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# The Geometry of Right-Invariant Metrics

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abstract

## Statements and Declarations

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## 1 Introduction

**Erratum:** *Previous versions of this paper were entitled 'The Geometry of Right-Invariant Metrics on Lie Groups: Towards a BCH Formula on Diffeomorphism Groups'. They had a section deriving an expression for  $D \exp$ . These were based on what was then Lemma 1, which was incorrect. This version of the paper removes all of that material, which leaves relatively little novel content. However, we hope that the exposition of the problem may remain of interest to some.*

In his much-celebrated geometric interpretation of hydrodynamics Arnold [1] showed that geodesics of the right-invariant  $L^2$  metric on vector fields correspond to solutions of the Euler equations of hydrodynamics on the underlying manifold  $M$  on which the diffeomorphism group acts. The right-invariance is natural in the sense that the  $L^2$  metric on vector fields becomes right-invariant when restricted to the group of volume-preserving diffeomorphisms  $\mathcal{D}_\mu$  and its tangent space of divergence-free vector fields  $T_e \mathcal{D}_\mu$ . Ebin and Marsden [2] exploited Arnold's observation to show that the  $L^2$  metric is indeed a smooth Riemannian metric on  $\mathcal{D}_\mu$  that admits a smooth

right-invariant Levi-Civita connection and, consequently, a smooth right-invariant exponential map that is a diffeomorphism in a neighbourhood of the identity.

The geometry and analysis of Riemannian exponential maps of right-invariant metrics on diffeomorphism groups have been of interest ever since, particularly as they are related to many equations from continuum mechanics.

Misiołek and Preston [3] showed that many right-invariant metrics of Sobolev type admit “well behaved” exponential maps that are, themselves, diffeomorphisms in a neighbourhood of the identity element of the group on which they are defined. They are also nonlinear Fredholm maps of index zero in the sense of Smale, i.e., smooth maps  $f$  with Frechét derivative  $df(p)$  being Fredholm operators at each  $p$ , where by Fredholm operators we mean bounded linear operators between Banach spaces with closed range and finite-dimensional kernel and co-kernel.

The particular importance of these metrics is that when equipped with a particular right-invariant metric the diffeomorphism group and its algebra are reconnected, and a neighbourhood of the identity may be parameterised by infinitesimal transformations via an exponential map whose topological and analytic properties are similar to those of a finite-dimensional exponential map.

In this paper we explore the geometry of right-invariant metrics on Lie and pseudo-Lie groups, particularly the diffeomorphism groups. We begin with an accessible introduction to the algebra and right-invariant geometry of a finite-dimensional Lie group, and highlight the difficulties of working on diffeomorphism groups. We describe Arnold’s framework for studying geodesics of one-sided metrics on Lie groups and derive the basic results, along with a new form of the geodesic equations.

Following this introductory material, we derive an explicit formula for the derivative of the exponential map defined by a right-invariant metric on a Lie group  $G$  (Theorem 3) in the spirit of Poincaré. The formula we obtain reduces to the known formula for the group exponential map when  $\langle \cdot, \cdot \rangle$  is bi-invariant (assuming  $G$  allows for one), but our derivation is fundamentally different from the original and operator theoretic in nature.

One benefit of our formulation is that we can give a dynamical characterisation of the conjugate locus of a right-invariant metric, which provides a new geometric description of conjugate points (Theorem ??). In addition, we use the formula to show how in/stability of the geodesic system measured through linearisation of the geodesic equation (Eulerian stability) is related to in/stability measured through linearisation of the corresponding flow. When  $G$  is compact and the metric is bi-invariant, for example in a symmetric space, these two notions of stability coincide; our two examples (Theorems ?? and ??) illustrate the topological and algebraic influence on the disconnect between these two in/stability measures when the symmetry is broken by a one-sided invariant metric. The conclusion is that if one has control over the algebra and topology then it is enough to study the first order stability problem (Eulerian stability) to understand the second order stability problem (Lagrangian stability) that describes the geometry of  $G$ .

Moving to reformulate our finite-dimensional results in the context of right-invariant Sobolev metrics on the infinite-dimensional diffeomorphism groups we use the theory of [3], and show that our finite-dimensional results persist whenever the

exponential maps have the Fredholm property (Theorems ?? to ??). These results become immediately relevant to the stability of the Euler equations of hydrodynamics and show that  $L^2$  Eulerian and Lagrangian stabilities, as well as instabilities, coincide for large times; thus the  $L^2$  metric on the volume-preserving diffeomorphism group mimics the behaviour of a bi-invariant metric on a compact Lie group.

Our conjugate point and stability results seem directly relevant to Problem 11 of [4]. However, there is another problem that our work has the potential to contribute to, namely the desire for a BCH formula on diffeomorphism groups, and we dedicate the last section to exploring this further. In a normal coordinate neighbourhood of the identity in a *finite-dimensional* Lie group  $G$  with Lie algebra  $\mathfrak{g}$  the Baker-Campbell-Hausdorff (BCH) formula expresses group multiplication as a power series in the Lie algebra whose terms are given by iterated Lie brackets. While some infinite-dimensional Lie groups readily admit a BCH formula (known as BCH-Lie groups or exponential-Lie groups), there is an important class of Lie groups that are not BCH: the diffeomorphism groups. This is because there exist diffeomorphisms arbitrarily close to the identity diffeomorphism that do not embed in any flow, thus the intrinsic group exponential map is not even locally surjective – see [5],[6], [7], [8], [9] for the earliest and original examples, and [10], [11], or [12] for extensions and collected results. The situation is actually worse: there exist arc-wise connected free subgroups on infinitely many generators which, except for the identity, do not lie in the image of the exponential map [13].

