v \in M_k$ the function $g_v|_{C^{an}}$ is harmonic on $\{g_v > 0\}$.

**Proof.** Note that $h_f(C) \geq 0$. By Theorem 1.5, our assumption implies that $\text{essmin}_C(h_f) \leq 0$, therefore $h_f(C) = 0$. Then the result follows from (4) and the fact that a point $p$ is $f$-preperiodic if and only if $h_f(p) = 0$. \qed

### 2. Super-stable manifolds and local estimates


In this section we work under the following hypothesis: $(k, |\cdot|)$ is a complete metrized field of characteristic 0 (which may be either Archimedean or non-Archimedean).

**Theorem 2.1.** Suppose $f: (\mathbb{A}^2_k, 0) \to (\mathbb{A}^2_k, 0)$ is a germ of analytic map fixing the origin of the form

$$f(x, y) = \left(\lambda x + \mu y + g(x, y), y^d(1 + h(x, y))\right),$$

where $d \geq 2$, $\lambda \neq 0$, $\mu \in k$, $h(0, 0) = 0$, and $g(x, y) = O(|(x, y)|^2)$. Then there exists a unique smooth analytic curve which is transverse to $\{y = 0\}$ and $f$-invariant.

We shall call this curve the **local super-stable manifold** of the origin, and denote it by $W_{\text{ss}}^0(0)$. After a linear change of coordinates of the form $(x, y) \mapsto (x + \frac{\mu}{\lambda} y, y)$, we may and will assume from now on that $\mu = 0$. Expressing the invariant curve as a graph of the form $x = \varphi(y)$ and making a change of coordinates of the form $(x, y) \mapsto (x - \varphi(y), y)$, $f$ takes the form

$$f(x, y) = (\lambda x + x\tilde{g}(x, y), y^d(1 + \tilde{h}(x, y))).$$

It follows that $f$ is analytically conjugate to $y \mapsto y^d$ on $W_{\text{ss}}^0(0)$, hence the terminology.
The result is classical when \( k \) is Archimedean and/or \( f \) is locally invertible (see e.g. [21, Appendix]). For convenience we include a proof that works simultaneously in the Archimedean and non-Archimedean settings, and is adapted to \( \{ y = 0 \} \) being superattracting.

**Proof.** As explained above, we look for an analytic map \( y \mapsto \varphi(y) \), with \( \varphi(0) = 0 \) such that the change of coordinates \( \Phi(x, y) = (x + \varphi(y), y) \) satisfies

\[
\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x \tilde{g}(x, y), y^d(1 + \tilde{h}(x, y)))
\]

with \( \tilde{g}, \tilde{h} \) analytic and vanishing at 0. This property is equivalent to the identities:

\[
\begin{align*}
\lambda \varphi(y) + g(\varphi(y), y) &= \varphi(y^d(1 + \tilde{h}(0, y))) \\
\tilde{h}(0, y) &= h(\varphi(y), y)
\end{align*}
\]

so that we aim at finding some analytic function \( \varphi \) satisfying

\[
\lambda \varphi(y) + g(\varphi(y), y) = \varphi(y^d(1 + h(\varphi(y), y)))
\]

For any \( r > 0 \), let us introduce the Banach space \( B_r \) that consist of those power series \( \varphi(y) := \sum_{j \geq 1} a_j y^j \) which are convergent in the disk of radius \( r \), and satisfy \( \| \varphi \| := \sup_{|y| < r} |\varphi(y)| < \infty \). Note that in the non-Archimedean case, we have \( \| \varphi \|_r := \sup_{|y| < r} |\varphi(y)| \).

For any \( \varphi \in B_r \), we set

\[
T \varphi(y) := \frac{1}{\lambda} \left( \varphi(y^d(1 + h(\varphi(y), y))) - g(\varphi(y), y) \right).
\]

We claim that for \( r > 0 \) and \( \rho > 0 \) sufficiently small, \( T \) is a well-defined strictly contracting map on \( B(0, \rho) \subset B_r \). Then, applying the Banach fixed point theorem implies the existence of the desired \( \varphi \).

First, we may suppose that \( g \) is analytic in the polydisk of radius \( r \), and since \( g \) vanishes up to order 2 at the origin, we have

\[
|g(x, y)| \leq C \max\{|x|, |y|\}^2
\]

for some \( C > 0 \) and all \( |x|, |y| < r \). Similarly, we may suppose that \( h \) is analytic in the polydisk of radius \( r > 0 \), and that \( |h(x, y)| \leq \frac{1}{2} \) for all \( |x|, |y| < r \).

Pick any \( \varphi(y) = \sum_{j \geq 1} a_j y^j \in B(0, \rho) \subset B_r \). Reduce \( r > 0 \) if necessary so that \( \frac{3}{2} r^d < r \). Then \( \tilde{\varphi} : y \mapsto \varphi(y^d(1 + h(\varphi(y), y))) \) is well-defined and analytic on the disk of radius \( r \).

In the non-Archimedean case, \( |1 + h(\varphi(y), y)| = 1 \), so that one has

\[
\| \tilde{\varphi}(y) \|_r = \sup_{j \geq 1} |a_j| r^d \leq r \sup_{j \geq 1} |a_j| r^{d-1} \leq r |\varphi|_r.
\]

In the Archimedean case, the Schwarz lemma yields \( |\varphi(y)| \leq \frac{|y|}{r} |y| \) for all \( |y| < r \), hence \( \| \tilde{\varphi} \|_r \leq \frac{3}{2} r^{d-1} |\varphi|_r \). Note that we also have:

\[
\| g(\varphi(y), y) \|_r \leq C \max\{\| \varphi \|_r, r\}^2
\]

so for \( \varphi \in B(0, \rho) \) we deduce

\[
|T \varphi|_r \leq \frac{1}{\lambda} \left( \frac{3}{2} r^{d-1} \rho + C \max\{\rho, r\}^2 \right).
\]
By choosing \( \rho = r \) and then \( r \) small enough, this estimate shows that \( T \varphi \) is well-defined on the ball \( B(0, r) \subset B_r \) and \( T(B(0, r)) \subset B(0, r) \).

