Three remarks on the convergence of some discretized second order gradient-like systems

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The first author dedicates this paper to the memory of his colleague Naceur Manai.

Abstract

We study several discretizations of a second order gradient-like system with damping. We first consider an explicit scheme with a linear damping in finite dimension. We prove that every solution converges if the nonlinearity satisfies a global Lojasiewicz inequality. Convergence rates are also established. In the case of a strong nonlinear damping, we prove convergence of every solution for a fully implicit scheme in the one-dimensional case, even if the nonlinearity does not satisfy a Lojasiewicz inequality. The optimality of the damping is also established. Numerical simulations illustrate the theoretical results.

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1 Introduction

In this paper, we study the asymptotic behaviour of solutions to several discretizations of the second order ordinary differential system

$$u'' + \|u'\|^\alpha u' + \nabla F(u) = 0, \quad t \geq 0, \tag{1}$$

where $F : \mathbb{R}^d \to \mathbb{R}$ is a C^1 function and $\alpha \in [0, 1]$.

The asymptotic behaviour of solutions to (1) has been extensively studied. A crucial tool is the energy estimate

$$\frac{d}{dt}E(t) = -\|u'(t)\|^{2+\alpha} \leq 0,$$

where E is the energy defined for every solution u by

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + F(u(t)).$$

The system (1) is known as a gradient-like system. In particular, the ω-limit set of a bounded solution consists only of equilibrium points. A natural question is the study of convergence of such a solution.

Haraux [10] proved under quite general assumptions on F that if $d = 1$ and $0 \leq \alpha < 1$, every bounded solution converges to a critical point of F as time goes to infinity. This is no longer true if the damping is too weak ($\alpha \geq 1$), as pointed out in [10], or if $d \geq 2$, as proved in [14].

However, if F satisfies additional properties, more can be said. If F is real analytic, $\alpha = 0$ and d is arbitrary, Haraux and Jendoubi [12] proved that every bounded solution to (1) converges to a single point. Their result is based on the celebrated Lojasiewicz inequality [15, 16]. A similar convergence result was obtained by Chergui [7] in the case $0 \leq \alpha < 1$, where α depends also on the global Lojasiewicz exponent of the function F. We refer the reader to the book [11] for details on this matter. Alvarez [3] proved convergence to an equilibrium for convex functions in the case $\alpha = 0$ (see also [6]). Generalizations of (1) have been considered in [5].

Discretizations of the system (1) have also been studied. In [3] mentioned above, Alvarez also studied the asymptotic behaviour for convex functions of a fully implicit discretization in the case $\alpha = 0$. Convergence to equilibrium was also proved in [9] for a fully implicit scheme in the case of real analytic functions and linear damping ($\alpha = 0$). A semi-implicit scheme was considered in [2] for functions satisfying a Lojasiewicz inequality. Horsin and Jendoubi [13] studied the nonlinear damping $0 < \alpha < 1$ for a fully implicit scheme. Related schemes were considered in [4].

Our main purpose in this paper is to study an explicit discretization of (1) for a linear damping and a function F which satisfies a global Lojasiewicz inequality. In contrast to the

fully implicit discretization, we do not prove that the energy is nonincreasing but instead, we prove that a modified energy is nonincreasing. This is achieved in Section 2 and the proof of convergence is based on an angle condition introduced in [1]. Convergence rates are also established.

In Section 3, we consider an implicit discretization of (1) in the one-dimensional case ($d = 1$) and we prove that the sequence generated by the scheme converges for $0 \leq \alpha < 1$, without assuming that F satisfies a Lojasiewicz inequality. The optimality of α is established in Section 4, where a non convergence result is shown to hold for $\alpha = 1$ and $d = 1$, even if F satisfies a Lojasiewicz inequality with optimal exponent $\theta = 1/2$. We note that altogether, in Sections 2-4, the discrete case mimicks the situation for the continuous system (1). Numerical simulations illustrate the theoretical results in the last section.

2 Convergence to equilibrium for an explicit scheme

In this section, we prove convergence to equilibrium for an explicit discretization of the system (1). We first recall a general convergence result.

2.1 A discrete angle condition

Let $\Phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a C^1 function, $\sigma > 0$ and let us consider a sequence (x_n) satisfying for all $n \in \mathbb{N}$

$$\Phi(x_n) - \Phi(x_{n+1}) \geq \sigma \|\nabla \Phi(x_n)\| \|x_{n+1} - x_n\|, \tag{2}$$

$$[\Phi(x_{n+1}) = \Phi(x_n)] \Longrightarrow [x_{n+1} = x_n]. \tag{3}$$

Theorem 1 (Absil et al. [1]) *We assume that there exists $\theta \in (0, \frac{1}{2}]$ such that*

$$\begin{aligned} &\forall a \in \mathbb{R}^N \ \exists c_a > 0 \ \exists r_a > 0 / \ \forall x \in \mathbb{R}^N : \\ &\quad \|x - a\| < r_a \Longrightarrow \|\nabla \Phi(x)\| \geq c_a |\Phi(x) - \Phi(a)|^{1-\theta}. \end{aligned} \tag{4}$$

Let (x_n) be a sequence satisfying (2) and (3). Then either $\lim_{n \to +\infty} \|x_n\| = +\infty$, or there exists $x^\infty \in \mathbb{R}^N$ such that $\nabla \Phi(x^\infty) = 0$ and

$$\lim_{n \to +\infty} x_n = x^\infty.$$

Precisely, in this case we have

$$\|x_n - x^\infty\| = \begin{cases} O(e^{-cn}) & \text{for some } c > 0 \text{ if } \theta = \frac{1}{2} \\ O(n^{-\frac{\theta)}{1-2\theta}}) & \text{if } \theta \in (0, \frac{1}{2}). \end{cases} \tag{5}$$

Remark 2 If there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $\|x_n\| \leq M$, then the assumption (4) may merely apply to those $a \in \mathbb{R}^N$ such that $\|a\| \leq M$.

2.2 The explicit scheme

We consider a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\left\{\begin{array}{l} \dfrac{u_{n+1}-u_n}{\Delta t} = v_n \\ \dfrac{v_{n+1}-v_n}{\Delta t} = -v_n - \nabla F(u_n) \\ \quad u_0,\ v_0 \in \mathbb{R}^d \end{array}\right. \tag{6}$$

where $\alpha > 0$ and $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a C^1 function such that

$$\exists L_F > 0/\ \forall u, v \in \mathbb{R}^d \quad \|\nabla F(u) - \nabla F(v)\| \leq L_F \|u - v\|, \tag{7}$$

$$\exists c_1, c_2 > 0/\ \forall u \in \mathbb{R}^d \ \ F(u) \geq c_1\|u\|^2 - c_2. \tag{8}$$

It is easy to check (see for example [2, 13]) that hypothesis (7) on F implies the following inequalities ($< ., >$ denoting the standard scalar product on $\mathbb{R}^d$)

$$\forall u, v \in \mathbb{R}^d \quad < \nabla F(u) - \nabla F(v), u - v > \geq -L_F\|u - v\|^2, \tag{9}$$

$$\forall u, v \in \mathbb{R}^d \quad F(v) \geq F(u) + < \nabla F(u), v - u > -\frac{L_F}{2}\|u - v\|^2. \tag{10}$$

The existence and uniqueness of a sequence satisfying (6) are obvious.