## 2 Background in Algebra and Geometry

Let  $G$  be a finite-dimensional Lie group with group operation  $\star$  and identity element  $e$ . The classical operations of right and left translation are:

$$\begin{aligned} R : G \times G &\rightarrow G & L : G \times G &\rightarrow G \\ R(g, h) &:= R_g(h) = h \star g & L(g, h) &:= L_g(h) = g \star h. \end{aligned}$$

For a fixed  $g \in G$  both  $R_g$  and  $L_g$  are smooth diffeomorphisms of  $G$ . Vector fields  $U$  and  $V$  on  $G$  are said to be (respectively) left- or right-invariant if the derivative  $d_g$  satisfies:

$$dL_g \cdot U(h) = U(L_g(h)) \quad \text{or} \quad dR_g \cdot V(h) = V(R_g(h)).$$

In what follows we will denote a vector field by a capital letter, for example  $V$ , and the value of the vector field at  $e$  by the corresponding lower case letter:  $V(e) = v$ .

We can deduce several facts about left/right-invariant vector fields: (1) the set of left/right-invariant vector fields on  $G$  is isomorphic to the tangent space at the identity  $T_e G =: \mathfrak{g}$  via the evaluation at  $e$  map; (2) left/right-invariant vector fields are smooth (this can be seen by writing the map  $g \mapsto v(f \circ R_g)$  as  $\left. \frac{d}{dt} \right|_{t=0} f \circ R(g, \eta(t))$  for any curve  $\eta(t)$  with  $\eta(0) = e$  and  $\left. \frac{d}{dt} \right|_{t=0} \eta(t) = v$ , and any  $f \in C^\infty(G)$ ); (3) the Lie bracket of two left/right-invariant vector fields is itself left/right-invariant, which follows from general principles of diffeomorphisms and the Lie bracket; and (4) the vector space  $\mathfrak{g}$  forms a Lie algebra under the Lie bracket of left/right-invariant vector

fields. Specifically, the bracket operation  $[\cdot, \cdot]_{\mathfrak{g}}$  in  $\mathfrak{g}$  is determined by:

$$[V, W] = [dR_g \cdot v, dR_g \cdot w] = dR_g \cdot [v, w]_{\mathfrak{g}} \quad (1)$$

$$[V, W] = [dL_g \cdot v, dL_g \cdot w] = dL_g \cdot [v, w]_{\mathfrak{g}} \quad (2)$$

depending on whether one is considering left-, or right-, invariant vector fields, respectively.

An operator that will feature significantly in later sections is the group adjoint operator  $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ , which gives a representation of  $G$  through an action on its Lie algebra  $\mathfrak{g}$  by:

$$\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{Ad}_g v = dL_g dR_{g^{-1}} \cdot v. \quad (3)$$

The left and right translation maps commute, and so their differentials commute as well. Hence the ordering of the differentials in (3) will not always be respected, but rather chosen according to what is convenient in the context (see the proof of Proposition 1 for example).

The differential of  $\text{Ad}$  is the map  $\text{ad} : \mathfrak{g} \rightarrow \text{Aut}(\mathfrak{g})$  and gives a complementary representation of  $\mathfrak{g}$  on itself:

$$\text{ad}_v : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \text{ad}_v w = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\eta(t)} w, \quad (4)$$

where  $\eta(t)$  is any smooth curve in  $G$  passing through the identity  $e$  at  $t = 0$  with velocity  $v$ .

To relate these operators to the Lie bracket in  $\mathfrak{g}$  we need to understand the flows of left/right-invariant vectors. It is convenient to introduce the group exponential map  $\exp_e^G$  for this. Following Warner [14], the map  $\lambda \cdot \frac{d}{dt} \mapsto \lambda \cdot v \in \mathfrak{g}$  ( $\lambda \in \mathbb{R}$ ) is a  $C^\infty$  Lie algebra homomorphism from  $\mathbb{R}$  into  $\mathfrak{g}$ . Since  $\mathbb{R}$  is simply connected there exists a *unique*  $C^\infty$  homomorphism from  $\mathbb{R}$  into  $G$  which we call  $\exp_v : \mathbb{R} \rightarrow G$  and has the property (see Theorem 3.27 [14]):

$$d \exp_v \left( \lambda \cdot \frac{d}{dt} \right) = \lambda \cdot v.$$

In other words, the map  $t \mapsto \exp_v(t)$  is the unique one-parameter subgroup of  $G$  whose tangent vector at the identity  $e$  is  $v$ . Define the group exponential map:

$$\exp_e^G : \mathbb{R} \times \mathfrak{g} \rightarrow G, \quad \exp_e^G(tv) = \exp_v(t). \quad (5)$$

The group homomorphism property implies two things about  $\exp_e^G(tv)$ :

$$\exp_e^G(sv) \star \exp_e^G(tv) = \exp_e^G((s+t)v), \quad \exp_e^G(tv)^{-1} = \exp_e^G(-tv),$$

i.e.,  $\exp_e^G(tv)$  is a one-parameter subgroup of  $G$ .

