Orthogonality of invariant measures for weighted shifts

Sophie Grivaux, Etienne Matheron, Quentin Menet

ORTHOGONALITY OF INVARIANT MEASURES FOR WEIGHTED SHIFTS

by

Sophie Grivaux, Étienne Matheron & Quentin Menet

Abstract. — We introduce and study the notion of *orthogonality* for two operators in the context of weighted backward shifts on $\ell_p(\mathbb{Z}_+)$, $1 \leqslant p < \infty$. Two continuous linear operators T_1 and T_2 acting on a Polish topological vector space X are said to be orthogonal if any two Borel probability measures m_1 and m_2 on X which are respectively T_1 - invariant and T_2 - invariant and satisfy $m_1(\{0\}) = m_2(\{0\}) = 0$ must be orthogonal. In this note, we provide several conditions on the weights $\boldsymbol{u}$ and $\boldsymbol{v}$ implying orthogonality or non-orthogonality of the associated weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$, and we investigate in some detail the case where the invariant measures are product measures.

To Gilles Godefroy, with affection and admiration

1. Introduction

1.1. Weighted shifts. — This note is a contribution to the study of the dynamics of weighted backward shifts acting on ℓ_p - spaces, $1 \leqslant p < \infty$. More specifically, we will be concerned with *invariant measures* for such weighted shifts.

Let X be one of the (real or complex) Banach spaces $\ell_p = \ell_p(\mathbb{Z}_+)$, $1 \leqslant p < \infty$, and denote by $(e_n)_{n\geqslant 0}$ the canonical basis of X. Let $\boldsymbol{w} = (w_n)_{n\geqslant 1}$ be a *weight sequence*, *i.e.* a bounded sequence of non-zero scalars.The backward shift associated with $\boldsymbol{w}$ is the operator $B_{\boldsymbol{w}} : X \to X$ defined by $B_{\boldsymbol{w}}e_0 := 0$ and $B_{\boldsymbol{w}}e_n := w_n e_{n-1}$ for $n \geqslant 1$. Weighted shift operators occupy a central place in operator theory. On the one hand, their explicit form permits a systematic study, so that they are natural "test" operators for any question that comes to mind. On the other hand, their theory is quite rich, and their behaviour remains in some aspects rather mysterious. We refer the reader to the classical paper [22], which proposes a systematic study of weighted shifts including their representations as multiplication operators on certain Banach spaces of holomorphic functions and their spectral properties, and to the authoritative book [18], which deals with the study of cyclic vectors for the unweighted backward shift B on $\ell_2(\mathbb{Z}_+)$.

Many important dynamical properties are completely characterized for weighted shifts $B_{\boldsymbol{w}}$ and can be expressed in a simple way in terms of the weights w_n, see e.g [19], [20],

2020 ***Mathematics Subject Classification.*** — 47B01, 47A16, 28C20, 28A35.
Key words and phrases. — Weighted shifts, invariant measures, product measures, ℓ_p - spaces.

The third author is a Research Associate of the Fonds de la Recherche Scientifique - FNRS.

[12], [7], [8]. For instance, $B_{\boldsymbol{w}}$ is *hypercyclic*, *i.e.* it admits a vector with a dense orbit, if and only if $\overline{\lim}_{n\to\infty} |w_1 \cdots w_n| = \infty$, and it is *topologically mixing* if and only if $|w_1 \cdots w_n| \to \infty$ as $n \to \infty$. See [16] for proofs of these results as well as extensions to a large class of sequence spaces. Observe that these characterizations are the same on any ℓ_p. By contrast, *chaoticity* of a weighted shift on $X = \ell_p$ depends explicitely on p (recall that an operator is chaotic if it is hypercyclic with a dense set of periodic points): $B_{\boldsymbol{w}}$ is chaotic on ℓ_p if and only $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$. For the class of weighted shift operators of ℓ_p, chaos is an especially important notion since it turns out to be equivalent to strong dynamical properties such as *frequent hypercyclicity* (the existence of a vector whose orbit visits each non-empty open set along a set of integers having positive lower density) or *$\mathcal{U}$-frequent hypercyclicity* (same as frequent hypercyclicity with lower density replaced by upper density). Indeed, by an important result of Bayart and Ruzsa [6], frequent hypercyclicity and $\mathcal{U}$-frequent hypercyclicity of a weighted shift $B_{\boldsymbol{w}}$ on ℓ_p are both equivalent to the condition $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$. We refer the reader to the books [4] and [16] for more on frequent hypercyclicity and related questions, and to the paper [11] for a study of the relation between chaos and frequent hypercyclicity for weighted shifts on a large class of Fréchet sequence spaces.

