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Jean-Pierre Delmas, Habti Abeida

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***Abstract*—This letter extends the well-known Whittle's formula for the asymptotic Fisher information matrix (FIM) on the power spectrum parameters of zero-mean stationary Gaussian processes to compound Gaussian processes (CGP). This new formula involves the usual Gaussian term plus a corrective factor that depends on the considered CG distribution.**

***Index Terms*—Whittle's formula, Slepian-Bangs's formula, Cramér-Rao bound, compound-Gaussian process, Elliptical symmetric distributions, Student's t processes.**

I. Introduction

Parametric discrete-time CGP, also called spherically invariant random processes (SIRP), are very commonly used in many statistical and engineering applications [1], and the problem attracting considerable interest is that of estimating its dependent parameters from a set of n consecutive measurements. The Cramér-Rao bound (CRB) usually computed as the inverse of the FIM, allows us to assess the performance of many parameter estimation algorithms. Recall that closed-form expressions of the FIM, called Slepian-Bangs (SB) formulas have been derived for the real, circular and noncircular complex Gaussian distributions in [2] [3], [4] and [5], respectively, then recently extended to circular and noncircular complex elliptically symmetric (ES) distributions in [6], [7] and [8], respectively. These latter formulas have been later extended when the density generator is considered as an infinite-dimensional nuisance parameter [9] or parameterized by a nuisance parameter [10].

However, in terms of complexity, direct computation of these SB formulas require a number of operations proportional to n^3. Furthermore, these formulas only allow numerical values to be given without providing any engineering insight into the role of the different parameters. Alternatively, the FIM was approximated by Whittle's asymptotic formula [11] for zero-mean non-deterministic stationary Gaussian processes. This approximation has a reduced computational complexity and can be interpreted thanks to its spectral expression. However, to the best of our knowledge, there is no work so far addressing Whittle's asymptotic formula for zero-mean stationary CGPs. In this letter, we profitably use the Toeplitz structure of the covariance matrix of the measurement for zero-mean stationary CGPs to derive an alternative approximate expression of the FIM and a limit of the FIM rate, which is a generalization of Whittle's formula [11].

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The remainder of this letter is organized as follows. In Section II, some preliminaries on stationary CGPs are given. In Section III, Bangs's formula for ES distributed r.v.s. is briefly recalled and new results are brought for CG distributed r.v.s. After proving the EVD of the covariance matrix under different assumptions on the sequence of correlation in Section IV, an approximation and a limit of the FIM rate, which is a generalization of Whittle's formula, are given for zero-mean stationary CGPs. Finally, this letter is concluded in Section VI.

II. Preliminaries on stationary CGPs

Let us first recall that a discrete-time random process $(x_k)_{k\in\mathbb{Z}}$ is said to be Gaussian if every finite collection of $\{x_{k_1},...,x_{k_n}\}$ forms a Gaussian random vector $\mathbf{x} \overset{\text{def}}{=} (x_{k_1},...,x_{k_n})^T$, denoted $\mathbf{x}\sim\mathcal{N}_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$, with p.d.f.

$$p(\mathbf{x}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\exp(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})). \quad (1)$$

It is also natural to consider more general families of ES processes [1] by replacing the exponential function in (1) by an arbitrary function $g_n(.) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\int_0^\infty t^{n/2-1}g_n(t)dt < \infty$ (called density generator) to form the p.d.f. of an n-dimensional ES distributed r.v.

$$p(\mathbf{x}) = |\boldsymbol{\Sigma}|^{-1/2}g_n[(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})]. \quad (2)$$

In (2), the first and second-order moments are assumed to exist, with $\mathrm{E}(\mathbf{x}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{x}) = \boldsymbol{\Sigma}$ to avoid scale ambiguity in (2). Then, discrete ES random processes can be defined similarly. But far from all ES distributed r.v.s allow to define discrete ES processes in this way. In fact, the sequence of density generators $g_n(.)$ cannot be arbitrary because it must satisfy the Kolmogorov consistency condition that ensures that the ES distribution is closed under marginalization. This condition boils down to whether (2) is the p.d.f. of $\mathbf{x}_n$, the p.d.f. of any marginal $\mathbf{x}_n$ of $\mathbf{x}_m$ (with $m > n$) is always given by (2). A necessary and sufficient condition is given in [12, Th. 2.2] for which this property holds, which is the existence of a positive r.v. τ of cumulative distribution function $F\tau(.)$ such that:

$$g_n(t) = (2\pi)^{-n/2}\int_0^\infty \tau^{-n/2}\exp(-t/2\tau)dF_\tau(\tau). \quad (3)$$