In order to prove that \( T \) is strictly contracting, observe that we can write

\[
g(x, y) - g(x', y) = (x - x') \hat{g}(x, x', y)
\]

where \( \hat{g}(x, x', y) \) is again analytic in the polydisk of radius \( r \), and

\[
|\hat{g}(x, x', y)| \leq C' \max\{|x|, |x'|, |y|\}
\]

for some constant \( C' > 0 \). For any pair of analytic functions \( \varphi_1, \varphi_2 \in B(0, r) \subset B_r \) we infer:

\[
|g(\varphi_1(y), y) - g(\varphi_2(y), y)| \leq C' \|\varphi_1 - \varphi_2\|_r \max\{|\varphi_1|_r, |\varphi_2|_r, r\}
\]

hence

\[
\|T \varphi_1 - T \varphi_2\|_r \leq \frac{1}{\lambda}(3r^d + C'r) \|\varphi_1 - \varphi_2\|_r.
\]

Again, by choosing \( r \) sufficiently small, we obtain that \( T \) is strictly contracting and we are done. \( \Box \)

2.2. The rescaling argument in the repelling case. We work in \( \mathbb{A}^2_\kappa \) where \( \kappa \) is an arbitrary complete metrized field of characteristic 0. We start with a preparation lemma.

Lemma 2.2. Suppose \( f \) is an analytic map of the form

\[
f(x, y) = \left( \lambda x + xyg(x, y), y^d(1 + h(x, y)) \right)
\]

where \( |\lambda| > 1, d \geq 2 \) and \( g(0) = h(0) = 0 \).

Then there exists an analytic change of coordinates \( \Phi \) such that

\[
\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x(1 + xy\hat{g}(x, y)), y^d(1 + x\hat{h}(x, y)))
\]

for some analytic functions \( \hat{g}, \hat{h} \).

Recall that the form \( (7) \) is what is obtained from \( (5) \) after conjugating to get \( \mu = 0 \) and declaring that the stable manifold of Theorem 2.1 is \( \{x = 0\} \).

Proof. By Böttcher’s theorem (see [2, Chapter 4] for the non-Archimedean case) applied to \( y \mapsto f(0, y) \) we may suppose that \( x \) divides \( h \). Similarly, since \( |\lambda| > 1 \), by we may linearize \( x \mapsto f(x, 0) \), and suppose that \( f \) is of the form \( f(x, y) = (\lambda x(1 + g_1(y)) + x^2gh_1(x, y), y^d(1 + \mathcal{O}(x))) \) for some analytic functions \( g_1, h_1 \) with \( g_1(0) = 0 \).

We claim that there exists \( \Phi(x, y) = (x(1 + \varphi(y)), y) \) with \( \varphi(0) = 0 \) such that

\[
\Phi^{-1} \circ f \circ \Phi(x, y) = (\lambda x + x^2\varphi(y), y^d(1 + \mathcal{O}(x)))
\]

for some analytic function \( h_2 \). Indeed, \( \varphi \) is then characterized by the equation

\[
\lambda x(1 + \varphi(y))(1 + g_1(y)) + \mathcal{O}(x^2) = (\lambda x + \mathcal{O}(x^2))(1 + \varphi(y^d))
\]

that is, \((1 + \varphi(y))(1 + g_1(y)) = (1 + \varphi(y^d))\), a solution of which is given by the infinite product

\[
1 + \varphi(y) = \prod_{k=0}^{\infty} \left( 1 + g_1 \left( y^d^k \right) \right)^{-1}
\]

and we are done. \( \Box \)
The next result is similar to [11, Lemma 4.2].

**Proposition 2.3.** Suppose $f$ is an analytic map of the form

$$f(x, y) = (\lambda x(1 + xyg(x, y)), y^{d}(1 + xh(x, y)))$$

where $|\lambda| > 1$, $d \geq 2$ and $g, h$ are analytic functions.

Then $f^n\left(\frac{x}{\lambda^n}, y\right) \to (x, 0)$ when $n \to \infty$, uniformly on a polydisk of sufficiently small radius centered at the origin.

**Proof.** Fix $0 < r \leq 1/4$ small enough so that $g, h$ are both analytic on the polydisk of radius $r$ and $|g(x, y)|, |h(x, y)| \leq 1$ for $|x|, |y| < r$. Let us first show that if $|x| \leq \frac{r}{2|\lambda|^n}$ and $|y| \leq r$, then the $n$ first iterates of $(x, y)$ remain in $\mathbb{D}^2_r$. We argue by induction. So assume that $(x_0, y_0) \in \mathbb{D}_{r|\lambda|^{-n/2}} \times \mathbb{D}_r$, put $f^j(x_0, y_0) = (x_j, y_j)$, let $k \leq n$ and assume that $(x_j, y_j) \in \mathbb{D}^2_r$ for $j \leq k - 1$. Observe that for $j \leq k - 1$, $|y_j| \leq 2|y_j|^d$ from which it follows that

$$|y_k| \leq \left(2^{1/(d-1)}|y_0|\right)^d \leq \left(2^{1/(d-1)}r\right)^d \leq r.$$

Observe that the first inequality together with $r \leq 1/4$ also yield $|y_j| \leq 2^{-d^j}$. For the first coordinate, we use recursively the relation $x_{j+1} = \lambda x_{j}(1 + x_j y_j g(x_j, y_j))$ to get

$$|x_k| \leq |\lambda|^{k} |x_0| \prod_{j=0}^{k-1} (1 + |x_j||y_j|) \leq \frac{r}{2} |\lambda|^{k-n} \prod_{j=0}^{k-1} \left(1 + 2^{-d^j}\right) \leq r |\lambda|^{k-n},$$

where the last inequality follows from

$$\prod_{j=0}^{k-1} \left(1 + 2^{-d^j}\right) \leq \prod_{j=0}^{k-1} \left(1 + 2^{-2^j}\right) = \frac{2^{2^k} - 1}{2^{2^k} - 1} < 2,$$

in which the middle equality is easily obtained by induction.