If  $V$  is a right-invariant vector field on  $G$  then the fundamental theorem on existence and uniqueness of ODEs says that there exists a one-parameter subgroup of diffeomorphisms  $\varphi_t^V$  acting on  $G$  and satisfying  $\partial_t \varphi_t^V(g) = V(\varphi_t^V(g))$ . Since  $\exp_e^G(tv)$  is the unique one-parameter subgroup (5) of  $G$  passing through the identity in the direction  $v = V(e)$  the flow  $\varphi_t^V(e)$  through the identity is given by  $\exp_e^G(tv)$ . Right-translating  $\exp_e^G(tv)$  by an element  $g \in G$  we see that  $R_g \exp_e^G(tv)$  satisfies the ODE:

$$\partial_t R_g \exp_e^G(tv) = dR_g V(\exp_e^G(tv)) = V(R_g \exp_e^G(tv)),$$

where we have used the right-invariance of  $V$ . So  $R_g \exp_e^G(tv)$  satisfies the same ODE as  $\varphi_t^V(g)$  and has the same initial conditions and therefore the two curves must coincide by uniqueness of solutions to ODEs. Consequently, the flow  $\varphi_t^V(g)$  is given by:

$$\varphi_t^V(g) = R_g(\exp_e^G(tv)) = \exp_e^G(tv) \star g = L_{\exp_e^G(tv)}(g). \quad (6)$$

An identical argument shows that the flow  $\psi_t^U$  of a left-invariant vector field  $U$  is given by:

$$\psi_t^U(g) = R_{\exp_e^G(tv)}(g). \quad (7)$$

**Remark 1.** Normally the group exponential map is defined by:

$$\exp_e^G : \mathfrak{g} \rightarrow G, \quad \exp_e^G(v) = \exp_v(1)$$

and it is then proved, using only the homomorphism property of  $\exp_v$ , that  $\exp_v(t)$  is the unique integral curve of the left/right-invariant vector field  $V$  whose value at the identity is  $v$ . From this it can be deduced that  $\exp_e^G(tv) = \exp_v(t)$  (see Theorem 3.31 of [14]). In contrast, we have chosen to take  $\exp_e^G(tv) = \exp_v(t)$  as the definition and work backwards.

**Proposition 1.** Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ .

1. If  $X$  and  $Y$  are two left-invariant vector fields on  $G$  then the algebra adjoint operator (4) is given by the Lie bracket in  $\mathfrak{g}$ :

$$\text{ad}_x y = dL_{g^{-1}} \cdot [X, Y] = [x, y]_{\mathfrak{g}}$$

2. If  $V$  and  $W$  are two right-invariant vector fields on  $G$  then the algebra adjoint operator (??) is given by minus the Lie bracket in  $\mathfrak{g}$ :

$$\text{ad}_v w = -dR_{g^{-1}} \cdot [V, W] = -[v, w]_{\mathfrak{g}}$$

*Proof.* For Part (1) we are going to show that

$$\text{ad}_{X_e} Y_e = \left. \frac{d}{dt} \right|_{t=0} d\psi_{-t}^X Y(\psi_t^X(e)) =: [X, Y](e) = dL_{g^{-1}} \cdot [X, Y](g) = [X_e, Y_e]_{\mathfrak{g}},$$

where  $\psi_t^X$  is the flow of the left-invariant vector field  $X$  given in (7).

Using the definition of the algebra adjoint (4), the definition of the group adjoint (3), left-invariance, the flow (7), and the definition of the Lie bracket of vector fields we compute:

$$\begin{aligned}
\text{ad}_{X_e} Y_e &= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp_e^G(tX_e)} Y_e = \left. \frac{d}{dt} \right|_{t=0} dR_{\exp_e^G(-tX_e)} dL_{\exp_e^G(tX_e)} Y_e \\
&= \left. \frac{d}{dt} \right|_{t=0} dR_{\exp_e^G(-tX_e)} Y(L_{\exp_e^G(tX_e)}(e)) = \left. \frac{d}{dt} \right|_{t=0} d\psi_{-t}^X \cdot Y(R_{\exp_e^G(tX_e)}(e)) \\
&= \left. \frac{d}{dt} \right|_{t=0} d\psi_{-t}^X \cdot Y(\psi_t^X(e)) =: [X, Y](e) \\
&= dL_{g^{-1}} \cdot [X, Y](g) = [X_e, Y_e]_{\mathfrak{g}}.
\end{aligned}$$

The statement is completed by using left-invariance of the vector fields and their Lie bracket.

