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Incompressible limit of porous media equation with chemotaxis and growth

Qingyou He, Hai-Liang Li, and Benoît Perthame

Abstract We revisit the problem of proving the incompressible limit for the compressible porous media equation with Newtonian drift and growth. The question is motivated by models of living tissues development including chemotaxis. We extend the problem, already treated by the authors and several other contributions, in using a simplified approach, in treating dimensions two or higher, and in incorporating the pressure driven growth term. We also complete the analysis with stronger L^4 estimates on the pressure gradient. The major difficulty is to prove the strong convergence of the pressure gradient which is obtained here by a new observation on an algebraic relation involving the pressure gradient for weak limits.

1 Compressible porous media with chemotaxis and growth

Setting the problem. Several recent mechanical approaches propose to describe the development of living tissues by flows through a porous media formed by the extracellular matrix [3, 44, 18]. Including chemotaxis, this leads to write the compressible porous media equation combined with the Keller-Segel model

$$\begin{cases} \partial_t \rho_m - \operatorname{div}[\rho_m[\nabla p_m - \nabla \phi_m]] = \rho_m G(p_m), & x \in \mathbb{R}^n, t \geq 0, \\ p_m = (\rho_m)^m, & \phi_m = \mathcal{N} \star \rho_m, \end{cases} \quad (1)$$

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where $\rho_m(x, t)$ denotes the cell number density, p_m is the pressure which here is taken as an homogeneous law for simplicity (see [15, 4] for variants and comments on this issue), $G(\cdot)$ is the growth/death rate of cells. The chemotactic term is based on the Newtonian (attractive) potential

$$\begin{cases} \mathcal{N}(x) = -\alpha_n \frac{1}{|x|^{n-2}}, & \nabla \mathcal{N}(x) = (n-2)\alpha_n \frac{x}{|x|^n}, & -\Delta \mathcal{N} = \delta \quad (n \geq 3), \\ \mathcal{N}(x) = \alpha_2 \ln(|x|), & \nabla \mathcal{N}(x) = \alpha_2 \frac{x}{|x|^2}, & -\Delta \mathcal{N} = \delta \quad (n = 2). \end{cases}$$

However, the interested reader can check that the drift term can be much more general in the class

$$\nabla \phi(x) = \nabla \phi_0(x) + \int_{\mathbb{R}^n} K(x, y) \rho(y) dy,$$

as long as it satisfies at least a control (with compactness), for some $1 \leq q < \infty$,

$$\|\nabla \phi\|_{L^2(\mathbb{R}^n)} \leq C[1 + \|\rho\|_{L^1(\mathbb{R}^n)} + \|\rho\|_{L^q(\mathbb{R}^n)}], \quad m \geq q,$$

see Remark 3. All the equations in the present paper are understood in the weak sense as in [20].

We consider the incompressible limit, that is the limit $m \rightarrow \infty$. Departing from the following version of Eq. (1),

$$\partial_t \rho_m - \Delta \frac{m}{m+1} (\rho_m)^{\frac{m+1}{m}} + \operatorname{div}[\rho_m \nabla \phi_m] = \rho_m G(p_m), \quad (2)$$

we may formally pass to the limits ρ_∞ of ρ_m , p_∞ of $(p_m)^{\frac{m+1}{m}}$. We obtain that these limits satisfy the equation

$$\begin{cases} \partial_t \rho_\infty - \Delta p_\infty + \operatorname{div}[\rho_\infty \nabla \phi_\infty] = \rho_\infty G(p_\infty), & x \in \mathbb{R}^n, t \geq 0, \\ p_\infty(1 - \rho_\infty) = 0, & \rho_\infty \leq 1, \quad \phi_\infty = \mathcal{N} \star \rho_\infty, \end{cases} \quad (3)$$

and we show later that the limiting equation can also be written equivalently

$$\partial_t \rho_\infty - \operatorname{div}[\rho_\infty \nabla p_\infty] + \operatorname{div}[\rho_\infty \nabla \phi_\infty] = \rho_\infty G(p_\infty). \quad (4)$$

One can also establish the so-called *complementarity relation*, for almost all $t > 0$

$$p_\infty \Delta [p_\infty - \phi_\infty + G(p_\infty)] = 0, \quad x \in \mathbb{R}^n. \quad (5)$$

System (3) with (5) is a weak and global form of the geometric Hele-Shaw free boundary problem (see [38, 27]).

As always, the equation for the pressure p_m is playing a central role in the analysis; it is written

$$\partial_t p_m - |\nabla p_m|^2 + \nabla p_m \cdot \nabla \phi_m = m p_m [\Delta p_m - \Delta \phi_m + G(p_m)]. \quad (6)$$

The dominant term on the right hand side of this equation explains formally the complementarity relation (5), the difficulty being to pass to the limit in the quadratic term $p_m \Delta p_m$ which amounts to prove the strong convergence of ∇p_m .

What is new. We complement our previous study [20] by taking into account the source term on the right hand side of Eq. (1), treating all dimensions $n \geq 2$ (rather than $n \geq 3$), allowing more general kernels to define ϕ_m and reducing the assumptions on the initial data (in terms of regularity, because we do not use the Aronson-Bénilan estimate here) and removing the compact support assumption.

The purpose of the present paper is also to give a simple proof of the limit $m \rightarrow \infty$ in Eq. (1) and recover the limit formulations (3) as well as (5). The difficulty is to prove strong convergence of the gradient ∇p_m and we show how to use the recent method proposed in [43, 34] and extended in [16, 8]. We base our analysis of the observation in [8] that compensated compactness arguments permit to identify certain limits. This however is not enough and we develop a new idea, namely that the relation $\rho_\infty \nabla p_\infty = \nabla p_\infty$ can be proved a priori.

Let us stress that we prove the strong convergence of ∇p_m and not only $\nabla(p_m)^{\frac{m+1}{m}}$ as in the previous works using this method.