Proposition 3 *For any $(u_0, v_0) \in \mathbb{R}^{2d}$ the sequence (u_n, v_n) given by (6) is well defined.*

We define

$$E(u, v) = \frac{1}{2}\|v\|^2 + F(u).$$

Proposition 4 *Assume F satisfies (7). Let (u_n, v_n) be a sequence satisfying (6), then we have*

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -\Delta t \left[1 - \left(1 + \frac{3L_F}{2}\right)\Delta t\right] \|v_n\|^2 + (\Delta t)^2\|\nabla F(u_n)\|^2.$$

Proof. By taking the scalar product of the second relation of (6) with $\Delta t v_n$, it comes

$$\begin{array}{l} < \dfrac{v_{n+1}-v_n}{\Delta t}, \Delta t v_n > = < -v_n - \nabla F(u_n), \Delta t v_n >, \\ < v_n, v_{n+1} > -\|v_n\|^2 = -\Delta t\|v_n\|^2 - < \nabla F(u_n), \Delta t v_n >, \\ \dfrac{1}{2}\|v_{n+1}\|^2 - \dfrac{1}{2}\|v_n\|^2 - \dfrac{1}{2}\|v_{n+1} - v_n\|^2 = -\Delta t\|v_n\|^2 - < \nabla F(u_n), u_{n+1} - u_n > . \end{array}$$

Then we obtain

$$\begin{array}{rl} & E(u_{n+1}, v_{n+1}) - E(u_n, v_n) - \dfrac{1}{2}\|v_{n+1} - v_n\|^2 \\ \leq & -\Delta t\|v_n\|^2 + F(u_{n+1}) - F(u_n) - < \nabla F(u_n), u_{n+1} - u_n > \\ \leq & -\Delta t\|v_n\|^2 + F(u_{n+1}) - F(u_n) - < \nabla F(u_{n+1}), u_{n+1} - u_n > \\ & \quad + < \nabla F(u_{n+1}) - \nabla F(u_n), u_{n+1} - u_n > . \end{array}$$

By using (7) and (10), we get

$$\begin{aligned} & E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \\ \leq\ & -\Delta t\|v_n\|^2 + \frac{1}{2}\|v_{n+1} - v_n\|^2 + \frac{L_F}{2}\|u_{n+1} - u_n\|^2 + L_F\|u_{n+1} - u_n\|^2 \\ \leq\ & -\Delta t\left[1 - \frac{3L_F}{2}\Delta t\right]\|v_n\|^2 + \frac{1}{2}\|v_{n+1} - v_n\|^2. \end{aligned}$$

Since

$$\frac{1}{2}\|v_{n+1} - v_n\|^2 = \frac{(\Delta t)^2}{2}\|v_n + \nabla F(u_n)\|^2 \leq (\Delta t)^2\|v_n\|^2 + (\Delta t)^2\|\nabla F(u_n)\|^2,$$

we deduce

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -\Delta t\left[1 - \left(1 + \frac{3L_F}{2}\right)\Delta t\right]\|v_n\|^2 + (\Delta t)^2\|\nabla F(u_n)\|^2.$$

Let $\mathcal{S} = \{a \in \mathbb{R}^d \ / \nabla F(a) = 0\}$.
Now we assume also that there exists $\theta \in (0, \frac{1}{2}]$ such that

$$\forall a \in \mathcal{S}\ \exists \delta_a > 0\ \exists \nu_a > 0 / \ \forall u \in \mathbb{R}^d :\ \|u - a\| < \delta_a \Longrightarrow \|\nabla F(u)\| \geq \nu_a |F(u) - F(a)|^{1-\theta}. \quad (11)$$

□

Proposition 5 *([15, 16, 8, 7]) Assumption* (11) *is satisfied if one of the following two cases holds:*
- F is a polynomial, or
- F is analytic and $\mathcal{S}$ is compact.

The first part of this proposition is a result of D'Acunto and Kurdyka [8]. The proof of the second part can be found in [7].

Theorem 6 *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^2 function satisfying* (7), (8) *and* (11). *Assume also that Δt is small enough. Let (u_n, v_n) be a sequence satisfying* (6). *Then (u_n, v_n) is bounded and there exists $a \in \mathcal{S}$ such that*

$$\lim_{n \to +\infty} \|v_n\| + \|u_n - a\| = 0.$$

In addition as $n \to +\infty$ we have

$$\|v_n\| + \|u_n - a\| = \begin{cases} O(e^{-cn}) & \text{for some } c > 0 \text{ if } \theta = \frac{1}{2} \\ O(n^{-\frac{\theta)}{1-2\theta}}) & \text{if } \theta \in (0, \frac{1}{2}). \end{cases} \quad (12)$$

Remark 7 We are able to give an upper bound of Δt namely

$$0 < \Delta t < \min\left(\frac{1}{4+6L_F}, \frac{1}{4+3L_F(1+\eta)}, \frac{\eta}{2+L_F\eta}\right)$$

with $\eta = \min\left(\frac{1}{2}, \frac{c_1}{2L_F^2}, \frac{1}{2L_F+1}, \frac{1}{2C}\right)$, where $C = \max\left(1, \max_{\|u\|\leq R} \|\nabla^2 F(u)\|\right)$ and R is given by (17).

Proof. Let ε be a positive real. We define for all $u, v \in \mathbb{R}^d$

$$\Phi_\varepsilon(u,v) = E(u,v) + \varepsilon < \nabla F(u), v > .$$

First of all we will prove that for ε small enough, Φ_ε will be coercive. In fact, using (7), we see that

$$\forall u \in \mathbb{R}^d \ \ \|\nabla F(u)\| \leq L_F\|u\| + \|\nabla F(0)\|,$$

and then

$$\forall u \in \mathbb{R}^d \ \ \|\nabla F(u)\|^2 \leq 2L_F^2\|u\|^2 + 2\|\nabla F(0)\|^2. \tag{13}$$

Now for all $(u,v) \in \mathbb{R}^d$, we have by Cauchy-Schwarz inequality

$$\begin{aligned}\Phi_\varepsilon(u,v) \quad &\geq \quad \frac{1}{2}\|v\|^2 + F(u) - \frac{\varepsilon}{2}\|v\|^2 - \frac{\varepsilon}{2}\|\nabla F(u)\|^2 \\ &\underset{\text{by (8) and (13)}}{\geq} \quad \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\|v\|^2 + (c_1 - \varepsilon L_F^2)\|u\|^2 - c_2 - \varepsilon\|\nabla F(0)\|^2.\end{aligned}$$

We impose to ε to be such that $\varepsilon \leq \min\left(\frac{1}{2}, \frac{c_1}{2L_F^2}\right)$. We see that with this choice, we have

$$\Phi_\varepsilon(u,v) \geq \frac{1}{4}\|v\|^2 + \frac{c_1}{2}\|u\|^2 - c_2 - \frac{1}{4}\|\nabla F(0)\|^2, \tag{14}$$

so that Φ_ε is coercive.