Another equivalent condition [13] is given by the stochastic representation of the ES distribution:

$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\mathcal{Q}_n}\boldsymbol{\Sigma}^{1/2}\mathbf{u}, \tag{4}$$

where $\mathcal{Q}_n$ is a positive r.v. whose distribution depends on n which is independent of the r.v. $\mathbf{u}$ uniformly distributed on the unit n-sphere, which takes the particular form [14]:

$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\tau}\boldsymbol{\Sigma}^{1/2}\mathbf{n}_0, \tag{5}$$

where the r.v τ is independent of the r.v. $\mathbf{n}_0 \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$. This is equivalent to state that the r.v.s $\mathcal{Q}_n$ and τ in (4)-(5) are related by $\mathcal{Q}_n = \tau\chi^2_n$ where τ and χ^2_n are independent. This stochastic representation (5) characterizes the subclass of CG distributions, whose associated discrete random processes are called SIRP in the engineering literature. Other equivalent conditions on $g_n(t)$ in (3) are given in [14]–[16].

Throughout this paper, we only consider zero-mean stationary CG processes $(x_k)_{k\in\mathbb{Z}} \in \mathbb{R}$. Similarly to Gaussian processes, this property is ensured i.f.f. these processes are zero-mean wide-sense stationary. Their distribution is thus characterized by the distribution of the v.a. τ and by the covariance matrix of the r.v. $\mathbf{x}_n = (x_{k+1}, x_{k+2}, ..., x_{k+n})$: $\mathrm{Cov}(\mathbf{x}_n) = \boldsymbol{\Sigma}_{\mathbf{x}_n}$, which is a symmetric Toeplitz positive definite matrix with $(\boldsymbol{\Sigma}_{\mathbf{x}_n})_{k,\ell} = r_x(|k-\ell|)$ where the sequence $r_x(k) \overset{\text{def}}{=} \mathrm{E}(x_\ell x_{\ell+k})$ is supposed absolutely summable. We assume that the spectrum $S_x(f) = \sum_k r_x(k)e^{-i2\pi kf}$ is parameterized by $\boldsymbol{\theta} = (\theta_1, .., \theta_q)^T \in \mathbb{R}^q$, which is omitted by simplicity.

III. Bangs's formulas

We give here a brief reminder of the Slepian-Bangs formula for zero-mean ES distributed data $\mathbf{x}_n \in \mathbb{R}^n$ which reduces here to the Bang's formula, where the density generator $g_n(t)$ is known or unknown [10]. This Bang's formula is given by the elementwise FIM under usual regularity conditions on $g_n(t)$:

$$\begin{aligned}(\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta}))_{k,\ell} &= a_{1,n}\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n}\boldsymbol{\Sigma}^{'}_{\mathbf{x}_n,k}\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n}\boldsymbol{\Sigma}^{'}_{\mathbf{x}_n,\ell}) \\ &+ a_{2,n}\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n}\boldsymbol{\Sigma}^{'}_{\mathbf{x}_n,k})\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n}\boldsymbol{\Sigma}^{'}_{\mathbf{x}_n,\ell}) \end{aligned} \tag{6}$$

with $\boldsymbol{\Sigma}^{'}_{\mathbf{x}_n,k} \overset{\text{def}}{=} \frac{\partial\boldsymbol{\Sigma}_{\mathbf{x}_n}}{\partial\theta_k}$. The weights $a_{1,n}$ and $a_{2,n}$ are free from scalar ambiguity and given by

$$a_{1,n} = \frac{1}{2}\xi_{1,n} \tag{7}$$

for both classic and semiparametric Bangs's formulas, and

$$a_{2,n} = a^{\text{Clas}}_{2,n} = \frac{1}{4}(\xi_{1,n} - 1), \quad a_{2,n} = a^{\text{SePa}}_{2,n} = -\frac{1}{2n}\xi_{1,n} \tag{8}$$

for classic and semiparametric Bangs's formulas, respectively where

$$\xi_{1,n} = \frac{\mathrm{E}[Q_n^2\phi_n^2(\mathcal{Q}_n)]}{n(n+2)}, \tag{9}$$

with $\phi_n(t) \overset{\text{def}}{=} \frac{2}{g_n(t)}\frac{dg_n(t)}{dt}$.