Related studies. This problem has attracted a lot of attention recently and various methods have been used. The problem of the incompressible limit of the compressible porous media equations with a growth/death term controlled by pressure was introduced in [41] using strong regularity assumptions and the Aronson-Bénilan estimate. This strategy has been extended successfully to various situations as a singular pressure law $p = \frac{\varepsilon p}{1-p}$ in [21, 15], chemotaxis and a given drift [13], coupling with an equation for nutrients [11], cell active motion (including a diffusion) in [40],

Another route to study the problem is through viscosity solutions and this was performed in [32, 28]. The case of an additional drift term is treated in [29, 31]. Still another approach is based on the obstacle problem, [19], a method which has been extended to bi-stable terms on the right hand side in [26]. Recently a Lagrangian formulation is given in [23].

Because the geometric form of the Hele-Shaw problem uses the set $\Omega(t) = \{\rho_\infty(t, \cdot) = 1\}$, several authors have studied the question to know, when the initial data is the indicator function of $\Omega(0)$, if this property is propagated even if $\Omega(t)$ has little regularity. For this question it is convenient to work on time integrated variables, which leads to the obstacle problem, we refer to [38, 6, 24, 27]. The regularity and stability of such patches is studied in [25, 5].

In the conservative case, that is $G \equiv 0$, the Hele-Shaw problem has been handled under the name of *congested flows* by [35, 37, 36], using a new method based on Wasserstein distance and optimal transport. See also for systems [36, 33]. From

this point of view, the pressure p_∞ is interpreted as a Lagrange multiplier associated with the constraint $\rho_\infty \leq 1$. The derivation departing from the compressible porous media equation has been achieved in [2, 6]. These papers combine the gradient flow approach with methods based on viscosity solutions which also allow to define another notion of solutions.

Eq. (1) assumes that cells are pushed passively, when including active movement, one arrives at the equation

$$\partial_t \rho_m - \mu \Delta \rho_m - \operatorname{div}[\rho_m[\nabla p_m - \nabla \phi_m]] = \rho_m G(p_m).$$

This problem is studied in [40] and in the conservative case [27]. There is still a set $\Omega(t)$ where $p_\infty > 0$, but ρ_∞ is smooth and has a positive tail because the equation is parabolic non-degenerate.

Error estimates for $p_m - p_\infty$ have also been established in [2, 6] for the conservative case with Newtonian potential in Wasserstein distance. With a growth term (and no drift) error estimates in H^{-1} are proved in [10].

Related to the limit $m \rightarrow \infty$, we also mention that the growth term allows for traveling waves. The existence and Hele-Shaw limit are established in [7], see also the references therein and, for the problem with a necrotic core [17]. Traveling waves in the case with nutrients are also built analytically in [42]. Also Darcy's law for the porous media can be generalized to other rules as Brinkman's law, see [14, 9, 30], or as full Navier-Stokes system [47].

Outline of the paper. In the next section, we present the basic estimates necessary to analyze the problem. With these estimates we can establish, in Section 3, several relations between weak limits of ρ_m and p_m . Those are used in Section 4 to prove that ∇p_m converges strongly in L^2 and this establishes the complementarity condition. In Section 5, we complete our analysis with some additional bounds on p_m in particular $\nabla p_m \in L^4$.

2 Estimates

We complete system (1) with an initial data $\rho_m^0 \geq 0$ and set $p_m^0 = (\rho_m^0)^m$. We assume that for some constant K^0 independent of m ,

$$\int_{\mathbb{R}^n} \rho_m^0 [1 + |x|^2] dx \leq K^0, \quad \int_{\mathbb{R}^n} p_m^0 [1 + (\rho_m^0)^4] dx \leq K^0 m. \quad (7)$$

For the right hand side of Eq. (1), we assume there is an 'homeostatic pressure' $p_H > 0$ and a constant G_M such that

$$G \in C^2([0, \infty); \mathbb{R}), \quad G(p) \leq 0 \quad \text{for } p \geq p_H, \quad |G(\cdot)| \leq G_M. \quad (8)$$

We recall standard estimates.

Proposition 1 (A priori estimates)

With the assumptions (7), (8), the solution of Eq. (1) satisfies the following bounds for constant $C(T)$ independent of m , and for $t \leq T$,

- (i) $\int_{\mathbb{R}^n} \rho_m(t) dx \leq M(t) := e^{tG(0)} \int_{\mathbb{R}^n} \rho_m^0 dx$,
- (ii) $\int_0^T \int_{\mathbb{R}^n} |\nabla p_m(t)|^2 dx dt \leq C(T)$, $\int_{\mathbb{R}^n} p_m(t) dx \leq C(T)m$,
- (iii) $\int_0^T \int_{\mathbb{R}^n} [|\nabla p_m^{\frac{m+1}{m}}|^2 + |\nabla p_m^{\frac{m+2}{m}}|^2] dx dt \leq C(T)$, $\int_0^T \int_{\mathbb{R}^n} (\rho_m)^{m+5} dx dt \leq C(T)$,
- (iv) $\int_{\mathbb{R}^n} |x|^2 \rho_m(t) dx \leq C(T)$,

As usual, several further conclusions follow from this proposition. For instance, the second estimate in (ii) and interpolation with (i) give for $0 \leq t \leq T$,

$$\|\rho_m(t)\|_{L^m(\mathbb{R}^n)} \leq (C(T)m)^{\frac{1}{m}} \leq C(T), \quad \|\rho_m(t)\|_{L^q(\mathbb{R}^n)} \leq C(T), \quad 1 \leq q \leq m.$$

Furthermore, the first estimate in (ii), (iii) and the Sobolev inequality also give

$$\begin{cases} \|p_m + (p_m)^{\frac{m+1}{m}}\|_{L^2((0,T); L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \leq C(T), & n \geq 3, \\ \|p_m\|_{L^r((0,T); L^q(\mathbb{R}^n))} \leq C(T), \quad \frac{m+1}{m} \leq q < \infty, \quad \frac{2}{r} = 1 + \frac{m+1}{mq}, & n = 2, \end{cases} \quad (9)$$

where, for $n = 2$, the inequality follows from the Gagliardo-Nirenberg-Sobolev inequality and the third estimate in (ii).