Next, we prove that for ϵ small enough, $(\Phi_\varepsilon(u_n, v_n))$ is nonincreasing. Let us define $x_n = (u_n, v_n)$. According to Proposition 4, for all $n \in \mathbb{N}$:

$$\begin{aligned}&\Phi_\varepsilon(x_{n+1}) - \Phi_\varepsilon(x_n) \\ \leq \ & -\Delta t\left[1 - \left(1 + \frac{3L_F}{2}\right)\Delta t\right]\|v_n\|^2 + (\Delta t)^2\|\nabla F(u_n)\|^2. \\ & +\varepsilon\left[\underbrace{< \nabla F(u_{n+1}), v_{n+1} > - < \nabla F(u_n), v_n >}_{T_n}\right].\end{aligned}$$

We have

$$\begin{aligned}
T_n &= < \nabla F(u_{n+1}) - \nabla F(u_n) + \nabla F(u_n), v_n - \Delta t v_n - \Delta t \nabla F(u_n) > - < \nabla F(u_n), v_n > \\
&= < \nabla F(u_{n+1}) - \nabla F(u_n), v_n - \Delta t v_n - \Delta t \nabla F(u_n) > - \Delta t \|\nabla F(u_n)\|^2 \\
&\quad - \Delta t < \nabla F(u_n), v_n > \\
&\leq \|\nabla F(u_{n+1}) - \nabla F(u_n)\| \|v_n - \Delta t v_n - \Delta t \nabla F(u_n)\| - \Delta t \|\nabla F(u_n)\|^2 \\
&\quad - \Delta t < \nabla F(u_n), v_n > \\
&\underset{\text{by (7)}}{\leq} L_F \|u_{n+1} - u_n\| \|v_n - \Delta t v_n - \Delta t \nabla F(u_n)\| - \Delta t \|\nabla F(u_n)\|^2 - \Delta t < \nabla F(u_n), v_n > \\
&\underset{\text{by (6)}}{\leq} L_F \Delta t \|v_n\| \|v_n - \Delta t v_n - \Delta t \nabla F(u_n)\| - \Delta t \|\nabla F(u_n)\|^2 - \Delta t < \nabla F(u_n), v_n > \\
&\leq L_F \Delta t \|v_n\|^2 + L_F (\Delta t)^2 \|v_n\|^2 + L_F (\Delta t)^2 \|v_n\| \|\nabla F(u_n)\| - \Delta t \|\nabla F(u_n)\|^2 + \\
&\quad \Delta t \|\nabla F(u_n)\| \|v_n\| \\
&\leq \Delta t \left[\left(L_F + \frac{1}{2} \right) + \frac{3}{2} L_F \Delta t \right] \|v_n\|^2 - \frac{\Delta t}{2} \left(1 - L_F \Delta t \right) \|\nabla F(u_n)\|^2 \qquad (15)
\end{aligned}$$

where we used the triangle and the Cauchy-Schwarz inequalities. Then we get

$$\begin{aligned}
&\Phi_\varepsilon(x_{n+1}) - \Phi_\varepsilon(x_n) \\
\leq\ & -\Delta t \left[1 - \left(1 + \frac{3L_F}{2} \right) \Delta t \right] \|v_n\|^2 + (\Delta t)^2 \|\nabla F(u_n)\|^2 \\
& + \varepsilon \Delta t \left[\left(L_F + \frac{1}{2} \right) + \frac{3}{2} L_F \Delta t \right] \|v_n\|^2 - \varepsilon \frac{\Delta t}{2} \left(1 - L_F \Delta t \right) \|\nabla F(u_n)\|^2 \\
\leq\ & -\Delta t \left[1 - \varepsilon \left(L_F + \frac{1}{2} \right) - \left(1 + \frac{3L_F}{2} (1 + \varepsilon) \right) \Delta t \right] \|v_n\|^2 \\
& - \Delta t \left[\frac{\varepsilon}{2} - (1 + \frac{\varepsilon}{2} L_F) \Delta t \right] \|\nabla F(u_n)\|^2
\end{aligned}$$

We choose $\varepsilon \leq \overline{\varepsilon} = \min \left(\frac{1}{2}, \frac{c_1}{2L_F^2}, \frac{1}{2L_F + 1} \right)$. Then we obtain

$$\begin{aligned}
&\Phi_\varepsilon(x_{n+1}) - \Phi_\varepsilon(x_n) \\
\leq\ & -\Delta t \left[\frac{1}{2} - \left(1 + \frac{3L_F}{2} (1 + \varepsilon) \right) \Delta t \right] \|v_n\|^2 - \Delta t \left[\frac{\varepsilon}{2} - (1 + \frac{\varepsilon}{2} L_F) \Delta t \right] \|\nabla F(u_n)\|^2.
\end{aligned}$$

We assume that Δt is small enough such that $\frac{1}{2} - \left(1 + \frac{9L_F}{4} \right) \Delta t > 0$ (then $\frac{1}{2} - \left(1 + \frac{3L_F}{2} (1 + \varepsilon) \right) \Delta t > 0$ since $\varepsilon \leq \frac{1}{2}$) and $\frac{\varepsilon}{2} - (1 + \frac{\varepsilon}{2} L_F) \Delta t > 0$.
Then there is a constant $\delta = \delta(\Delta t, L_F) > 0$ such that

$$\forall n \in \mathbb{N} \quad \Phi_\varepsilon(x_{n+1}) - \Phi_\varepsilon(x_n) \leq -\delta \left(\|v_n\|^2 + \|\nabla F(u_n)\|^2 \right). \qquad (16)$$

Then the sequence $(\Phi_\varepsilon(x_n))$ is nonincreasing. Using (14), we see that (u_n) and (v_n) are

bounded:

$$\begin{aligned}
& \Phi_\varepsilon(x_n) \leq \Phi_\varepsilon(x_0) \\
\Longrightarrow\ & \frac{1}{4}\|v_n\|^2 + \frac{c_1}{2}\|u_n\|^2 - c_2 - \frac{1}{4}\|\nabla F(0)\|^2 \leq \frac{1}{2}\|v_0\|^2 + F(u_0) + \varepsilon < \nabla F(u_0), v_0 > \\
\Longrightarrow\ & \frac{1}{4}\|v_n\|^2 + \frac{c_1}{2}\|u_n\|^2 \leq \frac{1}{2}\|v_0\|^2 + F(u_0) + \|\nabla F(u_0)\|\|v_0\| + c_2 + \frac{1}{4}\|\nabla F(0)\|^2 \\
\Longrightarrow\ & \min(\frac{1}{8}, \frac{c_1}{4})[\|v_n\| + \|u_n\|]^2 \leq \frac{1}{2}\|v_0\|^2 + F(u_0) + \|\nabla F(u_0)\|\|v_0\| + c_2 + \frac{1}{4}\|\nabla F(0)\|^2 \\
\Longrightarrow\ & \|v_n\| + \|u_n\| \leq R
\end{aligned}$$

where

$$R = \left[\frac{1}{\min(\frac{1}{8}, \frac{c_1}{4})}\left(\frac{1}{2}\|v_0\|^2 + F(u_0) + \|\nabla F(u_0)\|\|v_0\| + c_2 + \frac{1}{4}\|\nabla F(0)\|^2\right)\right]^{\frac{1}{2}}. \tag{17}$$

At this stage, it is important to note that if we choose ε smaller, (16) remains true. Note also that R is independent of Δt and ε.

Obviously we have $\sum(\|v_n\|^2 + \|\nabla F(u_n)\|^2)$ converges. Then $v_n \longrightarrow 0$ and the $\omega-$limit set $\omega((u_n)_{n\in\mathbb{N}}) := \{a \in \mathbb{R}^d : \exists n_k \to \infty / u_{n_k} \longrightarrow a\}$ is a nonempty compact connected subset of $\mathcal{S}$.