For Part (2) we will show that:

$$\text{ad}_v w = \left. \frac{d}{dt} \right|_{t=0} d\psi_t^V W(\psi_{-t}^V(e)) =: [-V, W](e) = -dR_{g^{-1}} \cdot [V, W](g) = -[v, w]_{\mathfrak{g}}.$$

Note that the time parameter has the opposite sign. If  $\varphi_t^V$  is the flow of the right-invariant vector field  $V$  then  $\varphi_{-t}^V$  is the flow of the right-invariant vector field  $-V$ . The difference between the algebra adjoint operator for left-invariant vector fields and the algebra adjoint operator for right-invariant vector fields is due to the way in which left/right-invariance absorbs the respective factors appearing in the group adjoint operator  $\text{Ad}$ . So using the definition of the algebra adjoint (4), the definition of the group adjoint (3), right-invariance, the flow (6) and the definition of the Lie bracket:

$$\begin{aligned}
\text{ad}_v w &= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp_e^G(tv)} w = \left. \frac{d}{dt} \right|_{t=0} dL_{\exp_e^G(tv)} dR_{\exp_e^G(-tv)} w \\
&= \left. \frac{d}{dt} \right|_{t=0} dL_{\exp_e^G(tv)} W(R_{\exp_e^G(-tv)}(e)) = \left. \frac{d}{dt} \right|_{t=0} d\varphi_t^V \cdot W(L_{\exp_e^G(-tv)}(e)) \\
&= \left. \frac{d}{dt} \right|_{t=0} d\varphi_t^V \cdot W(\varphi_{-t}^V(e)) =: [-V, W](e) \\
&= -dR_{g^{-1}} \cdot [V, W](g) = -[v, w]_{\mathfrak{g}}.
\end{aligned}$$

Analogously to Part (1), the statement is completed by using right-invariance of the vector fields and their Lie bracket. □

**Remark 2.** *The relationship between the contents of Proposition 1 and the differential geometry of a left/right-invariant metric on a Lie group has been opaque for quite some time in the literature. Proposition 1 is purely algebraic and has absolutely nothing to do with a choice of Riemannian metric on  $G$ ; rather, it is analogous to a choice of orientation on the group. We will refer to the “algebraic orientation” in (1) as the*

left-handed algebra and the “algebraic orientation” in (2) as the right-handed algebra. When choosing a left/right-invariant metric on a Lie group it makes sense to pair the left/right-invariant geometry with the corresponding left/right-handed algebra, but only because it simplifies the relationship between the geometry and the algebra: there is no mathematical implication in either direction.

From now on we will restrict to right-invariant objects, but the reader can make the necessary changes for left-invariance using the above algebra.

Turning to the geometric viewpoint, we endow  $G$  with a right-invariant Riemannian metric  $\langle \cdot, \cdot \rangle$ . There is no obstruction in doing so since we can give  $\mathfrak{g}$  any inner product and smoothly extend it to the rest of the group via:

$$\langle V_g, W_g \rangle_g = \langle dR_{g^{-1}} \cdot v_g, dR_{g^{-1}} \cdot W_g \rangle_e \quad \forall V_g, W_g \in T_g G.$$

We have the usual geometric objects associated with  $\langle \cdot, \cdot \rangle$ ; namely, a smooth Levi-Civita connection  $\nabla$  and smooth Riemann curvature tensor  $R$ . Right translation  $R_g : G \rightarrow G$  is a global isometry of the metric  $\langle \cdot, \cdot \rangle$ , so both  $\nabla$  and  $R$  are invariant under the pullback action of  $R_g$ , which follows from general properties of Levi-Civita connections (see Lee [15], for example):  $R_g^* \nabla = \nabla$  and  $R_g^* R = R$ , i.e., for any  $g \in G$  and any vector fields  $U, V$  and  $W$  on  $G$ :

$$dR_g \cdot \nabla_V W = \nabla_{dR_g \cdot V} dR_g \cdot W \quad (8)$$

$$dR_g \cdot R(U, V)W = R(dR_g \cdot U, dR_g \cdot V) dR_g \cdot W. \quad (9)$$

So just like the metric  $\langle \cdot, \cdot \rangle$ , both  $\nabla$  and  $R$  are themselves right-invariant.

Linking this back to the algebra we define the group coadjoint operator  $\text{Ad}_g^* : \mathfrak{g} \rightarrow \mathfrak{g}$  as the metric adjoint of the group adjoint operator:

$$\langle \text{Ad}_g^* v, w \rangle = \langle v, \text{Ad}_g w \rangle, \quad \forall v, w \in \mathfrak{g}; \quad (10)$$

the algebra coadjoint operator  $\text{ad}_u^* : \mathfrak{g} \rightarrow \mathfrak{g}$  is the metric adjoint of the algebra adjoint operator:

$$\langle \text{ad}_u^* v, w \rangle = \langle v, \text{ad}_u w \rangle, \quad \forall v, w \in \mathfrak{g}. \quad (11)$$

**Proposition 2.** *Let  $G$  be a Lie group with a right-handed Lie algebra  $\mathfrak{g}$ , and right-invariant metric  $\langle \cdot, \cdot \rangle$ . Suppose  $V$  and  $W$  are any two right-invariant vector fields on  $G$ . Then:*

$$\nabla_V W = -\frac{1}{2} dR_g \cdot (\text{ad}_v w - \text{ad}_v^* w - \text{ad}_w^* v)$$