These estimates on $\rho_m \in L^\infty((0, T); L^1 \cap L^m(\mathbb{R}^n))$ also have consequences on the field ϕ_m when it is given by the Newtonian field. Using the Young inequalities for convolutions, or Sobolev inequalities, we obtain, for $0 \leq t \leq T$,

$$\|\nabla \phi_m(t)\|_{L^q(\mathbb{R}^n)} \leq C(T), \quad \frac{d}{d-1} < q \leq \infty, \quad \text{for } m > n. \quad (10)$$

Also because

$$\partial_t \phi_m(t) = -\frac{m}{m+1} (\rho_m)^{\frac{m+1}{m}} - \Delta^{-1} \operatorname{div}[\rho_m \nabla \phi_m] + \Delta^{-1} \rho_m G(p_m),$$

we conclude bounds as

$$\partial_t \phi_m(t) \quad \text{is bounded in } L^2((0, T); L^{\frac{2n}{n-2}}(\mathbb{R}^n)). \quad (11)$$

These further estimates are not necessarily sharp but are enough for our purposes in the next sections. Lemma 2 furnishes other estimates by interpolation.

Proof The first bound is immediate by integrating Eq. (1).

For the bound (ii), we compute, for $t \leq T$, integrating Eq. (6)

$$\frac{d}{dt} \int_{\mathbb{R}^n} p_m(t) dx + \int_{\mathbb{R}^n} (m-1) |\nabla p_m|^2 dx = \int_{\mathbb{R}^n} [(m-1) \nabla p_m \cdot \nabla \phi_m + m p_m G(p_m)] dx.$$

Because $\min(p_m, p_H) \leq (p_H)^{\frac{m-1}{m}} \rho_m$, and integrating by parts the potential term, we find

$$\frac{d}{dt} \int_{\mathbb{R}^n} p_m(t) dx + \int_{\mathbb{R}^n} (m-1) |\nabla p_m|^2 dx \leq (m-1) \int_{\mathbb{R}^n} p_m \rho_m dx + m p_H^{\frac{m-1}{m}} M(T) G_M.$$

This shows the estimates (ii).

The estimate (iii) is just a variant of (ii) obtained multiplying Eq. (6) by the power $(\rho_m)^2 = (p_m)^{\frac{2}{m}}$ or $(\rho_m)^4 = (p_m)^{\frac{4}{m}}$, and integrating by parts. We find, for instance for the first case

$$\begin{aligned} \frac{m}{m+2} \frac{d}{dt} \int_{\mathbb{R}^n} p_m(t)^{\frac{m+2}{m}} dx + \int_{\mathbb{R}^n} (1+m) (\rho_m)^2 |\nabla p_m|^2 dx \\ = \int_{\mathbb{R}^n} \left[m \frac{m+1}{m+2} (\rho_m)^{m+3} + m p_m \rho_m^2 G(p_m) \right] dx. \end{aligned}$$

As before, we obtain the inequality

$$\begin{aligned} \frac{m}{m+1} \frac{d}{dt} \int_{\mathbb{R}^n} p_m(t)^{\frac{m+2}{m}} dx + \frac{m^2}{1+m} \int_{\mathbb{R}^n} |\nabla (p_m)^{\frac{1+m}{m}}|^2 dx \\ \leq m \frac{m+1}{m+2} \int_{\mathbb{R}^n} (\rho_m)^{m+3} dx + m p_H^{\frac{m+1}{m}} M(T) G_M. \end{aligned}$$

Then, we use (12) with $\tilde{m} = m+1$ and $k = 1$ to control the term $(\rho_m)^{m+3}$ by the term $|\nabla (p_m)^{\frac{1+m}{m}}|^2$ and we conclude the first estimate of (iii). The same computation gives the estimate on $|\nabla (p_m)^{\frac{2+m}{m}}|^2$ and on $(\rho_m)^{m+5}$.

The bound on $(\rho_m)^{m+2}$ is also a conclusion of (12) with $\tilde{m} = m+2$ and $k = 0$.

For proving (iv), we compute

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \frac{|x|^2}{2} \rho_m dx &\leq - \int_{\mathbb{R}^n} \rho_m x \cdot [m(\rho_m)^{m-1} \nabla \rho_m - \nabla \phi_m] dx + G(0) \int_{\mathbb{R}^n} |x|^2 \rho_m dx \\ &\leq \frac{nm}{m+1} \int_{\mathbb{R}^n} (\rho_m)^{m+1} dx - \frac{n-2}{2} \alpha_n \int_{\mathbb{R}^{2n}} \frac{1}{|x-y|^{n-2}} \rho_m(x) \rho_m(y) dx dy \\ &\quad + G(0) \int_{\mathbb{R}^n} |x|^2 \rho_m dx. \end{aligned}$$

It remains to use the second bound in (iii) and Proposition 1 is proved.

For this proof, we use the following general result.

Lemma 2 Let $\rho \geq 0$ and set $M := \int_{\mathbb{R}^n} \rho(x) dx < \infty$. Assume that for some $\tilde{m} > 1$, $\nabla \rho^{\tilde{m}} \in L^2(\mathbb{R}^n)$. Then, for all $0 \leq \frac{k+1}{\tilde{m}} < 1$

$$\int_{\mathbb{R}^n} \rho(x)^{\tilde{m}+k+1} dx \leq C\left(\frac{k+1}{\tilde{m}}, n, M\right) + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho(x)^{\tilde{m}}|^2 dx. \quad (12)$$

The constant $C(\frac{k+1}{\tilde{m}}, n, M)$ blows-up as $\frac{k+1}{\tilde{m}} \rightarrow 1$, $\tilde{m} \rightarrow \infty$ and is uniformly bounded on each closed subinterval.

The difficulty here is to obtain controls independent of m , which forbids direct use of the Sobolev inequalities on ρ .