Let us show that $(x_n) = (u_n, v_n)$ satisfies (2) and (3) with the function Φ_ε. Indeed, if for some n we have $\Phi_\varepsilon(x_{n+1}) = \Phi_\varepsilon(x_n)$, then by (16) we deduce that $v_n = 0$ and $\nabla F(u_n) = 0$. Using system (6), we get that $u_{n+1} = u_n$ and $v_{n+1} = v_n$, that is $x_{n+1} = x_n$ and (3) is satisfied. On the other hand, a simple computation gives

$$\nabla\Phi_\varepsilon(u, v) = \begin{pmatrix} \nabla F(u) + \varepsilon\nabla^2 F(u)\cdot v \\ v + \varepsilon\nabla F(u) \end{pmatrix}.$$

Since (u_n) and (v_n) are bounded, there exists a constant $\eta_1 > 0$ such that

$$\forall n \in \mathbb{N} \quad \|\nabla\Phi_\varepsilon(x_n)\| \leq \eta_1[\|v_n\| + \|\nabla F(u_n)\|]. \tag{18}$$

On the other hand, there exists a constant $\eta_2 > 0$ such that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(u_{n+1} - u_n, v_{n+1} - v_n)\| \\
&= \|(\Delta t v_n, -\Delta t v_n - \Delta t\nabla F(u_n))\| \\
&\leq \eta_2[\|v_n\| + \|\nabla F(u_n)\|].
\end{aligned} \tag{19}$$

By combining (16) $-$ (18) and (19), we get that $(\Phi_{\bar{\varepsilon}}(x_n))$ satisfies (2) with $\sigma = \frac{\delta}{2\eta_1\eta_2}$. Now we will show that under the hypothesis (11) on F, Φ_ε satisfies (4).

From the remark 2, let $B = \{(u,v) \in \mathbb{R}^d \times \mathbb{R}^d,\ \|u\| + \|v\| \leq R\}$ where R is given by (17). Clearly B contains the sequence (u_n, v_n). Now, for all $(u,v) \in B$

$$\begin{aligned} & \|\nabla \Phi_\varepsilon(u,v)\| \\ = \ & \|\nabla F(u) + \varepsilon \nabla^2 F(u) \cdot v\| + \|v + \varepsilon \nabla F(u)\| \\ \geq \ & \|\nabla F(u)\| - \varepsilon \|\nabla^2 F(u) \cdot v\| + \|v\| - \varepsilon \|\nabla F(u)\| \\ \geq \ & (1 - \varepsilon \max_{\|u\| \leq R} \|\nabla^2 F(u)\|)\|v\| + (1-\varepsilon)\|\nabla F(u)\| \\ \geq \ & (1 - \varepsilon C)[\|v\| + \|\nabla F(u)\|] \end{aligned}$$

where $C = \max\left(1, \max_{\|u\| \leq R} \|\nabla^2 F(u)\|\right)$.
By possibly taking $\varepsilon > 0$ smaller ($\varepsilon = \min(\overline{\varepsilon}, \frac{1}{2C})$), there exists $\rho > 0$ such that

$$\forall (u,v) \in B \qquad \|\nabla \Phi_\varepsilon(u,v)\| \geq \rho[\|v\| + \|\nabla F(u)\|]. \tag{20}$$

If (a,b) is not a critical point of Φ_ε, then Φ_ε satisfies (4) with $\theta = \frac{1}{2}$ as best exponent, thanks to the continuity of Φ_ε.
Let $(a,b) \in B$ be a critical point of Φ_ε. Then $\nabla F(a) = 0$ and $b = 0$. From (11)

$$\exists \delta_a > 0\ \exists \nu_a > 0 /\ \forall u \in \mathbb{R}^d :\ \|u - a\| < \delta_a \Longrightarrow \|\nabla F(u)\| \geq \nu_a |F(u) - F(a)|^{1-\theta}. \tag{21}$$

On the other hand, by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} [\Phi_\varepsilon(u,v) - \Phi_\varepsilon(a,0)]^{1-\theta} &= \left[\tfrac{1}{2}\|v\|^2 + F(u) - F(a) + \varepsilon < \nabla F(u), v > \right]^{1-\theta} \\ &\leq \|v\|^{2(1-\theta)} + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\|^{1-\theta}\|v\|^{1-\theta}. \end{aligned} \tag{22}$$

Thanks to Young's inequality we obtain

$$\|\nabla F(u)\|^{1-\theta}\|v\|^{1-\theta} \leq \|\nabla F(u)\| + \|v\|^{\frac{1-\theta}{\theta}}.$$

Then (22) becomes

$$[\Phi_\varepsilon(u,v) - \Phi_\varepsilon(a,0)]^{1-\theta} \leq \|v\|^{2(1-\theta)} + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\| + \|v\|^{\frac{1-\theta}{\theta - \alpha(1-\theta)}}.$$

Since $2(1-\theta)$ and $\frac{1-\theta}{\theta}$ are bigger then 1, using also (21), we get for all $(u,v) \in B$ with $\|v\| \leq 1$ and $\|u - a\| < \delta_a$

$$\begin{aligned} [\Phi_\varepsilon(u,v) - \Phi_\varepsilon(a,0)]^{1-\theta} &\leq \|v\| + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\| + \|v\| \\ &\leq \left(2 + \frac{1}{\nu_a}\right)[\|\nabla F(u)\| + \|v\|] \\ &\leq \frac{1}{\rho}\left(2 + \frac{1}{\nu_a}\right)\|\nabla \Phi_\varepsilon(u,v)\| \quad \text{by (20)}. \end{aligned}$$

Therefore Φ_ε satisfies (4). Theorem 1 is proved. □

3 A convergence result in the one dimensional case

We consider the case of a strong nonlinear damping ($0 \leq \alpha < 1$) in the one-dimensional situation ($d = 1$). This time, we use an implicit discretization of (1). The proof is adapted from the continuous case. We recall it for the reader's convenience.

3.1 The continuous version

Let $u \in W^{1,\infty}(\mathbb{R})$ be a bounded solution of the ODE

$$u'' + |u'|^\alpha u' + f(u) = 0,$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$. We know that (see, e.g., [11])

$$\omega(u, u') \subset f^{-1}(\{0\}) \times \{0\},$$

where $\omega(u, u')$ is the ω-limit set of the solution defined by

$$\omega(u, u') = \left\{(u^\star, v^\star) \in \mathbb{R}^2,\ \exists t_n \to +\infty,\ (u(t_n), u'(t_n)) \to (u^\star, v^\star) \text{ as } n \to +\infty\right\}.$$

Theorem 8 ([10, 11]) *There exists $a \in f^{-1}(\{0\})$ such that*

$$\lim_{t\to\infty} |u'(t)| + |u(t) - a| = 0.$$

Proof. For the sake of completeness, we give the proof here. If $\omega(u, u')$ is not reduced to a single point, then there are $a, b \in f^{-1}(\{0\})$ $(a < b)$ such that $\omega(u, u') = [a, b] \times \{0\}$. Let $c = \frac{a+b}{2}$. Since $\lim_{t\to\infty} u'(t) = 0$ and $(c, 0) \in \omega(u, u')$, then there exists $T > 0$ such that

$$\begin{aligned} &|u(T) - c| < \frac{b-a}{8}, \\ &\frac{|u'(t)|^{1-\alpha}}{1-\alpha} < \frac{b-a}{8} \quad \forall t \geq T. \end{aligned}$$