*Proof.* Let  $Z$  be any other right-invariant vector field on  $G$ . Since the metric  $\langle \cdot, \cdot \rangle$  is right-invariant the functions  $\langle V, W \rangle$ ,  $\langle V, Z \rangle$ ,  $\langle W, Z \rangle$  are all constant, and Koszul’s formula for the connection gives:

$$2\langle \nabla_V W, Z \rangle = \langle [V, W], z \rangle - \langle [W, Z], V \rangle - \langle [V, Z], W \rangle.$$



Since  $V$ ,  $W$ , and  $Z$  are right-invariant their Lie brackets are right-invariant and since the connection is right-invariant, Koszul's formula collapses to the Lie algebra as:

$$2\langle \nabla_v w, z \rangle = \langle [v, w]_{\mathfrak{g}}, z \rangle - \langle [w, z]_{\mathfrak{g}}, v \rangle - \langle [v, z]_{\mathfrak{g}}, w \rangle.$$

Using Part (2) of Proposition 1 to write the Lie bracket in the algebra in terms of the adjoint operator  $\text{ad}_v w = -[v, w]_{\mathfrak{g}}$  and taking metric adjoints we conclude:

$$2\langle \nabla_v w, z \rangle = -\langle \text{ad}_v w, z \rangle + \langle z, \text{ad}_w^* v \rangle + \langle z, \text{ad}_v^* w \rangle,$$

and since this holds for any vector  $z \in \mathfrak{g}$  we obtain:

$$\nabla_v w = -\frac{1}{2} (\text{ad}_v w - \text{ad}_w^* v - \text{ad}_v^* w).$$

Multiplying both sides by  $dR_g$  and using (8) once more gives the result.  $\square$

**Remark 3.** Proposition 2 does not hold for arbitrary smooth vector fields on  $G$ . If  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $\mathfrak{g}$  then we can right-translate these vectors around the group  $G$  to obtain a global right-invariant orthonormal frame  $\{E_1, \dots, E_n\}$ . Writing the vector fields  $V$  and  $W$  in terms of the global frame  $(E_i)$ :

$$V = v^1(g)E_1 + \dots + v^n(g)E_n, \quad W = w^1(g)E_1 + \dots + w^n(g)E_n.$$

we see that the covariant derivative  $(\nabla_V W)(e)$  depends on the values of the functions  $w^i(g)$  in a neighbourhood of the identity due to the Leibniz rule. In comparison, c.f. Proposition 3.1 and Corollary 3.2 of [16] and Proposition 1 of [17], where it is shown that any connection is uniquely determined by its values on right-invariant vector fields, albeit through a different relationship.

The tangent vector of a curve  $\eta : I \rightarrow G$  defined on an interval  $I \subset \mathbb{R}$  containing zero such that  $\eta(0) = e$  can be right-translated back to the identity  $e$  to give a curve  $v(t) \in \mathfrak{g}$  and we may write  $\partial_t \eta(t) = dR_{\eta(t)} \cdot v(t)$ . If  $U : I \rightarrow TG$  is any vector field along  $\eta(t)$ , i.e.,  $U(t) \in T_{\eta(t)}G$ , then it too can be right-translated back to the identity to give a curve  $u(t) \in \mathfrak{g}$ :

$$U(t) = dR_{\eta(t)} \cdot u(t).$$

The  $t$ -derivative (covariant derivative) of  $U(t)$  along  $\eta(t)$  is given as:

$$\begin{aligned} \partial_t U(t) &= dR_{\eta(t)} \cdot \partial_t u(t) + \nabla_{\partial_t \eta} dR_{\eta(t)} \cdot u(t) \\ &= dR_{\eta(t)} \cdot (\partial_t u(t) + \nabla_{v(t)} u(t)) \end{aligned} \tag{12}$$

where right-invariance of the connection (8) has been used and the covariant derivative in the Lie algebra has the interpretation given by Proposition 2.

Propositions 1 and 2 also give us a way of computing the Lie bracket of vector fields along curves in  $\mathfrak{g}$ . If  $U : I \rightarrow G$  and  $V : I \rightarrow G$  are two vector fields along

a curve  $\eta : I \rightarrow G$  with  $dR_{\eta(t)} \cdot U(t) = u(t) \in \mathfrak{g}$  and  $dR_{\eta(t)} \cdot V(t) = v(t) \in \mathfrak{g}$  then subtracting the formulas given in Proposition 2 for  $\nabla_{v(t)}u(t)$  and  $\nabla_{u(t)}v(t)$  gives:

$$\nabla_{u(t)}v(t) - \nabla_{v(t)}u(t) = [u(t), v(t)]_{\mathfrak{g}} = -\text{ad}_{u(t)}v(t), \quad (13)$$

which is consistent with the symmetry of the Levi-Civita connection  $\nabla$ .

### 3 The Geodesic Equations of a Right-Invariant Metric

Arnold developed a general framework in which to study geodesic equations of both left- and right-invariant metrics on arbitrary Lie groups as Euler equations on the associated Lie algebras. In everything that follows we will assume a right-handed algebra.