Now take $(x, y) \in \mathbb{D}_{r/2}$, and consider $f^n\left(\frac{x}{\lambda^n}, y\right)$. Denote as before $(x_0, y_0) = (\frac{x}{\lambda^n}, y)$ and $(x_j, y_j) = f^j(x_0, y_0)$. The first part of the proof shows that $(x_j, y_j)$ is well-defined for all $j \leq n$, and that $y_n \to 0$. Now we have

$$x_n = \lambda^n x_0 \prod_{j=0}^{n-1} (1 + x_j y_j h(x_j, y_j)) = x \prod_{j=0}^{n-1} (1 + x_j y_j h(x_j, y_j)).$$

The inequality $|\prod (1 + z_j) - 1| \leq \exp(\sum |z_j|) - 1$ shows that to establish the convergence $x_n \to x$ it is enough to show that $\sum_{j=0}^{n-1} |x_j y_j h(x_j, y_j)|$ tends to 0. But by (9), $|x_j| \leq r|\lambda|^{j-n}$, thus

$$\sum_{j=0}^{n-1} |x_j y_j h(x_j, y_j)| \leq \sum_{j=0}^{n-1} r |\lambda|^{j-n} 2^{-d^j} \leq r |\lambda|^{-n} \sum_{j=0}^{\infty} |\lambda|^{j-2^{-d^j}},$$

and we are done. \qed
2.3. Graph transform for $\lambda = 1$. In this paragraph we assume that $k = \mathbb{C}$ and $f$ is of the form

$$(10) \quad f(x, y) = \left( x + g(x, y), y^d(1 + h(x, y)) \right),$$

with $g(x, y) = \mathcal{O}(|x|^2)$, $h(0, 0) = 0$, and $g(x, 0) = cx^{k+1} + \mathcal{O}(x^{k+2})$ for some $k \geq 1$ and $c \neq 0$. Observe that $f|_{y=0}$ has a parabolic point at the origin with $k$ attracting and $k$ repelling petals (see e.g. [1, §6.5]). An attracting petal is a $f$-invariant (connected and simply-connected) open subset $U$ containing $0$ in its boundary, and such that, for all $z \in U$, $f^n(z) \to 0$ tangentially to some real direction (in our case, to the normalized $k$-th roots of $-c$). A repelling petal is an attracting petal for $f^{-1}$ (they are tangent to the normalized $k$-th roots of $c$). One can chose the $k$ attracting petals and $k$ repelling petals so that their union fills up a punctured neighborhood of the origin.

A vertical graph $V$ in a domain of the form $\Omega \times \mathbb{D}_r$ is a submanifold of the form $V := \{(\varphi(y), y), y \in \mathbb{D}_r\}$ for some holomorphic function $\varphi : \mathbb{D}_r \to \Omega$. In the next theorem we consider pull backs of such graphs in $\mathbb{D}_r^2$ in the graph transform sense, that is, when pulling back some vertical graph under $f$, we keep only the component of $f^{-1}(V) \cap \mathbb{D}_r^2$ containing $f^{-1}(V \cap \{y = 0\})$. Abusing notation we simply denote it by $f^{-1}(V)$.

**Theorem 2.4.** Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be of the form (10) with $h(0) = 0$, $g(x, 0) \neq 0$ and $g(x, y) = \mathcal{O}(|(x, y)|^2)$. Let $U$ be any repelling petal of $f|_{y=0}$ and consider a germ $V$ of analytic curve transverse to $\{y = 0\}$, intersecting $U$.

Then there exists $r > 0$ depending only on $f$, such that for large enough $n$, the analytic sets $f^{-n}(V)$ are vertical graphs in $\mathbb{D}_r^2$ converging to $W_{\text{loc}}^0(0)$ in the $C^1$ topology.

**Lemma 2.5.** Suppose $f$ is a holomorphic map of the form (10) as in Theorem 2.4. Then there exist an integer $k \geq 1$, and an analytic change of coordinates $\Phi$ such that

$$\Phi^{-1} \circ f \circ \Phi(x, y) = (x + x^{k+1} + x^{2k+1}\tilde{g}(x, y), y^d(1 + x\tilde{h}(x, y)))$$

for some analytic functions $\tilde{g}, \tilde{h}$.

**Proof.** The proof is essentially contained in [20, Proposition 2.3]. We provide the details for the sake of completeness. By Theorem 2.4 and by applying the Böttcher theorem to $y \mapsto f(0, y)$ we may assume that both $g$ and $h$ are divisible by $x$, so that we may write

$$(11) \quad f(x, y) = \left( x + xg(x, y), y^d(1 + xh(x, y)) \right),$$

with $g(0, 0) = 0$.

By a local change of coordinates involving only $x$, we can arrange so that $f(x, 0) = (x + x^{k+1} + \mathcal{O}(x^{2k+1}), 0)$. Expand the first coordinate of $f$ in power series of $x$ as follows:

$$x \circ f(x, y) = x(1 + g_0(y)) + x^{k+1}(1 + g_k(y)) + \sum_{j \neq 0,k}^{\infty} x^{j+1}g_j(y),$$

with $g_j(0) = 0$ for $0 \leq j \leq 2k - 1$.

We claim that for all $n \leq 2k - 1$, we can conjugate $f$ by a germ of invertible holomorphic map such that $g_j \equiv 0$ for every $j \leq n$. Applied to $n = 2k - 1$, this claim implies the proposition.
For \( n = 0 \) this is done by a change of coordinates of the form \( \Phi_0(x, y) := (x(1 + \varphi_0(y)), y) \) such that \((1 + g_0)(1 + \varphi_0) = 1 + \varphi_0(y^d)\). This equation can be solved exactly as in Lemma 2.2.

Now assume that \( n > 0 \), and that the result has been achieved up to \( j = n - 1 \). Put \( \Phi_n(x, y) = (x(1 + \varphi_n(y)x^n), y) \), so that \( \Phi_n^{-1}(x, y) = (x(1 - \varphi_n(y)x^n + O(x^{n+1})), y) \).

Depending on the position of \( n \) and \( k \), we obtain:

\[
x \circ (\Phi_n^{-1} \circ f \circ \Phi_n) = \begin{cases} 
  x + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n < k, \\
  x + x^{n+1}(\varphi_k(y) + 1 + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n = k, \\
  x + x^{k+1} + x^{n+1}(\varphi_n(y) + g_n(y) - \varphi_n(y^d)) + O(x^{n+2}) & \text{if } n > k.
\end{cases}
\]

Therefore, to render the term in \( x^{n+1} \) constant, it is enough to solve the equation \(-g_n(y) = \varphi_n(y) - \varphi_n(y^d)\) which can be done by setting \( \varphi_n(y) = -\sum_{m=0}^{\infty} g_n(y^{dm}) \).