We will prove that

$$\forall t \geq T \quad u(t) \in [a, b].$$

Assume that this is not the case and let

$$\theta = \inf\{t \geq T,\ u(t) \notin [a, b]\}.$$

Clearly we have $\theta < \infty$ and

$$\forall t \in [T, \theta] \quad u(t) \in [a, b].$$

Then

$$\forall t \in [T, \theta] \quad u''(t) + |u'(t)|^\alpha u'(t) = 0.$$

If there is a $s \in [T, \theta]$ such that $u'(s) = 0$, then $u'(t) = 0$ for all $t \in [T, \theta]$, in this case we conclude easily. Otherwise, solving this ODE, we get

$$\forall t \in [T, \theta] \quad |u'(t)| = \frac{1}{[|u'(T)|^{-\alpha} + \alpha(t-T)]^{\frac{1}{\alpha}}}. \tag{23}$$

$$\begin{aligned} |u(\theta) - u(T)| &\leq \int_T^\theta |u'(t)|\, dt \\ &\leq \int_T^\theta \frac{dt}{[|u'(T)|^{-\alpha} + \alpha(t-T)]^{\frac{1}{\alpha}}} \\ &\leq \frac{|u'(T)|^{1-\alpha}}{1-\alpha} < \frac{b-a}{8}. \end{aligned}$$

$$\begin{aligned} |u(\theta) - c| &\leq |u(\theta) - u(T)| + |u(T) - c| \\ &< \frac{b-a}{8} + \frac{b-a}{8} = \frac{b-a}{4}, \end{aligned}$$

which contradicts the definition of θ and then $\theta = \infty$.
Now from (23), we deduce that $u' \sim \frac{1}{\alpha^{\frac{1}{\alpha}} t^{\frac{1}{\alpha}}}$, and then $u' \in L^1(\mathbb{R}_+)$. This proves that u converges and the ω-limit set $\omega(u, u')$ is reduced to a single point. □

3.2 The discrete version

The scheme reads:

$$\begin{cases} \dfrac{u_{n+1} - u_n}{\Delta t} = v_{n+1} \\ \dfrac{v_{n+1} - v_n}{\Delta t} = -|v_{n+1}|^\alpha v_{n+1} - f(u_{n+1}) \\ u_0,\ v_0 \in \mathbb{R} \end{cases} \tag{24}$$

and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that

$$\exists c_F > 0 / \ \forall u, v \in \mathbb{R}^d \quad (f(u) - f(v))(u - v) \geq -c_F |u - v|^{\alpha+2}, \tag{25}$$

$$\exists L_F > 0 / \ \forall u, v \in \mathbb{R}^d \quad |f(u) - f(v)| \leq L_F \|u - v\|. \tag{26}$$

Let $\mathcal{S} = \{a \in \mathbb{R}^d \ / \ f(a) = 0\}$ and F be an antiderivative of f. We define

$$E(u, v) = \frac{1}{2}|v|^2 + F(u).$$

We recall the following results.

Proposition 9 *[13] Assume that F is of class $C^1(\mathbb{R}^d)$, coercive and that (25) and (26) hold, then for any $(u_0, v_0) \in \mathbb{R}^{2d}$, provided Δt is small enough, the sequence (u_n, v_n) given by (24) is well defined, and we have*

$$\forall n \in \mathbb{N} \quad E(u_{n+1}, v_{n+1}) \leq E(u_n, v_n). \tag{27}$$

Moreover we have

$$E(u_{n+1}, v_{n+1}) - E(u_n, v_n) \leq -\Delta t \left[1 - \frac{c_F}{2}(\Delta t)^{\alpha+1}\right] |v_{n+1}|^{\alpha+2}.$$

Corollary 10 *[13] Let F satisfy (25) and assume that $0 < \Delta t < \left(\frac{2}{c_F}\right)^{\frac{1}{\alpha+1}}$. Let (u_n, v_n) be a sequence satisfying (24). If (u_n) is bounded, then $\lim_{n\to\infty} E(u_n, v_n)$ exists and $v_n \longrightarrow 0$. Moreover the $\omega-$limit set $\omega((u_n)_{n\in\mathbb{N}})$ given by*

$$\omega((u_n)_{n\in\mathbb{N}}) = \{a \in \mathbb{R} : \exists n_k \to \infty / u_{n_k} \longrightarrow a\}$$

is a nonempty compact connected subset of $\mathcal{S}$.

The main result of this paper is the following

Theorem 11 *Under the assumptions of the corollary 10, there exists $a \in \mathcal{S}$ such that $u^n \longrightarrow a$.*

Proof. As in the continuous case, by connectedness we have either $\omega((u_n)_{n\in\mathbb{N}}) = \{a\} \times \{0\}$ for some $a \in f^{-1}(\{0\})$ and the result is established, or $\omega((u_n)_{n\in\mathbb{N}}) = [a, b] \times \{0\}$ for some $a, b \in f^{-1}(\{0\})$ $(a < b)$. In this case, let $c = \frac{a+b}{2}$. Using corollary 10, we can choose n_0 large enough such that

$$|u_{n_0} - c| < \frac{b-a}{8}, \tag{28}$$

$$\Delta t|v^{n_0}| + \frac{2^{\alpha+1}}{1-\alpha}|v^{n_0}| < \frac{b-a}{8} \quad \forall n \geq n_0, \tag{29}$$

$$\Delta t|v_n|^\alpha \leq 1, \quad \forall n \geq n_0. \tag{30}$$

Assume by contradiction that the set $\{n \geq n_0 \,:\, u^{n+1} \notin [a, b]\}$ is nonempty and let

$$n_1 = \inf\left\{n \geq n_0 \,:\, u^{n+1} \notin [a, b]\right\}.$$

Note that $n_1 > n_0 + 2$. In fact by (29), we have $\Delta t|v_{n_0+1}| < \frac{b-a}{8}$ and $\Delta t|v_{n_0+2}| < \frac{b-a}{8}$. Using (24) and tringle inequality, we find

$$\begin{aligned}
|u_{n_0+1} - c| &\leq |u_{n_0+1} - u_{n_0}| + |u_{n_0} - c| \\
&= \Delta t|v_{n_0+1}| + |u_{n_0} - c| < \frac{b-a}{8} + \frac{b-a}{8} = \frac{b-a}{4}; \\
|u_{n_0+2} - c| &\leq |u_{n_0+2} - u_{n_0+1}| + |u_{n_0+1} - c| \\
&= \Delta t|v_{n_0+2}| + |u_{n_0} - c| < \frac{b-a}{8} + \frac{b-a}{4} = 3\frac{b-a}{8}.
\end{aligned}$$

It is clear that

$$\forall n \in \{n_0, \cdots, n_1 - 1\},\ u^{n+1} \in [a, b]$$

and then $f(u^{n+1}) = 0$ since $[a, b] \subset \mathcal{S}$. Using (24), we get

$$\forall n \in \{n_0, \cdots, n_1 - 1\},\ \frac{v_{n+1} - v_n}{\Delta t} = -|v_{n+1}|^\alpha v_{n+1}.$$

which can also be written

$$\forall n \in \{n_0, \cdots, n_1 - 1\},\ v_{n+1}(1 + \Delta t|v_{n+1}|^\alpha) = v_n.$$

Obviously we have

$$\forall n \in \{n_0, \cdots, n_1 - 1\},\ |v_n| - |v_{n+1}| = \Delta t|v_{n+1}|^{\alpha+1} \geq 0. \tag{31}$$

