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Carleman-Based Reconstruction Algorithm on a wave Network

Lucie Baudouin* Maya de Buhan† Emmanuelle Crépeau‡
Julie Valein§

Abstract

In the context of a network of vibrating strings, modelled by interconnected linear partial differential equations, we are interested in the reconstruction of a zeroth order term of each one-dimensional wave equation involved, using some appropriate external boundary measurements. More precisely, we are interested in an inverse problem set on a tree shaped network where each edge behaves according to the wave equation with potential, external nodes have Dirichlet boundary conditions and internal nodes follow the Kirchoff law. The main goal is the reconstruction of the potential everywhere on the network, from the Neumann boundary measurements at all but one external vertices. Leveraging from the Lipschitz stability of this inverse problem, we aim at providing an efficient reconstruction algorithm based on the use of a specific global Carleman estimate. The proof of the main tool and of the convergence of the algorithm are provided; along with a detailed description of the numerical illustrations given at the end of the article.

Keywords: network of partial differential equations, wave/string equation, inverse problem, Carleman estimate, numerical simulations.

AMS subject classifications: 35R30, 93C20, (34B45)

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1 Introduction and main result

The goal of this article is to present a complete overview of a coefficient recovery inverse problem in a network of one-dimensional wave equations (also called string equations), from the identifiability questions, such as uniqueness and stability of the solution of the inverse problem, to the identification of the coefficient (here, a zeroth order term, called potential) by a well suited reconstruction algorithm. If the well-posedness of such an inverse problem was already addressed in [6] with a Lipschitz stability result, the actual reconstruction of the coefficient of the partial differential equations system at stake is still very challenging and we will provide a global analysis of these two connected issues.

Taking advantage of the Carleman-based Reconstruction Algorithm proposed in [7] (C-bRec in short) for the reconstruction of the potential in a multidimensional wave equation set in a bounded domain with Dirichlet boundary condition and appropriate Neumann boundary measurements, this article aims at considering the quite challenging case of a tree-shaped network of string equations. In this kind of planar graph context, the simplicity of string equation considered on each branch of the network is balanced by the technicality of the global setting stemming from the interconnection laws that describe the communication at the nodes of the network.

On the mathematical and engineering points of view, the study of mechanical systems coupling flexible, elastic, or vibrating elements as strings or

beams, are very relevant as well as demanding. Concerning some control theory studies in this framework, one can mention the books [14] and [2] where these kind of systems, also called multi-link structures, are studied according to their observation, control and stability for various partial differential equations.

To be more precise, the goal of this article is to provide a detailed study of the design of an effective and efficient reconstruction algorithm for an unknown (time independent) coefficient of the string equation involved everywhere on the tree-shaped network, from measurements only localized at the *leaves* of the tree (all but one external node, called the *root*), as one can picture for instance with Figure 1. The rough idea is to use the C-bRec method to effectively solve this inverse problem on a network. But its concrete realisation means overcoming several difficulties such as having the appropriate global Carleman estimate over the whole network, that cannot be the one from [6] for numerical efficiency questions detailed in [7]. Moreover, as one will understand below, this article will not be only the tailoring of the C-bRec approach to network's specificities, but rather a complete analysis of a difficult reconstruction goal, from abstract tools and results of identifiability, to acute algorithmic and numerical issues related to identification.

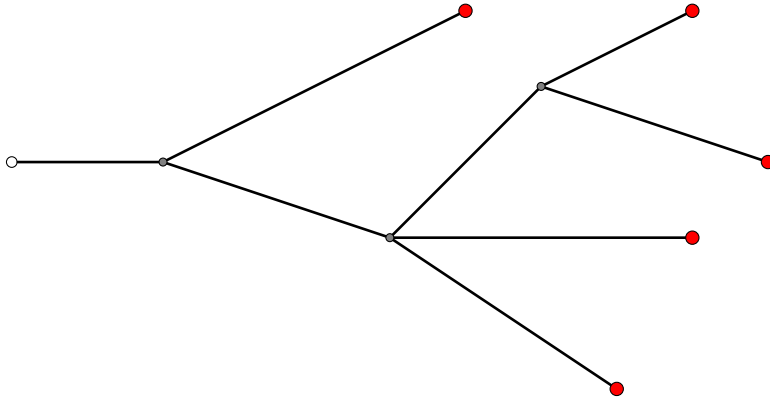


Figure 1: An 8 branches tree-shaped network \mathcal{R} , with an unobserved root node and 5 observed leaf nodes \bullet .

Before detailing in the rest of this introduction the statement of the problem, some elements of context, and the founding idea of the proposed reconstruction method, one should know that Section 2 will be devoted to the Carleman estimate, Section 3 to the description of the reconstruction algorithm and its proof of convergence and Section 4 to the numerical setting and simulations.

1.1 Problem statement

Tree-shaped networks are planar connected graphs that do not contain even a single cycle. They represent a hierarchical structure and despite their simplicity, they allow to model a lot of specific contexts and have a rich structure to exploit. The edges of such a network are known as *branches* connected by nodes, also called vertices. Except for the *root*, any node has a mother branch and the nodes without child branches are called *leaf nodes*.

Let us thus consider a finite tree-shaped network \mathcal{R} (see e.g. Figure 1). We define the name of the tree's branches by recurrence, with nested subscripts, the finite set \mathcal{J} being the set of names of all branches of the network. To the root branch, named 1, we associate its N_1 children branches denoted by $1_i \in \mathbb{N}$ for $i = 1..N_1$.

Then by recurrence, from a branch named $j \in \mathcal{J}$ we define the names of its N_j children branches by j_i for $i = 1..N_j$. This definition only holds locally in the network, as we will not need more until the end of the article, for the numerical simulation's section. Besides, we denote by ℓ_j the length of the branch j and we identify the branch j with the segment $(0, \ell_j)$.

We denote by $\mathcal{J}_{ext} = \{j \in \mathcal{J}, N_j = 0\}$ the set of the branches having a final leaf node and $\mathcal{J}_{int} = \{j \in \mathcal{J}, N_j > 0\}$ the other branches. We define a function f on the network \mathcal{R} through its restriction f_j to each of the branches $j \in \mathcal{J}$ and we introduce the following notations:

$$\int_{\mathcal{R}} f(x)dx := \sum_{j \in \mathcal{J}} \int_0^{\ell_j} f_j(x)dx, \quad (1.1)$$

$$[f]_j := f_j(\ell_j) - \sum_{i=1}^{N_j} f_{j_i}(0), \quad \forall j \in \mathcal{J}_{int}. \quad (1.2)$$

Let $T > 0$. On each branch $j \in \mathcal{J}$ of the network, we consider the one-dimensional wave equation system

$$\begin{cases} \partial_{tt}u_j(t, x) - \partial_{xx}u_j(t, x) + p_j(x)u_j(t, x) = g_j(t, x), & \forall (t, x) \in (0, T) \times (0, \ell_j), \\ u_j(0, x) = u_j^0(x), \quad \partial_t u_j(0, x) = u_j^1(x), & \forall x \in (0, \ell_j), \end{cases} \quad (1.3)$$

where the time-independant potential p_j is an unknown coefficient.

In order to complete system (1.3) by enough boundary conditions, we then equip each branch $j \in \mathcal{J}$ with boundary conditions: Dirichlet boundary conditions at external nodes (root and leaves), and continuity condition and Kirchhoff law at any internal node. So that for all $t \in (0, T)$:

$$\begin{cases} \text{for } j = 1, & u_1(t, 0) = h_1(t), \\ \text{if } j \in \mathcal{J}_{ext}, & u_j(t, \ell_j) = h_j(t), \\ \text{if } j \in \mathcal{J}_{int}, & u_j(t, \ell_j) = u_{j_i}(t, 0), \quad \forall i \in \llbracket 1, N_j \rrbracket, \\ & [\partial_x u]_j(t) = 0, \end{cases} \quad (1.4)$$

assuming additionally some compatibility conditions between Dirichlet boundary and initial data: $u_1^0(0) = h_1(0)$ and $u_j^0(\ell_j) = h_j(0)$ for all $j \in \mathcal{J}_{ext}$.

Let us now make clear the ‘not so classical’ functional spaces we are working in. We define:

$$\begin{aligned}
L^\infty(\mathcal{R}) &= \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in L^\infty(0, \ell_j) \forall j \in \mathcal{J}\}; \\
L^2(\mathcal{R}) &= \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in L^2(0, \ell_j) \forall j \in \mathcal{J}\}; \\
H^1(\mathcal{R}) &= \{f : \mathcal{R} \rightarrow \mathbb{R}, f_j \in H^1(0, \ell_j) \forall j \in \mathcal{J}, \\
&\quad f_j(\ell_j) = f_{j_i}(0) \forall i \in \llbracket 1, N_j \rrbracket \forall j \in \mathcal{J}_{int}\}; \\
H_0^1(\mathcal{R}) &= \{f \in H^1(\mathcal{R}), f_1(0) = 0, f_j(\ell_j) = 0 \forall j \in \mathcal{J}_{ext}\}; \\
\mathcal{C}^2((0, T) \times \mathcal{R}) &= \{\eta : (0, T) \times \mathcal{R} \rightarrow \mathbb{R}, \eta_j \in \mathcal{C}^2((0, T) \times (0, \ell_j)) \forall j \in \mathcal{J}, \\
&\quad \eta_j(t, \ell_j) = \eta_{j_i}(t, 0) \forall i \in \llbracket 1, N_j \rrbracket, [\partial_x \eta]_j(t) = 0 \forall j \in \mathcal{J}_{int}\}.
\end{aligned}$$

We assume basically that the potential p belongs to $L^\infty(\mathcal{R})$, the source data g belongs to $L^1(0, T; L^2(\mathcal{R}))$, the initial conditions satisfy $u^0 \in H^1(\mathcal{R})$ and $u^1 \in L^2(\mathcal{R})$ and are compatible (as already mentioned) with the external boundary data h_1, h_j to $H^1(0, T)$. Then, based for instance on [22, Chapter 3], the Cauchy problem (1.3)-(1.4) is well posed and has a unique solution

$$u \in C([0, T]; H^1(\mathcal{R})) \cap C^1([0, T]; L^2(\mathcal{R})).$$

Details can be found also in [6, Lemma 3.2] in particular for the homogeneous boundary data case.

We are studying the following inverse problem:

Knowing, for each branch $j \in \mathcal{J}$, the source term g_j and the initial data (u_j^0, u_j^1) , for the root and for each leaf $j \in \{1\} \cup \mathcal{J}_{ext}$ the boundary source term h_j , is it possible to identify the unknown potentials $p_j^*(x)$ for any $x \in (0, \ell_j)$, from the only extra knowledge of the flux of the solutions through the leaf nodes of the network, meaning:

$$d_i^*(t) = \partial_x u_i^*(t, \ell_i), \quad \text{for } i \in \mathcal{J}_{ext} \text{ and } t \in (0, T), \quad (1.5)$$

where u_i^* is the solution of (1.3)-(1.4) associated to potential p_i^* ?

Before getting any further, let us underline that we are interested here in the case of a unique measurement (during a time T) associated to the given data set (g_j, u_j^0, u_j^1) for any $j \in \mathcal{J}$ and h_j for $j \in \{1\} \cup \mathcal{J}_{ext}$. Several usual answers are expected regarding the inverse problem at stake: uniqueness and stability of the solution of inverse problem will answer to the Hadamard well-posedness notion, and pave the way for the reconstruction of the unknown coefficient that is our specific final goal.

The following Lipschitz stability result will answer both the first two questions (of uniqueness and stability of the inverse problem). It also has the interest of detailing the assumptions under which we can hope for reconstruction as well.

Theorem 1. *There exist a time $T_0 > 0$ and a scalar $\alpha_0 > 0$ such that if we assume the following*

(Ha) *Time condition: $T > T_0$,*

(Hb) *Regularity condition: $u \in H^1(0, T; L^\infty(\mathcal{R}))$,*

(Hc) *Sign condition: $|u^0| \geq \alpha^0 > 0$ on the whole network \mathcal{R} ,*

then for a fixed $m > 0$, there exists a positive constant $C = C(\mathcal{R}, T, m)$ such that, if p and p^ belong to $L_m^\infty(\mathcal{R}) = \{p \in L^\infty(\mathcal{R}), \|p\|_{L^\infty(\mathcal{R})} \leq m\}$, we have*

$$\|p - p^*\|_{L^2(\mathcal{R})}^2 \leq C \sum_{i \in \mathcal{J}_{ext}} \|\partial_x u_i(\cdot, \ell_i) - \partial_x u_i^*(\cdot, \ell_i)\|_{H^1(0, T)}^2.$$

This Lipschitz stability result can be read in [6] and is proved using an ad-hoc two-parameters Carleman estimate as in [4] for the corresponding basic multidimensional setting. The approach we used was first sketched by Bukgheim and Klibanov in [13] for uniqueness purposes, pursued in [27] for the stability of an inverse problem in the wave equation similar to ours, and popularized by Imanuvilov and Yamamoto [15, 16] and a large literature afterward for various modifications in the setting. We can refer also to Klibanov's survey article [19] in this area, concerning uniqueness, stability, and reconstruction of coefficient inverse problems for evolution partial differential equations.

In this specific result, the minimal time T_0 can be evaluated from [6, Lemma 2.1], or in our forthcoming Lemma 2, and is necessarily more restrictive than the optimal observability time for the corresponding controllability problem, for consistency reasons (the same Carleman estimate can bring an observability inequality).

The stability result of Theorem 1 is at the root of the reconstruction procedure we are going to present. However, we will need to build another Carleman estimate than the one used to prove it in [6], mainly for numerical efficiency purposes. It will be a one-parameter Carleman estimate (see e.g. [16, 7]) and this will be detailed later in Section 2.

1.2 Globally converging reconstruction methods

Given the stability result of Theorem 1, an intuitive method to compute p^* would consist in minimizing the cost functional:

$$J(p) = \frac{1}{2} \sum_{i \in \mathcal{J}_{ext}} \|\partial_x u_i(\cdot, \ell_i) - d_i^*\|_{H^1(0, T)}^2. \quad (1.6)$$

The problem is that the functional J is in general not convex. The presence of local minima implies that traditional minimization algorithms generally do not converge to the global minimum of J , unless they can be initialized in a neighborhood of this minimum.

A method is said to be globally convergent if its convergence towards the solution can be guaranteed from any initial data. Globally convergent methods of reconstruction remain rare, even if decisive works in this direction have been done since the 1990s, in particular the ones of Klivanov and his co-authors, based on Carleman inequalities. The original idea of introducing Carleman weights in the cost function to make the problem convex goes back to [18]. For a state of the art on these questions, we refer to the book by Beilina and Klivanov [10].

