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Stability analysis of sampled-data systems with sector-bounded input nonlinearity^{*}

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Abstract: This work proposes a method for the stability analysis of aperiodic sampled-data control systems with sector and slope bounded input nonlinearities. The stability conditions are derived by using a hybrid system representation and a timer-dependent Lur'e type Lyapunov function. Considering a polynomial timer-dependence, the stability conditions are cast in sum-of-squares optimization problems aiming at computing the largest range of sampling intervals or the largest sector bounds on the nonlinearity for which the origin of the closed-loop system is globally asymptotically stable.

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Keywords: sampled-data systems; input nonlinearity; hybrid systems

1. INTRODUCTION

Motivated by the advent of control loops implemented in a shared network, the literature about stability of sampled-data systems with aperiodic sampling has grown rapidly in the last three decades (see Hétel et al. (2017) and references therein). For instance, Cloosterman et al. (2010) have developed a discrete representation for networked control systems subject to packet losses, aperiodic sampling and uncertain delays for the synthesis of stabilizing controllers. Based on the partition of the admissible sampling interval, Fujioka (2009) has proposed a technique using norm-bounded uncertainties to model the effects of aperiodic sampling. In Kao and Fujioka (2013), a method based on integral quadratic constraints is developed. There is also the modeling technique based on time-varying delays (Fridman (2010)) to express the aperiodic sampling. In Seuret (2012), an approach based on looped-functionals is established. In Naghshtabrizi et al. (2008) and Briat (2013), the sampled-data system is modeled by impulsive differential equations. This approach is also referred as the hybrid approach (Naghshtabrizi et al. (2006)), because the sampled-data system may be represented as a particular case of a hybrid dynamical system (Goebel et al., 2012).

It is worth noting that the references of the previous paragraph consider sampled-data systems composed by a continuous-time system with linear inputs. In practice, however, some nonlinearities can affect the input of the

controlled system. This can in fact model the effect of nonlinear actuators. Common actuator nonlinearities are saturation (Tarbouriech et al. (2011)), deadzone, and quantization (Ferrante et al., 2015), which are all sector-bounded.

This paper investigates the stability analysis of sampled-data systems with sector-bounded input nonlinearities, and aperiodic sampling in intervals that range from a lower to an upper bound, similarly to Ferrante and Tarbouriech (2022). In this case, the hybrid system representation is combined with a Lur'e type Lyapunov function candidate in order to reduce conservatism of the stability analysis. Assuming a polynomial timer dependence of the Lur'e type function, stability conditions are developed and cast as sum-of-squares (SOS) constraints. This combined approach, however, requires the development of relaxed conditions for the jump trajectories, to guarantee the decrease of the Lur'e type function candidate, in a similar problem found for discrete-time control systems subject to actuator amplitude saturation (Gomes da Silva Jr. et al. (2001) Haddad and Kapila (1995)). Based on the stability conditions, optimization problems are proposed to estimate the maximum sampling interval and the largest sector bounds for which the global asymptotic stability of the sampled-data system is ensured.

Notation: The vector v has Euclidean norm given by $|v|$. The distance of a vector v to a closed set \mathcal{A} is denoted $|v|_{\mathcal{A}}$ and is defined by $|v|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |v - y|$. The induced 2-norm of a matrix M is represented by $|M|$. \mathbb{S}^n is the set of symmetric matrices of size n , and for a symmetric matrix $S \in \mathbb{S}^n$, $S \succ 0$ means that S is positive definite. \mathbb{D}^n and $\mathbb{D}_{\geq 0}^n$ are the sets of the diagonal and positive semidefinite diagonal matrices of order n , respectively. M^{\top} denotes the transpose of M , and $\text{He}\{M\} = M^{\top} + M$. The matrices I_n

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and $0_{n \times p}$ denote an identity matrix of size n and a $n \times p$ matrix of zeros, respectively.

2. PROBLEM STATEMENT

2.1 Closed-loop system modeling

Consider the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ u(t) = \phi(y(t)) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the plant state, $y \in \mathbb{R}^p$ is the control signal, and $u \in \mathbb{R}^p$ is the plant input, resulting from the control signal subject to the actuator nonlinearity ϕ . The nonlinear function $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is decentralized, sector-bounded and slope-restricted, that is, it satisfies the following properties element-wise:

$$\phi_{(i)}(y) = \phi_{(i)}(y_{(i)}) \quad (2a)$$

$$\phi_{(i)}(0) = 0 \quad (2b)$$

$$\frac{\phi_{(i)}(y)}{y_{(i)}} \in [\underline{\delta}_{(i)}, \bar{\delta}_{(i)}] \quad (2c)$$

$$0 \leq \frac{\phi_{(i)}(y_b) - \phi_{(i)}(y_a)}{y_{b(i)} - y_{a(i)}} \leq \bar{\gamma}_{(i)}, \quad y_{b(i)} \neq y_{a(i)} \quad (2d)$$

for $i = 1, \dots, p$. Notice that in (2c), because of (2d), one has that $\underline{\delta}_{(i)} \geq 0$. It is also assumed that $\bar{\delta}_{(i)} \leq \bar{\gamma}_{(i)}$. For the next lemmas and the rest of the text, we define the diagonal matrices $\underline{\Delta} \triangleq \text{diag}\{\underline{\delta}_{(1)}, \dots, \underline{\delta}_{(p)}\}$, $\bar{\Delta} \triangleq \text{diag}\{\bar{\delta}_{(1)}, \dots, \bar{\delta}_{(p)}\}$ and $\bar{\Gamma} \triangleq \text{diag}\{\bar{\gamma}_{(1)}, \dots, \bar{\gamma}_{(p)}\}$, and the following function:

$$S_{\Delta}(U, y) \triangleq (\phi(y) - \underline{\Delta}y)^{\top} U (\bar{\Delta}y - \phi(y)) \quad (3)$$