The condition $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$ also turns out to be of special importance in the study of invariant measures for the weighted backward shift $B_{\boldsymbol{w}}$. Recall that a Borel probability measure m on X is said to be $B_{\boldsymbol{w}}$-invariant if $m\big(B_{\boldsymbol{w}}^{-1}(A)\big) = m(A)$ for every Borel set $A \subseteq X$. Under the condition $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$, the operator $B_{\boldsymbol{w}}$ acting on $X = \ell_p$ admits plenty of invariant measures, of quite different kinds: discrete measures associated with periodic points, Gaussian measures with full support (with respect to some of which $B_{\boldsymbol{w}}$ is ergodic or even strongly mixing), continuous measures with full support which are very far from being Gaussian, ... See [2] or [4, Chapter 5] for more information on (shift-invariant) Gaussian measures; and [15], which testifies of the richness of the class of invariant measures for backward shifts by showing that under the condition $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$, the backward shift $B_{\boldsymbol{w}}$ is *universal for ergodic system* in the sense of Glasner and Weiss [14]: for every ergodic transformation T on a standard Lebesgue probability space $(Z, \mathcal{B}, \mu)$, there exists a $B_{\boldsymbol{w}}$-invariant Borel probability measure m on $X = \ell_p$ with full support such that the two dynamical systems $(Z, \mathcal{B}, \mu; T)$ and $(X, \mathcal{B}_X, m; B_{\boldsymbol{w}})$ are isomorphic (where $\mathcal{B}_X$ is the Borel σ-algebra of X).

Invariant measures for weighted backward shifts are thus far from being completely understood (and in some sense, they will never be). In this note, our aim is to contribute to their understanding by studying possible links between invariant measures for *two different* weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$. More specifically, we will be concerned with the notion of *orthogonality* of two weighted shifts.

1.2. Orthogonality. — Before giving the definition we will be playing with, we fix some notation. The scalar field is denoted by $\mathbb{K}$. All measures on $X = \ell_p(\mathbb{Z}_+)$ are understood to be Borel probability measures. Given a weight sequence $\boldsymbol{w}$, we denote by $\mathcal{P}_{\boldsymbol{w}}(X)$ the set of all $B_{\boldsymbol{w}}$-invariant measures on X. If m and m' are two measures on X, we write $m \perp m'$ to indicate that m and m' are orthogonal (*i.e.* mutually singular), and $m \ll m'$ to indicate that m is absolutely continuous with respect to m'. If $m \ll m'$ and $m' \ll m$, we say that m and m' are equivalent and we write $m \sim m'$.

We now define orthogonality of two weighted backward shifts.

Definition 1.1. — Two weighted backward shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ on $X = \ell_p$ are said to be *orthogonal* if the following holds true: whenever $m_{\boldsymbol{u}} \in \mathcal{P}_{\boldsymbol{u}}(X)$, $m_{\boldsymbol{v}} \in \mathcal{P}_{\boldsymbol{v}}(X)$ and $m_{\boldsymbol{u}}(\{0\}) = 0 = m_{\boldsymbol{v}}(\{0\})$, it follows that $m_{\boldsymbol{u}} \perp m_{\boldsymbol{v}}$.

Removing the δ_0 - parts of the measures, we see that this definition can be reformulated as follows: $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal if and only if whenever $m_{\boldsymbol{u}} \in \mathcal{P}_{\boldsymbol{u}}(X)$ and $m_{\boldsymbol{v}} \in \mathcal{P}_{\boldsymbol{v}}(X)$ are non-orthogonal, it must be that $m_{\boldsymbol{u}}(\{0\}) > 0$ *and* $m_{\boldsymbol{v}}(\{0\}) > 0$. Equivalently: there exists no (Borel probability) measure on X which is absolutely continuous with respect to both a $B_{\boldsymbol{u}}$ - invariant measure and a $B_{\boldsymbol{v}}$ - invariant measure, except the Dirac mass δ_0. Thus, informally speaking, $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal if their invariant measures "have nothing to say to each other", unless they charge the singleton $\{0\}$. The main question we want to investigate is the following.