Using $\mathcal{Q}_n =_d \tau\chi^2_n$ for CG distributions, one can alternatively express $\xi_{1,n}$ (9) as:

$$\begin{aligned}\xi_{1,n} &= \frac{1}{n(n+2)}\int_0^\infty\int_0^\infty u^2v^2 \\ &\left(\frac{\int_0^\infty \tau^{-n/2-1}\exp(-uv/2\tau)dF_\tau(\tau)}{\int_0^\infty \tau^{-n/2}\exp(-uv/2\tau)dF_\tau(\tau)}\right)^2 dF_\tau(v)p_{\chi^2_n}(u)du,\end{aligned} \tag{10}$$

depending on n and the distribution of the r.v. τ, under the constraint $\mathrm{E}(\tau) = 1$ to assure that $\mathrm{Cov}(\mathbf{x}_n) = \boldsymbol{\Sigma}_{\mathbf{x}_n}$. Note that a direct calculation of the FIM was carried out in the case of complex circular CG distributions [17, Eqs. (19b), (20), (21b), (21c)] which leads to a much less compact expression. The asymptotic behavior of the sequence $\xi_{1,n}$ is generally difficult to analyze except for the following three distributions. For Gaussian distributions $\tau = 1$ gives $\xi^{\text{G}}_{1,n} = 1$, and for Student's t-distributions $\tau^{-1} \sim \mathrm{Gam}(\nu/2, 2/\nu)$ for which $\xi^{\text{St}}_{1,n}$ has been derived in the circular complex case in [6], and in the real case in [10] using the real-to-complex representation, yielding

$$\xi^{\text{St}}_{1,n} = \frac{n+\nu}{n+\nu+2}, \tag{11}$$

where $\nu > 2$ is the degree of freedom parameter. For the generalized Gaussian (GG) distributions, it was proved in [18] that if the exponent s satisfies $s \in (0, 1]$, these distributions belong to the subset of the CG distributions, whose p.d.f. of $\sqrt{\tau}$ is given by [18, Th. 1]. The corresponding $\xi^{\text{GG}}_{1,n}$ has been derived in the circular complex case in [7], and in the real case in [10] using the real-to-complex representation, giving

$$\xi^{\text{GG}}_{1,n} = \frac{n+2s}{n+2}. \tag{12}$$

It follows from [19, Appendix A] for complex circular case that for real-valued ES distributions

$$\frac{\mathrm{E}[Q_n^2\phi_n^2(\mathcal{Q}_n)]}{n(n+2)} = 1 - 2\frac{\mathrm{E}[Q_n^2\phi_n^{'}(\mathcal{Q}_n)]}{n(n+2)} \tag{13}$$

with $\phi_n^{'}(t) \overset{\text{def}}{=} \frac{d\phi_n(t)}{dt}$. We get for CG distributions, using $\phi_n^{'}(t) \geq 0$ proved in the Appendix, that $0 < \xi_{1,n} \leq 1$, and $\xi_{1,n} = 1$ is equivalent to $\mathrm{E}[Q_n^2\phi_n^{'}(\mathcal{Q}_n)] = 0 \Leftrightarrow Q_n^2\phi_n^{'}(\mathcal{Q}_n) = 0$ a.s. $\Leftrightarrow \phi_n^{'}(\mathcal{Q}_n) = 0$ because $Q_n > 0$ a.s., i.e. $\mathbf{x}$ is Gaussian distributed. This allows us to state the new result:

Result 1: For CG distributions, the parameters of the FIM satisfy the relations: $0 \leq a_{1,n} \leq 1/2$ and $a_{2,n} \leq 0$, and $\mathbf{x}$ is Gaussian distributed i.f.f. $(a_{1,n}, a_{2,n}) = (1/2, 0)$.

This property proves that the Gaussian distribution always leads to the smallest stochastic CRB for CG distributions, but not in the larger family of the ES distributions as shown by the GG distributions associated with $s > 1$ (12). This contrasts with the case of parameterized mean, for which the Gaussian distribution always leads to the largest stochastic CRB for all second-order distributions [20].

IV. Whittle's formula for CG processes

A. Approximation formula

The following approximate eigenvalue decomposition (EVD) of the Toeplitz structured matrix $\mathbf{\Sigma}_{\mathbf{x}_n}$ has been derived and used for large n compared to the correlation length of x_k without rigorous theoretical support (see e.g., [21, eq. (9)], [22, eq. (5)] [23, p.186]):

$$\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n \approx \mathrm{Diag}(S_x(0), S_x(\frac{1}{n}), ..., S_x(\frac{n-1}{n})), \quad (14)$$

where $\mathbf{W}_n$ is the discrete Fourier transform (DFT) unitary matrix defined by $\mathbf{W}_n \stackrel{\text{def}}{=} (\mathbf{w}_0, .., \mathbf{w}_k, .., \mathbf{w}_{n-1})$ with $\mathbf{w}_k \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}}(1, e^{i2\pi k/n}, ..., e^{i2\pi k(n-1)/n})^T$.