Since then, many other works have been proposed on different models, among which we can cite in particular Beilina-Klivanov [11] for the recovery of a density in a wave equation, Klivanov-Li-Zhang [21] and Smirnov-Klivanov-Nguyen [23] in the more delicate case where the initial datum is a Dirac mass, Kamburg-Klivanov [20] for inverse problems for parabolic equations, Thanh-Beilina-Klivanov-Fiddy [25] on real data, Bakushinskii-Klivanov-Koshev [3] to solve Cauchy problems for general PDEs.

The C-bRec method belongs to these globally convergent reconstruction methods, sometimes called *convexification methods*. The main novelty of our approach is that it consists in minimizing a sequence of quadratic functionals, for which a wide range of efficient minimization algorithms exist, while the works cited above consider a single functional which is strictly convex but which is not quadratic and is therefore only suitable for classical gradient descent strategies.

1.3 Founding idea of the C-bRec method

Let us here give a more precise idea of the design of the Carleman-based Reconstruction method in order to explain basically how it works.

The C-bRec algorithm is considering the unknown potential p^* to be recovered as the fix point of a contracting application. Indeed, an iterative process constructs a sequence $(p^k)_{k \in \mathbb{N}}$ converging towards p^* . At step k , the idea is to linearize the inverse problem, considering the system satisfied by

$$v^k = \eta^\varphi \partial_t (u^k - u^*) \text{ in } (0, T) \times \mathcal{R},$$

where u^k is the solution of (1.3)-(1.4) associated with potential p^k , u^* corresponding to p^* , and $\eta^\varphi \in \mathcal{C}^2((0, T) \times \mathcal{R})$ is a smooth cut-off function. Actually η^φ , who depends on a Carleman weight φ designed later in Lemma 2, is constructed to vanish on a given subdomain of $(0, T) \times \mathcal{R}$ where unknown information may lie (namely, later, the subdomains \mathcal{O} and \mathcal{O}_{T_j} of Theorem 3

and Figure 2). Therefore, we have,

$$\begin{cases} \partial_{tt}v^k(t, x) - \partial_{xx}v^k(t, x) + p^k(x)v^k(t, x) = f^k(t, x), & \text{in } (0, T) \times \mathcal{R}, \\ v^k(0, x) = 0, \quad \partial_t v^k(0, x) = \eta^\varphi(0, x)(p^*(x) - p^k(x))u^0(x), & \text{in } \mathcal{R}, \end{cases} \quad (1.7)$$

where $f^k := \eta^\varphi(p^* - p^k)\partial_t u^* - [\eta^\varphi, \partial_{tt} - \partial_{xx}]\partial_t(u^k - u^*)$, using the Lie bracket notation $([A, B])(z) = A(Bz) - B(Az)$. Beware that the cut-off function η^φ will be designed to be differentiable everywhere on the network \mathcal{R} in order to be able to ensure continuity and Kirchhoff law (1.4) for v^k at the internal nodes.

This construction is also called the Bukhgeim-Klibanov method, that goes back to [13], and is the approach developed in the proof of the Lipschitz stability result [6] recalled in Theorem 1.

One should then notice that for all $k \in \mathbb{N}$ and all $s \in \mathbb{R}$, v^k was built to be the unique minimizer of the functional

$$\begin{aligned} F_s[p^k, f^k, \mu^k](z) &= \frac{1}{2} \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt}z - \partial_{xx}z + p^kz - f^k|^2 dxdt \\ &+ \frac{s}{2} \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} |\partial_x z_i(t, \ell_i) - \mu_i^k(t)|^2 dt + \frac{s^3}{2} \mathcal{I}(z, z), \end{aligned} \quad (1.8)$$

where we set, for all $i \in \mathcal{J}_{ext}$, $\mu_i^k(t) = \eta_i^\varphi(t, \ell_i)\partial_t(\partial_x u_i^k(t, \ell_i) - d_i^*(t))$ on $(0, T)$ and where $\mathcal{I}(z, z)$ is a residual quadratic term to be defined later that has the specificity to be erased by the cut-off η^φ . Recall also that $d_i^*(t) = \partial_x u_i^*(t, \ell_i)$, for $i \in \mathcal{J}_{ext}$ and $t \in (0, T)$ is the measured extra information we have to recover p^* . The specific shape of φ insures good properties to the functional F_s , in particular its strict convexity stated in Lemma 4 and proved thanks to the Carleman estimate (3) of Theorem 3. F_s thus has a unique minimizer.

The value of the initial speed in (1.7) allows to recover p^* from v^k , as soon as the cut-off η^φ is constructed so that $\eta_j^\varphi(0, x) = 1$, by the formula

$$p^* = p^k + \frac{\partial_t v^k(0)}{u^0}, \quad \text{on } \mathcal{R}, \quad (1.9)$$

where we assumed that u^0 satisfies (Hc).

Of course, one should argue that v^k is unknown since the source term f^k of the equation depends on the unknown p^* . The idea is actually to minimize another functional $F_s[p^k, 0, \mu^k]$ associated to $f^k = 0$. In this case the unique minimizer is another function we denote w^k . And the error made by replacing v^k by w^k can be estimated with respect to f^k . If we apply the initial speed formula corresponding to (1.9) to w^k , of course we do not obtain p^* anymore but another potential we denote

$$p^{k+1} := p^k + \frac{\partial_t w^k(0)}{u^0}, \quad \text{on } \mathcal{R}. \quad (1.10)$$

And the key point of this design is that we are able to show that the potential p^{k+1} constructed that way is closer to p^* than p^k was, thus leading to construct a convergent sequence of potentials.

We are going to explain this construction scheme in a completely rigorous way later below, and for that we first need to state and prove the appropriate Carleman inequality.

2 One parameter Carleman Estimate

We will prove here a global Carleman estimate with one parameter s and a weight function φ , that can be compared to the previous result presented in [6] where the Carleman estimate was built with two parameters s and λ and the weight function $e^{\lambda\varphi}$. Indeed, the convergence of the C-bRec algorithm is stemming from the use of this Carleman weight φ through the terms $e^{2s\varphi}$ in the cost functional (1.8), and numerical efficiency, however, will benefit a lot that the term is not like $e^{se^{\lambda\varphi}}$.

As already mentioned, this new Carleman estimate is a mandatory step towards an efficient C-bRec algorithm, and as one can read below, even if some structural similarities with the previous article can be pointed at, the challenges and solutions brought here highlight the need of this new technical result.

Lemma 2. *Let us define the Carleman weight function φ on each edge of the network \mathcal{R} as follows:*

$$\forall j \in \mathcal{J}, \varphi_j(t, x) = (x - x_j)^2 - \beta t^2 + M_j, \quad (t, x) \in \mathbb{R} \times (0, \ell_j). \quad (2.1)$$

There exist $(x_j)_{j \in \mathcal{J}} \in \mathbb{R}^-$, $(M_j)_{j \in \mathcal{J}} \in \mathbb{R}^+$, $\beta \in (0, 1)$ and $T > 0$ satisfying

$$\beta T > \sup_{j \in \mathcal{J}} (\ell_j - x_j) \quad (2.2)$$

such that it holds

(i) *The continuity at each internal node is satisfied:*

$$\forall j \in \mathcal{J}_{int}, \varphi_{j_i}(t, 0) = \varphi_j(t, \ell_j), \quad \forall i \in \llbracket 1, N_j \rrbracket. \quad (2.3)$$

(ii) *The following matrices $A_j^\varphi(t)$ satisfy some estimates for any $j \in \mathcal{J}_{int}$:*

$\exists \alpha_j^0 > 0, \beta_j > 0$ such that, for all $\xi = (\xi_1, \dots, \xi_{N_j+1}) \in \mathbb{R}^{N_j+1}$,

$$\begin{aligned} (A_j^\varphi(t)\xi, \xi) &\geq \alpha_j^0 \|\xi\|^2, & \forall t, \quad |t| \leq T_j &:= \frac{\ell_j - x_j}{\beta}; \\ (A_j^\varphi(t)\xi, \xi) &\geq \alpha_j^0 \|\xi\|^2 - \beta_j |\xi_{N_j+1}|^2, & \forall t, \quad T_j \leq |t| \leq T; \end{aligned} \quad (2.4)$$

where $A_j^\varphi(t)$ are $(N_j + 1) \times (N_j + 1)$ symmetric matrices defined by

$$A_j^\varphi(t) := \begin{pmatrix} \phi_{j_1}(0) - \phi_j(\ell_j) & -\phi_j(\ell_j) & \cdots & -\phi_j(\ell_j) & -\phi_j(\ell_j)[\phi]_j \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & -\phi_j(\ell_j) & \vdots \\ & & & \phi_{j_{N_j}}(0) - \phi_j(\ell_j) & -\phi_j(\ell_j)[\phi]_j \\ & & & & a_j(t) \end{pmatrix} \quad (2.5)$$

with $\phi(x) := \partial_x \varphi(t, x)$ and $a_j(t) = -\phi_j(\ell_j)[\phi]_j^2 + [(|\partial_t \varphi(t)|^2 - |\phi|^2)\phi]_j$.

Proof. We prove this Lemma by induction on $j \in \mathcal{J}$. Let us fix $\beta \in (0, 1)$. We decompose $M_j = C_j + M$ where M will be determined at the end of the proof in order to be sure that $M_j \geq 0$. First let us choose $x_1 < 0$ and $C_1 = 0$.

The iteration goes as follows: let $j \in \mathcal{J}_{int}$ and suppose that $x_j < 0$ and $C_j \in \mathbb{R}$ are given, thus $T_j = (\ell_j - x_j)/\beta$ is fixed. Then we choose to take the same value for all $x_{j_i} := -c_j$ for $i \in \llbracket 1, N_j \rrbracket$. The problem is now to find $c_j > 0$ such that the matrix $A_j^\varphi(t)$ satisfies (2.4). We can rewrite it as follows

$$A_j^\varphi(t) = 2 \begin{pmatrix} c_j - (\ell_j - x_j) & -(\ell_j - x_j) & \cdots & -(\ell_j - x_j) & 2(\ell_j - x_j)(N_j c_j - \ell_j + x_j) \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & -(\ell_j - x_j) & \vdots \\ & & & c_j - (\ell_j - x_j) & 2(\ell_j - x_j)(N_j c_j - \ell_j + x_j) \\ & & & & a_j(t)/2 \end{pmatrix}$$

with

$$\begin{aligned} a_j(t)/2 &= -4(\ell_j - x_j)(\ell_j - x_j - N_j c_j)^2 \\ &\quad + 4(\beta^2 t^2 - (\ell_j - x_j)^2)(\ell_j - x_j) - 4N_j(\beta^2 t^2 - c_j^2)c_j. \end{aligned}$$

The first N_j leading principal minors of the matrix $A_j^\varphi(t)$ can be computed easily and do not depend on the time variable. Indeed, for $k \in \llbracket 1, N_j \rrbracket$ we obtain $2^k c_j^{k-1} (c_j - k(\ell_j - x_j))$. Thus by taking

$$c_j > N_j(\ell_j - x_j), \quad (2.6)$$

we get the positivity of the first N_j leading principal minors.

Besides, $\det(A_j^\varphi(T_j)) = P(c_j, (\ell_j - x_j), T_j)$ is a polynomial in c_j of degree $N_j + 3$ and its leading order term is $N_j(2c_j)^{N_j+3}$. Moreover $a_j(t) - a_j(T_j) = 4\beta^2(t^2 - T_j^2)(\ell_j - x_j - c_j N_j) \geq 0$ for c_j sufficiently large and $|t| \leq T_j$.

On the one hand, using Laplace expansion, by choosing c_j sufficiently large, we can get for all $|t| \leq T_j$, $\det(A_j^\varphi(t)) \geq \det(A_j^\varphi(T_j)) > 0$. Then, as soon as $|t| \leq T_j$, since all its leading principal minors are positive, $A_j^\varphi(t)$ is a definite positive matrix, namely we get the first estimate of (2.4):

$$(A_j^\varphi(t)\xi, \xi) \geq \alpha_j^0 \|\xi\|^2, \text{ for all } t \text{ such that } |t| \leq T_j.$$

On the other hand, for $T_j \leq |t| \leq T$, it is easy to calculate that we get the second estimate of (2.4):

$$(A_j^\varphi(t)\xi, \xi) = (A_j^\varphi(T_j)\xi, \xi) + (a_j(t) - a_j(T_j))|\xi_{N_j+1}|^2 \geq \alpha_j^0 \|\xi\|^2 - \beta_j |\xi_{N_j+1}|^2,$$

with $\beta_j = 8\beta^2(N_j c_j - (\ell_j - x_j))(T^2 - T_j^2) > 0$, stemming from (2.6) and the fact that $N_j \geq 2$ bring indeed

$$N_j c_j > N_j^2(\ell_j - x_j) > (\ell_j - x_j). \quad (2.7)$$

Furthermore, for each $i \in \llbracket 1, N_j \rrbracket$, after choosing the $c_j = -x_{j_i}$, we take $C_{j_i} := (\ell_j - x_j)^2 + C_j - (c_j)^2$, in order to have the continuity of the weight functional at the internal nodes (2.3). The last step is to choose $M > 0$ large enough such that all the M_j are non-negative, which is possible as the tree \mathcal{R} has a finite number of edges. \square

Thanks to this construction of weight functions, we are now in position to prove the following global Carleman estimate on the whole network \mathcal{R} .