Proof For a constant A to be chosen later, we decompose ρ as follows

$$\rho = \min(A^{\frac{1}{\tilde{m}+k}}, \rho) + \psi(\rho), \quad \psi'(\rho) = 0 \quad \text{for } \rho < A^{\frac{1}{\tilde{m}+k}}, \quad 0 \leq \psi'(\rho) \leq 1.$$

Then, we use the Gagliardo-Nirenberg-Sobolev inequality for $u = \psi(\rho)^{\frac{\tilde{m}+k+1}{2}}$ and the Cauchy-Schwarz inequality. We find distinguishing the subsets where $\rho \leq A^{\frac{1}{\tilde{m}+k}}$ or not,

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^{\tilde{m}+k+1} dx &\leq AM + C(n) \left(\int_{\mathbb{R}^n} \psi(\rho)^{\frac{\tilde{m}+k+1}{2}} dx \right)^{\frac{4}{n+2}} \left(\int_{\mathbb{R}^n} |\nabla \psi(\rho)^{\frac{\tilde{m}+k+1}{2}}|^2 dx \right)^{\frac{n}{n+2}} \\ &\leq AM + C(n) \left(\int_{\mathbb{R}^n} \psi(\rho)^{\tilde{m}+k+1} dx \text{meas}(\{\rho > A^{\frac{1}{\tilde{m}+k}}\}) \right)^{\frac{2}{n+2}} \\ &\quad \left(\int_{\mathbb{R}^n} \left| \frac{\tilde{m}+k+1}{2} \psi(\rho)^{\frac{\tilde{m}+k-1}{2}} \psi'(\rho) \nabla \rho \right|^2 dx \right)^{\frac{n}{n+2}}. \end{aligned}$$

Because $A^{\frac{1}{\tilde{m}+k}} \text{meas}(\{\rho > A^{\frac{1}{\tilde{m}+k}}\}) \leq M$, the last term can be further controlled as

$$\begin{aligned} C(n) \left(A^{\frac{-1}{\tilde{m}+k}} M \int_{\mathbb{R}^n} \rho^{\tilde{m}+k+1} dx \right)^{\frac{2}{n+2}} \left(\int_{\mathbb{R}^n} \left| \frac{\tilde{m}+k+1}{2} A^{-\frac{\tilde{m}-k-1}{2(\tilde{m}+k)}} \rho^{\tilde{m}-1} \nabla \rho \right|^2 dx \right)^{\frac{n}{n+2}} \\ \leq A^{-\frac{2+n(\tilde{m}-k-1)}{(\tilde{m}+k)(n+2)}} C(n) \left(M \int_{\mathbb{R}^n} \rho^{\tilde{m}+k+1} dx \right)^{\frac{2}{n+2}} \left(\int_{\mathbb{R}^n} \left| \frac{\tilde{m}+k+1}{2\tilde{m}} \nabla \rho^{\tilde{m}} \right|^2 dx \right)^{\frac{n}{n+2}} \end{aligned}$$

because $\psi(\rho) > 0$ only when $\rho > A^{\frac{1}{\tilde{m}+k}}$. Consequently, choosing A large enough independently of \tilde{m} , we may write.

$$\int_{\mathbb{R}^n} \rho^{\tilde{m}+k+1} dx \leq AM + \left(\int_{\mathbb{R}^n} \rho^{\tilde{m}+k+1} dx \right)^{\frac{2}{n+2}} \left(\frac{1}{2} \int_{\mathbb{R}^n} |\nabla \rho^{\tilde{m}}|^2 dx \right)^{\frac{n}{n+2}}.$$

This directly gives the estimate (12). \square

Remark 3 Another way to perform the estimate (ii) in Proposition 1 is as follows. The control of $\int_{\mathbb{R}^n} p_m dx$ is changed as

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^n} p_m(t) dx + \int_{\mathbb{R}^n} (m-1) |\nabla p_m|^2 dx \\
& \leq (m-1) \int_{\mathbb{R}^n} \nabla p_m \cdot \nabla \phi_m dx + m p_H^{\frac{m-1}{m}} M(T) G_M \\
& \leq \frac{m-1}{2} \int_{\mathbb{R}^n} [|\nabla p_m|^2 + |\nabla \phi_m|^2] dx + m p_H^{\frac{m-1}{m}} M(T) G_M.
\end{aligned}$$

Consequently any control of $\int_{\mathbb{R}^n} |\nabla \phi_m|^2 dx$ by a fixed norm L^q of ρ_m will allow to close the bound thanks to (12) for $m \geq q$ (not optimal in view of (9)).

Remark 4 Because we also have $\int_{\mathbb{R}^n} (p_m^0)^{\frac{m+1}{2m}} \leq K_0 m^{\frac{1}{2}}$, a similar argument as in the proof of (ii) arrives to the conclusion that we can control lower powers of p_m

$$\int_0^T \int_{\mathbb{R}^n} |\nabla p_m(t)^{\frac{m+1}{2m}}|^2 dx dt \leq C(T).$$

3 Fundamental relations for the weak limits

Departing from the integrability properties in Proposition 1, after extraction, there are weak limits

$$\begin{cases} \rho_m \rightharpoonup \rho_\infty & \text{in } w-L^q((0, T) \times \mathbb{R}^n), & 1 \leq q < \infty, \\ \rho_m p_m \rightharpoonup p_\infty & \text{in } w-L^2((0, T); \dot{H}^1(\mathbb{R}^n)). \end{cases} \quad (13)$$

For the second statement, the existence of a weak limit is a consequence of the bound on $\rho_m p_m$ stated in Proposition 1 (iii). We call it p_∞ because of one of the statements below.