**Theorem 3.** *If  $G$  is a Lie group with a right-invariant metric  $\langle \cdot, \cdot \rangle$ , then a curve  $\eta(t)$  is a geodesic if and only if the curve  $v(t)$  in  $\mathfrak{g}$ , defined by the flow equation*

$$\dot{\eta}(t) = dR_{\eta(t)}v(t) \quad (14)$$

satisfies the Euler equations

$$\dot{v}(t) = -\text{ad}_{v(t)}^*v(t). \quad (15)$$

*Proof.* To derive the Euler equations on  $\mathfrak{g}$  we write the energy of a one-parameter family of curves  $(-\epsilon, \epsilon) \ni s \mapsto \eta(s, t)$  with fixed endpoints  $t = a, b$  in the form:

$$E(s) = \frac{1}{2} \int_a^b \|\eta(s, t)\|^2 dt = \frac{1}{2} \int_a^b \langle dR_{\eta(s, t)^{-1}}\partial_t\eta(s, t), dR_{\eta(s, t)^{-1}}\partial_t\eta(s, t) \rangle dt$$

using right-invariance of the metric. To calculate  $\partial_s|_{s=0} (dR_{\eta(s, t)^{-1}}\partial_t\eta(s, t))$  use the right-translation of the variation vector field induced by  $\eta(s, t)$  to define:

$$\begin{aligned} z(t) &= \partial_s|_{s=0}v(s, t) = \partial_s|_{s=0} (dR_{\eta(s, t)^{-1}}\partial_t\eta(s, t)) \\ y(t) &= dR_{\eta(t)^{-1}}\partial_s\eta(s, t)|_{s=0}. \end{aligned}$$

Then using (12):

$$\partial_t\partial_s|_{s=0}\eta(s, t) = \partial_t dR_{\eta(t)}y(t) = dR_{\eta(t)} (\partial_t y(t) + \nabla_{v(t)}y(t)).$$

On the other hand, using (12) again

$$\partial_s|_{s=0}\partial_t\eta(s, t) = \partial_s|_{s=0}dR_{\eta(s, t)} \cdot v(s, t) = dR_{\eta(t)} \cdot z(t) + dR_{\eta(t)} \cdot (\nabla_{y(t)}v(t)).$$

Since the left-hand sides of each of the above equations are equal, putting the two together and rearranging using (13) we have:

$$\partial_s|_{s=0} (dR_{\eta(s,t)^{-1}} \partial_t \eta(s,t)) = z(t) = \partial_t y(t) - \text{ad}_{v(t)} y(t).$$

Differentiating the energy  $E$  with respect to  $s$ , then integrating by parts and using the fact that  $y(a) = y(b) = 0$  we obtain:

$$\partial_s|_{s=0} E(s) = \int_a^b \langle v, \partial_t y - \text{ad}_v y \rangle dt = - \int_a^b \langle \partial_t v + \text{ad}_v^* v, y \rangle dt,$$

which is zero for any variation field  $y$  if and only if (15) holds.  $\square$

**Corollary 1.** *If  $\eta(t)$  is a curve in  $G$  with velocity field  $v(t) \in \mathfrak{g}$  defined through (14) and satisfying (15) with initial conditions  $\eta(0) = e$  and  $v(0) = v_o$  then we have the conservation law*

$$v(t) = \text{Ad}_{\eta^{-1}(t)}^* v_o. \quad (16)$$

*Proof.* Since  $\frac{d}{dt} \text{Ad}_{\eta(t)} w = \text{ad}_v \text{Ad}_{\eta(t)} w$  we have, by general properties of adjoints,

$$\frac{d}{dt} \text{Ad}_{\eta(t)}^* w = \text{Ad}_{\eta(t)}^* \text{ad}_v^* w.$$

Multiplying both sides of the Euler equations (15) by  $\text{Ad}_{\eta(t)}^*$ , reversing the product rule, and integrating in time gives the result.  $\square$   $\square$

Let us introduce one more operator

$$\begin{aligned} K_v &: \mathfrak{g} \rightarrow \mathfrak{g} \\ K_v(w) &= \text{ad}_w^* v \end{aligned}$$

for later use.

**Corollary 2.** *If  $\eta(t)$  is a curve in  $G$  with velocity field  $v(t) \in \mathfrak{g}$  defined through (14) and satisfying (15) with initial conditions  $\eta(0) = e$  and  $v(0) = v_o$  then:*

$$K_{v(t)} = \text{Ad}_{\eta^{-1}(t)}^* K_{v_o} \text{Ad}_{\eta^{-1}(t)}. \quad (17)$$