Theorem 3. *Let φ be a weight function defined as in Lemma 2, with appropriate $(x_j, M_j)_{j \in \mathcal{J}}$, $\beta \in (0, 1)$ and T satisfying assumption (2.2). Then there exist $C > 0$, $s_0 > 0$ such that for all $s \geq s_0$, for all $p \in L_m^\infty(\mathcal{R})$, we have the following Carleman estimate*

$$\begin{aligned} s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0,x)} |\partial_t z(0,x)|^2 dx + s \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} (|\partial_t z|^2 + |\partial_x z|^2 + s^2 |z|^2) dx dt \\ \leq C \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + pz|^2 dx dt \\ + Cs \sum_{i \in \mathcal{J}_{ext}} \int_{-T}^T e^{2s\varphi_i(t, \ell_i)} |\partial_x z_i(t, \ell_i)|^2 dt + Cs^3 \mathcal{I}(z, z), \end{aligned} \quad (2.8)$$

satisfied by all $z \in H^1((-T, T); H_0^1(\mathcal{R}))$ such that $\partial_{tt} z - \partial_{xx} z \in L^2((0, T) \times \mathcal{R})$, under internal continuity from the definition of $H_0^1(\mathcal{R})$ and Kirchhoff node condition in (1.4), and the additional null initial condition $z(0, \cdot) = 0$ in \mathcal{R} , and where

$$\mathcal{I}(z, z) = \iint_{(|t|, x) \in \mathcal{O}} e^{2s\varphi} |z|^2 dx dt + \sum_{j \in \mathcal{J}_{int}} \int_{|t| \in \mathcal{O}_{T_j}} e^{2s\varphi_j(t, \ell_j)} |z_j(t, \ell_j)|^2 dt \quad (2.9)$$

with

$$\mathcal{O} = \cup_{j \in \mathcal{J}} \mathcal{O}_j \text{ where } \mathcal{O}_j = \{(t, x) \in (0, T) \times (0, \ell_j), |x - x_j| - \beta|t| < 0\} \quad (2.10)$$

and

$$\mathcal{O}_{T_j} = \{t \in (0, T), |\ell_j - x_j| - \beta|t| < 0\},$$

defined only for $x = \ell_j$ and any $j \in \mathcal{J}_{int}$. (2.11)

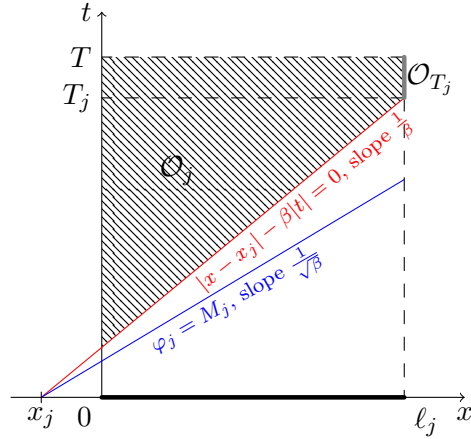


Figure 2: Illustration of domains \mathcal{O}_j and \mathcal{O}_{T_j} for the branch $(0, \ell_j)$, denoting $T_j = |l_j - x_j|/\beta$.

There is a huge literature on proof of Carleman estimates for a lot of different partial differential equations. Let us only suggest here the books [17] (for the proof and the use of Carleman estimates for uniqueness and stability of inverse problems) and [26] (for Carleman estimates of elliptic and heat operators, for observability or unique continuation purposes) where one can find some landmarks, or the seminal and technical work [24] among many other articles.

Proof. Even if the structure of the proof is rather classical, one will need to pay a specific attention here to the networked setting we are facing, bearing in mind that a global Carleman estimate has to be constructed. A great part of the work was done with the design of the Carleman weight φ in Lemma 2. The goal here will be mainly to deal with the internal node terms and to explain how the domains \mathcal{O} , \mathcal{O}_{T_j} end up to be part of the extra right hand side term $\mathcal{I}(z, z)$ in (3).

- **Step 1.** Let us first work on a generic edge of length ℓ omitting as often as possible the index j . We set $y = ze^{s\varphi}$ on $(-T, T) \times (0, \ell)$ and the conjugate operator

$$L_s(y) = e^{s\varphi}(\partial_{tt} - \partial_{xx})(e^{-s\varphi}y). \quad (2.12)$$

Easy calculations bring

$$L_s(y) = \underbrace{(\partial_{tt}y - \partial_{xx}y + s^2(|\partial_t\varphi|^2 - |\partial_x\varphi|^2)y)}_{P_1y} + \underbrace{2s\partial_x\varphi\partial_xy - 2s\partial_t\varphi\partial_ty}_{P_2y} - \underbrace{s(\partial_{tt}\varphi - \partial_{xx}\varphi)y}_{Ry}. \quad (2.13)$$

Thanks to our choice of weight φ according to (2.1), we have $\partial_{xt}\varphi = 0$, $\partial_{xx}\varphi = 2$ and $\partial_{tt}\varphi = -2\beta$. The main work of the proof consists in the computation and bound from below of the cross-term

$$I = \int_{-T}^T \int_0^\ell P_1 y P_2 y dx dt. \quad (2.14)$$

Tedious, yet rather classical computations and integrations by parts (e.g. [7], [6]) yield

$$\begin{aligned} I &= 2(1-\beta)s \int_{-T}^T \int_0^\ell (|\partial_t y|^2 + |\partial_x y|^2) dx dt \\ &+ s^3 \int_{-T}^T \int_0^\ell [\partial_t ((|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_t \varphi) - \partial_x ((|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_x \varphi)] |y|^2 dx dt \\ &+ s \left[\int_0^\ell (2\partial_t y \partial_x \varphi \partial_x y - (|\partial_t y|^2 + |\partial_x y|^2) \partial_t \varphi - s^2 (|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_t \varphi |y|^2) dx \right]_{-T}^T \\ &+ s \left[\int_{-T}^T (2\partial_x y \partial_t \varphi \partial_t y - (|\partial_t y|^2 + |\partial_x y|^2) \partial_x \varphi + s^2 (|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_x \varphi |y|^2) dt \right]_0^\ell. \end{aligned} \quad (2.15)$$

Let us explain briefly how to estimate each term of (2.15).

- The terms in $s|\partial_t y|^2 + s|\partial_x y|^2$ are strictly positive thanks to the assumption $\beta \in (0, 1)$.
- The terms in $s^3|y|^2$ can be rewritten as follows, after calculations using the definition (2.1) of φ :

$$\begin{aligned} &\partial_t ((|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_t \varphi) - \partial_x ((|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_x \varphi) \\ &= -24\beta^3 t^2 + 8\beta|x - x_j|^2 - 8\beta^2 t^2 + 24|x - x_j|^2 \\ &= 8(3\beta + 1)(|x - x_j|^2 - \beta^2 t^2) + 16|x - x_j|^2(1 - \beta). \end{aligned}$$

This term can be bounded by below by a strictly positive constant only where $|x - x_j| - \beta|t| \geq 0$, which is the case in the complementary of the set \mathcal{O} defined by (2.10). The remaining term will appear as a part of $\mathcal{I}(z, z)$ in the right hand side of (3).

- Terms at $t = T$ and $t = -T$. Let us only explain what to do at time $t = T$, since the case $t = -T$ can be handled similarly.
 - The term in $s^3|y(T)|^2$ is positive if and only if

$$-(|\partial_t \varphi(T, x)|^2 - |\partial_x \varphi(T, x)|^2) \partial_t \varphi(T, x) = 8\beta T(\beta^2 T^2 - |x - x_j|^2) \geq 0,$$

which is obviously true from the assumption (2.2) on T in Lemma 2.

– We also get the positiveness of the other terms

$$\begin{aligned}
& 2\partial_t y(T, x) \partial_x \varphi(T, x) \partial_x y(T, x) \\
& \quad - (|\partial_t y(T, x)|^2 + |\partial_x y(T, x)|^2) \partial_t \varphi(T, x) \\
& = 4(x - x_j) \partial_t y(T, x) \partial_x y(T, x) + 2\beta T (|\partial_t y(T, x)|^2 + |\partial_x y(T, x)|^2) \\
& \geq 2\beta T (|\partial_t y(T, x)| - |\partial_x y(T, x)|)^2 \geq 0,
\end{aligned}$$

because $x - x_j \geq -|x - x_j| \geq -\beta T$ from (2.2) again.

Summarizing, none of these terms should appear in the final below estimates as they are all positive.

- Terms at $x = 0$ and $x = \ell$. Here, we need to reintroduce the subscript j to explain the calculations with respect to the concerned branches. This leads us to the next step of the proof.

- **Step 2.** Let us work on the sum of the boundary terms at $x \in \{0, \ell_j\}$ over the whole network \mathcal{R} .

We first recall (2.3) that says the weight function φ is continuous at each internal node, so that we have the following boundary conditions for $y = ze^{s\varphi}$, for any $t \in (-T, T)$:

$$\begin{cases} \text{if } j = 1, & y_1(t, 0) = 0, \\ \text{if } j \in \mathcal{J}_{ext}, & y_j(t, \ell_j) = 0, \\ \text{if } j \in \mathcal{J}_{int}, & y_j(t, \ell_j) = y_{j_i}(t, 0), \forall i \in \llbracket 1, N_j \rrbracket. \end{cases} \quad (2.16)$$

We sum the last term of (2.15) on all edges $j \in \mathcal{J}$ of the network to get

$$s \sum_{j \in \mathcal{J}} \left[\int_{-T}^T (2\partial_x y_j \partial_t \varphi_j \partial_t y_j - (|\partial_t y_j|^2 + |\partial_x y_j|^2) \partial_x \varphi_j + s^2 (|\partial_t \varphi_j|^2 - |\partial_x \varphi_j|^2) \partial_x \varphi_j |y_j|^2) dt \right]_0^{\ell_j}.$$

We decompose this sum on \mathcal{J} into three different sums depending on the position of the node in the network.

- At the root node ($x = 0$ on the edge $j = 1$), since $y_1(t, 0) = 0, \forall t \in (-T, T)$:

$$\begin{aligned}
& s \int_{-T}^T - \left(2\partial_x y_1 \partial_t \varphi_1 \partial_t y_1 - (|\partial_t y_1|^2 + |\partial_x y_1|^2) \partial_x \varphi_1 \right. \\
& \quad \left. + s^2 (|\partial_t \varphi_1|^2 - |\partial_x \varphi_1|^2) \partial_x \varphi_1 |y_1|^2 \right) (t, 0) dt \\
& = s \int_{-T}^T |\partial_x y_1(t, 0)|^2 \partial_x \varphi_1(t, 0) dt,
\end{aligned}$$

and since $\partial_x \varphi_1(t, 0) = -2x_1$ and $x_1 < 0$, this term is positive, thus, can be omitted.

- At the leaf nodes ($x = \ell_j$ on the edges $j \in \mathcal{J}_{ext}$), since $y_j(t, \ell_j) = 0, \forall t \in (-T, T)$:

$$\begin{aligned} \sum_{j \in \mathcal{J}_{ext}} s \int_{-T}^T & \left(2\partial_x y_j \partial_t \varphi_j \partial_t y_j - (|\partial_t y_j|^2 + |\partial_x y_j|^2) \partial_x \varphi_j \right. \\ & \left. + s^2 (|\partial_t \varphi_j|^2 - |\partial_x \varphi_j|^2) \partial_x \varphi_j |y_j|^2 \right) (t, \ell_j) dt \\ & = - \sum_{j \in \mathcal{J}_{ext}} s \int_{-T}^T |\partial_x y_j(t, \ell_j)|^2 \partial_x \varphi_j(t, \ell_j) dt. \end{aligned}$$

This term is strictly negative since $\partial_x \varphi_j(t, \ell_j) = 2(\ell_j - x_j) > 0$, so that it will appear in the end at the right hand side of (3).

- At the internal nodes ($x = \ell_j$ on the edge $j \in \mathcal{J}_{int}$ and $x = 0$ on the daughter edges j_i for $1 \leq i \leq N_j$), recalling $y_j(t, \ell_j) = y_{j_i}(t, 0)$ and the definition (1.2) of $[\cdot]_j$:

$$\begin{aligned} \sum_{j \in \mathcal{J}_{int}} s & \left(\int_{-T}^T (2\partial_x y_j \partial_t \varphi_j \partial_t y_j - (|\partial_t y_j|^2 + |\partial_x y_j|^2) \partial_x \varphi_j \right. \\ & \left. + s^2 (|\partial_t \varphi_j|^2 - |\partial_x \varphi_j|^2) \partial_x \varphi_j |y_j|^2) (t, \ell_j) dt \right. \\ & \left. - \sum_{i=1}^{N_j} \int_{-T}^T (2\partial_x y_{j_i} \partial_t \varphi_{j_i} \partial_t y_{j_i} - (|\partial_t y_{j_i}|^2 + |\partial_x y_{j_i}|^2) \partial_x \varphi_{j_i} \right. \\ & \left. + s^2 (|\partial_t \varphi_{j_i}|^2 - |\partial_x \varphi_{j_i}|^2) \partial_x \varphi_{j_i} |y_{j_i}|^2) (t, 0) dt \right) \\ & = \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \left(2[\partial_x y]_j(t) \partial_t \varphi_j(t, \ell_j) \partial_t y_j(t, \ell_j) - |\partial_t y_j(t, \ell_j)|^2 [\partial_x \varphi]_j(t) \right. \\ & \left. - [|\partial_x y|^2 \partial_x \varphi]_j(t) + s^2 [(|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_x \varphi]_j(t) |y_j(t, \ell_j)|^2 \right) dt \\ & = B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{2.17}$$

We are going to study the sign of each of the four terms in (2.17), keeping in mind that the positive ones can be omitted and the negative ones are meant to be moved to the right hand side of the final estimate. In order to lighten the writing we adopt the notations given in Lemma 2, namely $\phi_j(x) = \partial_x \varphi_j(t, x)$ for $x \in (0, \ell_j)$ and $x_{j_i} = -c_j$ for all $i \in \llbracket 1, N_j \rrbracket$.