Passing to the weak limit in the equation under the form (2), we obtain, as in Eq. (3), the relation between ρ_∞ and p_∞

$$\partial_t \rho_\infty - \Delta p_\infty + \operatorname{div}[\rho_\infty \nabla \phi_\infty] = R(t, x) \in L^\infty((0, T); L^1 \cap L^\infty(\mathbb{R}^n)). \quad (14)$$

Indeed,

- the drift term $\rho_m \nabla \phi_m$ passes to the limit by weak-strong limit since $\nabla \phi_m$ converges strongly to $\nabla \phi_\infty$ because of (10), (11) and $D^2 \phi_m \in L^q$ by singular integral theory (see also [20] for details).
- the right hand side passes to the weak limit because G is bounded and thus $\rho_m G(p_m)$ is bounded in the same Lebesgue spaces as ρ_m , following the bounds in Proposition 1. In fact, by compensated compactness, [39, 46] and following [8], and as it is also used to prove (iii) in the proposition below, it follows that

$$R(t, x) = \rho_\infty G(p_\infty).$$

Also, we insist that the weak limits of p_m and $\rho_m p_m$ are the same, as stated in the

Proposition 5 (Fundamental relations)

With the assumptions of Proposition 1, these limits satisfy

- (i) $\rho_\infty \leq 1$ for almost all $t \in (0, T)$ and $x \in \mathbb{R}^n$,
- (ii) $p_m \rightharpoonup p_\infty$ in $w-L^p((0, T) \times \mathbb{R}^n)$,
- (iii) $p_\infty = \rho_\infty p_\infty$ for almost all $t \in (0, T)$ and $x \in \mathbb{R}^n$,
- (iv) $\rho_\infty \nabla p_\infty = \nabla p_\infty \in L^2((0, T) \times \mathbb{R}^n)^n$,
- (v) $\partial_t \rho_\infty p_\infty = 0$, in duality $L^2((0, T); H^{-1}(\mathbb{R}^n)), L^2((0, T); H^1(\mathbb{R}^n))$.

The new information that we bring here is the direct proof of the identity in (iv) based on the estimates of Proposition 1 only. It is instrumental to treat the drift term. Also it establishes that both equations (3) and (4) hold.

Notice also that the product $\partial_t \rho_\infty p_\infty$ is well defined by duality $L_t^2(H^{-1}), L_t^2(H^1)$.

Proof For (i), we recall that the inequality $\rho_\infty \leq 1$ is a consequence of the bound in Proposition 1 (ii) since

$$\|\rho_m(t)\|_{L^m(\mathbb{R}^n)} \leq (C(T)m)^{\frac{1}{m}} \xrightarrow{m \rightarrow \infty} 1, \quad 0 \leq t \leq T,$$

which implies that for all $1 \leq q \leq m$, by interpolation between L^1 and L^m ,

$$\|\rho_m(t)\|_{L^q(\mathbb{R}^n)} \leq M(T)^{1-\theta} (mC(T))^{\frac{\theta}{m}}, \quad \frac{1}{q} = (1-\theta) + \frac{\theta}{m}.$$

In the weak limit we find

$$\|\rho_\infty(t)\|_{L^q(\mathbb{R}^n)} \leq M(T)^{\frac{1}{q}} \rightarrow 1 \quad \text{as } q \rightarrow \infty.$$

Next, we prove (ii), i.e., that the weak limits of p_m and $\rho_m p_m$ are the same. This follows from the two inequalities

$$\begin{aligned} p_m &\leq (\rho_m)^m = \frac{1}{m+1} + \frac{m}{m+1} (\rho_m)^{m+1}, \\ (\rho_m)^{m+1} &\leq \frac{m+1}{m} (\rho_m)^m + \frac{m+1}{2m^2} (\rho_m)^{2m}, \end{aligned}$$

thanks to the bounds in Proposition 1 and estimate (9).

For (iii), i.e., the identity $p_\infty = \rho_\infty p_\infty$ is a consequence of the above convergence and compensated compactness, following [8]

$$\rho_\infty p_\infty = w\text{-}\lim \rho_m p_m \tag{15}$$

because ∇p_m is bounded in L^2 , see Proposition 1 (ii), and $\partial_t \rho_m$ is bounded in $L_t^2(H^{-1})$ thanks to Eq. (2) and the bound in Proposition 1 (ii) (see [20, 8] for details and other arguments).

For proving (iv), we use Proposition 1 (ii) and that the chain rule holds in $W_{\text{loc}}^{1,1}$. For A large, consider a non-decreasing 'truncation' function which satisfies $\chi_A(p) = p$ for $0 \leq p \leq \frac{A}{2}$, $\chi_A(p) = A$ for $p \geq 2A$. We may write

$$p_\infty = \chi_A(p_\infty) + [p_\infty - \chi_A(p_\infty)].$$

On the one hand, because $\chi_A(p_\infty)$ is bounded, the mapping $\chi \mapsto \chi^{1+\varepsilon}$ is Lipschitz and we may use the chain rule to write

$$\nabla \chi_A(p_\infty)^{1+\varepsilon} = (1+\varepsilon)\chi_A(p_\infty)^\varepsilon \nabla \chi_A(p_\infty) = \rho_\infty(1+\varepsilon)\chi_A(p_\infty)^\varepsilon \nabla \chi_A(p_\infty),$$

still using (iii) and because when $\rho_\infty \neq 1$, then $p_\infty = \chi_A(p_\infty)^\varepsilon = 0$. Therefore, we have obtained that

$$\nabla \chi_A(p_\infty)^{1+\varepsilon} = \rho_\infty \nabla \chi_A(p_\infty)^{1+\varepsilon}$$

and as $\varepsilon \rightarrow 0$ we find $\nabla \chi_A(p_\infty) = \rho_\infty \nabla \chi_A(p_\infty)$.

On the other hand, $\nabla p_\infty - \nabla \chi_A(p_\infty) \rightarrow 0$ in $L^2((0, T) \times \mathbb{R}^n)$ as $A \rightarrow \infty$. All together, we have proved (iv).

The identity (v) can be obtained in different ways, see [43, 34, 22]. A simple argument is given in [8] and uses that $\partial_t \rho_\infty \in L^2((0, T); H^{-1}(\mathbb{R}^n))$ (see Eq. (14)) and $p_\infty \in L^2((0, T); H^1(\mathbb{R}^n))$ (see Proposition 1 (ii)). Therefore $\partial_t \rho_\infty(t) p_\infty(t)$ can be approximated successively as

$$\begin{aligned} \frac{\rho_\infty(t+h) - \rho_\infty(t)}{h} p_\infty(t) &= \frac{\rho_\infty(t+h) - 1}{h} p_\infty(t) \leq 0, \\ \frac{\rho_\infty(t) - \rho_\infty(t-h)}{h} p_\infty(t) &= \frac{1 - \rho_\infty(t-h)}{h} p_\infty(t) \geq 0, \end{aligned}$$

where we have only used again (iii). □

4 Strong convergence of ∇p_m and the complementarity condition

A simple procedure to pass to the limit $m \rightarrow \infty$ and recover the complementarity relation (5) has been elaborated when chemotaxis is ignored. We follow the most advanced form in [8], based on the ideas in [43, 34, 16].