Now assume that there exists $p \in \{n_0, \cdots, n_1 - 1\}$ such that $v^p = 0$. Then for all $n \in \{n_0, \cdots, n_1 - 1\}$, $v^n = 0$. In this case $n_1 = \infty$ and (u^n) is a constant sequence.
Otherwise for all $n \in \{n_0, \cdots, n_1 - 1\}$, $v^n \neq 0$. Now from (30) and (31) we deduce that

$$\forall n \in \{n_0, \cdots, n_1 - 1\},\ |v_n| \leq 2|v_{n+1}|. \tag{32}$$

Then we get from (31)

$$\begin{aligned}
\Delta t &= \frac{|v_n| - |v_{n+1}|}{|v_{n+1}|^{\alpha+1}} \\
&\underset{\text{by (31) and (32)}}{\leq} 2^{\alpha+1} \frac{|v_n| - |v_{n+1}|}{|v_n|^{\alpha+1}} \\
&\leq 2^{\alpha+1} \int_{|v_{n+1}|}^{|v_n|} \frac{dt}{t^{\alpha+1}} \\
&= \frac{2^{\alpha+1}}{\alpha} \left[-|v^n|^{-\alpha} + |v^{n+1}|^{-\alpha} \right].
\end{aligned}$$

Summing from n_0 to $n \in \{n_0, \cdots, n_1 - 1\}$, we obtain

$$\frac{\alpha}{2^{\alpha+1}} \Delta t(n - n_0) \leq |v^{n+1}|^{-\alpha} - |v^{n_0}|^{-\alpha}$$

or

$$|v^{n+1}| \leq \frac{1}{\left[|v^{n_0}|^{-\alpha} + \frac{\alpha}{2^{\alpha+1}} \Delta t(n - n_0) \right]^{\frac{1}{\alpha}}}. \tag{33}$$

Now let $n \in \{n_0, \cdots, n_1 - 1\}$, we have

$$\begin{aligned}
|u^{n+1} - u^{n_0}| &\leq \sum_{k=n_0}^{n} |u^{k+1} - u^k| \\
&\underset{\text{by (24)}}{=} \Delta t \sum_{k=n_0}^{n} |v^{k+1}| \\
&\underset{\text{by (33)}}{\leq} \Delta t \sum_{k=n_0}^{n} \frac{1}{\left[|v^{n_0}|^{-\alpha} + \frac{\alpha}{2^{\alpha+1}} \Delta t (k - n_0)\right]^{\frac{1}{\alpha}}} \\
&\leq \Delta t |v^{n_0}| + \Delta t \int_{n_0}^{n} \frac{dx}{\left[|v^{n_0}|^{-\alpha} + \frac{\alpha}{2^{\alpha+1}} \Delta t (x - n_0)\right]^{\frac{1}{\alpha}}} \\
&\leq \Delta t |v^{n_0}| + \frac{2^{\alpha+1}}{1-\alpha} |v^{n_0}| \\
&\underset{\text{by (29)}}{<} \frac{b-a}{8}.
\end{aligned} \tag{34}$$

In particular with $n = n_1 - 1$, we get $|u^{n_1} - u^{n_0}| < \frac{b-a}{8}$. By the triangle inequality we deduce

$$\begin{aligned}
|u^{n_1} - c| &\leq |u^{n_1} - u^{n_0}| + |u^{n_0} - c| \\
&\underset{\text{by (28)}}{<} \frac{b-a}{8} + \frac{b-a}{8} = \frac{b-a}{4},
\end{aligned}$$

and this contradicts $u^{n_1} \notin [a, b]$. Then the set $\{n \geq n_0 \,:\, u^{n+1} \notin [a, b]\}$ is empty and $n_1 = \infty$. Now (34) holds for all $n \geq n_0$:

$$|u^{n+1} - u^{n_0}| < \frac{b-a}{4}$$

and by triangle inequality, we get

$$\forall n \geq n_0 \quad |u^{n+1} - c| \leq |u^{n+1} - u^{n_0}| + |u^{n_0} - c| < \frac{b-a}{4} + \frac{b-a}{8} = \frac{3(b-a)}{8}.$$

By choosing a subsequence such that $u^{n_k} \longrightarrow b$, we obtain

$$|b - c| \leq \frac{3(b-a)}{8},$$

a contradiction. We conclude that $\omega((u^n))$ is a reduced to a single point u^* and that $u^n \longrightarrow u^*$. □

4 A non convergence result

We adapt to the discrete case a result from [10, 11]. Let $\Delta t > 0$ denote the time step. We consider a sequence defined by

$$\left\{\begin{array}{l} \dfrac{u_{n+1}-u_n}{\Delta t} = v_{n+1}, \\ \dfrac{v_{n+1}-v_n}{\Delta t} = -|v_{n+1}|v_{n+1} - f(u_{n+1}), \\ \quad u_0,\ v_0 \in \mathbb{R}, \end{array}\right. \tag{35}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally Lipschitz continuous and satisfies

$$f \text{ is nondecreasing}, \tag{36}$$
$$f(s) < 0, \quad \forall s < a, \tag{37}$$
$$f(s) = 0, \quad \forall s \in [a, b], \tag{38}$$
$$f(s) > 0, \quad \forall s > b. \tag{39}$$

Here a, $b \in \mathbb{R}$ are such that $a < b$.

We denote by F an antiderivative of f. The assumptions on f imply that F is convex and that there exist two positive constants c_1, c_2 such that

$$F(s) \geq c_1|s| - c_2, \quad \forall s \in \mathbb{R}. \tag{40}$$

Since F is convex, we have

$$F(u) - F(\tilde{u}) \geq f(\tilde{u})(u - \tilde{u}), \quad \forall u, \tilde{u} \in \mathbb{R}. \tag{41}$$

Since F is coercive, we have:

Proposition 12 *For every $(u_0, v_0) \in \mathbb{R}^2$, there exists at least one sequence (u_n, v_n) which complies with* (35)*.*

Proof. Assume by induction that for some $n \in \mathbb{N}$, the values (u_n, v_n) are defined in $\mathbb{R}^2$. We consider the function

$$G_n : u \mapsto \frac{(u-u_n)^2}{2\Delta t^2} - \frac{uv_n}{\Delta t} + \frac{|u-u_n|^3}{3\Delta t^2} + F(u).$$

It is clear that G_n is continuous on $\mathbb{R}$ and $G_n(u) \to +\infty$ as $|u| \to +\infty$ (cf. (40)). Thus, G_n has at least one minimizer $u^\star$ in $\mathbb{R}$, which solves $G_n'(u^\star) = 0$, that is

$$\frac{u^\star - u_n}{\Delta t^2} - \frac{v_n}{\Delta t} + \frac{|u^\star - u_n|(u^\star - u_n)}{\Delta t^2} + f(u^\star) = 0.$$

We set $u_{n+1} = u^\star$ and $v_{n+1} = (u_{n+1} - u_n)/\Delta t$. We see that u_{n+1}, v_{n+1} complies with (35) and we proceed by induction. □

Remark 13 *Let $(u_0, v_0) \in \mathbb{R}^2$. By a fixed point argument, it is possible to prove that if Δt is small enough, there exists a unique sequence (u_n, v_n) which complies with* (35) *(see, e.g., [13]).*