From property (2c), the following sector condition can be stated (Valmorbidia et al., 2018):

Lemma 1. If ϕ satisfies (2c), then the relation

$$S_{\Delta}(U, y) \geq 0 \quad (4)$$

is satisfied for all $y \in \mathbb{R}^m$, with any matrix $U \in \mathbb{D}_{\geq 0}^m$.

We consider a sampled-data control law with sampling instants given by t_k , $k \in \mathbb{N}$ occurring in intervals that are possibly aperiodic, ranging from a lower value of T_m to an upper value of T_M , that is

$$0 < T_m \leq t_{k+1} - t_k \leq T_M \quad (5)$$

with $t_0 = 0$. Hence, we consider that system (1) is stabilized by a control law given as follows:

$$\begin{aligned} y(t) &= K_x x(t_k) + K_u u(t_{k-1}) \\ &= K_x x(t_k) + K_u \phi(y(t_{k-1})) \quad \forall t \in [t_k, t_{k+1}) \end{aligned} \quad (6)$$

where $K_x \in \mathbb{R}^{p \times n}$ and $K_u \in \mathbb{R}^{p \times p}$ are the controller gains. The control law (6) can be seen as the combination of a linear state feedback with a nonlinear feedback of the last computed control signal. The term depending on the control signal is in particular useful to the design when a hybrid system representation is considered (see, for instance, Huff et al. (2021), Huff et al. (2022), Fagundes and Gomes da Silva Jr. (2022)). Note that the particular case of a pure state feedback sampled-data control corresponds to (6) with $K_u = 0$. As in this paper we are concerned only by stability analysis, we assume that the gains K_x and K_u have been previously computed.

From the above setup, the analysis problems we intend to solve in this paper are described as follows:

P1. Given the control law (6), sector and slope bounds $\underline{\delta}_{(i)}$, $\bar{\delta}_{(i)}$, and $\bar{\gamma}_{(i)}$ for $i = 1, \dots, p$, and the lower interval of sampling T_m , find an estimate for the maximum T_M such that the closed-loop system given by (1) and (6) is globally asymptotically stable.

P2. Given the control law (6), the sampling interval limits T_m and T_M , and the lower sector bounds $\underline{\delta}_{(i)}$, find estimates for the maximum $\bar{\delta}_{(i)}$, and/or $\bar{\gamma}_{(i)}$ for $i = 1, \dots, p$ such that the closed-loop system given by (1) and (6) is globally asymptotically stable.

To tackle problems P1 and P2, we will consider a hybrid system framework to model the sampling behavior

2.2 Hybrid system modeling

The closed-loop system described by (1) and (6) is now represented as a hybrid dynamical system \mathcal{H} , with state given by $\eta = [z^{\top} \tau]^{\top}$, where $z = [x^{\top} u^{\top}]^{\top}$, and τ is a timer variable:

$$\mathcal{H} \begin{cases} \dot{\eta} = \begin{bmatrix} \dot{z} \\ \dot{\tau} \end{bmatrix} = f(\eta), & \forall \eta \in \mathcal{C} \\ \eta^+ = \begin{bmatrix} z^+ \\ \tau^+ \end{bmatrix} = g(\eta), & \forall \eta \in \mathcal{D} \end{cases} \quad (7)$$

The flow and jump sets are defined as $\mathcal{C} = \mathbb{R}^q \times [0, T_M]$ and $\mathcal{D} = \mathbb{R}^q \times [T_m, T_M]$, with $q = n + p$, while the flow and jump maps $f : \mathbb{R}^h \rightarrow \mathbb{R}^h$ and $g : \mathbb{R}^h \rightarrow \mathbb{R}^h$, with $h = q + 1$, are defined as follows:

$$f(\eta) = \begin{bmatrix} A_F z \\ 1 \end{bmatrix} \quad \text{and} \quad g(\eta) = \begin{bmatrix} A_J z + K_J \phi(K_z z) \\ 0 \end{bmatrix} \quad (8)$$

with:

$$\begin{aligned} A_F &= \begin{bmatrix} A & B \\ 0_{p \times n} & 0_{p \times p} \end{bmatrix}, \quad K_z = [K_x \quad K_u] \\ A_J &= \begin{bmatrix} I_n & 0_{n \times p} \\ 0_{p \times n} & 0_{p \times p} \end{bmatrix}, \quad K_J = \begin{bmatrix} 0_{n \times p} \\ I_p \end{bmatrix} \end{aligned}$$

The system \mathcal{H} satisfies the *hybrid basic conditions* (see (Goebel et al., 2012), chapter 6): the sets \mathcal{C} and \mathcal{D} are closed and the functions f and g are continuous. The solutions to \mathcal{H} are given by an hybrid arc $\eta(t, k)$ with hybrid domain $\text{dom } \eta = \cup_{k=0}^{\infty} ([t_k, t_{k+1}], k)$. In particular, for any $\eta(0, 0) = [z(0, 0)^{\top}, 0]^{\top} \in \mathcal{C} \cup \mathcal{D}$, the corresponding solution starting from $\eta(0, 0)$ is well posed, maximal, and, from assumption (5), without Zeno behavior.