The first step is to pass to the limit in Eq. (2) and, as already established in Section 3, we find that

$$\begin{cases} \partial_t \rho_\infty - \Delta p_\infty + \text{div}[\rho_\infty \nabla \phi_\infty] = \rho_\infty G(p_\infty), & x \in \mathbb{R}^n, t \geq 0, \\ p_\infty = \rho_\infty p_\infty, & \phi_\infty = \mathcal{N} \star \rho_\infty, & \rho_\infty \leq 1. \end{cases} \quad (16)$$

Thanks to the relations proved in Proposition 5, one can establish the

Theorem 6 (Strong convergence of ∇p_m)

With the assumptions of Proposition 1, we have after extraction of a subsequence

$$\nabla(p_m)^{\frac{m+1}{m}} \rightarrow \nabla p_\infty \quad \text{in } L^2((0, T) \times \mathbb{R}^n).$$

Therefore we have

$$(p_m)^{\frac{m+1}{m}} \rightarrow p_\infty \quad \text{in } L^2((0, T) L^{\frac{2n}{n-2}}(\mathbb{R}^n)), \quad n \geq 3,$$

and this holds for $n = 2$ in interpolated Lebesgue spaces.

Assuming additionally $\int_{\mathbb{R}^n} (p_m^0)^{\frac{1}{2}} \leq K_0 m$, we also have

$$\nabla p_m \rightarrow \nabla p_\infty \quad \text{in } L^2((0, T) \times \mathbb{R}^n).$$

Therefore the complementarity relation (5) holds true.

Proof Strong convergence of $\nabla(p_m)^{\frac{m+1}{m}}$. We subtract Eq. (16) to Eq. (1) and we find

$$\begin{aligned} \partial_t(\rho_m - \rho_\infty) - \Delta\left(\frac{m}{m+1}\rho_m^{m+1} - p_\infty\right) + \operatorname{div}(\rho_m \nabla \phi_m - \rho_\infty \nabla \phi_\infty) \\ = \rho_m G(p_m) - \rho_\infty G(p_\infty). \end{aligned}$$

Multiplying by $\frac{m}{m+1}\rho_m^{m+1} - p_\infty$ and integrating by parts gives

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{m}{m+1}\rho_m^{m+1} - p_\infty\right) \partial_t(\rho_m - \rho_\infty) dx + \int_{\mathbb{R}^n} \left|\nabla\left(\frac{m}{m+1}\rho_m^{m+1} - p_\infty\right)\right|^2 dx \\ - \int_{\mathbb{R}^n} \left(\nabla\left(\frac{m}{m+1}\rho_m^{m+1} - p_\infty\right)\right) (\rho_m \nabla \phi_m - \rho_\infty \nabla \phi_\infty) dx = \int_{\mathbb{R}^n} [\rho_m G(p_m) - \rho_\infty G(p_\infty)] dx. \end{aligned}$$

This serves to show that the term $\int_{\mathbb{R}^n} \left|\nabla\left(\frac{m}{m+1}\rho_m^{m+1} - p_\infty\right)\right|^2 dx$ converges to 0 as $m \rightarrow \infty$ because the three other terms do so.

As already mentioned, and following [8], the last term vanishes as $m \rightarrow \infty$ by compensated compactness.

The first term (with ∂_t) vanishes also as in [43, 34, 8]. This is a simple consequence of the identity $p_\infty \partial_t \rho_\infty = 0$ stated in Proposition 5 (v).

The new difficulty comes from the drift term. We consider successively the four terms of this product and show we can pass to the limit. For the first product, we consider

$$\int_{\mathbb{R}^n} \nabla \frac{m}{m+1} \rho_m^{m+1} \rho_m \nabla \phi_m dx = \int_{\mathbb{R}^n} \frac{m}{m+2} \nabla \rho_m^{m+2} \nabla \phi_m dx \rightarrow \int_{\mathbb{R}^n} \nabla p_\infty \nabla \phi_\infty dx,$$

here is the only place where we use estimate $\nabla \rho_m^{m+2}$ in Proposition 1 (iii).

The second product we consider is

$$\int_{\mathbb{R}^n} \nabla p_\infty \rho_m \nabla \phi_m dx \rightarrow \int_{\mathbb{R}^n} \nabla p_\infty \rho_\infty \nabla \phi_\infty dx$$

where the convergence is obtained by weak-strong limits because $\nabla \phi_m$ converges strongly as used before (or again by compensated compactness).

The third product is

$$\int_{\mathbb{R}^n} \nabla \frac{m}{m+1} \rho_m^{m+1} \rho_\infty \nabla \phi_\infty dx \rightarrow \int_{\mathbb{R}^n} \nabla p_\infty \rho_\infty \nabla \phi_\infty dx$$

as a weak limit tested against a given function.

The fourth product does contain terms which pass to the limit and thus, we conclude the limit

$$\int_{\mathbb{R}^n} (\nabla (\frac{m}{m+1} \rho_m^{m+1} - p_\infty) (\rho_m \nabla \phi_m - \rho_\infty \nabla \phi_\infty)) dx \rightarrow \int_{\mathbb{R}^n} \nabla p_\infty (\rho_\infty - 1) \nabla \phi_\infty dx = 0$$

thanks to Proposition 5 (iv).

This concludes the strong convergence of $\nabla (p_m)^{\frac{m+1}{m}}$.