- ★ We first calculate, using $y = ze^{s\varphi}$, (2.3) and the assumption $[\partial_x z]_j = 0$:

$$[\partial_x y]_j(t) = [(\partial_x z + sz\phi)e^{s\varphi}]_j(t) = sy_j(t, \ell_j)[\phi]_j. \tag{2.18}$$

Thus the first term of (2.17) after integration by parts in time is

$$\begin{aligned}
B_1 &= \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T 2[\partial_x y]_j(t) \partial_t \varphi_j(t, \ell_j) \partial_t y_j(t, \ell_j) dt \\
&= -4\beta s^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T t [\phi]_j y_j(t, \ell_j) \partial_t y_j(t, \ell_j) dt \\
&= -2\beta s^2 T \sum_{j \in \mathcal{J}_{int}} [\phi]_j (|y_j(T, \ell_j)|^2 + |y_j(-T, \ell_j)|^2) \\
&\quad + 2\beta s^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T [\phi]_j |y_j(t, \ell_j)|^2 dt
\end{aligned}$$

Since c_j obeys (2.6) one gets (2.7) so that $[\phi]_j = 2(\ell_j - x_j) - 2N_j c_j \leq 0$. Thus, using a generic positive constant C that may change from line to line in what follows, we get

$$\begin{aligned}
B_1 &\geq -4\beta s^2 \sum_{j \in \mathcal{J}_{int}} (N_j c_j - (\ell_j - x_j)) \int_{-T}^T |y_j(t, \ell_j)|^2 dt \\
&\geq -C s^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T |y_j(t, \ell_j)|^2 dt,
\end{aligned}$$

- ★ The second term in (2.17) is positive thanks to $[\phi]_j \leq 0$: $B_2 \geq 0$.
- ★ From (2.18) and the definition (1.2), we get

$$\partial_x y_j(t, \ell_j) = s y_j(t, \ell_j) [\phi]_j + \sum_{i=1}^{N_j} \partial_x y_{j_i}(t, 0).$$

Thus for all $j \in \mathcal{J}_{int}$, the integrand of the third term in (2.17) is

$$\begin{aligned}
&-[|\partial_x y|^2 \partial_x \varphi]_j(t) \\
&= -|\partial_x y_j(t, \ell_j)|^2 \phi_j(\ell_j) + \sum_{i=1}^{N_j} |\partial_x y_{j_i}(t, 0)|^2 \phi_{j_i}(0) \\
&= -\left| s y_j(t, \ell_j) [\phi]_j + \sum_{i=1}^{N_j} \partial_x y_{j_i}(t, 0) \right|^2 \phi_j(\ell_j) + \sum_{i=1}^{N_j} |\partial_x y_{j_i}(t, 0)|^2 \phi_{j_i}(0) \\
&= -s^2 |y_j(t, \ell_j)|^2 [\phi]_j^2 \phi_j(\ell_j) - 2s y_j(t, \ell_j) [\phi]_j \left(\sum_{i=1}^{N_j} \partial_x y_{j_i}(t, 0) \right) \phi_j(\ell_j) \\
&\quad + \sum_{i=1}^{N_j} |\partial_x y_{j_i}(t, 0)|^2 (\phi_{j_i}(0) - \phi_j(\ell_j)) - 2\phi_j(\ell_j) \sum_{i,k=1, i \neq k}^{N_j} \partial_x y_{j_i}(t, 0) \partial_x y_{j_k}(t, 0).
\end{aligned}$$

Therefore we can rewrite the third and fourth term in (2.17) as:

$$\begin{aligned} B_3 + B_4 &= \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \left(- [|\partial_x y|^2 \partial_x \varphi]_j(t) \right. \\ &\quad \left. + s^2 [(|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_x \varphi]_j(t) |y_j(t, \ell_j)|^2 \right) dt \\ &= \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \langle A_j^\varphi(t) W_j(t), W_j(t) \rangle dt, \end{aligned}$$

where $W_j(t) \in \mathbb{R}^{N_j+1}$ is defined by

$$W_j(t) = \left(\partial_x y_{j_1}(t, 0) \quad \dots \quad \partial_x y_{j_{N_j}}(t, 0) \quad s y_j(t, \ell_j) \right)^\top,$$

and A_j^φ is the matrix given in (2.5).

Now, gathering the previous estimates, one can write

$$\begin{aligned} B_1 + B_2 + B_3 + B_4 &\geq \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \langle A_j^\varphi(t) W_j(t), W_j(t) \rangle dt - C s^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T |y_j(t, \ell_j)|^2 dt. \end{aligned} \quad (2.19)$$

- **Step 3.** From the results of the two previous steps to build an estimation by below of (2.14), by summation of all the branches' integrals, we get, for s chosen again large enough:

$$\begin{aligned} \int_{-T}^T \int_{\mathcal{R}} P_1 y P_2 y \, dx dt &\geq C s \int_{-T}^T \int_{\mathcal{R}} (|\partial_t y|^2 + |\partial_x y|^2 + s^2 |y|^2) \, dx dt \\ &\quad - C s \sum_{j \in \mathcal{J}_{ext}} \int_{-T}^T |\partial_x y_j(t, \ell_j)|^2 dt - C s^3 \iint_{\mathcal{O}} |y|^2 \, dx dt \\ &\quad + C \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \langle A_j^\varphi(t) W_j(t), W_j(t) \rangle dt - C s^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T |y_j(t, \ell_j)|^2 dt. \end{aligned}$$

Thus, since one easily computes

$$2|L_s(y)|^2 + 2|Ry|^2 \geq |L_s(y) - Ry|^2 = |P_1 y + P_2 y|^2 = |P_1 y|^2 + |P_2 y|^2 + 2P_1 y P_2 y,$$

and using that

$$\int_{-T}^T \int_{\mathcal{R}} |Ry|^2 \, dx dt \leq C s^2 \int_{-T}^T \int_{\mathcal{R}} |y|^2 \, dx dt,$$

we obtain, for s large enough, the preliminary estimate

$$\begin{aligned}
& s \int_{-T}^T \int_{\mathcal{R}} (|\partial_t y|^2 + |\partial_x y|^2 + s^2 |y|^2) dx dt + \int_{-T}^T \int_{\mathcal{R}} (|P_1 y|^2 + |P_2 y|^2) dx dt \\
& \quad + C \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \langle A_j^\varphi(t) W_j(t), W_j(t) \rangle dt \\
& \leq C \int_{-T}^T \int_{\mathcal{R}} |L_s(y)|^2 dx dt + C s \sum_{j \in \mathcal{J}_{ext}} \int_{-T}^T |\partial_x y_j(t, \ell_j)|^2 dt \\
& \quad + C s^3 \iint_{\mathcal{O}} |y|^2 dx dt + C s^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T |y_j(t, \ell_j)|^2 dt. \quad (2.20)
\end{aligned}$$

- Step 4. Let us take one step further to explain how a term $s^{1/2} \int_{\mathcal{R}} |\partial_t y(0, \cdot)|^2 dx$ is obtained in the left hand side of the final estimate. Indeed, if we suppose $z(0, \cdot) = 0$ in \mathcal{R} , we get $y(0, \cdot) = 0$ in \mathcal{R} . Choosing an increasing smooth function $\rho : t \mapsto \rho(t)$ such that $\rho(0) = 1$ and ρ vanishes close to $t = -T$, we multiply $P_1 y$ by $\rho \partial_t y$ and integrate over $(-T, 0) \times \mathcal{R}$, to get (from integrations by parts),

$$\begin{aligned}
& \int_{-T}^0 \int_{\mathcal{R}} P_1 y \rho \partial_t y dx dt = \int_{-T}^0 \int_{\mathcal{R}} (\partial_{tt} y - \partial_{xx} y + s^2 (|\partial_t \varphi|^2 - |\partial_x \varphi|^2) y) \rho \partial_t y dx dt \\
& = \frac{1}{2} \int_{-T}^0 \int_{\mathcal{R}} \rho \partial_t (|\partial_t y|^2 + |\partial_x y|^2) dx dt \\
& \quad + \frac{s^2}{2} \int_{-T}^0 \int_{\mathcal{R}} \rho (|\partial_t \varphi|^2 - |\partial_x \varphi|^2) \partial_t (y^2) dx dt - \sum_{j \in \mathcal{J}} \int_{-T}^0 \rho [\partial_t y_j \partial_x y_j]_0^{\ell_j} dt.
\end{aligned}$$

We focus on the last term. Thanks to the branches boundary conditions (2.16), the equation (2.18), the properties of ρ and the fact that $y_j(0, \ell_j) = 0$, we can calculate, using an integration by parts, that

$$\begin{aligned}
& - \sum_{j \in \mathcal{J}} \int_{-T}^0 \rho [\partial_t y_j \partial_x y_j]_0^{\ell_j} dt \\
& = - \int_{-T}^0 \rho \sum_{j \in \mathcal{J}_{int}} \left(\partial_t y_j(t, \ell_j) \partial_x y_j(t, \ell_j) - \sum_{i=1}^{N_j} \partial_t y_{j_i}(t, 0) \partial_x y_{j_i}(t, 0) \right) dt \\
& = - \int_{-T}^0 \rho \sum_{j \in \mathcal{J}_{int}} [\partial_x y_j]_j \partial_t y_j(t, \ell_j) dt = - \frac{1}{2} \int_{-T}^0 s \rho \sum_{j \in \mathcal{J}_{int}} [\phi]_j \partial_t |y_j(t, \ell_j)|^2 dt \\
& = - \frac{1}{2} \sum_{j \in \mathcal{J}_{int}} \left[s \rho [\phi]_j |y_j(t, \ell_j)|^2 \right]_{-T}^0 + \frac{1}{2} \int_{-T}^0 s \partial_t \rho \sum_{j \in \mathcal{J}_{int}} [\phi]_j |y_j(t, \ell_j)|^2 dt \\
& = \frac{s}{2} \int_{-T}^0 \partial_t \rho \sum_{j \in \mathcal{J}_{int}} [\phi]_j |y_j(t, \ell_j)|^2 dt.
\end{aligned}$$

Thus we have, from $\rho(0) = 1$, $\rho(-T) = 0$, $[\phi]_j \leq 0$, and an other integration by parts:

$$\begin{aligned}
\int_{-T}^0 \int_{\mathcal{R}} P_1 y \rho \partial_t y \, dx dt &= \frac{1}{2} \int_{\mathcal{R}} |\partial_t y(0, \cdot)|^2 \, dx + \frac{s}{2} \sum_{j \in \mathcal{J}_{int}} \int_{-T}^0 \partial_t \rho [\phi]_j |y_j(t, \ell_j)|^2 \, dt \\
&\quad - \frac{1}{2} \int_{-T}^0 \int_{\mathcal{R}} (\partial_t \rho (|\partial_t y|^2 + |\partial_x y|^2) + s^2 \partial_t (\rho (|\partial_t \varphi|^2 - |\partial_x \varphi|^2)) y^2) \, dx dt \\
&\geq \frac{1}{2} \int_{\mathcal{R}} |\partial_t y(0, \cdot)|^2 \, dx - C \int_{-T}^0 \int_{\mathcal{R}} (|\partial_t y|^2 + |\partial_x y|^2 + s^2 |y|^2) \, dx dt \\
&\qquad\qquad\qquad - Cs \sum_{j \in \mathcal{J}_{int}} \int_{-T}^0 |y_j(t, \ell_j)|^2 \, dt.
\end{aligned}$$

By Cauchy-Schwarz inequality, this implies

$$\begin{aligned}
s^{1/2} \int_{\mathcal{R}} |\partial_t y(0, \cdot)|^2 \, dx &\leq \int_{-T}^T \int_{\mathcal{R}} |P_1 y|^2 \, dx dt \\
+ Cs \int_{-T}^T \int_{\mathcal{R}} (|\partial_t y|^2 + |\partial_x y|^2 + s^2 |y|^2) \, dx dt &+ Cs^{3/2} \sum_{j \in \mathcal{J}_{int}} \int_{-T}^0 |y_j(t, \ell_j)|^2 \, dt.
\end{aligned}$$

Therefore, this estimate and (2.20) lead to

$$\begin{aligned}
s^{1/2} \int_{\mathcal{R}} |\partial_t y(0, \cdot)|^2 \, dx + s \int_{-T}^T \int_{\mathcal{R}} (|\partial_t y|^2 + |\partial_x y|^2 + s^2 |y|^2) \, dx dt \\
+ \int_{-T}^T \int_{\mathcal{R}} (|P_1 y|^2 + |P_2 y|^2) \, dx dt + C \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \langle A_j^\varphi(t) W_j(t), W_j(t) \rangle \, dt \\
\leq C \int_{-T}^T \int_{\mathcal{R}} |L_s(y)|^2 \, dx dt + Cs \sum_{j \in \mathcal{J}_{ext}} \int_{-T}^T |\partial_x y_j(t, \ell_j)|^2 \, dt \\
+ Cs^3 \iint_{\mathcal{O}} |y|^2 \, dx dt + Cs^2 \sum_{j \in \mathcal{J}_{int}} \int_{-T}^T |y_j(t, \ell_j)|^2 \, dt. \quad (2.21)
\end{aligned}$$

- **Step 5.** Thanks to the properties (2.4) of Lemma 2 concerning $A_j^\varphi(t)$ and $W_j(t)$, for $|t| \leq T_j$, one has

$$\langle A_j^\varphi(t) W_j(t), W_j(t) \rangle \geq \alpha_j^0 \|W_j(t)\|^2,$$

and for $|t| \geq T_j$, there exist $\alpha_j^0 > 0$ and $\beta_j > 0$ such that

$$\langle A_j^\varphi(t) W_j(t), W_j(t) \rangle \geq \alpha_j^0 \|W_j(t)\|^2 - \beta_j s^2 |y_j(t, \ell_j)|^2.$$

This writes finally

$$\begin{aligned} \sum_{j \in \mathcal{J}_{int}} s \int_{-T}^T \langle A_j^\varphi(t) W_j(t), W_j(t) \rangle dt \\ \geq C s^3 \sum_{j \in \mathcal{J}_{int}} \int_{|t| < T_j} |y_j(t, \ell_j)|^2 dt - C s^3 \sum_{j \in \mathcal{J}_{int}} \int_{|t| > T_j} |y_j(t, \ell_j)|^2 dt \end{aligned}$$

and this last boundary term remains in the right hand side of (2.21) for s large enough.

- **Step 6.** Going back to $z = ye^{-s\varphi}$, one can check that

$$\begin{aligned} e^{2s\varphi} |\partial_t z|^2 &\leq 2|\partial_t y|^2 + 2s^2 |\partial_t \varphi|^2 |y|^2, & \text{in } (-T, T) \times \mathcal{R}, \\ e^{2s\varphi} |\partial_x z|^2 &\leq 2|\partial_x y|^2 + 2s^2 |\partial_x \varphi|^2 |y|^2, & \text{in } (-T, T) \times \mathcal{R}, \\ e^{2s\varphi(0, \cdot)} |\partial_t z(0, \cdot)|^2 &= |\partial_t y(0, \cdot)|^2, & \text{in } \mathcal{R}, \\ e^{2s\varphi(\cdot, \ell_j)} |\partial_x z_j(\cdot, \ell_j)|^2 &= |\partial_x y_j(\cdot, \ell_j)|^2, & \text{in } (-T, T), \forall j \in \mathcal{J}_{ext} \end{aligned}$$

using $z(t=0, \cdot) = 0$ and $z(\cdot, \ell_j) = 0, \forall j \in \mathcal{J}_{ext}$, and

$$\begin{aligned} |\partial_{tt} z - \partial_{xx} z|^2 &\leq 2|\partial_{tt} z - \partial_{xx} z + pz|^2 + 2\|p\|_{L^\infty(\mathcal{R} \times (-T, T))}^2 |z|^2 \\ &\leq 2|\partial_{tt} z - \partial_{xx} z + pz|^2 + 2m^2 |z|^2. \end{aligned}$$

Recalling that $L_s(y) = e^{s\varphi}(\partial_{tt} - \partial_{xx})z$, (2.21) becomes

$$\begin{aligned} s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0, \cdot)} |\partial_t z(0, \cdot)|^2 dx + s \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} (|\partial_t z|^2 + |\partial_x z|^2 + s^2 |z|^2) dx dt \\ \leq C \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + pz|^2 dx dt + C s \sum_{j \in \mathcal{J}_{ext}} \int_{-T}^T e^{2s\varphi(t, \ell_j)} |\partial_x z_j(t, \ell_j)|^2 dt \\ + C s^3 \iint_{\mathcal{O}} e^{2s\varphi} |z|^2 dx dt + C s^3 \sum_{j \in \mathcal{J}_{int}} \int_{T_j < |t| < T} e^{2s\varphi(t, \ell_j)} |z_j(t, \ell_j)|^2 dt \\ + 2C m^2 \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} |z|^2 dx dt. \end{aligned}$$

By taking s sufficiently large, we can absorb the last term

$$2C m^2 \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} |z|^2 dx dt$$

by the left hand side of the inequality, which ends the proof of the Carleman inequality (3). □

3 Carleman based Reconstruction algorithm

This section is devoted to the rigorous presentation of the C-bRec algorithm, as well as its convergence proof.