**Proof of Theorem 2.4.** By the previous lemma, we may suppose that

\[
f(x, y) = \left( x + x^{k+1} + x^{2k+1}g(x, y), y^d(1 + xh(x, y)) \right)
\]

with \( g \) and \( h \) holomorphic near the origin. Note that \( f|_{y=0} \) has a repelling petal along the positive real axis. Fix \( r > 0 \) such that \( f \) is holomorphic and injective on a neighborhood of \( \overline{D}_r \). Reducing and rotating the petal if necessary, we may assume that

\[
U := \left\{ x : \text{arg}(x) \in \left( -\frac{\pi}{4k}, \frac{\pi}{4k} \right), \ |x| < r \right\}.
\]

The holomorphic map \( z = (kx)^{-1} \) is univalent on \( U \times \overline{D}_r \), and takes its values in

\[
\Omega_R := \left\{ z : \text{arg}(z) \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right), \ |z| > R \right\},
\]

where \( R = (kr)^{-1} \). The expression of \( f \) in the coordinates \((z, y)\) is of the form

\[
f(z, y) = \left( z - 1 + \frac{1}{z}a(z, y), y^d \left( 1 + \frac{1}{z^{1/k}}b(z, y) \right) \right) .
\]

so that \( f \) is now defined in \( \Omega_R \times D_r \). Fix \( M > 0 \) such that

\[
|a|, \ |b|, \ \left| \frac{\partial a}{\partial z} \right|, \ \left| \frac{\partial b}{\partial z} \right|, \ \left| \frac{\partial a}{\partial y} \right|, \ \left| \frac{\partial b}{\partial y} \right| \leq M \text{ on } \Omega_R \times D_r,
\]

and reduce \( r \) if necessary so that \( r < \frac{1}{100} \) and \( MR^{-1/k} \leq \frac{1}{100} \).

For any \( \rho < r \) and \( \sigma > 0 \) we let

\[
\mathcal{G}(\rho, \sigma) = \left\{ \varphi : D_\rho \to \Omega_R \text{ holomorphic s.t. } \sup_{D_\rho} |\varphi'| \leq \sigma \right\}.
\]

**Lemma 2.6.** Suppose that \( \sigma \rho < \frac{1}{1000} \). For any vertical graph \( \Gamma \) determined by \( \varphi \in \mathcal{G}(\rho, \sigma) \), then \( f^{-1}\Gamma \) is a vertical graph determined by a function \( \psi \in \mathcal{G}(\rho_1, \frac{1}{10}) \) where \( \rho_1 = \min \left( (\rho/2)^{1/d}, r \right) \) and \( \Re(\psi) \geq \Re(\varphi(0)) + 9/10 \).

Assuming this result for the moment, let us conclude the proof of the theorem. Let \( V \) be any germ of curve intersecting transversally \( \{ y = 0 \} \) at \( (x_0, 0) \in U \). Then \( V \) is a graph of slope \( \sigma \) over some disk \( D_\rho \) in the second coordinate, for some \( \rho > 0 \). Reduce \( \rho \) by trimming \( V \) if necessary so that \( \sigma \rho < 1/100d \). Since \( r/10 < 1/100d \), the previous
Theorem 3.1. Let $\mathbb{Q}$ be a number field. Suppose that

$$V_n := f^{-1}V_{n-1},$$

where $V_n$ is defined by a holomorphic function $z = \varphi_n(y)$ of uniformly bounded slope over $\mathbb{D}_{\rho_n}$, and furthermore $\rho_n = r$ for large enough $n$. Moreover, we have $\text{Re}(\varphi_n) \geq \text{Re}(\varphi(0)) + 9n/10$, hence, coming back to the $(x,y)$ coordinates, we see that $V_n = \{x = (k\varphi_n(y))^{1/k}\}$ converges in the $C^1$-topology to the curve $W_{\text{loc}}(0) = \{x = 0\}$. □

Proof of Lemma 2.6. Let $\Gamma$ be a vertical graph of equation $z = \varphi(y)$ in the $(z,y)$ coordinates, with $\varphi \in \mathcal{G}(\rho,\sigma)$. Then the equation of $f^{-1}\Gamma$ is given by $z = \ell(z,y)$, where

$$\ell(z,y) = \varphi\left(y^d\left(1 + \frac{1}{z^{1/k}}b(z,y)\right)\right) + 1 - \frac{1}{z}a(z,y).$$

Fix $y_0 \in \mathbb{D}_{\rho_1}$. We show that the equation $z = \ell(z,y_0)$ admits a unique solution $z \in \Omega_R$. First, observe that by the estimates [13] on $|a|$ and $|b|$, for $z \in \Omega_R$ we have

$$|\ell(z,y_0) - \varphi(0)| + 1| \leq \left|\varphi\left(y_0^d\left(1 + \frac{1}{z^{1/k}}b(z,y)\right)\right) - \varphi(0)\right| + \frac{M}{R} \leq 2\rho_1^d|\sigma - \frac{1}{100}| \leq 2\rho_1^d|\sigma + \frac{1}{100} | \leq \frac{1}{10},$$

hence $\ell(\cdot,y_0)$ maps $\Omega_R$ to itself. Next, we see that

$$\left|\frac{\partial \ell}{\partial z}(z,y_0)\right| \leq \left|y_0^d\left(1 + \frac{1}{z^{1/k}}\right)\frac{\partial b}{\partial z} - \frac{1}{z^{1/k+1}}b\right| \cdot \left|\varphi'\left(y_0^d\left(1 + \frac{1}{z^{1/k}}b\right)\right)\right| + \left|\frac{1}{z} - \frac{1}{z^{1/k}}\right| \leq \rho_1^d\frac{2M}{R^{1/k}} + \frac{2M}{R} \leq \frac{2}{100} |\sigma + \frac{1}{100} | \leq \frac{1}{10},$$

so $\ell(\cdot,y_0)$ is a contraction and the equation $z = \ell(z,y_0)$ has a unique solution. This means that $f^{-1}\Gamma$ is a vertical graph over $\mathbb{D}_{\rho_1}$ determined by a holomorphic function $\psi$ satisfying $\psi(y) = \ell(\psi(y),y)$. The slope of this graph can be estimated as above:

$$\left|\psi' (y)\right| \leq \frac{\left|\frac{\partial \ell}{\partial y}\right|}{\left|1 - \frac{\partial \ell}{\partial z}\right|} \leq \frac{10}{9} \left(\sigma \left(\frac{M}{R^{1/k}} + \frac{\rho_1^d M}{R^{1/k}}\right)\right) \leq \frac{1}{10}.$$

Finally, the estimate $\text{Re}(\psi) \geq \text{Re}(\varphi(0)) + 9/10$ follows from [14] and we are done. □

3. Proof of Theorem A when $k$ is a number field

Here we establish the following more precise form of our main theorem, in the number field case.