Question 1.2. — Characterize the pairs of weight sequences $(\boldsymbol{u}, \boldsymbol{v})$ such that the associated weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on $X = \ell_p$ are orthogonal.

The motivation for looking at this question comes from a recent work of Charpentier, Ernst, Mestiri and Mouze [10]. Let B be the unweighted backward shift acting on the complex Hilbert space $\ell_2(\mathbb{Z}_+)$, and let $\Lambda \subseteq \mathbb{C}$. In [10], it is shown that the operators λB, $\lambda \in \Lambda$ admit a *common frequently hypercyclic vector* if and only if the set $\{|\lambda|;\ \lambda \in \Lambda\}$ is a countable relatively compact subset of $(1, \infty)$. In particular, if $a, b \in \mathbb{C}$ and $|a|, |b| > 1$, then the operators aB and bB have a common frequently hypercyclic vector. This is by no means obvious if $|a| \neq |b|$. So it is tempting to ask whether this result could be retrieved in a "soft" way by measure-theoretic arguments. Indeed, imagine that it were possible to find two measures m_a and m_b on ℓ_2, with full support, invariant and *ergodic* for aB and bB respectively, such that one of them is absolutely continuous with respect to the other, say $m_a \ll m_b$. By the pointwise ergodic theorem, it would follow that m_a - almost every $x \in \ell_2$ is frequently hypercyclic for aB and m_b - almost every $x \in \ell_2$ is frequently hypercyclic for bB; hence m_a - almost every x would be frequently hypercyclic for both aB and bB since $m_a \ll m_b$, so one could conclude in particular that aB and bB have a common frequently hypercyclic vector. However, such measures m_a and m_b simply do not exist: we will see below that the operators aB and bB are in fact orthogonal.

1.3. Organization of the paper. — In Section 2, we complement existing results by showing that a weighted backward shift $B_{\boldsymbol{w}}$ on $X = \ell_p(\mathbb{Z}_+)$ admits a non-trivial invariant measure *if and only if* $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$. In Section 3, we define orthogonality in a very general context, we give a simple condition ensuring orthogonality (Theorem 3.2), and we use this to show, among other things, that if T is any continuous linear operator acting on a Polish topological vector space X and $a, b \in \mathbb{K}$ are such that $|a| \neq |b|$, then the operators aT and bT are orthogonal (Corollary 3.6). We also deduce from our criterion that two weighted shifts that are "far from being similar" must be orthogonal (Theorem 3.10). In Section 4, we use *periodic vectors* to give examples of non-orthogonal weighted shifts; in particular, we show that there exist non-orthogonal weighted shifts which are not similar (Example 4.5). Section 5 is devoted to invariant *product measures* for weighted shifts acting on ℓ_p, and their role in our study of orthogonality. We give a necessary and sufficient condition for two weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ to admit equivalent invariant Gaussian product measures (Corollary 5.8), and then we examine to what extent the existence of non-orthogonal invariant product measures implies the existence of equivalent Gaussian product measures (Theorem 5.13 and Proposition 5.22). Section 6 contains some

additional facts. Some of these facts are actually used in earlier proofs, but we prefered to postpone them for the sake of fluid reading. We conclude the paper with a few natural questions.

2. Existence of non-trivial invariant measures

It is, of course, not very interesting to study orthogonality of two weighted shifts if one does not know at least that each of them admits non-trivial invariant measures, *i.e.* invariant measures different from the Dirac mass δ_0. The following proposition says exactly when this happens for a weighted shift acting on ℓ_p.

Proposition 2.1. — *Let $B_{\boldsymbol{w}}$ be a weighted backward shift acting on $X = \ell_p$. Then $B_{\boldsymbol{w}}$ admits a non-trivial invariant measure if and only if $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$.*

One can give a short proof of Proposition 2.1 by using a (quite non-trivial) lemma inspired by [6], which can be extracted from the proof of [11, []Theorem 2.1]. Since this lemma might be useful in other situations, we state it explicitely. Let us denote by e_0^* the first coordinate functional on $X = \ell_p$.