3.1 Cut-off and cost functional properties

Let us begin by giving some details about two tools of the iterative process briefly presented in section 1.3, namely the cut-off function $\eta^\varphi \in \mathcal{C}^2((0, T) \times \mathcal{R})$ and the cost functional F_s given in (1.8).

Properties of the cut-off function η^φ .

This cut-off function η^φ will be used in the algorithm to truncate appropriately the difference between the state at current step and the measured objective, in order to get something denoted by $v^k = \eta^\varphi \partial_t (u^k - u^*)$ on $(0, T) \times \mathcal{R}$ in section 1.3. Their construction is a challenge when working on a network. A natural way to construct η^φ could be to define each component η_j^φ of η^φ over the branches $(0, \ell_j)$ of \mathcal{R} by $\eta_j^\varphi = \eta_j \circ \varphi_j$, where each $\eta_j \in \mathcal{C}^2(\mathbb{R}; [0, 1])$ is a smooth cut-off function that has to be defined such that v^k satisfies some mandatory properties. It turns out that it is necessary to design it differently, precisely in order to ensure the Kirchhoff law for v^k at the internal nodes. Let us begin by recalling the properties expected from v^k :

- Encoding $(p^k - p^*)$, which is the information we seek, through the initial speed data $\partial_t v^k(0, \cdot) = \eta^\varphi(0, \cdot)(p^* - p^k)u^0$. It means for instance that $\eta_j^\varphi(0, \cdot) = 1$ would be convenient.
- Vanishing in the domains \mathcal{O} and \mathcal{O}_{T_j} so that $\mathcal{I}(v^k, v^k) = 0$ (with \mathcal{I} appearing in F_s (1.8) and defined by (2.9)). Thus one needs $\eta_j^\varphi = 0$ on some domain greater than $\mathcal{O} \cup (\cup_{j \in \mathcal{J}_{int}} \mathcal{O}_{T_j} \times \{\ell_j\})$.
- Allowing the source term f^k of equation (1.7) solved by v^k to be manageable. Since one has a Lie bracket term in

$$f^k = \eta^\varphi(p^* - p^k)\partial_t u^* - [\eta^\varphi, \partial_{tt} - \partial_{xx}]\partial_t (u^k - u^*),$$

we will ask for η^φ to vary (between 0 and 1) only in a small region of $(0, T) \times \mathcal{R}$. Actually, on each $(0, T) \times (0, \ell_j)$, it will be specifically possible (meaning *manageable*) where $M_j < \varphi_j < x_j^2 + M_j$. But it also has to be done properly across each internal node to ensure continuity and Kirchhoff law for v^k at those nodes.

Thus, the cut-off function $\eta^\varphi \in \mathcal{C}^2((0, T) \times \mathcal{R})$ is designed as follows:

$$\begin{aligned} 0 \leq \eta^\varphi(t, x) \leq 1, \quad \forall (t, x) \in (0, T) \times \mathcal{R}, \text{ and} \\ \eta_j^\varphi(t, x) = \begin{cases} 0, & \text{if } \varphi_j(t, x) < M_j, \\ 1, & \text{if } \varphi_j(t, x) \geq x_j^2 + M_j > 0, \end{cases} \quad \forall j \in \mathcal{J}, \end{aligned} \quad (3.1)$$

where M_j and x_j are defined in Lemma 2 and are used in the construction of $\varphi_j(t, x) = (x - x_j)^2 - \beta t^2 + M_j$. The situation for two connected branches j and j_i is described in Figure 3 and allows to see that the variation of η^φ from 0 to 1 has to be done between an upper straight line and a lower curved line (blue or red) on each branch, paying attention to the need of a perpendicular variation at the junction of the branches.

The goal of this last recommendation is to have specifically, for any $j \in \mathcal{J}_{int}$ and $t > 0$, both $v_j^k(t, \ell_j) = v_{j_i}^k(t, 0)$, $\forall i \in \llbracket 1, N_j \rrbracket$, and $[\partial_x v^k]_j(t) = 0$. On the one hand, since $v^k = \eta^\varphi \partial_t (u^k - u^*)$, we need $\eta_j^\varphi(t, \ell_j) = \eta_{j_i}^\varphi(t, 0)$ for continuity. On the other hand, one observes that $\partial_x v^k = \partial_x \eta^\varphi \partial_t (u^k - u^*) + \eta^\varphi \partial_{xt} (u^k - u^*)$ indicating from the boundary assumptions (1.4) on u^k and u^* , that $\partial_x \eta_j^\varphi(t, \ell_j) = \partial_x \eta_{j_i}^\varphi(t, 0) = 0$ would be sufficient for Kirchhoff condition. Hence the need of a ‘perpendicular variation’ at the internal nodes junctions mentioned above.

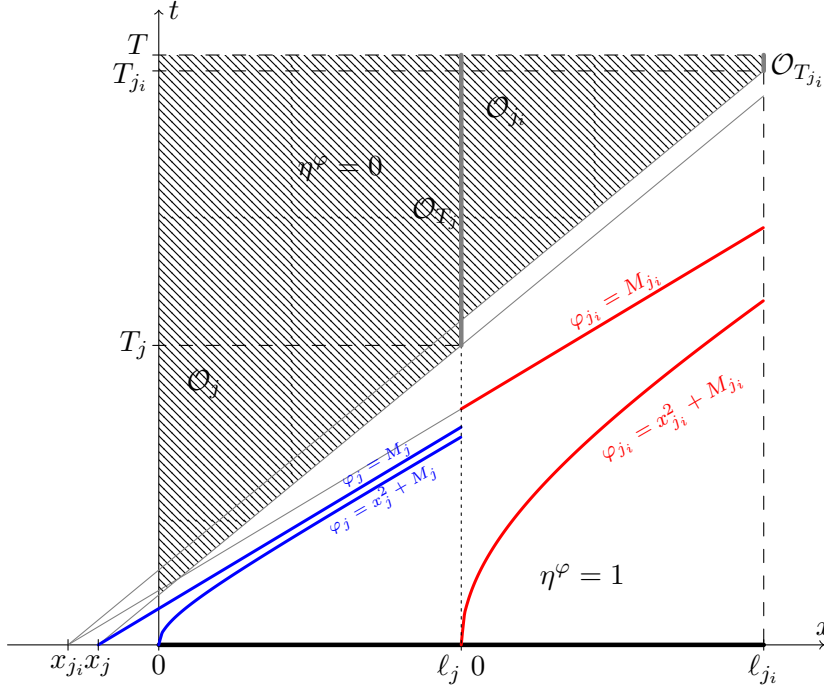


Figure 3: Context of application of the cut-off functions η^φ over two consecutive branches j and j_i .

Properties of the cost functional F_s .

Let us now give some insight about the chosen cost functional F_s presented in (1.8) and used in the algorithm.

Lemma 4. *Let us assume that the hypotheses of Theorem 1 are satisfied. Let φ be a weight function defined as in Lemma 2. Then for all $s > 0$ large enough, $p \in L^\infty(\mathcal{R})$, $f \in L^2(0, T; L^2(\mathcal{R}))$ and $\mu \in L^2(0, T)$, the functional $F_s[p, f, \mu]$ recalled here*

$$F_s[p, f, \mu](z) = \frac{1}{2} \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt}z - \partial_{xx}z + pz - f|^2 dxdt \\ + \frac{s}{2} \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} |\partial_x z_i(t, \ell_i) - \mu_i(t)|^2 dt + \frac{s^3}{2} \mathcal{I}(z, z),$$

(where $\mathcal{I}(z, z)$ is given in (2.9)) is continuous, strictly convex and coercive on \mathcal{T} defined by

$$\mathcal{T} = \left\{ z \in C^0([0, T]; H_0^1(\mathcal{R})) \cap C^1([0, T]; L^2(\mathcal{R})), \partial_{tt}z - \partial_{xx}z \in L^2((0, T) \times \mathcal{R}), \right. \\ \left. z(0, \cdot) = 0 \text{ in } \mathcal{R}, \text{ and } [\partial_x z]_j(t) = 0, \forall j \in \mathcal{J}_{int}, t \in (0, T) \right\} \quad (3.2)$$

and equipped with an appropriate weighed norm.

Thenceforth, the functional $F_s[p, f, \mu]$ admits a unique minimizer on the set \mathcal{T} .

Proof. Let us begin by defining the following weighed seminorm for any $z \in \mathcal{T}$:

$$\|z\|_{obs, s}^2 = 2F_s[p, 0, 0](z). \quad (3.3)$$

This is actually a well defined norm thanks to the Carleman estimate (3) applied to the extension of $z \in \mathcal{T}$ to negative times as an odd function. Indeed, since φ is a bounded function over the bounded domain $(0, T) \times \mathcal{R}$, as soon as s is large enough, one has

$$s^3 \|z\|_{L^2(-T, T; L^2(\mathcal{R}))}^2 \leq CF_s[p, 0, 0](z)$$

bringing $z = 0$ if $F_s[p, 0, 0](z) = 0$.

Moreover, the closed set \mathcal{T} is convex thanks to the linearity of the properties defining it. On the other hand, the functional $F_s[p, f, \mu]$ is continuous and differentiable on \mathcal{T} since one can calculate its gradient as follows, for any $z, Z \in \mathcal{T}$:

$$\langle \nabla F_s[p, f, \mu](z), Z \rangle_{\mathcal{T}} \\ = \int_0^T \int_{\mathcal{R}} e^{2s\varphi} (\partial_{tt}z - \partial_{xx}z + pz - f)(\partial_{tt}Z - \partial_{xx}Z + pZ) dxdt \\ + s \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} (\partial_x z_i(t, \ell_i) - \mu_i(t)) \partial_x Z_i(t, \ell_i) dt + s^3 \mathcal{I}(z, Z). \quad (3.4)$$

Now taking $y, z \in \mathcal{T}$ and $Z = y - z$ one obtains

$$\langle \nabla F_s[p, f, \mu](y) - \nabla F_s[p, f, \mu](z), y - z \rangle_{\mathcal{T}} = 2F_s[p, 0, 0](y - z).$$

Therefore, it shows the coercivity condition

$$\langle \nabla F_s[p, f, \mu](y) - \nabla F_s[p, f, \mu](z), y - z \rangle_{\mathcal{T}} \geq \|y - z\|_{obs, s}^2$$

yielding the strong convexity of $F_s[p, f, \mu]$ from basic properties of differentiable convex functions (see the book [1] for instance if needed). As a consequence, the functional $F_s[p, f, \mu]$ admits a unique minimizer on the set \mathcal{T} and allows to define the upcoming C-bRec algorithm. \square

3.2 The C-bRec algorithm

Let us now state the steps of the algorithm for the reconstruction of the potential p^* over the network \mathcal{R} in a wave equation described by (1.3)-(1.4) and of solution u^* .

Algorithm

Besides knowing, for each branch $j \in \mathcal{J}$, the source term g_j , the boundary term h_j and the initial data (u_j^0, u_j^1) of (1.3)-(1.4), we have the extra measured information at the leaves of the network \mathcal{R} :

$$d_i^*(t) = \partial_x u_i^*(t, \ell_i), \text{ for } i \in \mathcal{J}_{ext} \text{ and } t \in (0, T).$$

Initialisation: Choose any initial guess $p^0 \in L_m^\infty(\mathcal{R})$.

Iteration: Knowing $p^k \in L_m^\infty(\mathcal{R})$,

1. Calculate the solution u^k of (1.3)-(1.4) associated to the potential p^k , and for all $i \in \mathcal{J}_{ext}$ and $t \in (0, T)$, set

$$\mu_i^k(t) = \eta_i^\varphi(t, \ell_i) \partial_t \left(\partial_x u_i^k(t, \ell_i) - d_i^*(t) \right). \quad (3.5)$$

2. Minimize the functional $F_s[p^k, 0, \mu^k]$ defined by

$$\begin{aligned} F_s[p^k, 0, \mu^k](z) &= \frac{1}{2} \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + p^k z|^2 dx dt \\ &+ \frac{s}{2} \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} |\partial_x z_i(t, \ell_i) - \mu_i^k(t)|^2 dt + \frac{s^3}{2} \mathcal{I}(z, z), \end{aligned} \quad (3.6)$$

on the space \mathcal{T} and denote w^k its unique minimizer.

3. Then set

$$\tilde{p}^{k+1} = p^k + \frac{\partial_t w^k(0, \cdot)}{u^0}, \quad \text{on } \mathcal{R}. \quad (3.7)$$

4. Finally, construct

$$p^{k+1} = T_m(\tilde{p}^{k+1}) := \begin{cases} \tilde{p}^{k+1}, & \text{if } |\tilde{p}^{k+1}| \leq m, \\ \text{sign}(\tilde{p}^{k+1})m, & \text{if } |\tilde{p}^{k+1}| > m. \end{cases}$$

Stopping criterion: Choose some tolerance levels ε_1 and ε_2 and some maximum number of iterations K and stop the iterative loop as soon as one of the three following criteria is satisfied:

$$\sup_{j \in \mathcal{J}_{ext}} \frac{\|\partial_x u_i^k(t, \ell_i) - d_j^*\|_2}{\|d_j^*\|_2} \leq \varepsilon_1, \quad \text{or} \quad \sup_{j \in \mathcal{J}} \frac{\|p_j^{k+1} - p_j^k\|_\infty}{m} \leq \varepsilon_2, \quad (3.8)$$

or when the maximal number of iterations K is reached.

The final step of the iterative process is a projection to guarantee that the coefficient p^{k+1} remains appropriately bounded in the space $L_m^\infty(\mathcal{R})$ so that one can apply again, in the next iteration, the Carleman estimate (3).

This algorithm has the advantage to be convergent for any initial guess $p_0 \in L_m^\infty(\mathcal{R})$ without *a priori* knowledge on p^* except its bound m . It is therefore a globally convergent algorithm, as stated in the following result.