But this approximation (14) can be justified by the following limits for a (k,ℓ)-th element fixed of $\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n$ which are proved in the Appendix under the following conditions: For $r_x(k)$ absolutely summable, we get

$$\lim_{n\to\infty} [\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n]_{k,k} - S_x(\frac{k-1}{n}) = 0, \quad (15)$$

and additionally for banded Toeplitz matrices $\mathbf{\Sigma}_{\mathbf{x}_n}$

$$\lim_{n\to\infty} [\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n]_{k,\ell} - \delta_{k,\ell} S_x(\frac{k-1}{n}) = 0, \quad (16)$$

where is $\delta_{k,\ell}$ the Kronecker notation. Likewise for the strongest condition $kr_x(k)$ absolutely summable, (16) also remains valid. Using $\mathbf{W}_n^H \mathbf{\Sigma}'_{\mathbf{x}_n,k} \mathbf{W}_n \approx \mathrm{Diag}(S'_{x,k}(0), S'_{x,k}(\frac{1}{n}), ..., S'_{x,k}(\frac{n-1}{n}))$ deduced from (14) with $S'_{x,k}(f) \stackrel{\text{def}}{=} \frac{\partial S_x(f)}{\partial \theta_k}$, we get

$$\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k} \mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,\ell}) = \mathrm{Tr}[(\mathbf{W}_n \mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{W}_n^H)(\mathbf{W}_n \mathbf{\Sigma}'_{\mathbf{x}_n,k} \mathbf{W}_n^H)(\mathbf{W}_n \mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{W}_n^H) (\mathbf{W}_n^H \mathbf{\Sigma}'_{\mathbf{x}_n,\ell} \mathbf{W}_n)] \approx n \sum_{p=0}^{n-1} S_x^{-2}(\frac{p}{n}) S'_{x,k}(\frac{p}{n}) S'_{x,\ell}(\frac{p}{n}) \frac{1}{n}. \quad (17)$$

We obtain for n sufficiency large, by replacing the Riemann sum (17) with an integral, the approximation

$$\frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k} \mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,\ell}) \approx \int_0^1 S_x^{-2}(f) S'_{x,k}(f) S'_{x,\ell}(f) df. \quad (18)$$

Similarly, we have the following approximation

$$\frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k}) \approx \int_0^1 S_x^{-1}(f) S'_{x,k}(f) df. \quad (19)$$

Inserting (18) and (19) in (6), we immediately deduce the following result:

Result 2: The entries of the FIM for zero-mean stationary CG processes are given for large n by the approximation:

$$\frac{1}{n}\left(\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})\right)_{k,\ell} \approx a_{1,n} \int_0^1 S_x^{-2}(f) S'_{x,k}(f) S'_{x,\ell}(f) df + n a_{2,n} \int_0^1 S_x^{-1}(f) S'_{x,k}(f) df \int_0^1 S_x^{-1}(f) S'_{x,\ell}(f) df, \quad (20)$$

where $a_{1,n}$ and $a_{2,n}$ are respectively given by (7) and (8) which are expressed in terms of $\xi_{1,n}$ in (9).

B. Asymptotic formula

We consider here asymptotic properties of Toeplitz matrices depending on specific conditions on the spectra of x_k. For Gaussian ARMA processes, whose spectra $S_x(f) = \sigma^2 \left|\frac{A(e^{-i2\pi f})}{B(e^{-i2\pi f})}\right|^2$ is such that $A(z) = \sum_{k=0}^p a_k z^k$ and $B(z) = \sum_{k=0}^q b_k z^k$ are both bounded away for zero for $|z| \leq 1$, it is proved in [24, Th. 1] that

$$\frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k} \mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,\ell}) = \int_0^1 S_x^{-2}(f) S'_{x,k}(f) S'_{x,\ell}(f) df + O(\frac{1}{n}) \quad (21)$$

and

$$\frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k}) = \int_0^1 S_x^{-1}(f) S'_{x,k}(f) df + O(\frac{1}{n}). \quad (22)$$

But the proof requires only second-order properties of x_k, so (21) and (22) are also valid for CG ARMA processes.