Note that the timer variable τ is reset to zero at each jump instant (the jumps correspond to the sampling instants), i.e. $\tau(t, k) = t - t_k$. Taking into account that $t_0 = 0$, it follows that $\tau(0, 0) = 0$. Hence, we are interested in the behavior of the closed loop with respect to the initial conditions $\eta(0, 0) = [z(0, 0)^{\top} \ 0]^{\top}$.

3. STABILITY CONDITIONS

3.1 Stability of a hybrid system

The notion of stability of the system \mathcal{H} is characterized by the stability of a closed set containing the origin of the plant state space and the domain of the timer variable defined as follows:

$$\mathcal{A} = \{0\} \times [0, T_M] \quad (9)$$

The next theorem is found in Goebel et al. (2012) and provides sufficient conditions to ensure that \mathcal{A} is globally asymptotically stable for system \mathcal{H}

Theorem 1. If there exists a function $V : \mathbb{R}^h \rightarrow \mathbb{R}_{\geq 0}$ and class \mathcal{K}_∞ functions α_1 and α_2 such that

$$\alpha_1(|\eta|_{\mathcal{A}}) \leq V(\eta) \leq \alpha_2(|\eta|_{\mathcal{A}}) \quad , \quad \forall \eta \in (C \cup D) \quad (10)$$

$$\langle \nabla V(\eta), f(\eta) \rangle < 0 \quad , \quad \forall \eta \in C \setminus \mathcal{A} \quad (11)$$

$$V(g(\eta)) - V(\eta) < 0 \quad , \quad \forall \eta \in D \setminus \mathcal{A} \quad (12)$$

then the set \mathcal{A} is uniformly globally asymptotically stable.

The next corollary, in turn, states a particular consequence of this theorem regarding the original sampled data closed-loop system composed by (1) and (6):

Corollary 1. If conditions (10), (11) and (12) of Theorem 1 are satisfied, then the equilibrium point $x = 0$ of the closed-loop system(1)-(6) is globally asymptotically stable, provided that the sampling instants satisfy (5).

3.2 Stability of the closed-loop system

To derive stability conditions for the closed-loop system, we consider a generalized timer-dependent Lur’e-Postnikov function defined as follows:

$$V(\eta) = V_Q(\eta) + \sum_{i=1}^p \lambda_{(i)} \int_0^{K_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \quad (13)$$

where $V_Q : \mathbb{R}^h \rightarrow \mathbb{R}$ is a quadratic term defined as:

$$V_Q(\eta) \triangleq V_Q(z, \tau) = z^\top P(\tau)z \quad (14)$$

with $P : [0, T_M] \rightarrow \mathbb{S}^q$ and $\lambda_{(i)} \in \mathbb{R}$ for $i = 1, \dots, p$. Note that, differently from a classical Lur’e type function (Haddad and Kapila, 1995), matrix P and the scalars $\lambda_{(i)}$ are not required to be positive definite and nonnegative, respectively. In this case, the positivity of $V(\eta)$ can be enforced by a timer-dependent version of the lemma proposed in Valmorbidia et al. (2018) as proposed below:

Lemma 2. Consider $V(\eta)$ as in (13), where ϕ satisfies (2), and define $\Lambda \triangleq \text{diag}\{\lambda_{(1)}, \dots, \lambda_{(m)}\}$. If there exists a matrix function $\tilde{\Lambda} : [0, T_M] \rightarrow \mathbb{D}_{\geq 0}^p$ such that

$$\Psi_\Lambda(\tau) \succeq 0 \quad , \quad \forall \tau \in [0, T_M] \quad (15)$$

$$\Psi_P(\tau) \succ 0 \quad , \quad \forall \tau \in [0, T_M] \quad (16)$$

where

$$\Psi_\Lambda(\tau) = \Lambda + \tilde{\Lambda}(\tau)$$

$$\Psi_P(\tau) = P(\tau) - \frac{1}{2}K_z^\top (\bar{\Delta} - \underline{\Delta})\tilde{\Lambda}(\tau)K_z$$

then there exists class \mathcal{K}_∞ functions α_1 and α_2 such that the condition (10) is satisfied.