3.3 Convergence Result

Theorem 5. Assume the same hypothesis as in Theorem 1. Let φ be a weight function defined as in Lemma 2. Assume also that $p^* \in L_m^\infty(\mathcal{R})$. Then there exists a constant $C > 0$ such that for all s large enough and for all $k \in \mathbb{N}$, it holds

$$\int_{\mathcal{R}} e^{2s\varphi(0)} |p^k - p^*|^2 dx \leq \left(\frac{C}{s^{1/2}} \right)^k \int_{\mathcal{R}} e^{2s\varphi(0)} |p^0 - p^*|^2 dx. \quad (3.9)$$

In particular, if s is large enough, the sequence $(p^k)_{k \in \mathbb{N}}$ given by the algorithm converges towards p^* when k tends to infinity.

Proof. Let us introduce u^k and u^* the solutions of (1.3)-(1.4) with potential p^k , respectively p^* . Then if we set $z^k = \partial_t(u^k - u^*)$ in $(0, T) \times \mathcal{R}$, it satisfies

$$\begin{cases} \partial_{tt} z_j^k - \partial_{xx} z_j^k + p_j^k z_j^k = (p_j^* - p_j^k) \partial_t u_j^*, & \text{in } (0, T) \times (0, \ell_j), \\ z_j^k(0, \cdot) = 0, \quad \partial_t z_j^k(0, \cdot) = (p_j^* - p_j^k) u_j^0, & \text{in } (0, \ell_j), \end{cases} \quad (3.10)$$

along with the internal transmission conditions and external boundary conditions for all $t \in (0, T)$:

$$\begin{cases} \text{for } j = 1, & z_1(t, 0) = 0, \\ \text{if } j \in \mathcal{J}_{ext}, & z_j(t, \ell_j) = 0, \\ \text{if } j \in \mathcal{J}_{int}, & z_j(t, \ell_j) = z_{j_i}(t, 0), \quad \forall i \in \llbracket 1, N_j \rrbracket, \text{ and } [\partial_x z]_j(t) = 0. \end{cases} \quad (3.11)$$

Usual *a priori* energy estimates, as one can read in [6, Lemma 3.2] or in a more general framework in [22], for z^k solution of equation (3.10)-(3.11) yield

$$\begin{aligned} & \|z^k\|_{L^\infty(0,T;H_0^1(\mathcal{R}))} + \|\partial_t z^k\|_{L^\infty(0,T;L^2(\mathcal{R}))} \\ & \leq C\|p^k - p^*\|_{L^2(\mathcal{R})} (\|u^0\|_{L^\infty(\mathcal{R})} + \|\partial_t u^*\|_{L^1(0,T;L^\infty(\mathcal{R}))}). \end{aligned} \quad (3.12)$$

Let us now define the integral

$$I = s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0,x)} |p^{k+1}(x) - p^*(x)|^2 dx \quad (3.13)$$

that we will estimate succesively using the informations of the algorithm and the Carleman estimate.

One can first notice that T_m , the projection operator of Step 4 of the algorithm, is lipschitzian, and is also such that $p^{k+1} = T_m(\tilde{p}^{k+1})$ and $p^* = T_m(p^*)$, since we assumed that $p^* \in L_m^\infty(\mathcal{R})$, allowing therefore to write

$$I = s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0)} |T_m(\tilde{p}^{k+1}) - T_m(p^*)|^2 dx \leq s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0)} |\tilde{p}^{k+1} - p^*|^2 dx.$$

Moreover, the initial data u^0 satisfies the positivity assumption (Hc), so that

$$I \leq s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0)} |\tilde{p}^{k+1} - p^*|^2 \frac{|u^0|^2}{(\alpha^0)^2} dx = \frac{s^{1/2}}{(\alpha^0)^2} \int_{\mathcal{R}} e^{2s\varphi(0)} |(\tilde{p}^{k+1} - p^*)u^0|^2 dx.$$

Let us now split the gap between \tilde{p}^{k+1} and p^* using the previous value p^k , and use the optimization information of Step 2 and 3.

On the one hand, recall that w^k is the minimizer of $F_s[p^k, 0, \mu^k]$ on \mathcal{T} and is used to calculate $(\tilde{p}^{k+1} - p^k)u^0 = \partial_t w^k(0)$ from (3.7).

On the other hand, one can verify that $v^k = \eta^\varphi \partial_t (u^k - u^*)$ is the unique minimizer of $F_s[p^k, f^k, \mu^k]$, since $I(v_k, v_k) = 0$ thanks to the choice of η^φ (see (3.1) and around), and since it is indeed the solution of

$$\begin{cases} \partial_{tt} v_j^k - \partial_{xx} v_j^k + p_j^k v_j^k = f_j^k, & \text{in } (0, T) \times (0, \ell_j), \\ v_j^k(0, \cdot) = 0, \quad \partial_t v_j^k(0, \cdot) = (p_j^* - p_j^k)u_j^0, & \text{in } (0, \ell_j), \end{cases} \quad (3.14)$$

with $f_j^k := \eta_j(\varphi_j)(p_j^* - p_j^k)\partial_t u_j^* - [\eta_j(\varphi_j), \partial_{tt} - \partial_{xx}]\partial_t(u_j^k - u_j^*)$, using the Lie bracket notation $[\cdot, \cdot]$. Besides, from our design of η^φ , one has the boundary conditions (3.11) satisfied by v^k , so that we can ensure that v^k belongs to \mathcal{T} .

Summing up, we get

$$\begin{aligned} I & \leq \frac{s^{1/2}}{(\alpha^0)^2} \int_{\mathcal{R}} e^{2s\varphi(0)} |(\tilde{p}^{k+1} - p^k)u^0 + (p^k - p^*)u^0|^2 dx \\ & \leq \frac{s^{1/2}}{(\alpha^0)^2} \int_{\mathcal{R}} e^{2s\varphi(0)} |\partial_t w^k(0) - \partial_t v^k(0)|^2 dx. \end{aligned} \quad (3.15)$$

One can now apply the Carleman estimate (3) to $z = w^k - v^k$ after extending it to negative times as well, as an odd function that satisfies the required assumptions and in particular the fact that $z(0, x) = 0$ on the whole network \mathcal{R} . Therefore, there exists a positive constant C (generic, as it may change from line to line) depending on α^0 such that

$$\begin{aligned}
I &\leq C \int_{-T}^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt}(w^k - v^k) - \partial_{xx}(w^k - v^k) + p^k(w^k - v^k)|^2 dx dt \\
&\quad + Cs \sum_{i \in \mathcal{J}_{ext}} \int_{-T}^T e^{2s\varphi_i(t, \ell_i)} |\partial_x(w_i^k - v_i^k)(t, \ell_i)|^2 dt \\
&\quad + Cs^3 \mathcal{I}(w^k - v^k, w^k - v^k) \\
&\leq C \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt}w^k - \partial_{xx}w^k + p^kw^k|^2 dx dt + Cs^3 \mathcal{I}(w^k, w^k) \\
&\quad + Cs \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} |\partial_x w_i^k(t, \ell_i) - \mu_i^k|^2 dt \\
&\quad + C \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt}v^k - \partial_{xx}v^k + p^kv^k|^2 dx dt \\
&\quad + Cs \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} |\partial_x v_i^k(t, \ell_i) - \mu_i^k|^2 dt.
\end{aligned}$$

Indeed, we use here that $\mathcal{I}(w^k - v^k, w^k - v^k) = \mathcal{I}(w^k, w^k)$ since v^k vanish on $\mathcal{O} \cup \left(\bigcup_{j \in \mathcal{J}_{int}} \mathcal{O}_{T_j \times \{\ell_j\}} \right)$. This splitting, using also the measurement μ^k , allows to identify the functional $F_s[p^k, 0, \mu^k]$ defined in (3.6) that reaches its minimum at w^k :

$$\begin{aligned}
I &\leq C \left(F_s[p^k, 0, \mu^k](w^k) + F_s[p^k, 0, \mu^k](v^k) \right) \\
&\leq 2C F_s[p^k, 0, \mu^k](v^k) = C \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |f^k|^2 dx dt, \quad (3.16)
\end{aligned}$$

where $f^k = \eta^\varphi(p^* - p^k)\partial_t u^* - [\eta^\varphi, \partial_{tt} - \partial_{xx}]z^k$ is the source term of equation (3.14) satisfied by v^k that we can now estimate. Using therefore

$$[\eta^\varphi, \partial_{tt} - \partial_{xx}]z = -((\partial_{tt}\eta^\varphi - \partial_{xx}\eta^\varphi)z + 2\partial_t\eta^\varphi\partial_t z - 2\partial_x\eta^\varphi\partial_x z),$$

and the properties of φ and η^φ , it brings easily, from

$$\begin{aligned}
I &\leq C \int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\eta^\varphi(p^* - p^k)\partial_t u^*|^2 dx dt \\
&\quad + C \sum_{j \in \mathcal{J}} \int_0^T \int_0^{\ell_j} e^{2s\varphi_j} |[\eta_j^\varphi, \partial_{tt} - \partial_{xx}]z_j^k|^2 dx dt,
\end{aligned}$$

that

$$I \leq C \|u^*\|_{H^1(0,T;L^\infty(\mathcal{R}))}^2 \int_{\mathcal{R}} e^{2s\varphi(0)} |p^k - p^*|^2 dx \\ + C \sum_{j \in \mathcal{J}} e^{2s(M_j + x_j^2)} \int_0^T \int_0^{\ell_j} (|\partial_x z_j^k|^2 + |\partial_t z_j^k|^2 + |z_j^k|^2) dx dt.$$

Indeed, in the last estimate, we used both the fact that $\varphi(t, \cdot) \leq \varphi(0, \cdot)$ on \mathcal{R} and that the support of the term $[\eta_j^\varphi, \partial_{tt} - \partial_{xx}]$ lies where η_j^φ varies, meaning where one has $M_j < \varphi_j < M_j + x_j^2$, insuring that $e^{2s\varphi_j} \leq e^{2s(M_j + x_j^2)}$.

Together with the *a priori* estimate (3.12) on z^k that also writes

$$\|z^k\|_{L^\infty(0,T;H_0^1(\mathcal{R}))} + \|\partial_t z^k\|_{L^\infty(0,T;L^2(\mathcal{R}))} \leq C \|u^*\|_{H^1(0,T;L^\infty(\mathcal{R}))} \|p^k - p^*\|_{L^2(\mathcal{R})},$$

one gets

$$I \leq C \|u^*\|_{H^1(0,T;L^\infty(\mathcal{R}))} \left(\int_{\mathcal{R}} e^{2s\varphi(0)} |p^k - p^*|^2 dx \right. \\ \left. + \sum_{j \in \mathcal{J}} e^{2s(M_j + x_j^2)} \|p^k - p^*\|_{L^2(0,\ell_j)} \right).$$

Combining this estimate with the definition (3.13) of I and using that $\varphi(0, \cdot) \geq M_j + x_j^2$ on \mathcal{R} , we finally obtain

$$s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0)} |p^{k+1} - p^*|^2 dx \leq C \int_{\mathcal{R}} e^{2s\varphi(0)} |p^k - p^*|^2 dx \quad (3.17)$$

where $C > 0$ depends on $\|u^*\|_{H^1(0,T;L^\infty(\mathcal{R}))}$, α_0 and T .

By simple iteration on k , we conclude to the desired estimate (3.9) and the convergence result when k tends to infinity as soon as s is large enough to have $Cs^{-1/2} < 1$. \square

4 Numerics

Now, we present the discretization schemes and the technical solutions we have developed to implement numerically the C-bRec algorithm on a network.

4.1 Topology of the network

The only data concerning the network that the algorithm needs as input is the list of names (alpha-numeric character strings) of the branches, possibly in disorder, and their respective sizes.

For example, for the network shown in Figure 10, we can give the list

$$\{ac2, ab, ab4, a, ab1, aa, ac, ac5, ab3\}.$$

The only rule to respect is that all the children of the same branch bear the name of their mother to which a character has been added. Accordingly, for the network shown in Figure 10, the children of branch aa are for example branches $aa1$, $aa3$ and $aa4$. Thus, we can easily find the mother of a given branch by removing the last character in its name. Then the algorithm automatically takes care of:

- sort the branch's names alphabetically to build \mathcal{J} , so that the instance $\{ac2, ab, ab4, a, ab1, aa, ac, ac5, ab3\}$ becomes

$$\mathcal{J} = \{a, aa, ab, ab1, ab3, ab4, ac, ac2, ac5\};$$

- construct the sets \mathcal{J}_{int} and \mathcal{J}_{ext} , bringing here $\mathcal{J}_{int} = \{a, ab, ac\}$ and $\mathcal{J}_{ext} = \{aa, ab1, ab3, ab4, ac2, ac5\}$. One may notice that in practice, we only need the position in \mathcal{J} of the elements of \mathcal{J}_{int} and \mathcal{J}_{ext} , so that we only build the positions table $\mathcal{J}_{int} = \{0, 2, 6\}$ and $\mathcal{J}_{ext} = \{1, 3, 4, 5, 7, 8\}$.
- create the tables that give, for a given branch $j \in \mathcal{J}$, the number N_j of its children, its possible position in \mathcal{J}_{int} , and the position in \mathcal{J}_{int} of its mother branch.

Afterwards, following the proof of Lemma 2, we build the x_j , M_j and T_j by induction on $j \in \mathcal{J}$. We make sure that c_j is large enough to guarantee that the determinant of the matrix $A_j^\varphi(T_j)$ given in (2.5), that we calculate numerically, is indeed strictly positive.

In order to be consistent below with the explanations already given in the theoretical part, we will assume that the first branch bears the name 1.