Theorem 14 *Assume that f satisfies* (36)-(39). *For every non stationary sequence (u_n, v_n) which complies with* (35), *there exist a subsequence (u_{n_k}, v_{n_k}) such that $u_{n_k} < a$ for all $k \in \mathbb{N}$ and a subsequence $(u_{n_{k'}}, v_{n_{k'}})$ such that $u_{n_{k'}} > b$ for all $k' \in \mathbb{N}$.*

Proof. We multiply the second equation in (35) by $\Delta t v_{n+1}$. We find

$$(v_{n+1} - v_n)v_{n+1} + \Delta t|v_{n+1}|^3 = -f(u_{n+1})(u_{n+1} - u_n).$$

By (41),

$$\frac{1}{2}|v_{n+1}|^2 - \frac{1}{2}|v_n|^2 + \frac{1}{2}|v_{n+1} - v_n|^2 + \Delta t|v_{n+1}|^3 \leq F(u_n) - F(u_{n+1}).$$

Thus,

$$\Delta t|v_{n+1}|^3 \leq E(u_n, v_n) - E(u_{n+1}, v_{n+1}), \tag{42}$$

where

$$E(u, v) := \frac{1}{2}|v|^2 + F(u).$$

The sequence $(E_n) = (E(u_n, v_n))$ is nonincreasing and since $E(u, v)$ is coercive by (40), the sequence (u_n, v_n) is bounded. Morevover, (E_n) has a limit as n tends to $+\infty$, so $|v_{n+1}| \to 0$ by (42).

We assume that $u_n \geq a$ for al $n \geq n_0$, for some $n_0 \in \mathbb{N}$. We must prove that (u_n) is stationary.

- If $v_n \geq 0$ for all $n \geq n_0$, then (u_n) is nondecreasing and since (u_n) is bounded, it converges to some value $c \in [a, b]$. In particular, $u_n \in [a, b]$ for all $n \geq n_0$ and the second equation in (35) becomes

$$\frac{v_{n+1} - v_n}{\Delta t} = -|v_{n+1}|v_{n+1} \iff v_{n+1}(1 + \Delta t|v_{n+1}|) = v_n. \tag{43}$$

  If $v_n = 0$ for all $n \geq n_0$, then (u_n) is stationary. Otherwise, there exists $n_1 \geq n_0$ such that $v_{n_1} > 0$. Equation (43) shows that $v_{n_1+1} > 0$ and $v_{n_1} > v_{n_1+1}$ and by induction, $v_n > v_{n+1} > 0$ for all $n \geq n_1$. We have

$$\frac{1}{v_{n+1}} - \frac{1}{v_n} = \frac{v_n - v_{n+1}}{v_n v_{n+1}} \leq \frac{v_n - v_{n+1}}{v_{n+1}^2} = \Delta t, \quad \forall n \geq n_1.$$

  On summing this inequality, we find

$$\sum_{k=n_1}^{n} \left(\frac{1}{v_{k+1}} - \frac{1}{v_k} \right) \leq (n - n_1 + 1)\Delta t,$$

that is

$$\frac{1}{v_{n+1}} - \frac{1}{v_{n_1}} \leq (n - n_1 + 1)\Delta t.$$

Thus,

$$v_{n+1} \geq \frac{1}{(n - n_1 + 1)\Delta t + 1/v_{n_1}},$$

and so

$$u_{n+1} - u_n \geq \frac{\Delta t}{(n - n_1 + 1)\Delta t + 1/v_{n_1}}, \quad \forall n \geq n_1.$$

Thus, the series $\sum_{n=n_1}^{\infty}(u_{n+1} - u_n)$ diverges and (u_n) tends to $+\infty$. This is absurd since (u_n) is bounded.

- If there exists $n_1 \geq n_0$ such that $v_{n_1} < 0$, the second equation in (35) yields

$$\frac{v_{n+1} - v_n}{\Delta t} + |v_{n+1}|v_{n+1} = -f(u_{n+1}) \leq 0$$

that is

$$v_{n+1}(1 + \Delta t|v_{n+1}|) = v_n - \Delta t f(u_{n+1}), \quad \forall n \geq n_1.$$

In particular, $v_{n_1+1} < 0$ and by induction, $v_n < 0$ for all $n \geq n_1$. Moreover,

$$|v_{n+1}|(1 + \Delta t|v_{n+1}|) \geq |v_n|, \quad \forall n \geq n_1.$$

If $|v_n| \geq |v_{n+1}|$, then

$$\frac{1}{|v_{n+1}|} - \frac{1}{|v_n|} = \frac{|v_n| - |v_{n+1}|}{|v_n||v_{n+1}|} \leq \Delta t.$$

If $|v_n| < |v_{n+1}|$, then

$$\frac{1}{|v_{n+1}|} - \frac{1}{|v_n|} < 0.$$

In both cases, we have

$$\frac{1}{|v_{n+1}|} - \frac{1}{|v_n|} \leq \Delta t, \quad \forall n \geq n_1.$$

By summing on n, we find as previously that

$$|v_{n+1}| \geq \frac{1}{(n - n_1 + 1)\Delta t + 1/|v_{n_1}|},$$

Thus,

$$-(u_{n+1} - u_n) \geq \frac{\Delta t}{(n - n_1 + 1)\Delta t + 1/|v_{n_1}|}, \quad \forall n \geq n_1.$$

This shows that $\sum_{n=n_1}^{\infty}(u_{n+1} - u_n)$ diverges and so (u_n) converges to $-\infty$. Again, we obtain a contradiction since (u_n) is bounded.

This proves the existence of the subsequence (u_{n_k}). The existence of the subsequence $(u_{n_{k'}})$ follows similarly by reversing all signs. □

Remark 15 *The oscillations described in Theorem 14 occur even if F satisfies a Lojasiewicz inequality. Indeed, we may consider the function f defined by*

$$f(s) = \begin{cases} s-a & \forall s < a, \\ 0 & \forall s \in [a,b], \\ s-b & \forall s > b. \end{cases}$$

A Taylor expansion near a and b shows that F satisfies the Lojasiewicz inequality with optimal exponent $\theta = 1/2$.

5 Numerical simulations

We have performed numerical simulations with the `Scilab` software.

5.1 Slow oscillations for a weak damping

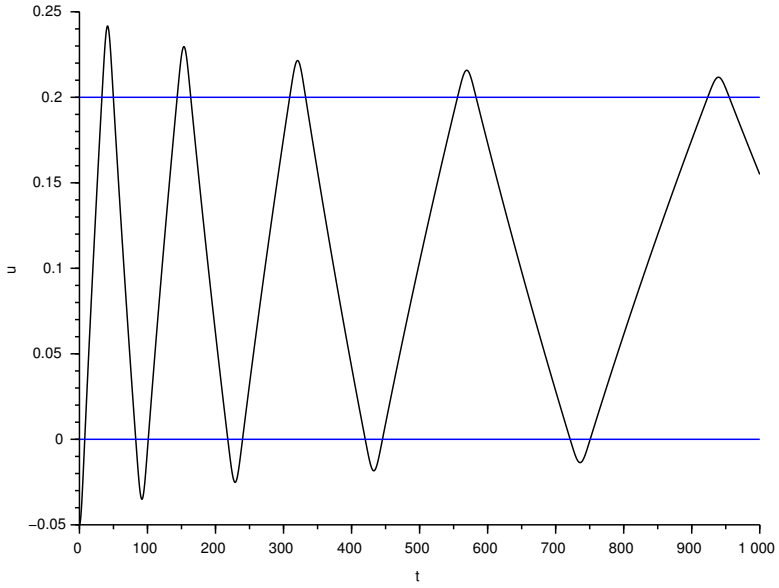

Figure 1: An oscillatory sequence (u_n) (in black) versus time with f defined by (44). The two horizontal blue lines represent the values of a and b.