Proof. To prove that class \mathcal{K}_∞ function α_2 exists, first note that

$$0 \leq \phi_{(i)}(s) - \underline{\delta}_{(i)}s \leq \bar{\delta}_{(i)}s - \underline{\delta}_{(i)}s = (\bar{\delta}_{(i)} - \underline{\delta}_{(i)})s$$

to obtain the following upper bound for (13):

$$V(\eta) \leq |P(\tau)||z|^2 + \frac{1}{2}|K_z^\top (\bar{\Delta} - \underline{\Delta})\Lambda K_z||z|^2 = \bar{\Psi}(\tau)|z|^2 \quad (17)$$

The upper bound (17) is, in turn, bounded by the maximum value of $\bar{\Psi}(\tau)|z|^2$ over the admissible interval for τ :

$$\bar{\Psi}(\tau)|z|^2 \leq \max_{\tau \in [0, T_M]} \{\bar{\Psi}(\tau)\}|z|^2 \quad (18)$$

Note that from the definition of \mathcal{A} in (9), one has the identity $|z| = |\eta|_{\mathcal{A}}$ for all $\tau \in [0, T_M]$. Therefore, one can rewrite (18) as follows:

$$\max_{\tau \in [0, T_M]} \{\bar{\Psi}(\tau)\}|z|^2 = \max_{\tau \in [0, T_M]} \{\bar{\Psi}(\tau)\}|\eta|_{\mathcal{A}}^2 = \alpha_2(|\eta|_{\mathcal{A}})$$

which proves the existence of a class \mathcal{K}_∞ function α_2 .

Now a lower bound must be found to prove the existence of a class \mathcal{K}_∞ function α_1 in (10). If the condition (15) is verified, a lower bound for $V(\eta)$ is provided as follows (Valmorbidia et al., 2018):

$$\begin{aligned} V(\eta) &= V_Q(z, \tau) + \sum_{i=1}^m \lambda_{(i)} \int_0^{K_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \\ &\geq V_Q(z, \tau) - \sum_{i=1}^m \tilde{\lambda}_{(i)}(\tau) \int_0^{K_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \end{aligned}$$

Then, taking into account that $\int_0^{K_{z(i)}z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds = \frac{1}{2}z^\top K_z^\top (\bar{\delta}_{(i)} - \underline{\delta}_{(i)})K_z z - \int_0^{K_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s)) ds$, the inequality above is equivalently expressed as:

$$V(\eta) \geq z^\top \Psi_P(\tau)z + \sum_{i=1}^m \tilde{\lambda}_{(i)}(\tau) \int_0^{K_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s)) ds \quad (19)$$

for $\tau \in [0, T_M]$. From (16), it follows that there exists $\tilde{\alpha}_1$ (the minimum eigenvalue of $\Psi_P(\tau)$ for $\tau \in [0, T_M]$) such that

$$z^\top \Psi_P(\tau)z \geq \tilde{\alpha}_1|z|^2 \quad (20)$$

Thus, as $\tilde{\lambda}_{(i)}(\tau) \geq 0$ and $\int_0^{K_{z(i)}z} (\bar{\delta}_{(i)}s - \phi_{(i)}(s)) ds > 0$, we can conclude from (19) and (20) that

$$V(\eta) \geq z^\top \Psi_P(\tau)z \geq \tilde{\alpha}_1|z|^2 = \alpha_1(|\eta|_{\mathcal{A}})$$

which proves the existence of a class \mathcal{K}_∞ function α_1 , finishing the proof. \blacksquare

Recalling from (7) that $g(\eta) = [(z^+)^\top \tau^+]^\top$ and considering $V(\eta)$ as in (13), the inequality (12) presents the term:

$$\sum_{i=1}^p \lambda_{(i)} \int_{K_{z(i)}z}^{K_{z(i)}z^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \quad (21)$$

In order to cast inequality (12) as a matrix inequality, the following lemma presents an upper bound for (21):

Lemma 3. If ϕ satisfies (2d), then

$$\int_{y_{(i)}}^{y_{(i)}^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \leq \quad (22)$$

$$\frac{1}{2}(2\phi_{(i)}(y) + \bar{\gamma}_{(i)}(y_{(i)}^+ - y_{(i)}) - \underline{\delta}_{(i)}(y_{(i)}^+ + y_{(i)}))(y_{(i)}^+ - y_{(i)})$$

is satisfied for all $i = 1, \dots, p$ and $y, y^+ \in \mathbb{R}^p$.

Proof. Note that

$$\begin{aligned} &\int_{y_{(i)}}^{y_{(i)}^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \\ &= \int_{y_{(i)}}^{y_{(i)}^+} (\phi_{(i)}(y) + \bar{\gamma}_{(i)}(s - y_{(i)}) - \underline{\delta}_{(i)}s) ds \quad (23) \\ &\quad - \int_{y_{(i)}}^{y_{(i)}^+} (\phi_{(i)}(y) + \bar{\gamma}_{(i)}(s - y_{(i)}) - \phi_{(i)}(s)) ds \end{aligned}$$

Based on the slope-restricted property (2d) of ϕ , the last term of (23) (including the signal) is non-positive, and thus the following inequality holds:

$$\int_{y(i)}^{y(i)^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \leq \tag{24}$$

$$(\phi_{(i)}(y) - \bar{\gamma}_{(i)}y(i))(y(i)^+ - y(i)) + \frac{1}{2}(\bar{\gamma}_{(i)} - \underline{\delta}_{(i)})((y(i)^+)^2 - y(i)^2)$$