4.2 Generation of the data and noise

We work here with synthetic data. To discretize equation (1.3)-(1.4) in $(0, T) \times \mathcal{R}$, we use finite differences (explicit centered scheme) in space and time. We introduce integers N_t and N_{x_j} and fix the time step Δt and the space step Δx_j in the branch $j \in \mathcal{J}$ such that $\ell_j = (N_{x_j} + 1)\Delta x_j$ and $T = (N_t + 1)\Delta t$. Then, we define, for $0 \leq i \leq N_{x_j} + 1$ and $0 \leq n \leq N_t + 1$, $(u_j^*)_i^n$ a numerical approximation of the solution $(u_j^*)(t^n, x_i)$ with $t^n = n\Delta t$ and $x_i = i\Delta x_j$. It satisfies the following system for $j \in \mathcal{J}$, $1 \leq i \leq N_{x_j}$ and $1 \leq n \leq N_t$:

$$\left\{ \begin{array}{l} \frac{(u_j^*)_i^{n+1} - 2(u_j^*)_i^n + (u_j^*)_i^{n-1}}{\Delta t^2} - \frac{(u_j^*)_{i+1}^n - 2(u_j^*)_i^n + (u_j^*)_{i-1}^n}{\Delta x_j^2} \\ \quad + p_j^*(x_i)(u_j^*)_i^n = g_j(t^n, x_i), \\ (u_j^*)_i^0 = u_j^0(x_i), \quad \text{and} \quad (u_j^*)_i^1 = u_j^0(x_i) + \Delta t u_j^1(x_i), \end{array} \right. \quad (4.1)$$

with the boundary conditions for $0 \leq n \leq N_t + 1$:

$$\left\{ \begin{array}{l} \text{for } j = 1, \quad (u_1^*)^n = h_1(t^n), \\ \text{if } j \in \mathcal{J}_{ext}, \quad (u_j^*)_{N_{x_j}+1}^n = h_j(t^n), \\ \text{if } j \in \mathcal{J}_{int}, \quad (u_j^*)_{N_{x_j}+1}^n = (u_{j_i}^*)_0^n, \quad \forall i \in \llbracket 1, N_j \rrbracket, \\ \\ \frac{(u_j^*)_{N_{x_j}+1}^n - (u_j^*)_{N_{x_j}}^n}{\Delta x_j} = \sum_{i=1}^{N_j} \frac{(u_{j_i}^*)_1^n - (u_{j_i}^*)_0^n}{\Delta x_{j_i}}. \end{array} \right. \quad (4.2)$$

Then, we compute the discrete counterpart of the continuous measurement d^* given in (1.5) as follows:

$$(d_j^*)^n = \frac{(u_j^*)_{N_{x_j}+1}^n - (u_j^*)_{N_{x_j}}^n}{\Delta x_j}, \quad j \in \mathcal{J}_{ext}, \quad 0 \leq n \leq N_t + 1.$$

On the measured data, we add a Gaussian noise:

$$(d_j^*)^n \leftarrow (d_j^*)^n + \theta(\max_m (d_j^*)^m) \mathcal{N}(0, 1), \quad j \in \mathcal{J}_{ext}, \quad 0 \leq n \leq N_t + 1 \quad (4.3)$$

where $\mathcal{N}(0, 1)$ satisfies a centered normal law with deviation 1 and θ is the level of noise.

Note that the chosen model of noise is additive. In the presence of noise, the first step of the inverse problem is to regularize the data in a way that depends on the nature of the noise and the physical phenomena. In this article, we use the Butterworth lowpass filter.

4.3 Discretization of the algorithm

Initialization process.

For each branch $j \in \mathcal{J}$, we choose a first guess function $p_j^0(x)$ satisfying

$$\|p_j^0(x)\|_{L^\infty(0, \ell_j)} \leq m$$

and set $(p_j^0)_i = (p_j^0)(x_i)$ for all $1 \leq i \leq N_{x_j}$.

Iterative steps.

Discretization of Step 1. It first consists in computing the solution u^k of system (4.1)-(4.2) associated to potential p^k .

Then, for $j \in \mathcal{J}_{ext}$ and $1 \leq n \leq N_t$, we compute a discrete approximation of $\mu_j^k(t^n)$ defined in (3.5) as:

$$(\mu_j^k)^n = \eta_j^\varphi(t^n, \ell_j) \frac{\left((d_j^k)^{n+1} - (d_j^*)^{n+1} \right) - \left((d_j^k)^{n-1} - (d_j^*)^{n-1} \right)}{2\Delta t},$$

where $(d_j^k)^n = \frac{(u_j^k)_{N_{x_j}+1}^n - (u_j^k)_{N_{x_j}}^n}{\Delta x_j}$ and η_j^φ is given by (3.1).

Discretization of Step 2. The minimization process for $F_s[p^k, 0, \mu^k]$ is equivalent to the resolution of the following variational formulation: Find $w^k \in \mathcal{T}$ such that for all $z \in \mathcal{T}$,

$$\begin{aligned} & \int_0^T \int_{\mathcal{R}} e^{2s\varphi} (\partial_{tt} w^k - \partial_{xx} w^k + p^k w^k) (\partial_{tt} z - \partial_{xx} z + p^k z) dx dt \\ & + s \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} \partial_x w_i^k(t, \ell_i) \partial_x z_i(t, \ell_i) dt + s^3 \mathcal{I}(w^k, z) \\ & = s \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t, \ell_i)} \mu_i^k(t) \partial_x z_i(t, \ell_i) dt. \quad (4.4) \end{aligned}$$

To discretize (4.4), we approximate the integrals using rectangle quadrature rules and we use standard centered finite differences for the various integrands. We do not detail here these classical developments.

Special attention must be paid to the discretization process of the minimization space \mathcal{T} defined by (3.2) in Lemma 4. We get some additional constraints on w^k , i.e. for $0 \leq n \leq N_t + 1$

$$\left\{ \begin{array}{l} \text{for } j = 1, \quad (w_1^k)_0^n = 0, \\ \text{if } j \in \mathcal{J}_{ext}, \quad (w_j^k)_{N_{xj}+1}^n = 0, \\ \text{if } j \in \mathcal{J}_{int}, \quad (w_j^k)_{N_{xj}+1}^n = (w_{j_i}^k)_0^n, \quad \forall i \in \llbracket 1, N_j \rrbracket, \\ \frac{(w_j^k)_{N_{xj}+1}^n - (w_j^k)_{N_{xj}}^n}{\Delta x_j} = \sum_{i=1}^{N_j} \frac{(w_{j_i}^k)_1^n - (w_{j_i}^k)_0^n}{\Delta x_{j_i}}. \end{array} \right. \quad (4.5)$$

Those constraints are expressed in the form of additional equations to (4.4). But we must not forget that it also has an impact on the discretized basis of \mathcal{T} .

Finally the minimizer w^k is obtained as the unique solution of a linear system associated with a sparse matrix of size $(\sum_{j \in \mathcal{J}} (N_{xj} + 2)) (N_t + 2)$ corresponding to the discretization of the variational problem (4.4)-(4.5).

As far as the discrete setting is concerned, we have to add new terms to the variational formulation (4.4) to guarantee its coercivity property uniformly with respect to the discretization parameters Δt and Δx_j . These additional terms (sometimes called viscosity terms) that help handling high frequency spurious waves generated by the numerical scheme take the form

$$\begin{aligned} & \sum_{j \in \mathcal{J}} s \Delta x_j \Delta t \sum_{n=0}^{N_t} \sum_{i=0}^{N_{xj}} e^{2s\varphi_j(t^n, x_i)} \left((w_j^k)_{i+1}^{n+1} - (w_j^k)_i^{n+1} - (w_j^k)_{i+1}^n + (w_j^k)_i^n \right) \\ & \quad \times (z_{i+1}^{n+1} - z_i^{n+1} - z_{i+1}^n + z_i^n). \quad (4.6) \end{aligned}$$

For explanations and proofs, we refer to the work [9] on discrete Carleman estimates for the wave operator, and see also [7] for other informations regarding this numerical issue.

Discretization of Step 3 and Step 4. We discretize equation (3.7) as:

$$(\tilde{p}_j^{k+1})_i = (p_j^k)_i + \frac{(w_j^k)_i^1}{u^0 \Delta t}, \quad j \in \mathcal{J}, \quad 1 \leq i \leq N_{xj} \quad (4.7)$$

and the discretization of Step 4 is straightforward.

Stopping criterion.

According to (3.8) and the data noise taken into account, the iterative loop is stopped when one of the following criteria is satisfied:

$$\sup_{j \in \mathcal{J}_{\text{ext}}} \frac{\|d_j^k - d_j^*\|_2}{\|d_j^*\|_2} \leq \max(\varepsilon_1, \theta), \quad (4.8)$$

or

$$\sup_{j \in \mathcal{J}} \frac{\|p_j^{k+1} - p_j^k\|_\infty}{m} \leq \varepsilon_2, \quad (4.9)$$

where $(\varepsilon_1, \varepsilon_2)$ is a fixed tolerance and θ is the level of noise (recall (4.3)), or when the maximal number K of iterations is reached.

If the exact solution p^* is known (which is the case in a synthetic data setting as ours), we can compute for each $j \in \mathcal{J}$ the final error

$$\text{err}_j = \frac{\|p_j^\infty - p_j^*\|_2}{\|p_j^*\|_2}, \quad (4.10)$$

with p_j^∞ the converged numerical solution of the sequence $(p_j^k)_{k \in \mathbb{N}}$. This error ratio is reported in Table 2.

4.4 Numerical challenges

The main drawback of the approach is related to the presence of large exponential factors in the functional $F_s[p^k, 0, \mu^k]$ that leads to severe numerical difficulties when performing the minimization for s large. This difficulty is solved thanks to the three following ideas:

- As already mentioned, we proved and used Carleman estimates with a single weight of the form $e^{2s\varphi}$ instead of the most frequently used double weight $e^{2se^{\lambda\varphi}}$ for $\lambda > 0$ large enough;

- At the discrete level, as in [7], we work on the conjugate variable

$$(y_j^k)_i^n = (w_j^k)_i^n e^{s\varphi_j(t^n, x_i)},$$

for $j \in \mathcal{J}$, $0 \leq n \leq N_t + 1$ and $0 \leq i \leq N_{x_j} + 1$. This change of unknown acts as a preconditioner of the linear system coming from (4.4) and decreases its condition number. Indeed, it allows to remove the exponential terms appearing in the matrix of the system.

- Unfortunately, there are still exponential factors in the right hand side vector. The idea to tackle those is to develop a progressive process to compute the solution as the aggregation of several problems localized in subdomains in which the exponential factors are all of the same order. See [7] for more details.

4.5 Numerical results

This subsection is devoted to the presentation of some numerical examples to illustrate the properties of the C-bRec algorithm constructed for the network of strings and its efficiency. All simulations are executed with PYTHON and the source codes are available on request. Table 1 gather the numerical values used for all the following examples, unless specified otherwise where appropriate. In all the figures showing reconstruction results, the exact coefficient that we want to recover is plotted by a red line, whereas the numerical coefficient recovered by the algorithm is represented by a dotted black line. The information about the convergence (number of iterations, running time, convergence error) is reported in Table 2.

u_0	u_1	g	h	m
(2,2,2)	(0,0,0)	(0,0,0)	(2,2,2)	2
ℓ_j	β	s in Theorem 3	ε_1 in (4.8)	ε_2 in (4.9)
(0.5,1,0.75)	0.99	1	10^{-3}	10^{-3}
x_j in Lemma 2	M_j	T given by (2.2)	N_{x_j}	N_t
(-0.3,-2.89,-2.89)	(7.71,0,0)	3.9	$100 * \ell_j$	$110 * T$

Table 1: Numerical values of the variables used for all the numerical examples over the network of Figure 4.

Simulations from data without noise

A first result of reconstruction obtained in the absence of noise is given for the network plotted in Figure 4. Figure 5 shows the evolution of the recovered coefficient p^k at each iteration k of the convergence process and below the final

Example	# of iterations	Running time in sec	err in (4.10)
Figure 5	4	226	(0.083,0.001,0.001)
Figure 6	4	167	(0.028, 0.057, 0.001)
	4	158	(0.095, 0.008, 0.007)
Figure 8	3	200	(0.25,0.05,0.09)
	3	190	(0.26,0.06,0.10)
	3	150	(0.27,0.11,0.15)
Figure 11	10	8261	(0.080,0.014,0.043,... ...,0.068,0.009,0.09... ...,0.032,0.020,0.004)

Table 2: Convergence results of the test cases. One can observe that in all cases of Figure 4 the convergence criteria (4.8) is met in less than 5 iterations. The algorithm is running on a personal laptop in a few minutes. The final error defined in (4.10) shows the effectiveness of the reconstruction. Figure 11 gives the convergence results for a more complex network given in 10.

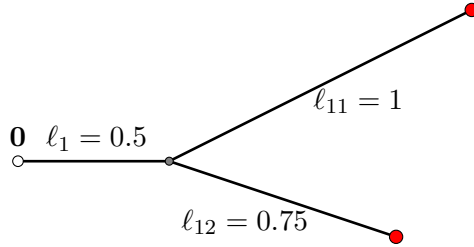


Figure 4: First setting - a 3 branches network, with observations at \bullet .

result p^∞ together with the exact solution p^* . In all the examples of Figure 5, the convergence is reached in less than 5 iterations and the solution after one iteration is almost the final one.

First line of Figure 6 represents the results for the same potential than in the previous case Figure 5 but on other branches. We show in particular that regular potential are well reconstructed regardless of their place in the network whereas rectangular pulse are less well reconstructed.

Second line of Figure 6 gives the results for potential without continuity at internal node.

Simulations with several levels of noise

In Figure 7, we plot an example of observations measured at the point $x = \ell_{11}$ during the time interval T corresponding to $p_1^*(x) = -\mathbb{1}_{[0.3,0.8]}(x/\ell_1)$, $p_{11}^*(x) = \sin(2\pi x/\ell_{11})$ and $p_{12}^*(x) = \sin(5\pi x/\ell_{12})$.

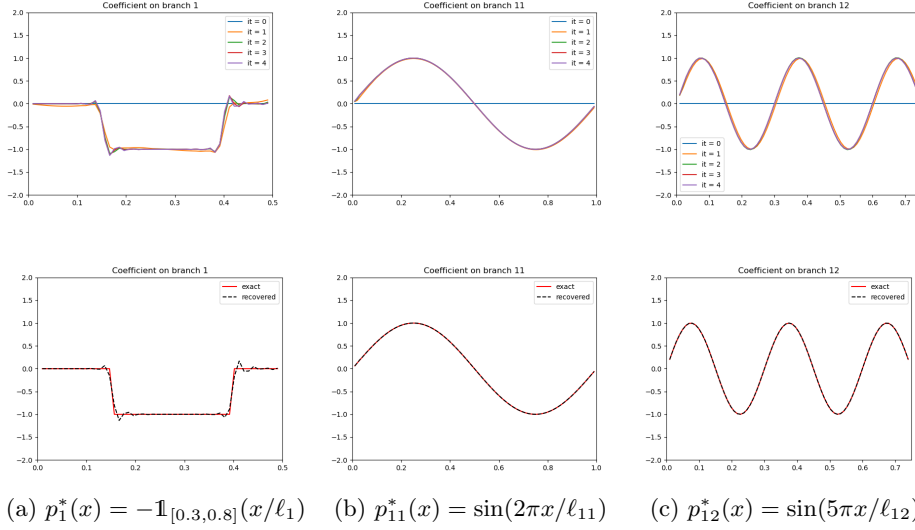


Figure 5: Top line: Convergence history of the reconstruction process. Bottom line: final reconstruction result (dotted black line) and exact coefficient (red line) for the three branches.