**Theorem 3.1.** Let $k$ be a number field and $f$ be a regular polynomial map of $\mathbb{A}_k^2$. Denote by $h_f$ the induced canonical height.

Suppose that $C \subset \mathbb{A}_k^2$ is an irreducible algebraic curve containing a sequence of distinct points $p_n \in C(k^{alg})$ such that $h_f(p_n) \to 0$. If there exists a point of $\mathbb{C} \cap L_\infty$ which is not eventually superattracting, then $C$ is preperiodic.

Let $f_\infty$ be the restriction to the line at infinity $L_\infty$ of the extension of $f$ to $\mathbb{P}^2$. By Proposition 1.7, all points in $\mathbb{C} \cap L_\infty$ are preperiodic, so we may replace $f$ by $f^N$ and $C$ by $f^N(C)$, to assume that $\mathbb{C} \cap L_\infty$ contains a fixed point $p$ which is not super-attracting.
Let $\lambda = f_\infty'(p)$ be the multiplier of $p$ along $L_\infty$. Then one of the two following mutually disjoint cases occur:

(a) either $\lambda$ is a root of unity;
(b) or there is a place $v \in M_\infty$ such that $|\lambda|_\infty > 1$.

In the remainder of this section we split the proof of the theorem according to these two cases.

3.1. When $\lambda$ is a root of unity. In this situation we iterate $f$ further so that $\lambda = 1$, and work over the complex numbers. Since $p$ belongs to the Julia set $J(f_\infty)$, which is a perfect set, the union of attracting and repelling petals cover a punctured neighborhood of $p$, and the attracting petals of $p$ are contained in the Fatou set, we see that there is a repelling periodic point $q$ of $f_\infty$ contained in some local repelling petal of $p$. Then the local (super-)stable manifold of $q$ is a disk transverse to $L_\infty$ at $q$, and by Theorem 2.4 the local truncated pull-backs $f^{-n}(W_{\text{loc}}^s(q))$ under $f^n$ converge to $W_{\text{loc}}^s(p)$ when $n \to \infty$.

Assume by way of contradiction that $\overline{C}$ does not locally coincide with $W_{\text{loc}}^s(p)$. We claim that $\overline{C}$ intersects $f^{-n}(W_{\text{loc}}^s(q))$ in $\mathbb{C}^2$ close to $p$ for large $n$. Indeed, locally near $p$, $\overline{C} \cap W_{\text{loc}}^s(p) = \{p\}$, so by the persistence of proper intersections, $\overline{C}$ intersects $f^{-n}(W_{\text{loc}}^s(q))$ close to $p$ for large $n$. But $p \notin f^{-n}(W_{\text{loc}}^s(q)) \cap L_\infty$, so these intersection points lie in $\mathbb{C}^2$, as claimed. If now $\Delta$ is a small disk in $C$ containing one of these intersection points, then by the Inclination Lemma, the derivative of $f^n$ in the direction of $\Delta$ tends to infinity, thus $(f^n|_{\Delta})$ is not a normal family. On the other hand, by Proposition 1.7 $g|_{\Delta}$ is harmonic, which implies that $(f^n|_{\Delta})$ is normal (see [13, Prop 5.10]). This contradiction shows that $\overline{C}$ locally coincides with $W_{\text{loc}}^s(p)$ near $p$, so it is fixed under $f$, and by irreducibility this property propagates to the whole of $C$. This completes the proof in this case.

Remark 3.2. This argument works essentially the same when $|\lambda|_v > 1$ at some Archimedean place, and may help understand the non-Archimedean argument below.

3.2. When $|\lambda|_v > 1$. We may assume that $p = [0 : 0 : 1]$, and by Lemma 2.2 find a local analytic isomorphism $(x,y) \mapsto \psi(x,y) = [z_0(x,y) : z_1(x,y) : 1]$ such that $\psi(0) = p$, $z_0(x,0) = 0$ so that $\{y = 0\}$ corresponds to the line at infinity,$^1$ and write

$$\tilde{f} := \psi^{-1} \circ f \circ \psi : (x,y) \mapsto \left(\lambda x(1 + xyg(x,y)), y^d(1 + xh(x,y))\right).$$

If $\overline{C}$ has an analytic branch at $p$ which coincides with $\{x = 0\}$ then $C$ is fixed as above, and we are done. Otherwise, we may find a Puiseux parameterization of a branch of $\overline{C}$ at $p$ in the $(x,y)$ coordinates of the form $\Gamma(t) = ((t^\theta, \gamma(t)))$, where $\gamma$ is analytic and defined in a small disk $\mathbb{D}_\delta$, and $q \in \mathbb{N}^*$. We seek a contradiction.

We lift $\psi$ to $\mathbb{A}^3_k$, and set $\Psi(x,y) = (z_0(x,y), z_1(x,y), 1)$. As in §1.3 we lift $f$ to a homogeneous polynomial map $F : \mathbb{A}^3_k \to \mathbb{A}^3_k$, of the form $F(z_0, z_1, z_2) = (z_0^d \tilde{P}, \tilde{Q})$. Since $\Psi \circ \psi^{-1}$ is a local section of the projection $\mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$, $\Psi \circ \tilde{f}$ must be a multiple of $F \circ \Psi$. From the expression of $F$ we obtain

$$\Psi \circ \tilde{f} = \frac{F \circ \Psi}{Q \circ \Psi}.$$

$^1$Beware that coordinates are swapped here: $\{z_0 = 0\}$ corresponds to $\{y = 0\}$.
To simplify notation, we write $F^n(z_0, z_1, z_2) = (x^n, \tilde{P}_n, \tilde{Q}_n)$. We consider the 1-homogeneous Green function $G_f : \mathbb{A}^2_{\text{an}} \to \mathbb{R}$ of Proposition 1.3; it satisfies $G_f \circ F = dG_f$ and $g_f(z_1, z_2) = G_f(1, z_1, z_2)$.

Observe that $h := G_f \circ \Psi \circ \Gamma$ is a continuous function on $\mathbb{D}_\delta$. Since $h(t) = g_f \circ \psi \circ \Gamma(t) + \log |z_0 \circ \Gamma(t)|$, by Proposition 1.7, $h$ is harmonic on $\{t \neq 0\}$. Since it is continuous at 0, it is also harmonic on $\mathbb{D}_\delta$, see, e.g., [12, Lemma 3.7]. Write $d^{\text{eq}} h(t) = G_f \circ F^{\text{eq}} \circ \Psi \circ \Gamma(t) = G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(t) + \log |\tilde{Q}_{nq} \circ \Psi \circ \Gamma(t)|$.