Strong convergence of $(p_m)^{\frac{m+1}{m}}$. This is just a consequence of the Sobolev inequality (we only deal with the dimensions larger than 3)

$$\| (p_m)^{\frac{m+1}{m}} - p_\infty \|_{L^2((0,T) L^{\frac{2n}{n-2}}(\mathbb{R}^n))} \leq \| \nabla (p_m)^{\frac{m+1}{m}} - \nabla p_\infty \|_{L^2((0,T) \times \mathbb{R}^n)}.$$

Strong convergence of ∇p_m . We fix a parameter $0 < \varepsilon \leq \frac{1}{2}$ and decompose

$$\begin{aligned} \nabla (p_m)^{\frac{m+1}{m}} - \frac{m+1}{m} \nabla p_m &= 2 \frac{m+1}{m} \nabla (p_m)^{\frac{1}{2}} \mathbf{I}_{\{\rho_m \leq 1-\varepsilon\}} [(p_m)^{\frac{1}{2}} - (p_m)^{\frac{1}{2} + \frac{1}{m}}] \\ &\quad + \nabla (p_m)^{\frac{m+1}{m}} \mathbf{I}_{\{\rho_m > 1-\varepsilon\}} \frac{1 - \rho_m}{\rho_m}. \end{aligned}$$

From this, we infer

$$\begin{aligned} |\nabla (p_m)^{\frac{m+1}{m}} - \frac{m+1}{m} \nabla p_m| &\leq 4 |\nabla (p_m)^{\frac{1}{2}}| (1 - \varepsilon)^{\frac{m}{2}} \\ &\quad + |\nabla (p_m)^{\frac{m+1}{m}}| [2\varepsilon + \mathbf{I}_{\{\rho_m > 1+\varepsilon\}} \frac{\rho_m - 1}{\rho_m}]. \end{aligned}$$

As $m \rightarrow \infty$, the first term converges to 0 in L^2 thanks to the bounds in Remark 4 and in Proposition 1 (ii), (iii). To treat the last one, we analyze the quantity Q_m defined as

$$Q_m := \mathbf{I}_{\{\rho_m > 1+\varepsilon\}} \frac{\rho_m - 1}{\rho_m}, \quad 0 \leq Q_m \leq 1, \quad Q_m \leq \rho_m \in L^\infty_{\text{bounded}}((0, \infty); L^1(\mathbb{R}^n)).$$

We also have, since $(p_m)^{\frac{m+1}{m}}$ is bounded in $L^1((0, T) \times \mathbb{R}^n)$,

$$0 \leq Q_m \leq \frac{(p^m)^{\frac{m+1}{m}}}{(1+\varepsilon)^m} \rightarrow 0 \quad \text{a.e. as } m \rightarrow \infty.$$

Therefore $Q_m \rightarrow 0$ in $L^2((0, T) \times \mathbb{R}^n)$.

Consequently, still using Proposition 1 (iii), we have obtained that, for all ε with $0 < \varepsilon \leq \frac{1}{2}$

$$\limsup_{m \rightarrow \infty} \left\| \nabla (p_m)^{\frac{m+1}{m}} - \frac{m+1}{m} \nabla p_m \right\|_{L^2((0, T) \times \mathbb{R}^n)} \leq 2\varepsilon C(T).$$

which proves the strong convergence of ∇p_m .

Then we can pass to the limit in (6) and obtain the complementarity relation. \square

5 More regularity

Several further regularity results can easily be obtained in the context of our assumptions.

Maximum principle for p_m . We complement the assumptions (8) on $G(\cdot)$ by

$$\exists P_M > 0 \quad \text{such that} \quad A + G(P_M) \leq 0 \quad \forall A \in [1, P_M]. \quad (17)$$

We also assume here that ϕ_m satisfies the bound (obvious for the Newtonian field)

$$-\Delta \phi_m \leq \|\rho_m(t)\|_{L^\infty(\mathbb{R}^n)}. \quad (18)$$

Lemma 7 *We assume (7), (8), (17), (18) and $p_m^0 \leq P_M$, then $p_m(t) \leq P_M$ for all $t > 0$.*

Proof Departing from Eq. (6), we may write

$$\begin{aligned} \partial_t p_m + \nabla p_m \cdot [\nabla \phi_m - \nabla p_m] - m p_m \Delta p_m &= m p_m [-\Delta \phi_m + G(p_m)] \\ &\leq p_m \left[\|\rho_m(t)\|_{L^\infty(\mathbb{R}^n)}^{\frac{1}{m}} + G(p_m) \right]. \end{aligned}$$

Because $A := P_M^{\frac{1}{m}} \in [1, P_M]$, the maximum principle applies and we obtain the conclusion. \square

The estimate $\nabla p_m \in L^4$. We continue with an extension of the L^4 estimate introduced in [11, 1], see also [12] for a recent Lipschitz bound. To simplify we only consider the case

$$-\Delta \phi_m = \rho_m.$$

Theorem 8 (L^4 estimate for ∇p_m) *With the assumptions (7), (8), and $|\nabla p_m^0|$ is bounded in $L^2(\mathbb{R}^n)$, we have for all $T > 0$ and some constant $C(T)$ independent of m*

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla p_m(T)|^2 dx + m \int_0^T \int_{\mathbb{R}^n} p_m (\Delta p_m + \rho_m + G(p_m))^2 dx dt \\ + \int_0^T \int_{\mathbb{R}^n} p_m |D^2 p_m|^2 dx dt \leq C(T). \end{aligned} \quad (19)$$

Additionally, if p_m is bounded in L^∞ (see Lemma 7), then for $0 \leq \alpha < 1$,

$$(1 - \alpha)^2 \int_0^T \int_{\mathbb{R}^n} \frac{|\nabla p_m|^4}{p_m^\alpha} dx dt \leq \|p_m\|_{L^\infty((0,T) \times \mathbb{R}^n)}^{1-\alpha} C(T). \quad (20)$$

Proof of Estimate (19). First of all, with $|D^2 p|^2 = \sum_{i,j=1}^n (D_{ij}^2 p)^2$, we recall the general identity

$$\int_{\mathbb{R}^n} |\nabla p|^2 \Delta p dx = \frac{2}{3} \int_{\mathbb{R}^n} p |D^2 p|^2 dx - \frac{2}{3} \int_{\mathbb{R}^n} p |\Delta p|^2 dx. \quad (21)$$