Finally, the expression (24) is equivalent to the right-hand side of (22), finishing the proof. ■

Considering $V(\eta)$ in (13), and defining the matrices $M_1 \triangleq [I_q \ 0_{q \times p}]$, $M_2 \triangleq [0_{p \times q} \ I_p]$, $\mathbb{A}_J \triangleq [A_J \ K_J]$, $\mathbb{A}_F \triangleq [A_F \ 0_{q \times p}]$, $\Omega_1 \triangleq \frac{1}{2}(A_J - I_q)^\top K_z^\top \Lambda \bar{\Gamma} K_z (A_J - I_q) - \frac{1}{4}\text{He}\{(A_J + I_q)^\top K_z^\top \Lambda \underline{\Delta} K_z (A_J - I_q)\}$, $\Omega_2 \triangleq \frac{1}{4}(A_J - I_q)^\top K_z^\top \Lambda \bar{\Gamma} K_z K_J - \frac{1}{4}(A_J + I_q)^\top K_z^\top \Lambda \underline{\Delta} K_z K_J + \frac{1}{2}(A_J - I_q)^\top K_z^\top \Lambda + \frac{1}{4}(A_J - I_q)^\top K_z^\top \Lambda (\bar{\Gamma} - \underline{\Delta}) K_z K_J$, $\Omega_3 \triangleq \frac{1}{2}K_J^\top K_z^\top \Lambda (\bar{\Gamma} - \underline{\Delta}) K_z K_J + \frac{1}{2}\text{He}\{K_J^\top K_z^\top \Lambda\}$ and $\Omega \triangleq \begin{bmatrix} \Omega_1 & \Omega_2 \\ \star & \Omega_3 \end{bmatrix}$, the next theorem proposes matrix inequalities that are sufficient for the satisfaction of (10), (11) and (12), and therefore for the stability of the closed-loop system (1)-(6).

Theorem 2. If there exist matrix functions $P : [0, T_M] \rightarrow \mathbb{S}^q$, $\Lambda \in \mathbb{D}^p$, $\bar{\Lambda} : [0, T_M] \rightarrow \mathbb{D}^p$ and $U_j : [0, T_M] \rightarrow \mathbb{D}^p$, $j = 0, 1, 2$ satisfying the following matrix inequalities

$$U_j(\tau) \succeq 0, \quad \forall \tau \in [0, T_M], \quad j \in \{0, 1, 2\} \tag{25}$$

$$\bar{\Lambda}(\tau) \succeq 0, \quad \forall \tau \in [0, T_M] \tag{26}$$

$$\Psi_\Lambda(\tau) \succeq 0, \quad \forall \tau \in [0, T_M] \tag{27}$$

$$\Psi_P(\tau) \succ 0, \quad \forall \tau \in [0, T_M] \tag{28}$$

$$\Psi_F(\tau) \prec 0, \quad \forall \tau \in [0, T_M] \tag{29}$$

$$\Psi_J(\tau) \prec 0, \quad \forall \tau \in [T_m, T_M] \tag{30}$$

where

$$\begin{aligned} \Psi_F(\tau) &= M_1^\top \frac{\partial P(\tau)}{\partial \tau} M_1^\top + \text{He}\{M_1^\top P(\tau) \mathbb{A}_F\} \\ &\quad - 0.5\text{He}\{(M_1^\top (\underline{\Delta} K_z)^\top - M_2^\top) \Lambda K_z (A_F M_1)\} \\ &\quad + \text{He}\{(M_2^\top - M_1^\top (\underline{\Delta})^\top) U_1(\tau) (\bar{\Delta} K_z M_1 - M_2)\} \\ \Psi_J(\tau) &= \mathbb{A}_J^\top P(0) \mathbb{A}_J - M_1^\top P(\tau) M_1 + \Omega \\ &\quad + \text{He}\left\{ \begin{bmatrix} (\underline{\Delta} K_z)^\top \\ -I_p \end{bmatrix} U_2(\tau) \begin{bmatrix} \bar{\Delta} K_z & -I_p \end{bmatrix} \right\} \end{aligned}$$

then the equilibrium point $x = 0$ of the closed-loop system given by (1) and (6) is globally and asymptotically stable, provided that the sampling instants satisfy (5).

Proof. From Lemma 2, it follows that (27) and (28) ensure that (10) is verified. Taking into account that

$$\frac{\partial}{\partial t} \left[\sum_{i=1}^m \lambda_{(i)} \int_0^{K_{z(i)} z} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \right] = \tag{31}$$

$$-(z^\top (\underline{\Delta} K_z)^\top - \phi^\top(K_z z)) \Lambda K_z \dot{z}$$

and denoting $\langle \nabla V(\eta), f(\eta) \rangle \triangleq \langle \nabla V, f \rangle$, one obtains

$$\begin{aligned} \langle \nabla V, f \rangle &= z^\top \frac{\partial P(\tau)}{\partial \tau} z \dot{\tau} + \text{He}\{z^\top P(\tau) \dot{z}\} \\ &\quad - (z^\top (\underline{\Delta} K_z)^\top - \phi^\top(K_z z)) \Lambda K_z \dot{z} \end{aligned} \tag{32}$$