By Proposition 2.3, $\tilde{f}[n](x, y) \to (x, 0)$ uniformly in a neighborhood of the origin, hence $G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(\frac{t}{x^n/q}) \to G_f(0, z_1(t^n, 0), 1)$ as $n \to \infty$. On the other hand, since $d^{\text{eq}} h(t) = G_f \circ \Psi \circ \tilde{f}[nq] \circ \Gamma(\frac{t}{x^n/q}) \to G_f(0, z_1(t^n, 0), 1)$ is a sequence of harmonic functions, it follows that $t \mapsto G_f(0, z_1(t^n, 0), 1)$ is harmonic as well.

Now, observe that the restriction of $f$ to the line at infinity is $f_\infty[z_1 : z_2] = [\tilde{P}(0, z_1, z_2) : \tilde{Q}(0, z_1, z_2)]$, so that $G_f(0, z_1, z_2)$ is the global Green function of $f_\infty$. The equilibrium measure of $f_\infty$ is the probability measure on the analytification of $L_\infty$ defined by $\mu_{f_\infty} := \Delta G_f(0, z_1, 1)$ in the chart $z_2 \neq 0$. Its support is the Julia set of $f_\infty$ (see [2, Theorem 13.39]), and it contains all repelling (rigid) fixed points, see [2, Theorem 8.7]. Therefore, $z_1 \mapsto G_f(0, z_1, 1)$ cannot be harmonic near 0, hence the function $t \mapsto G_f(0, z_1(t^n, 0), 1)$ cannot be harmonic either. This contradiction concludes the proof.

Remark 3.3. Under the assumptions of Theorem 3.1, the proof shows that the preperiodic points and whose closure $\overline{C} \cap L_\infty$ contains at least one point which is not eventually super-attracting.

4. Proof of Theorem A for arbitrary $\mathbb{k}$

In this section we use a specialization argument to deal with maps defined over arbitrary fields. It shares some arguments with [11, §5] (see also [5, §7]). Nevertheless, new ideas are needed to deal with preperiodic points instead of periodic ones.

We are in the setting of Theorem A, so we assume that $f$ is a regular polynomial map of $\mathbb{A}^2$ of degree $d \geq 2$ defined over a field $\mathbb{k}$ of characteristic 0, and $C$ is a curve containing an infinite set $\mathcal{P} = \{p_n, n \geq 0\}$ of preperiodic points and whose closure $\overline{C} \cap L_\infty$ contains at least one point which is not eventually super-attracting.

By enlarging $\mathbb{k}$ if necessary we may assume that it contains the algebraic closure $\mathbb{Q}^{\text{alg}}$ of its prime field. Let $R$ be the sub-$\mathbb{Q}^{\text{alg}}$-algebra of $\mathbb{k}$ generated by all coefficients defining $f$ and $C$. Its fraction field $K$ is finitely generated over $\mathbb{Q}^{\text{alg}}$. Let $S = \text{Spec } R$. This is an affine variety defined over $\mathbb{Q}^{\text{alg}}$, and elements of $R$ can be seen as regular functions on $S$.

Inverting some elements of $R$ if necessary, we may suppose that $C$ is flat over $S$, and $f$ extends as a morphism $f : \mathbb{P}^2_S \to \mathbb{P}^2_S$. We let $\pi : \mathbb{P}^2_S \to S$ be the canonical projection, and write $\mathbb{P}^2_S = \pi^{-1}(s)$. We also let $\mathbb{A}^2_S := \mathbb{A}^2_S \cap \pi^{-1}(s)$. 
For each (scheme theoretic) point \( s \in S \), we write \( C_s = C \cap \mathbb{A}^2_s \) and let \( \overline{C}_s \) be the closure of \( C_s \) in \( \mathbb{P}^2_s \). The flatness of the morphism \( C \to S \) implies \( \overline{C}_s \) (hence \( C_s \)) to be a curve. Similarly, we let \( f_s : \mathbb{P}^2_s \to \mathbb{P}^2_s \) be the induced map on the fibers: this is an endomorphism of degree \( d \).

We also denote by \( p_{n,s} \in \mathbb{A}^2_s \) the specialization of \( p_n \). Note that \( p_n \) is defined over some finite extension of \( K \) which depends on \( n \).

The first result does not use the assumption that our infinite set of preperiodic points lies on a curve.

**Proposition 4.1.** Let as above \( f : \mathbb{P}^2_S \to \mathbb{P}^2_S \) be a family of endomorphisms over an affine variety defined over \( \mathbb{Q}_{\text{alg}} \), and let \( \mathcal{P} = \{ p_n, \ n \geq 0 \} \) be an infinite family of preperiodic points. Then there exists a non-empty Zariski open and dense subset \( \mathcal{U} \subset S \) such that for any \( s \in \mathcal{U} \cap S(\mathbb{Q}_{\text{alg}}) \), \( \mathcal{P}_s = \{ p_{n,s}, \ n \geq 0 \} \subset \mathbb{A}^2_s \) is infinite.

Before starting the proof, let us fix some additional notation. For each \( n \geq 0 \), we denote by \( k_n \) the preperiod of \( p_n \), so that \( q_n := f^{k_n}(p_n) \) is the first periodic point in the orbit of \( p_n \). We let \( \ell_n \) be the (primitive) period of \( q_n \).

**Proof.** We may suppose that there exists a parameter \( s_0 \in S(\mathbb{Q}_{\text{alg}}) \) such that \( \mathcal{P}_{s_0} \) is finite (otherwise we take \( \mathcal{U} = S \) and the proof is complete).

**Lemma 4.2.** The family of periodic points \( (q_n) \) is finite.

**Proof.** We follow the arguments of [11, §5]. Set \( \mathcal{Q} = \{ q_n, \ n \geq 0 \} \). For each \( \ell \geq 1 \), we consider the subvariety \( \text{Per}_\ell \) of \( \mathbb{P}^2_S \) defined by the equation \( f^\ell(z) = z \). Since \( \mathbb{P}^2_S \to S \) is proper, the structure map \( \text{Per}_\ell \to S \) is also proper.