For long range dependent Gaussian processes, whose spectrum $S_x(f)$ and $(S'_{x,k}(f))_{k=1,..,q}$ satisfy some conditions [25, A2-A7], it is proved in [25, Th. 5.1], the following properties weaker than (21) and (22) which also apply for CG processes for the same reasons as previously.

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k} \mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,\ell}) = \int_0^1 S_x^{-2}(f) S'_{x,k}(f) S'_{x,\ell}(f) df \quad (23)$$

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1} \mathbf{\Sigma}'_{\mathbf{x}_n,k}) = \int_0^1 S_x^{-1}(f) S'_{x,k}(f) df. \quad (24)$$

Finally let us point out that thanks to the notion of asymptotically equivalence of sequences of Toeplitz and circulant structured matrices with absolutely summable elements [26], [27], it is straightforward to prove (23) and (24) under the additional condition of existence of m such that $S_x(f) \geq m > 0$, using [27, Th.5.2] and [27, Th.5.3] relative to inverses and products of Toeplitz matrices, respectively.

Furthermore since the limit of the FIM rate, i.e., $\frac{1}{n}\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})$, exists for stationary processes with finite Markov order [28, rel. 30], the limits of $a_{1,n}$ and $na_{2,n}$ (from (6) using the limits (18) (19)) when n tends to ∞), exist for a large class of stationary processes. For these processes, we deduce from $na_{2,n}^{\mathrm{Clas}} = \frac{n}{4}(\xi_{1,n} - 1)$ (8) that $\lim_{n\to\infty} \xi_{1,n} = 1$ which implies from (7) that $\lim_{n\to\infty} a_{1,n} = 1/2$, and from (8) that $\lim_{n\to\infty} na_{2,n}^{\mathrm{Clas}} = c \leq 0$ and $\lim_{n\to\infty} na_{2,n}^{\mathrm{SePa}} = -1/2$. And furthermore since more knowledge about the density generator results in a larger FIM, we get $c \geq -1/2$. This allows us to state the following result which is an extension of Whittle's formula [11], [29, Th. 5.3]:

Result 3: For stationary CG processes with finite Markov order, the limit of the FIM rate which is a generalization

of Whittle's formula for purely non-deterministic stationary Gaussian processes has the following expression:

$$\lim_{n\to\infty}\frac{1}{n}\left(\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})\right)_{k,\ell} = \frac{1}{2}\int_0^1 S_x^{-2}(f)S_{x,k}^{'}(f)S_{x,\ell}^{'}(f)df + c\int_0^1 S_x^{-1}(f)S_{x,k}^{'}(f)df\int_0^1 S_x^{-1}(f)S_{x,\ell}^{'}(f)df, \quad (25)$$

where $c=-1/2$ when $g_n(.)$ is unknown, and $-1/2\leq c\leq 0$, depending of the CG distribution when $g_n(.)$ is known.

From the values of $\xi_{1,n}$ given in Section **??**, $c=0$, $c=-1/2$, and $c=-(1-s)/2$ for the Gaussian, Student's t and GG distribution with $s\in(0,1]$, respectively. We will see in the next section that $c=0$ for the ϵ-contaminated Gaussian distribution.

V. Numerical illustrations

We consider in this section that x_k is zero-mean ϵ-contaminated Gaussian distributed with $P(\tau=\tau_1)=1-\epsilon$ et $P(\tau=\tau_0)=\epsilon$ under the constraint $\mathrm{E}(\tau)=(1-\epsilon)\tau_1+\epsilon\tau_0=1$, where (τ_0,ϵ) are parameters that control the heaviness of the tails as compared to the Gaussian distribution. We assume an AR(1) spectra, i.e., $S_x(f)=\frac{\sigma_x^2(1-a^2)}{|1-ae^{-i2\pi f}|^2}$ associated with $r_x(k)=\sigma_x^2a^{|k|}$ and $|a|<1$, where a is the only unknown parameter.

From numerical calculations, we can state that the approximations (18) and (19) are valid with a relative precision of 1% for arbitrary values of $|a|<1$, and that $\lim_{n\to\infty} na_{2,n}=0$ as shown e.g. in Fig.1 for $\epsilon=0.1$ and some values of τ_0. The more τ_0 increases, the more the distribution of $\mathbf{x}_n$ deviates from the Gaussian distribution and the slower the convergence of $na_{2,n}$ towards 0.