Define now $\zeta = [z^\top \ \phi^\top(K_z z)]^\top$. If (29) is verified, then

$$\begin{aligned} &\zeta^\top (M_1^\top \frac{\partial P(\tau)}{\partial \tau} M_1^\top + \text{He}\{M_1^\top P(\tau) \mathbb{A}_F\} - \\ &0.5\text{He}\{(M_1^\top (\underline{\Delta} K_z)^\top - M_2^\top) \Lambda K_z (A_F M_1)\} + \\ &\text{He}\{(M_2^\top - M_1^\top (\underline{\Delta} K_z z)^\top) U_1(\tau) (\bar{\Delta} K_z z M_1 - M_2)\}) \zeta < 0 \end{aligned} \tag{33}$$

which, taking into account (32) and recalling that $\dot{\tau} = 1$ and $\dot{z} = A_F z$, is equivalent to

$$\langle \nabla V, f \rangle + S_\Delta(U_1(\tau), K_z z) < 0 \tag{34}$$

Thus, from Lemma 1, it follows that $\langle \nabla V, f \rangle < 0$, that is, condition (29) implies the satisfaction of condition (11).

Now, consider the jump behavior. With the candidate function (13), the simplified notation $\phi = \phi(K_z z)$, and recalling the definition of the jump map g in (8), that is $z^+ = A_J z + K_J \phi$, one obtains the following expression:

$$\begin{aligned} \Delta V &\triangleq V(g(\eta)) - V(\eta) \\ &= \begin{bmatrix} z \\ \phi \end{bmatrix}^\top (\mathbb{A}_J^\top P(0) \mathbb{A}_J - M_1^\top P(\tau) M_1) \begin{bmatrix} z \\ \phi \end{bmatrix} \\ &\quad + \sum_{i=1}^p \lambda_{(i)} \int_{K_{z(i)} z}^{K_{z(i)} z^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \end{aligned} \tag{35}$$

Now consider the rightmost term of (35). From Lemma 3, replacing y with $K_z z$ and y^+ with $K_z z^+$, the following upper bound is obtained:

$$\begin{aligned} &\sum_{i=1}^p \lambda_{(i)} \int_{K_{z(i)} z}^{K_{z(i)} z^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \\ &\leq \sum_{i=1}^p \lambda_{(i)} \frac{1}{2} (2\phi_{(i)} + \bar{\gamma}_{(i)} K_{z(i)} (z^+ - z) \\ &\quad - \underline{\delta}_{(i)} K_{z(i)} (z^+ + z)) K_{z(i)} (z^+ - z) \\ &= \frac{1}{2} (2\phi + \bar{\Gamma} K_z (z^+ - z) \\ &\quad - \underline{\Delta} K_z (z^+ + z))^\top \Lambda K_z (z^+ - z) \end{aligned} \tag{36}$$

Recalling again that $z^+ = A_J z + K_J \phi$, one has that the right-hand side of inequality (36) is equivalent to

$$\begin{aligned} &\frac{1}{2} (\bar{\Gamma} K_z (A_J - I_q) z - \underline{\Delta} K_z (A_J + I_q) z + \\ &(2I_p + (\bar{\Gamma} - \underline{\Delta}) K_z K_J) \phi)^\top \Lambda K_z ((A_J - I_q) z + K_J \phi) \end{aligned} \tag{37}$$

In terms of z and ϕ , the upper bound (37) is equivalently expressed as:

$$\sum_{i=1}^p \lambda_{(i)} \int_{K_{z(i)} z}^{K_{z(i)} z^+} (\phi_{(i)}(s) - \underline{\delta}_{(i)}s) ds \leq \begin{bmatrix} z \\ \phi \end{bmatrix}^\top \Omega \begin{bmatrix} z \\ \phi \end{bmatrix} \tag{38}$$

and thus, a sufficient condition for the negativity of ΔV in (35) is given by

$$\begin{bmatrix} z \\ \phi \end{bmatrix}^\top (\mathbb{A}_J^\top P(0) \mathbb{A}_J - M_1^\top P(\tau) M_1 + \Omega) \begin{bmatrix} z \\ \phi \end{bmatrix} < 0 \tag{39}$$

Thus, considering $U_2 : [T_m, T_M] \rightarrow \mathbb{D}_{\geq 0}^m$ and Lemma 1, the condition (30) ensures that (12) is verified at the jumps. Hence, the conditions of Theorem 2 guarantee the verification of conditions of Theorem 1 with V defined in (13)-(14), which from Corollary 1 concludes the proof. ■

4. POLYNOMIAL DEPENDENCE ON τ

The issue in formulating an optimization problem with Theorem 2 is that its conditions have infinite dimension, since they must be verified for τ in continuous intervals. To overcome this problem, the variables with generic dependence on τ in Theorem 2 are specified now as polynomials of τ , with the intent of obtaining conditions as sum-of-squares (SOS) (Parrilo (2003), Peyrl and Parrilo (2008)).