To begin, we multiply Eq. (6), by $-(\Delta p_m + \rho_m + G(p_m))$ and integrate by parts. This gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla p_m(t)|^2 dx + m \int_{\mathbb{R}^n} p_m (\Delta p_m + \rho_m + G(p_m))^2 dx \\ = \int_{\mathbb{R}^n} [-|\nabla p_m|^2 + \nabla p_m \cdot \nabla \phi_m] (\Delta p_m + \rho_m + G(p_m)). \end{aligned}$$

Therefore, using Eq. (21), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla p_m(t)|^2 dx + m \int_{\mathbb{R}^n} p_m (\Delta p_m + \rho_m + G(p_m))^2 dx + \frac{2}{3} \int_{\mathbb{R}^n} p_m |D^2 p_m|^2 dx \\ = \frac{2}{3} \int_{\mathbb{R}^n} p_m |\Delta p_m|^2 dx + \text{Rem}(t), \end{aligned} \quad (22)$$

where

$$\text{Rem} = - \int_{\mathbb{R}^n} |\nabla p_m|^2 (\rho_m + G(p_m)) + \int_{\mathbb{R}^n} \nabla p_m \cdot \nabla \phi_m (\Delta p_m + \rho_m + G(p_m)).$$

Our next step is to prove that

$$\int_0^T \text{Rem}(t) dt \leq C(T) + \frac{1}{3} \int_0^T \int_{\mathbb{R}^n} p_m |D^2 p_m|^2 dx. \quad (23)$$

That is because, in the definition of Rem , the first integral is bounded thanks to the estimates in Proposition 1. This is also true for the term $\nabla p_m \cdot \nabla \phi_m (\rho_m + G(p_m))$.

The only term which requires a treatment is

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla p_m \cdot \nabla \phi_m \Delta p_m dx &= - \int_{\mathbb{R}^n} \partial_i p_m [\partial_{ij}^2 \phi_m \partial_j p_m + \partial_j \phi_m \partial_{ij}^2 p_m] dx \\ &= - \int_{\mathbb{R}^n} \partial_i p_m \partial_{ij}^2 \phi_m \partial_j p_m + \frac{1}{2} \int_{\mathbb{R}^n} \Delta \phi_m |\nabla p_m|^2 dx. \end{aligned}$$

After integration in time, the second term is bounded again thanks to the estimates in Proposition 1. We arrive at

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \nabla p_m \cdot \nabla \phi_m \Delta p_m dx dt &\leq C(T) + \int_0^T \int_{\mathbb{R}^n} p_m [\partial_{ii}^3 \phi_m \partial_j p_m + \partial_{ij}^2 \phi_m \partial_{ij}^2 p_m] dx dt \\ &\leq C(T) + \int_0^T \int_{\mathbb{R}^n} p_m [\partial_j p_m \partial_j p_m + \frac{1}{3} |D^2 p_m|^2 + \frac{2}{3} |D^2 \phi_m|^2] dx dt. \end{aligned}$$

The first term is also $\frac{m}{m+1} \int_0^T \int_{\mathbb{R}^n} \partial_j (p_m)^{\frac{m+1}{m}} \partial_j p_m dx dt$ and thus is bounded. The last term is also bounded because $|D^2 \phi_m|^2$ has the same regularity in L^p , $1 < p < \infty$ as p_m according to singular integral theory [45]. Therefore we have proved estimate (23).

We can now conclude the estimate (19). After time integration of Eq. (22) and using (23), we find

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla p_m(t)|^2 dx + m \int_0^T \int_{\mathbb{R}^n} p_m (\Delta p_m + \rho_m + G(p_m))^2 dx dt \\ + \frac{1}{3} \int_0^T \int_{\mathbb{R}^n} p_m |D^2 p_m|^2 dx \leq C(T) + \int_{\mathbb{R}^n} |\nabla p_m^0|^2 dx, \end{aligned}$$

and the estimate (19) is proved.

Proof of Estimate (20). Integrating by parts, we may write

$$\int_{\mathbb{R}^n} \frac{|\nabla p_m|^4}{p_m^\alpha} dx = - \int_{\mathbb{R}^n} p_m \left[-\alpha \frac{|\nabla p_m|^4}{p_m^{\alpha+1}} + \frac{\Delta p_m}{p_m^\alpha} |\nabla p_m|^2 + 2 \frac{\partial_{ij}^2 p_m \partial_i p_m \partial_j p_m}{p_m^\alpha} \right] dx.$$

Therefore, we also have

$$\begin{aligned} (1-\alpha) \int_{\mathbb{R}^n} \frac{|\nabla p_m|^4}{p_m^\alpha} dx &= - \int_{\mathbb{R}^n} p_m \left[\frac{\Delta p_m}{p_m^\alpha} |\nabla p_m|^2 + 2 \frac{\partial_{ij}^2 p_m \partial_i p_m \partial_j p_m}{p_m^\alpha} \right] dx \\ &\leq \frac{1-\alpha}{4} \int_{\mathbb{R}^n} \left[\frac{|\nabla p_m|^4}{p_m^\alpha} + \frac{4}{1-\alpha} p_m^{2-\alpha} |\Delta p_m|^2 \right. \\ &\quad \left. + \frac{1-\alpha}{4} \int_{\mathbb{R}^n} \frac{|\nabla p_m|^4}{p_m^\alpha} + \frac{16}{1-\alpha} \int_{\mathbb{R}^n} p_m^{2-\alpha} |D^2 p_m|^2 \right] dx. \end{aligned}$$

And finally, this is also written

$$\frac{1-\alpha}{2} \int_{\mathbb{R}^n} \frac{|\nabla p_m|^4}{p_m^\alpha} dx \leq \|p_m\|_{L^\infty(\mathbb{R}^n)}^{1-\alpha} \frac{C}{1-\alpha} \int_{\mathbb{R}^n} [p_m |\Delta p_m|^2 + p_m |D^2 p_m|] dx.$$

We obtain (20) because of the estimates (19) which also furnish a uniform bound on $\int_0^T \int_{\mathbb{R}^n} p_m |\Delta p_m|^2 dx$.

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