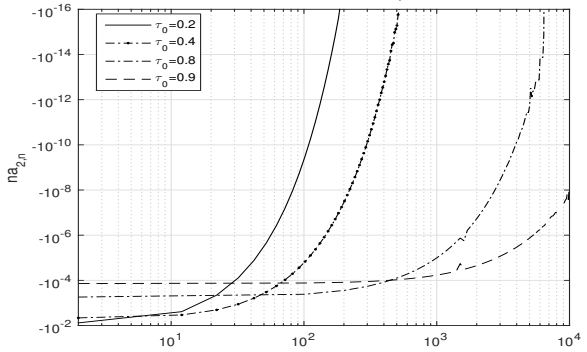

Fig.1 Coefficients $na_{2,n}$ as a function of n.

VI. Conclusion

In this letter, complementary results are brought on Bangs's formula for CG distributed r.v.s. An approximation and a limit of the FIM rate which is a generalization of Whittle's formula, are given for zero-mean stationary CG distributed random processes. Research is underway on extending Whittle's formula to continuous-times and multivariate stationary CG distributed random processes.

Appendix

A. Proof of $\phi_n^{'}(t)\geq 0$

Using (3), the 1st and 2nd derivatives of $g_n(t)$ are given by $g_n^{(k)}(t)=(-2)^{-k}(2\pi)^{-n/2}\int_0^\infty \tau^{-n/2-k}\exp(-t/2\tau)dF_\tau(\tau), k=1,2$. Thus, from the definition of $\phi_n(t)$, $\phi_n^{'}(t)\geq 0\Leftrightarrow g_n^{(2)}(t)g_n(t)\geq[g_n^{(1)}(t)]^2$, which holds thanks to Cauchy-Schwarz inequality. ■

B. Proof of (15)

We get straightforwardly

$$[\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n]_{k,k}=\sum_{|p|\leq n-1}\left(1-\frac{|p|}{n}\right)r_x(p)e^{-i2\pi p(k-1)/n}.$$

Consequently $[\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n]_{1,1}$ tends to $S_x(0)=\sum_p r_x(p)$ when n tends to ∞ according to Cesaro summability property [29, A10]. We then obtain for $k>1$

$$[\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n]_{k,k}-S_x(\frac{k-1}{n}) = -\sum_{|p|\leq n-1}\frac{|p|}{n}r_x(p)e^{-i2\pi p(k-1)/n}-\sum_{|p|\geq n}r_x(p)e^{-i2\pi p(k-1)/n}, \quad (26)$$

and the modulus of the two terms of (26) are respectively upper-bounded by $\frac{1}{n}\sum_{|p|\leq n-1}|p||r_x(p)|$ and $\sum_{|p|\geq n}|r_x(p)|$. The first bound tends to zero as a consequence of the Cesaro summability property [29, A10], and the second term also tends to zeros as a reminder of the convergent series $\sum_p|r_x(p)|$. ■

C. Proof of (16)

Suppose that $r_x(p)=0$ for $|p|>P$. We get straightforwardly for $k\neq\ell$ and $n>P$:

$$[\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n]_{k,\ell}=\frac{1}{n}r_x(0)e^{i2\pi(k-\ell)/n}\sum_{q=1}^{n}[e^{-i2\pi(k-\ell)/n}]^q + \frac{1}{n}\sum_{0<p\leq P}r_x(p)(e^{i2\pi p(1-\ell)/n}+e^{-i2\pi p(1-\ell)/n}) \left(\sum_{q=1}^{n-p}[e^{-i2\pi(k-\ell)/n}]^{q-1}\right), \quad (27)$$

where $\sum_{q=1}^{n}[e^{-i2\pi(k-\ell)/n}]^q=0$ and $|\sum_{q=1}^{n-p}[e^{-i2\pi(k-\ell)/n}]^{q-1}|=\frac{|\sin(\pi(k-\ell)(n-p)/n)|}{|\sin(\pi(k-\ell)/n)|}$ tends to p when n tends to ∞. ■

Suppose now that $pr_x(p)$ is summable. This naturally implies (15) and for $k\neq\ell$, the second sum of (27) must be replaced by the unbounded sum $\sum_{0<p<n}$ where

$$|r_x(p)(e^{i2\pi p(1-\ell)/n}+e^{-i2\pi p(1-\ell)/n}) \left(\sum_{q=1}^{n-p}[e^{-i2\pi(k-\ell)/n}]^{q-1}\right)|<2p|r_x(p)|(1+\epsilon), \quad (28)$$

for $n>N(\epsilon)$, $\forall\epsilon>0$. ■

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