Hence, we consider that the τ -dependent matrices of Theorem 2 can be generically cast as a matrix polynomial $M(\tau)$ of appropriate dimensions described as:

$$M(\tau) = \sum_{i=0}^d M_i \tau^i = M_0 + \tau M_1 + \dots + \tau^d M_d \quad (40)$$

where d is the order of the polynomial. The next theorem exploits this property and presents the stability conditions cast in the framework of the sum-of-squares programming.

Theorem 3. If there exist matrix polynomials $P : [0, T_M] \rightarrow \mathbb{S}^{q+p}$, $\Lambda \in \mathbb{D}^p$, $\tilde{\Lambda} : [0, T_M] \rightarrow \mathbb{D}_{\geq 0}^p$, $Q_i : [0, T_M] \rightarrow \mathbb{S}_{\geq 0}^p$, $i \in \{0, 1, 2, 3, 4\}$, $Q_5 : [0, T_M] \rightarrow \mathbb{S}_{\geq 0}^{q+p}$, $Q_6 : [0, T_M] \rightarrow \mathbb{S}_{\geq 0}^{q+p}$, $Q_7 : [0, T_M] \rightarrow \mathbb{S}_{\geq 0}^{q+p}$, $U_j : [0, T_M] \rightarrow \mathbb{D}_{\geq 0}^p$, $j \in \{0, 1, 2\}$, and a scalar $\xi > 0$ satisfying the following sum-of-squares (SOS) conditions

$$Q_0(\tau), Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau), Q_5(\tau), Q_6(\tau), Q_7(\tau) \text{ are SOS} \quad (41)$$

$$U_0(\tau) - Q_0(\tau)\tau(T_M - \tau) \text{ is SOS} \quad (42)$$

$$U_1(\tau) - Q_1(\tau)\tau(T_M - \tau) \text{ is SOS} \quad (43)$$

$$U_2(\tau) - Q_2(\tau)\tau(T_M - \tau) \text{ is SOS} \quad (44)$$

$$\tilde{\Lambda}(\tau) - Q_3(\tau)\tau(T_M - \tau) \text{ is SOS} \quad (45)$$

$$\Psi_\Lambda(\tau) - Q_4(\tau)\tau(T_M - \tau) \text{ is SOS} \quad (46)$$

$$\Psi_P(\tau) - Q_5(\tau)\tau(T_M - \tau) - \xi I_{q+p} \text{ is SOS} \quad (47)$$

$$-\Psi_F(\tau) - Q_6(\tau)\tau(T_M - \tau) - \xi I_{q+p} \text{ is SOS} \quad (48)$$

$$-\Psi_J(\tau) - Q_7(\tau)(\tau - T_m)(T_M - \tau) - \xi I_{q+p} \text{ is SOS} \quad (49)$$

then, the equilibrium point $x = 0$ of the closed-loop system (1) and (6) is globally asymptotically stable, provided that the sampling instants satisfy (5).

Proof. If the conditions in (41) are verified, then

$$Q_0(\tau), Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau), Q_5(\tau), Q_6(\tau), Q_7(\tau) \geq 0, \quad \forall \tau$$

Noting that $\tau(T_M - \tau) \geq 0$ for $\tau \in [0, T_M]$, provided that $Q_0(\tau) \geq 0, \forall \tau$, it follows that (42) ensures that

$$U_0(\tau) \geq 0, \quad \tau \in [0, T_M] \quad (50)$$

Similarly, as $Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau) \geq 0, \forall \tau$, the conditions (43), (44), (45) and (46) imply that $U_1(\tau), U_2(\tau), \tilde{\Lambda}(\tau)$ and $(\Lambda - \tilde{\Lambda}(\tau)) \geq 0, \forall \tau \in [0, T_M]$. Moreover, since $Q_5(\tau) \geq 0, \forall \tau$ and $\xi > 0$, the condition (47) implies that

$$\Psi_P(\tau) \geq \xi I_{q+p} \succ 0, \quad \forall \tau \in [0, T_M] \quad (51)$$

and therefore that (28) is satisfied. Similarly, since $Q_6(\tau) \geq 0, \forall \tau$, one has that (48) implies (29).

Finally, noting that $(\tau - T_m)(T_M - \tau) \geq 0$ for $\tau \in [T_m, T_M]$, provided that $Q_7(\tau) \geq 0, \forall \tau$, it follows that (49) ensures (30).

Hence, the conditions of Theorem 3 guarantee the verification of conditions of Theorem 2, which in turn guarantee the conditions of Theorem 1 with V as defined in (13)-(14). With Corollary 1, the proof is concluded. ■

5. OPTIMIZATION PROBLEMS

Here we formulate optimization problems to solve problems P1 and P2 defined in Section 2, based on the con-

straints of Theorem 3. Hence, we can consider the two following optimization problems:

$$P1 \begin{cases} \max T_M \\ \text{subject to:} \\ (46), (47), (48), (49) \end{cases} \quad (52)$$

$$P2 \begin{cases} \max f(\underline{\Delta}, \bar{\Gamma}) \\ \text{subject to:} \\ (46), (47), (48), (49) \end{cases} \quad (53)$$

In the optimization problem (53) a generic objective function depending on $\underline{\Delta}, \bar{\Gamma}$, i.e. on the bounds on the sector and slope of the nonlinearity, is considered. Recall that the basic idea is to enlarge the sector for which the stability can be certified. For instance, for a given $\underline{\Delta}$, we can consider $f(\underline{\Delta}, \bar{\Gamma}) = \sum_{i=1}^m (\bar{\delta}_{(i)} + \bar{\gamma}_{(i)})$ and $\bar{\delta}_{(i)} = \bar{\gamma}_{(i)} = \beta, \forall i$ and maximize β .