Let \( \mathcal{Q}_\ell \) be the union of the irreducible components of \( \text{Per}_\ell \) containing a point of \( \mathcal{P} \). Its underlying set is the Zariski closure of \( \mathcal{P} \cap \text{Per}_\ell \), hence \( \mathcal{Q}_\ell \to S \) is proper. Observe that for \( x \in \mathcal{Q}_\ell,s \), the multiplicity of \( x \) as a point of \( \mathcal{Q}_\ell,s \) equals its multiplicity as a fixed point of \( f^\ell_s \). By Nakayama’s lemma and the properness of \( \mathcal{Q}_\ell \) over \( S \), the function
\[
(15) \quad s \mapsto \sum_{x \in \mathcal{Q}_\ell,s} \text{mult}_x(\mathcal{Q}_\ell,s)
\]
is upper semi-continuous for the Zariski topology, hence
\[
(16) \quad \sum_{q \in \mathcal{Q}_\ell} \text{mult}_q(\mathcal{Q}_\ell) \leq \sum_{x \in \mathcal{Q}_\ell,s_0} \text{mult}_x(\mathcal{Q}_\ell,s_0),
\]
where the left hand side is the value of \((15)\) at the generic point. By assumption \( \mathcal{P}_{s_0} \) is a finite set, hence so does \( \mathcal{Q}_{s_0} = \{ q_{1,s_0}, \ldots, q_{r,s_0} \} \) and by the Shub-Sullivan theorem [29], there exists a uniform bound \( C > 0 \) such that for every \( j \), and for any \( \ell \), we have
\[
\text{mult}_{q_{j,s_0}}(\mathcal{Q}_\ell,s_0) \leq \text{mult}_{q_{j,s_0}}(\text{Per}_\ell,s_0) \leq C.
\]
It then follows from \((16)\) that
\[
\# \mathcal{Q}_\ell \leq \sum_{q \in \mathcal{Q}_\ell} \text{mult}_q(\mathcal{Q}_\ell) \leq rC
\]
hence \( \bigcup_\ell \mathcal{Q}_\ell \) is finite, as was to be shown. \( \square \)
By the previous lemma, replacing \( f \) by some iterate \( f^N \) we may assume that all periodic points \( q_s \) are fixed. Since \( \mathcal{P} \) is infinite, one of these fixed points, say \( q_1 \), admits infinitely many preimages in \( \mathcal{P} \). We may denote \( q = q_1 \) and suppose \( \mathcal{P} \) is made of an infinite set of preimages of \( q \), that is, (after possible reordering of \( \mathcal{P} \)) for any \( p_n \in \mathcal{P} \) there is a minimal \( k_n \geq 0 \) such that \( f^{k_n}(p_n) = q \), and that \( k_{n+1} > k_n \). We may adjoin to \( R \) the coordinates of \( q \) so that \( q \in \mathbb{A}^2(R) \), i.e., for any \( s \in S \), \( q_s \) is a single point (to say it differently, we replace \( S \) by a branched cover of a Zariski open dense subset of \( S \)).

Let \( d(s) \) be the local degree of \( f_s \) at \( q_s \), which is upper semicontinuous for the Zariski topology. Since \( f_s \) is a finite map of degree \( d^2 \), \( d(s) \leq d^2 \) for every \( s \). Thus there is an analytic hypersurface \( H \) such that \( d(s) = d_{\min} \) is constant for \( s \in S \setminus H \).

We claim that \( \mathcal{P}_{s_1} \) is infinite for any \( s_1 \in S \setminus H \). We argue again in the complex analytic category fixing an embedding \( \mathbb{Q}^{\text{alg}} \) into \( \mathbb{C} \). Observe that for any point \( s \in S(\mathbb{C}) \), \( p_{n,s} \) is a finite set included in the fiber \( \mathbb{A}^{\text{an}}_s \simeq \mathbb{C}^2 \) (not necessarily reduced to a single point since \( p_n \) lies in a finite extension of \( R \)).

Fix an analytic neighborhood \( V \) of \( q_{s_1} \) in \( \mathbb{P}^{\text{an}}_s(\mathbb{C}) \) such that \( f_{s_1}^{-1}(q_{s_1}) \cap V = \{ q_{s_1} \} \). Since \( d(s) \) is locally constant near \( s_1 \), there is an analytic neighborhood \( W \) of \( s_1 \) in \( S^{\text{an}}(\mathbb{C}) \) such that for \( s \in W \),

\[
(17) \quad f_s^{-1}(q_s) \cap V = \{ q_s \}
\]

Choose any \( n > m \), and suppose by contradiction that \( p_{m,s_1} = p_{n,s_1} \). Since \( k_n - 1 \geq k_m \), we have

\[
f_{s_1}^{-k_n}(p_{n,s_1}) = f_{s_1}^{-k_m}(p_{m,s_1}) = q_{s_1}.
\]

Thus, for \( s \) close to \( s_1 \), the finite set \( f_{s_1}^{-k_n}(p_{n,s_1}) \) is contained in \( V \), hence by (17), \( f_s^{-k_n}(p_{n,s}) = q_s \), and by analytic continuation this property holds throughout \( S \), which contradicts the definition of \( k_n \).

This shows that the \( p_{n,s} \) are all distinct for all \( s \in S \setminus H \), and concludes the proof of Proposition 4.1. \( \square \)

**Proposition 4.3.** Let \( f : \mathbb{A}^2_k \to \mathbb{A}^2_k \) be a regular polynomial map and \( C \subset \mathbb{A}^2_k \) be an algebraic curve containing infinitely many preperiodic points. Then every point of \( \overline{C} \cap L_\infty \) is preperiodic under \( f|_{L_\infty} \).

**Proof.** We keep the same formalism and notation as above, so that \( f \) is viewed as a family over \( S \). Write \( C \cap L_\infty = \{ c_1, \ldots, c_r \} \) and without loss of generality enlarge \( R \) so that the points at infinity \( c_i \) have their coordinates in \( R \). Fix \( i \in \{ 1, \ldots, r \} \) for the remainder of the proof and consider \( c = c_i \). By Proposition 4.1 there is a Zariski open subset \( U \) such that for any \( s \in U \cap S(\mathbb{Q}^{\text{alg}}) \), \( f_s \) admits infinitely many preperiodic points on \( C_s \). Therefore, by Proposition 1.7, for every such \( s \), \( c_s \) is preperiodic. Fix \( s_0 \in U \cap S(\mathbb{Q}^{\text{alg}}) \), then \( c_{s_0} \) eventually falls on a periodic point \( q_{s_0} \). Replacing \( f \) by \( f^N \) and \( C \) by \( f^N(C) \) for some \( N \), we may assume that \( q_{s_0} \) is fixed and \( c_{s_0} = q_{s_0} \). Enlarging \( R \) again if necessary we may assume that \( q_{s_0} \) is the specialization at \( s = s_0 \) of a fixed point \( q \in \mathbb{P}^2(R) \) of \( f \).