*Proof.* For any  $u, w \in \mathfrak{g}$  we compute using the relationship between the algebra adjoint and the Lie bracket, and the latter's behaviour under pushforward by diffeomorphisms:

$$\begin{aligned} \langle \text{Ad}_{\eta(t)}^* K_{v(t)} u, w \rangle &= \langle v(t), \text{ad}_u \text{Ad}_{\eta(t)} w \rangle = \langle v(t), \text{Ad}_{\eta(t)} \text{ad}_{\text{Ad}_{\eta^{-1}(t)} u} w \rangle \\ &= \langle \text{Ad}_{\eta(t)}^* v(t), \text{ad}_{\text{Ad}_{\eta^{-1}(t)} u} w \rangle \\ &= \langle v_o, \text{ad}_{\text{Ad}_{\eta^{-1}(t)} u} w \rangle \\ &= \langle K_{v_o} \text{Ad}_{\eta^{-1}(t)} u, w \rangle, \end{aligned}$$

where we have used Corollary 1 in the penultimate equality.  $\square$

## 4 The Exponential Map

We have not discussed the existence and uniqueness of solutions to the geodesic equations, nor their smooth dependence on initial conditions, but the standard analysis of geodesic systems on smooth finite-dimensional Riemannian manifolds applies (see [15] or [18], for example) and it can be shown that geodesics of a right-invariant metric on  $G$  exist, are unique, and do depend smoothly on initial conditions. Given this, we now introduce the exponential of the right-invariant metric.

In parallel with the group exponential map  $\exp_e^G$ , the (Riemannian) exponential map of the right-invariant metric  $\langle \cdot, \cdot \rangle$ ,  $\exp_e : \mathfrak{g} \rightarrow G$  maps a vector  $v \in \mathfrak{g}$  to the value of the (Riemannian) geodesic  $\eta(t)$  with initial conditions  $\eta(0) = e$ ,  $\eta'(0) = v$  at time  $t = 1$ :  $\exp_e(v) = \eta(1)$ .

The group exponential map  $\exp_e^G$  is *never* identically equal to the Riemannian exponential map  $\exp_e$  unless  $\langle \cdot, \cdot \rangle$  is both left- and right-invariant, although it is a Theorem of Kaiser [19] that there always exists a one-parameter subgroup of  $G$  that is a solution to the right-invariant geodesic system (15) so that the images of the two exponential maps do have a non-empty intersection. A standard rescaling procedure show that the exponential map is the data-to-solution map of the geodesic equations (15) assigning to each  $(e, v) \in \{e\} \times \mathfrak{g}$  the unique geodesic  $\eta(t)$  with those initial conditions:  $\exp_e(tv) = \eta(t)$ .

Since solutions of the geodesic system depend smoothly on the initial conditions, the exponential map is smooth, and since the derivative at  $t = 0$  is equal to the identity operator,  $\exp_e$  is a diffeomorphism between a neighbourhood of  $e \in G$  and a neighbourhood of the origin in  $\mathfrak{g}$ , by the inverse function Theorem.

As a map from the Lie algebra  $\mathfrak{g}$  – the model space – to the Lie group  $G$  – the manifold – the exponential map  $\exp_e$  contains the geometry of  $G$  with  $\langle \cdot, \cdot \rangle$  and it is therefore of interest to study its properties and singularities. Alternatively, one could ask in what way the singularities of  $\exp_e$  could restrict the algebraic, topological, and differentiable structure of  $G$ . See [20], [21], [22], or [23] for early examples of the latter line of questioning; we will proceed in the direction of the former.

Consider a geodesic  $\eta(t) = \exp_e(tv_o)$  starting from the identity  $e$  in the direction  $v_o \in \mathfrak{g}$ . Perturb the initial condition  $v_o$  by a vector  $z_o$ , and consider the one-parameter family of geodesics  $\eta(s, t) = \exp_e(t(v_o + sz_o))$ ; the Jacobi field  $J(t)$  along  $\eta(t)$  satisfying  $J(0) = 0$  and  $\dot{J}(0) = z_o$  is the variation field of  $\eta(s, t)$ :

$$J(t) = \partial_s|_{s=0}\eta(s, t) = D\exp_e(tv_o)tz_o.$$

**Proposition 4.** *Suppose  $G$  is any Lie group with a right-invariant metric  $\langle \cdot, \cdot \rangle$ . Let  $\eta(t)$  be a geodesic with  $\eta(0) = e$  and  $\dot{\eta}(0) = v_o$ . Then every Jacobi field  $J(t) = dR_{\eta(t)}y(t)$  along  $\eta(t)$  with initial conditions  $J(0) = 0$  and  $\dot{J}(0) = z_o$  satisfies the following system of equations on  $\mathfrak{g}$ :*

$$\partial_t y - \text{ad}_v y = z \tag{18}$$

$$\partial_t z + \text{ad}_v^* z + K_v z = 0, \tag{19}$$

where  $\dot{\eta}(t) = dR_{\eta(t)}v(t)$  as in (14).

*Proof.* Equation (18) is the linearisation of the flow equation (14) and was derived in the proof of Theorem 3. Indeed, if  $\eta(s, t)$  is a variation of a geodesic  $\eta(t)$  that continues to be a geodesic for each  $s$ , with  $v(s, t)$  the corresponding solution to the Euler equations (15), then after setting:

$$\begin{aligned} z(t) &= \partial_s|_{s=0} v(s, t) = \partial_s|_{s=0} (dR_{\eta(s, t)^{-1}} \partial_t \eta(s, t)) \\ y(t) &= dR_{\eta(t)^{-1}} \partial_s \eta(s, t)|_{s=0} \end{aligned}$$

we see that:

$$z(t) = \partial_s|_{s=0} (dR_{\eta(s, t)^{-1}} \partial_t \eta(s, t)) = \partial_t y - \text{ad}_v y.$$

Linearising the Euler equations (15) gives (19).  $\square$

We denote the solution operator to the linearised Euler equation (19) by:

$$S(t) : \mathfrak{g} \rightarrow \mathfrak{g}, \quad S(t)z_o = z(t). \quad (20)$$

Denote the solution operator to the Jacobi system (19) and (18) by:

$$\Phi(t)z_o := J(t) \quad (21)$$

which is a bounded linear operator from  $\mathfrak{g} = T_e G$  to  $T_{\eta(t)} G$ .