On the other hand, the objective variables T_M of problem P1 and $\underline{\Delta}, \bar{\Gamma}$ of P2 cannot be listed as decision variables, because they appear multiplying other decision variables. Thus, to solve P1 and P2, an SOS feasibility problem is solved repeatedly, each time increasing the value of the objective function, until the constraints become unfeasible. Hence, solutions to both P1 and P2 can be obtained with the SOSTools toolbox (Prajna et al., 2004).

5.1 Example 1

Consider system (1) with the following parameters

$$A = \begin{bmatrix} -4 & -1 & 1 & -1 \\ 2 & -1 & -1 & 1 \\ 3 & 1 & 0 & 1 \\ 4 & 3 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}, K_x = \begin{bmatrix} -2 \\ -1 \\ -1 \\ -2 \end{bmatrix}^T \quad (54)$$

Consider first that $K_u = 0, \bar{\Gamma} = 1$ and $\underline{\Delta} = \bar{\Delta} = 1$, which corresponds to a linear system. Notice that the system is open loop unstable, because A has one positive eigenvalue, and that $A + BK_x$ eigenvalues are all negative. With that in mind, problem (52) is solved with $d_{\tilde{\Lambda}} = d_{U_0} = d_{U_1} = d_{U_2} = 0$ and $d_P = 4$, and considering $T_m = T_M = T$, i.e. periodic sampling. Thus, maximizing T in problem (52), one obtains the maximum value of 0.355. This same value is also the maximum such that $e^{AT} + \int_0^T e^{A(T-s)} BK_x$ is Schur-Cohn, showing that the method is not conservative when applied to this particular case. However, when decreasing the degree of polynomial P to $d_P = 2$, the maximum found with (52) is 0.203. That is a decrease of 42.8% caused by reducing d_P , showing the importance of the timer dependence.

Now, setting $T_m = 0.1$ and considering again optimization problem (52), the results for $\bar{\Delta} = 1$ and different values of $\underline{\Delta}$ are exposed in Table 1:

Table 1.

$\underline{\Delta}$ (Given)	1	0.8	0.6	0.4
T_M (with $d_P = 4$)	0.355	0.324	0.237	0.104
T_M (with $d_P = 2$)	0.189	0.188	0.173	unfeasible

We can observe that the results obtained with $d_P = 4$ are less conservative than the ones with $d_P = 2$. For $d_P \geq 5$, the results are basically the same as obtained with $d_P = 4$. Furthermore, it can be noted that the larger is the considered sector (i.e. smaller is $\underline{\Delta}$), the smaller is the maximum value of T_M for which we can certify stability.

5.2 Example 2

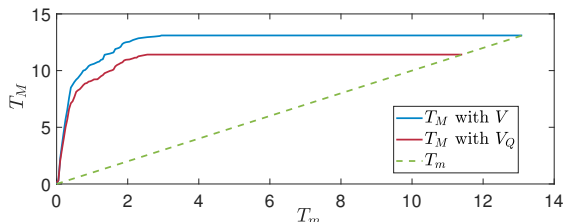
Consider system (1) and (6) with the following data borrowed from Huff et al. (2022):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -330.46 & -12.15 & -2.44 & 0 \\ 0 & 0 & 0 & 1 \\ -812.61 & -29.87 & -30.10 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2.71762 \\ 0 \\ 6.68268 \end{bmatrix}$$

$$K_x = [12.87 \ 0.24 \ -0.21 \ 0.032], \quad K_u = -0.00076$$

$$\underline{\Delta} = 0, \quad \bar{\Delta} = 1, \quad \bar{\Gamma} = 1 \quad (55)$$

First we compare conditions of Theorem 3 based on the candidate V with simplified conditions based on the quadratic candidate V_Q , i.e., V without the integral term. Solving (52) with the different sets of conditions and $d_P = 4$ in a range of values of T_m , the next figure shows the maximum estimates of T_M . It can be seen that, as in the previous example, the extra decision variables of the conditions based on V allow higher estimates to be found. Moreover, as T_m increases, the estimates obtained in both cases converge to the value of the maximum periodic sampling such that the system is globally asymptotically stable, intercepting the dashed line of T_m .



6. CONCLUSION

The stability analysis of sampled-data systems with sector and slope bounded input nonlinearities has been studied in this paper. Based on the hybrid system framework, and a Lur'e type function, conditions to assess the stability of the closed-loop system were developed. Based on these conditions, two problems were proposed: the first one consisted in finding the maximum allowable interval of sampling, and second consisted in finding the least restricted sector of the nonlinearity. Both problems were posed as sum-of-squares (SOS) optimization problems.

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