Our purpose is to show that \( c = q \). To simplify notation we write \( \hat{f} = f|_{L_\infty} \). Note that the multiplier \( \mu := \hat{f}'_{s_0}(q_{s_0}) \) belongs to \( \mathbb{Q}^{\text{alg}} \). It follows from Kronecker’s theorem that either \( \mu \) is a root of unity or there is a place \( v \) on \( \mathbb{Q}^{\text{alg}} \) such that \( |\mu|_v < 1 \).
Case 1. \( \mu \) is not a root of unity.

Fix a place \( v \) on \( \mathbb{Q}^{\text{alg}} \) such that \( |\mu|_v < 1 \), and consider the completion \( \mathbb{C}_v \) of \( (\mathbb{Q}^{\text{alg}}, |\cdot|_v) \). We then argue in the analytic topology in the Berkovich analytification of \( \mathbb{P}^2_{\mathbb{C}_v} \) and \( S_{\mathbb{C}_v} \).

Fix a neighborhood \( W \) of \( s_0 \in S_{\mathbb{C}_v}^{\text{an}} \) such that for \( s \in W \), \( q_s \) is attracting, and a neighborhood \( V \) of \( q_s \) in \( L_\infty \) independent of \( s \in W \) such that \( \hat{f}_s(V) \subset V \) and for any \( z \in V \), \( f_s^n(z) \) converges to \( q_s \) as \( n \to \infty \). Reducing \( W \) if necessary we may assume that for any \( s \in W \), \( c_s \) belongs to \( V \). For \( s \in W \cap S(\mathbb{Q}^{\text{alg}}) \), \( c_s \) is preperiodic and converges to \( q_s \), so it is preperiodic to \( q_s \), that is, there exists a minimal \( k = k(s) \) such that \( f^{k(s)}(c_s) = q_s \). Now we use an argument similar to that of Proposition 4.1 let \( H \subset S \) be a hypersurface such that the local degree of \( f_s \) at \( q_s \) is locally minimal outside \( H \) and fix \( s_1 \in W \setminus H \). Then, there is a neighborhood \( W_1 \) of \( s_1 \) in \( W \setminus H \) and a neighborhood \( V_1 \subset V \) of \( q_1 \) such that for any \( s \in W_1 \), \( f_s(V_1) \subset V_1 \) and \( f_s^{-1}(q_s) \cap V_1 = \{ q_s \} \). From this it follows that the only point eventually falling onto \( q_s \) in \( V_1 \) is \( q_s \) itself. Therefore if \( s \in W_1 \) is so close to \( s_1 \) that \( c_s \in V_1 \), we infer that \( c_s = q_s \), and finally \( c = q \) by analytic continuation.

Case 2. \( \mu \) is a root of unity.

To deal with this case we embed \( \mathbb{Q}^{\text{alg}} \) into \( \mathbb{C} \), and work at the complex place. Recall that a holomorphic family \( (g_s)_{\lambda \in \Lambda} \) of rational maps on \( \mathbb{P}^1(\mathbb{C}) \), parameterized by a connected complex manifold is trivial if any two members are conjugate by a Möbius transformation, depending holomorphically on \( \Lambda \). If \( c \) is persistently preperiodic we are done, so assume that \( c \) is not persistently preperiodic.

Under our assumptions, there is a dense set \( S(\mathbb{Q}^{\text{alg}}) \) of parameters such that \( c_s \) is preperiodic, but \( c \) is not persistently preperiodic. Thus by Chio-Roeder [7, Theorem 2.7] (see also [10, Theorem 4]) every such parameter belongs to the bifurcation locus of the marked family \( (f_s, c) \) (note that \( c \) is not a critical point here). As a consequence, the bifurcation locus of the family is equal to \( S_{\mathbb{C}_v}^{\text{an}} \). (Note that it is enough to work in a neighborhood of \( s_0 \), away from possible singularities of \( S_{\mathbb{C}_v}^{\text{an}} \).)

A first possibility is that the family \( (\hat{f}_s)_{s \in S_{\mathbb{C}_v}^{\text{an}}} \) is non-trivial. Then Gauthier [14, Theorem A] implies that \( J(\hat{f}_s) = L_\infty \) for all \( s \). But since \( \hat{f}_{s_0} \) has a rationally indifferent fixed point, it admits an attracting petal and \( J(\hat{f}_{s_0}) \neq L_\infty \). This contradiction shows that the family \( (\hat{f}_s)_{s \in S_{\mathbb{C}_v}^{\text{an}}} \) is trivial.

Now the situation is that there is a holomorphic family \( \varphi_s \) of Möbius transformations such that \( \varphi_s \hat{f}_s \varphi_s^{-1} = g \) is a fixed rational map \( g \) on \( \mathbb{P}^1 \) with a rationally indifferent fixed point at 0. After this conjugacy, the marked family \( (\hat{f}, c) \) becomes \( (g, \varphi(c)) \). Since \( c \) coincides with \( q \) at \( s_0 \) and by assumption it is not persistently preperiodic, there is an open set \( \Omega \) of parameters such that for \( s \in \Omega \), \( \varphi_s(c_s) \) belongs to some attracting petal associated to 0 for \( g \). This contradicts the fact that \( \varphi_s(c_s) \) must be preperiodic for a dense set of parameters, and the proof is complete.

Conclusion of the proof of Theorem A. By Proposition 4.3 any point at infinity of \( C \) is preperiodic, and, by assumption, one of these points, say \( c \), is preperiodic to a non-superattracting periodic point \( p \). Replace \( f \) by \( f^N \) and \( C \) by \( f^N(C) \) for some \( N \), so that \( c = p \) is fixed. By Proposition 4.1 there is a non-trivial Zariski open subset \( U \subset S \)
such that for every $s \in U \cap S(Q_{\text{alg}})$, $C_s$ contains infinitely many preperiodic points. Then, since $p_s$ is fixed and not superattracting for $f_s$, Theorem 3.1 asserts that $C_s$ is preperiodic, and more precisely fixed, under $f_s$ (see Remark 3.3). The density of $U \cap S(Q_{\text{alg}})$ in $S$ (for the Zariski or analytic topology) then implies that $f(C) = C$, and the proof is complete. \(\square\)

References