We first consider a sequence defined by (35) (the case $\alpha = 1$) with f defined by

$$f(s) = \begin{cases} -(a-s)^2 & \forall s < a, \\ 0 & \forall s \in [a,b], \\ (s-b)^2 & \forall s > b. \end{cases} \tag{44}$$

and $a = 0$, $b = 0.2$. The initial datum is $(u_0, v_0) = (-0.5, 0)$. The time step is $\Delta t = 0.01$ and we have computed 100 000 iterations so that the final time is 1000.

Figure 1 shows the oscillatory behaviour described in Theorem 14. This is described as *slow oscillations in the case of a weak damping* in [10].

5.2 Convergence to equilibrium for a "strong" damping

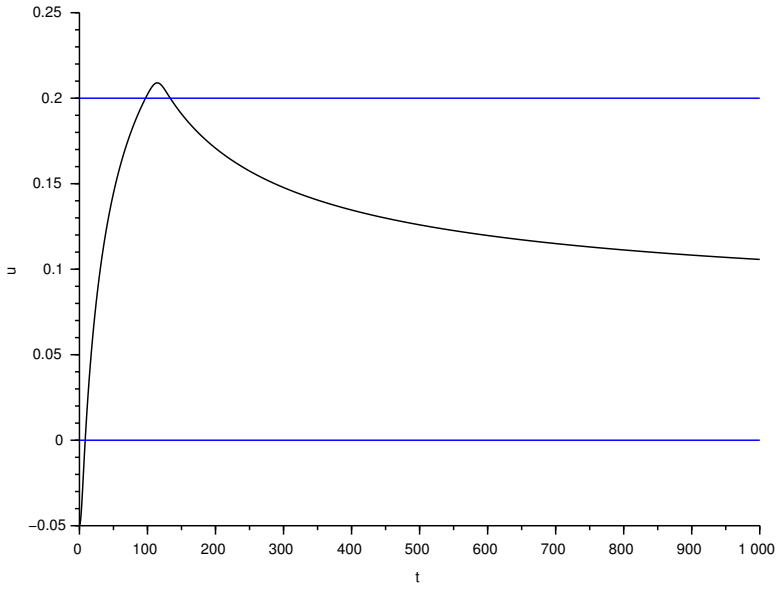

Figure 2: A convergent sequence (u_n) (in black) versus time with f defined by (44) in the case $\alpha = 0.6$. The two horizontal blue lines represent the values of a and b.

Next, we consider a sequence defined by (24) with $\alpha = 0.6$. The function f and the initial condition are the same as previously. The time step is $\Delta t = 0.01$ and we have computed 100 000 iterations. Figure 2 illustrates the convergence of the sequence proved in Theorem 11.

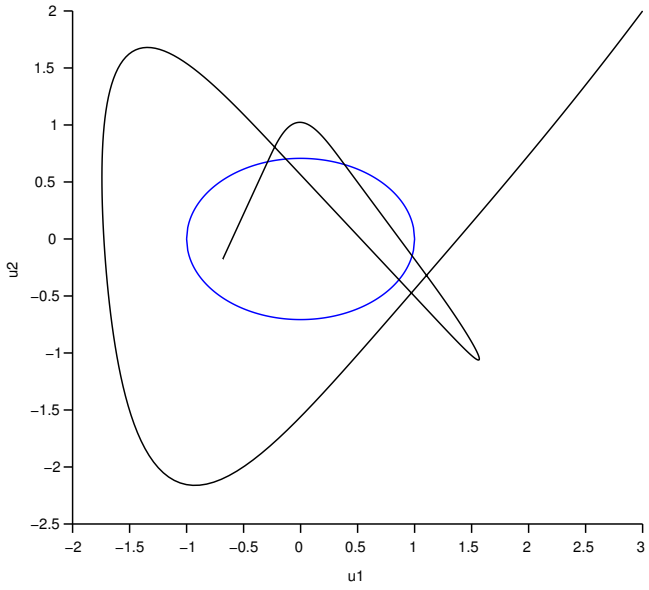

Figure 3: Behaviour of the sequence $(u^n) = (u_1^n, u_2^n)$ for the explicit scheme (6) with f defined by (45). The starting point is $u^0 = (3, 2)$. The blue line is the ellipse (46).

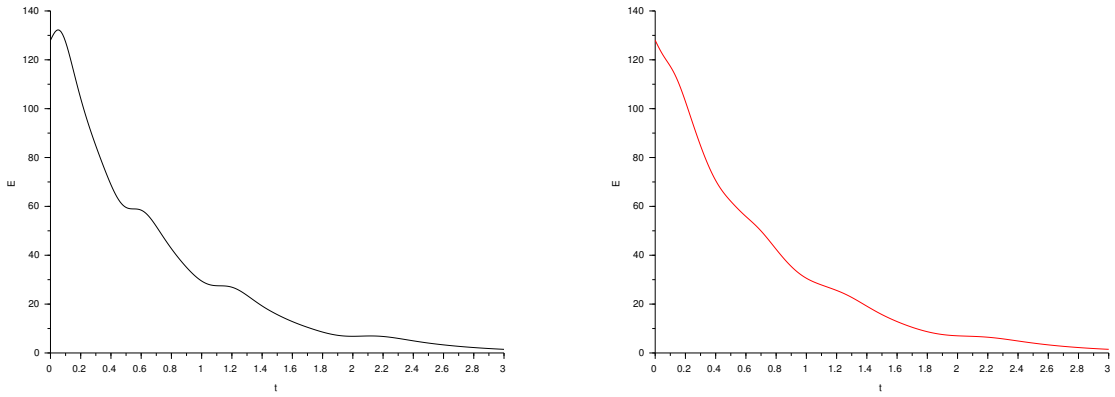

Figure 4: Energy vs. time (left) and modified energy vs. time (right).

5.3 Convergence to equilibrium for a system

We consider the sequence (u_n, v_n) defined by the explicit scheme (6). The function $F : \mathbb{R}^2 \to \mathbb{R}$ is

$$F(x, y) = \begin{cases} \frac{1}{2}(x^2 + 2y^2 - 1)^2 & \text{if } x^2 + 2y^2 - 1 \geq 0, \\ 0 & \text{if } x^2 + 2y^2 - 1 < 0. \end{cases} \tag{45}$$

We note that F is a convex function and its set of critical point is the ellipse

$$\mathcal{S} = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + 2y^2 - 1 \leq 0\right\}. \tag{46}$$

Moreover, F satisfies the Lojasiewicz inequality since it is a polynomial outside $\mathcal{S}$.

Figure 3 illustrates the convergence of the sequence (u^n) to a critical point which belongs to the interior of $\mathcal{S}$. The initial condition is $u^0 = (3, 2)$, $v^0 = (0, 0)$, the time step is $\Delta t = 0.01$ and we have performed 1000 iterations. The value of u^n is approximately $(-0.68, -0.17)$.

Figure 4 (left) shows the energy E^n versus time $t_n = n\Delta t$ over the first 300 iterations. We see especially at the beginning that E^n is not monotonic. In contrast, the modified energy Φ_ϵ shown in Figure 4 (right) is decreasing. The value chosen for ϵ is 0.01.

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