**Proposition 5.** *Let  $G$ ,  $\eta(t)$ , and  $v(t)$  be as in Proposition 4. Then the solution operator (20) to the linearised Euler equation (19) is related to the solution operator (21) to the Jacobi system (18) and (19) by:*

$$\text{Ad}_{\eta^{-1}(t)}^* = S(t) + K_{v(t)} \circ dR_{\eta^{-1}(t)} \circ \Phi(t) \quad (22)$$

*Proof.* If  $\eta(s, t)$  is a smooth variation of  $\eta(t)$  with corresponding variation of the velocity  $v(s, t)$  then  $z(t) = \partial_s|_{s=0} v(s, t)$ ,  $z_o = \partial_s|_{s=0} v(s, 0)$  and  $y(t) = dR_{\eta^{-1}(t)} \partial_s|_{s=0} \eta(s, t)$ . By Corollary 1:

$$v(s, t) = \text{Ad}_{\eta^{-1}(s, t)}^* v_o(s)$$

for each  $s$  and differentiating both sides with respect to  $s$  and setting  $s = 0$  we obtain:

$$z(t) = -\text{ad}_{y(t)}^* \text{Ad}_{\eta^{-1}(t)}^* v_o + \text{Ad}_{\eta^{-1}(t)}^* z_o.$$

Using Corollary 1 again and rewriting the expression in terms of the operator  $K_{v(t)}$  and the solution operators (20) and (21) we find:

$$\text{Ad}_{\eta^{-1}(t)}^* z_o = S(t)z_o + K_{v(t)} \circ dR_{\eta^{-1}(t)} \circ \Phi(t)z_o.$$

$\square$

Proposition 5 together with Corollary 1 give another form of the Euler-Arnold equations which describes the geodesic evolution as the time average of the linearised geodesic evolution.

**Theorem 6.** *The Euler-Arnold equations of Theorem 3 can be re-written as a feedback system*

$$\begin{aligned}\partial_t \tilde{v}(t) &= -\text{ad}_{v(t)}^* \tilde{v}(t) - K_{v(t)} \tilde{v}(t) \\ v(t) &= \frac{1}{t} \int_0^t \tilde{v}(s) ds \\ v(0) &= \tilde{v}(0) = v_o\end{aligned}$$

*Proof.* If  $v(t)$  is a solution to the Euler-Arnold equations with initial condition  $v_o$  then setting  $z_o = v_o$  and applying Corollary 1 to the left-hand side of equation (22) in Proposition 5, while making use of the fact that  $\Phi(t)v_o = t dR_{\eta(t)} \cdot v(t)$  for the right hand side of equation (22), we obtain:

$$v(t) = S(t)v_o + t \cdot K_{v(t)}v(t).$$

The curve  $v(t)$  does *not* solve the linearised Euler equation (19) *unless*  $v(t) = v_o$  is stationary (in which case the above equation reads  $v_o = v_o$ ). So if  $v(t)$  is non-stationary this equation says  $v(t)$  decomposes as the solution to the linearised Euler equation (19) with the same initial condition as  $v(t)$  plus an extra term  $K_v v$ . Making use of the Euler-Arnold equations once more we can substitute  $-\partial_t v(t)$  for  $K_v v$  and obtain a new form of the geodesic equations:

$$t \cdot \partial_t v(t) = -v(t) + S(t)v_o = -v(t) + \tilde{v}(t).$$

which can be written using the product rule as  $\partial_t (t \cdot v(t)) = \tilde{v}(t)$ .

Integrating both sides in  $t$ , we see that the geodesic equation of a right-invariant metric on a Lie group can be written as a feedback system:

$$\begin{aligned}\partial_t \tilde{v}(t) &= -\text{ad}_{v(t)}^* \tilde{v}(t) - K_{v(t)} \tilde{v}(t) \\ v(t) &= \frac{1}{t} \int_0^t \tilde{v}(s) ds \\ v(0) &= \tilde{v}(0) = v_o.\end{aligned}$$

where  $v(t)$  is the output - the geodesic dynamics - which, at every moment, is fed back into the first equation generating the dynamics.  $\square$

It is probably easiest to interpret these equations in terms of an Euler integration scheme: using the initial velocities  $v(0)$  and  $\tilde{v}(0)$  obtain the next  $\tilde{v}$  by taking a step in the direction of the right-hand side of the first equation, and then obtain the next  $v$  by averaging the two  $\tilde{v}$ 's. In this way one can see the solution  $v$  to the geodesic equations as a moving average of it's linearization.

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