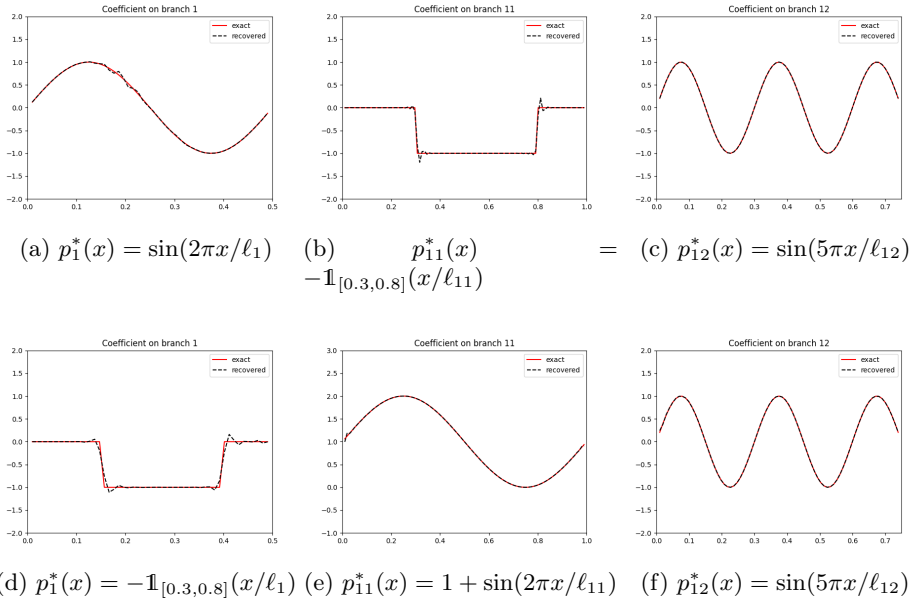


Figure 6: Final reconstruction result (dotted black line) and exact coefficient (red line) with other potentials.

Figures 8 show the results with different levels of noise in the measurements ($\theta = 1\%$, $\theta = 2\%$ and 5% in (4.3)).

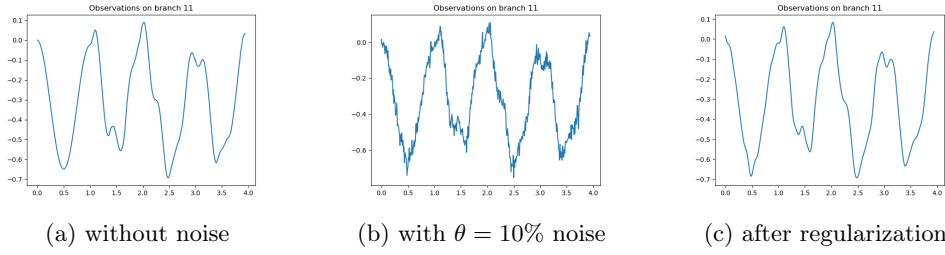


Figure 7: Examples of observations $d^*(t)$ at $x = \ell_{11}$. We show the impact of adding noise following (4.3) and the result obtained after applying the low-pass filter.

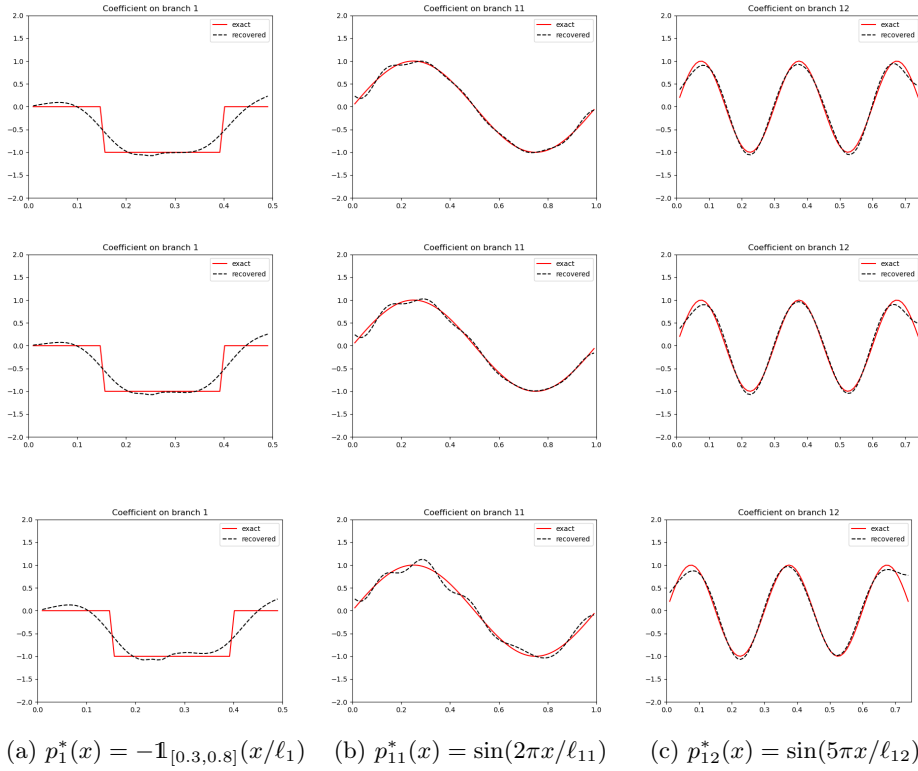


Figure 8: Recovery of the coefficient in presence of $\theta = 1\%$ (first line), $\theta = 2\%$ (second line) and $\theta = 5\%$ (third line) noise in the data. The final error on the coefficients is reported in Table 2.

Wrong choices of the parameters

The first two lines of Figure 9 present the reconstruction result when the assumption on the minimal observation time T given in (2.2) is not satisfied. On the first line, the observation time is taken equal to the intuitive physical

time $\ell_1 + \ell_{11} = 1.5$, corresponding to the time needed for the wave travelling at speed 1 to reach one of the observation nodes even if it starts from the root node of the network. In that case, the reconstruction is as accurate as the one obtained with the much larger time satisfying (2.2). Then, on the second line, we try the same reconstructions but with a shorter time $T = 1.25$. In that case, the result begins to deteriorate near the root node. This illustrates the fact that the minimal time to get the convergence of the algorithm is very likely the larger path in the network from the root node to a leaf, instead of the more restrictive time condition (2.2) (that is probably only technical).

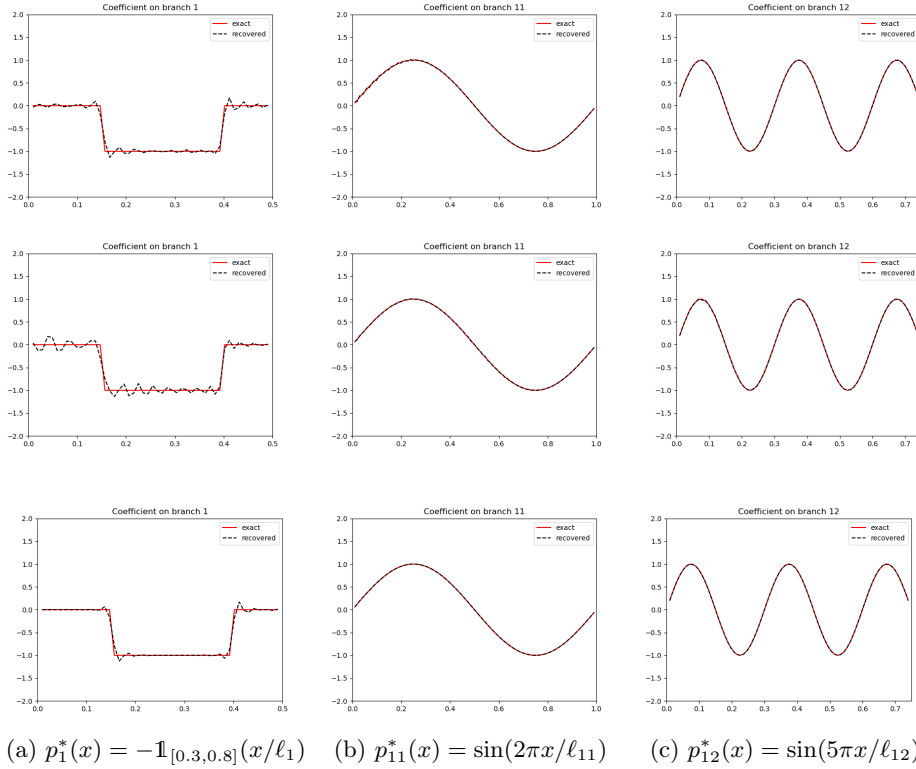


Figure 9: Some examples of reconstructions using the C-bRec algorithm when one assumption of the convergence Theorem 5 is not satisfied. Top line: The observation time $T=1.5$ is smaller than the minimal time given in (2.2). Middle line: The observation time $T=1.25$ is smaller than the minimal time given in (2.2). Bottom line: The result when the projection step 4 of the algorithm is discard.

In the last line of Figure 9, we also plot the results given by the algorithm when the projection Step 4 of the algorithm is disregarded. The numerical experiments seem to indicate good convergence of the algorithm in this case, although it is used in a setting in which we do not know how to provide a proof of convergence. This means that the knowledge of m such that the exact

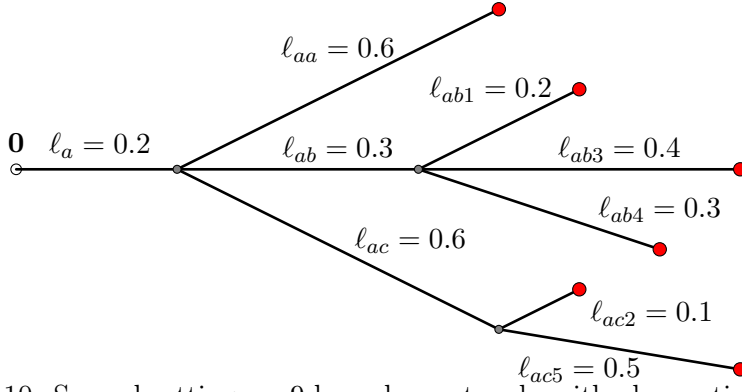


Figure 10: Second setting - a 9 branches network, with observation at \bullet .

coefficient to be recovered belongs to $L_m^\infty(\mathcal{R})$ is not mandatory.

A more complex network

Finally, we give some simulations in the case of a nine-branches network as the one sketched in Figure 10. One should mention that here, the complexity of the network brings the specific numerical difficulty that the $|x_j|$ are getting very large, following the assumptions of Lemma 2 and bringing large weights in the Carleman-based cost functional. The values used for the examples are reported in Table 3. The result is presented in Figure 11.

u_0	u_1	m
(2,2,2,2,2,2,2,2,2)	(0,0,0,0,0,0,0,0,0)	2
g	ℓ_j	β
(0,0,0,0,0,0,0,0,0)	(0.2,0.6,0.3,0.2,0.4,0.3,0.6,0.1,0.5)	0.99
h	x_j in Lemma 2	ε_1 in (4.8)
(2,2,2,2,2,2,2,2,2)	-(0.01,1.2,1.2,8.7,8.7,8.7,1.2,6.5,6.5)	10^{-3}
s in Theorem 3	M_j	ε_2 in (4.9)
1	(74.1,72.6,72.6,0,0,0,72.6,33.4,33.4)	10^{-2}
T given by (2.2)	N_{x_j}	N_t
9.15	$100 * \ell_j$	$110 * T$

Table 3: Numerical values of the variables used for the numerical examples over the network of Figure 10.

Conclusion

As concluding remarks, one can underline that this work has the ambition to be a technically sound and complete first step for testing on application

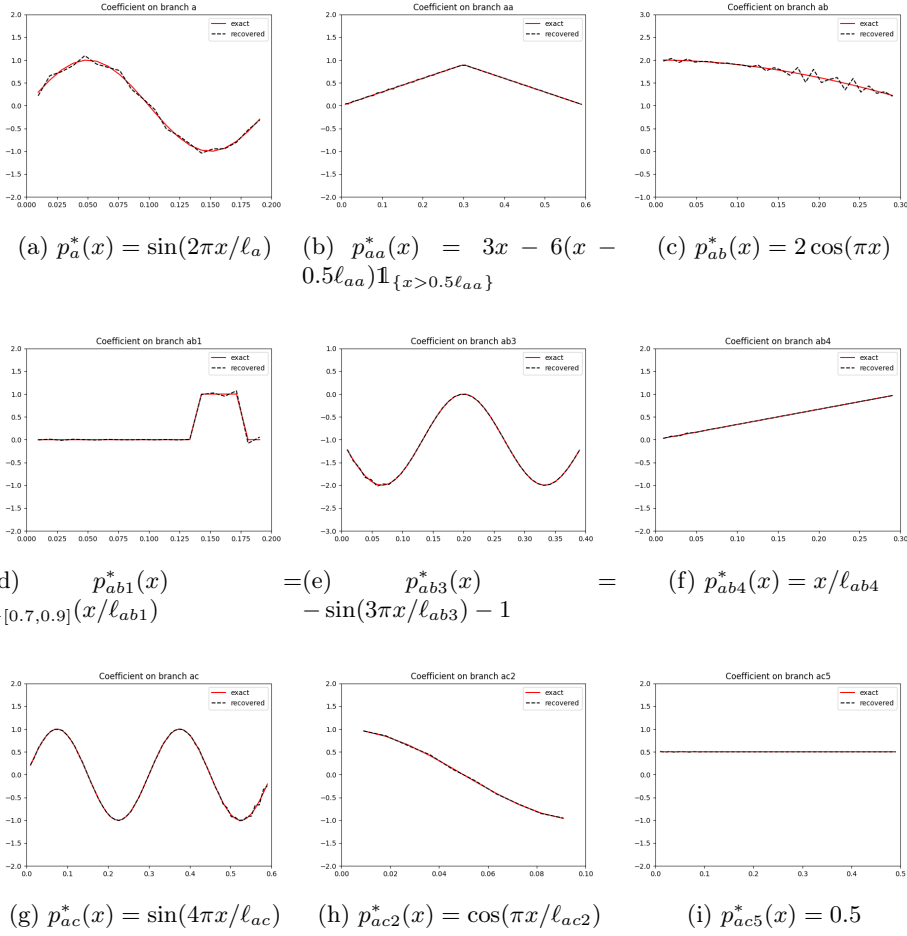


Figure 11: Final reconstruction results (black crosses) and exact coefficient (red line) for the nine branches of the network plotted in Figure 10.

data related to networks of string equations. We proposed here a competitive approach for the identification/recovery of coefficients in partial differential equations that takes its roots in the preliminary work [5] that also inspired results for source term identification in reaction-diffusion equations [12] and even for main coefficient in multi-dimensional wave equation [8]. Our C-bRec approach proves to be quite adaptable, even if it is to the price of appropriate one-parameter Carleman estimates. One can see challenging questions to explore concerning more applicative settings or various other partial differential equations, such as, for instance, some elasticity context.

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