Lemma 2.2. — *Let $B_{\boldsymbol{w}}$ be a weighted shift acting on $X = \ell_p(\mathbb{Z}_+)$. Assume that one can find a vector $x \in \ell_p$ and a set of integers $\mathcal{N}$ such that*

(i) *$\mathcal{N}$ has positive upper density,*
(ii) *$\sup_{n\in\mathcal{N}} \|B_{\boldsymbol{w}}^n x\| < \infty$,*
(iii) *$\inf_{n\in\mathcal{N}} |\langle e_0^*, B_{\boldsymbol{w}}^n x\rangle| > 0$.*

Then, one can conclude that $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$.

Sketch of the proof of Lemma 2.2. — Write $x = \sum_{n=0}^{\infty} x_n e_n$. Let $C_1 := \sup_{n\in\mathcal{N}} \|B_{\boldsymbol{w}}^n x\|$ and $C_2 := \inf_{n\in\mathcal{N}} |\langle e_0^*, B_{\boldsymbol{w}}^n x\rangle| = \inf_{n\in\mathcal{N}} |w_1 \cdots w_n\, x_n|$. If we let $v_n := (w_1 \cdots w_n)^{-1}$, we have for every $n \in \mathcal{N}$ and every integer $M \geqslant 0$,

$$\left\| \sum_{\substack{m\in\mathcal{N} \\ n\leqslant m\leqslant n+M}} v_{m-n} e_{m-n} \right\| \leqslant \frac{1}{C_2} \left\| \sum_{\substack{m\in\mathcal{N} \\ n\leqslant m\leqslant n+M}} v_{m-n} \frac{x_m}{v_m} e_{m-n} \right\| = \frac{1}{C_2} \left\| \sum_{\substack{0\leqslant k\leqslant M \\ k+n\in\mathcal{N}}} v_k \frac{x_{k+n}}{v_{k+n}} e_k \right\| \leqslant \frac{C_1}{C_2}.$$

We then conclude thanks to [11, []Lemma 2.5]. □

Proof of Proposition 2.1. — As mentioned in the introduction, it is well-known that if $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$, then $B_{\boldsymbol{w}}$ admits lots of non-trivial invariant measures. Perhaps the simplest such measure is the Dirac mass δ_x at the fixed point $x := e_0 + \sum_{n=1}^{\infty} \frac{1}{w_1 \cdots w_n} e_n$.

Conversely, assume that $B_{\boldsymbol{w}}$ admits a non-trivial invariant measure m. By the ergodic decomposition theorem, we may assume that m is an ergodic measure for $B_{\boldsymbol{w}}$. Since $m \neq \delta_0$, we may choose a point $u \neq 0$ in the support of m. Then $B_{\boldsymbol{w}}^n u \in \mathrm{supp}(m)$ for all $n \geqslant 0$ by the $B_{\boldsymbol{w}}$-invariance of m, so we may assume that $\langle e_0^*, u\rangle \neq 0$. Let $\alpha > 0$ be such that $|\langle e_0^*, u\rangle| > \alpha$, and consider the open set

$$U := \big\{ z \in X;\ \|z - u\| < 1 \quad \text{and} \quad |\langle e_0^*, z\rangle| > \alpha \big\}.$$

Since U is a neighbourhood of u, we have $m(U) > 0$. By the pointwise ergodic theorem, it follows that one can find $x \in X$ such that the set $\mathcal{N}_{B_{\boldsymbol{w}}}(x, U) := \{n \in \mathbb{Z}_+;\ B_{\boldsymbol{w}}^n x \in U\}$

admits a positive density, *i.e.* $\lim_{N\to\infty} \frac{1}{N+1}\#\big([\![0,N]\!] \cap \mathcal{N}_{B_{\boldsymbol{w}}}(x,U)\big)$ exists and is (strictly) positive; in particular, $\mathcal{N}_{B_{\boldsymbol{w}}}(x,U)$ has positive *upper* density. So we have found a vector $x \in \ell_p$ and a set of integers $\mathcal{N}$ with positive upper density such that $\|B_{\boldsymbol{w}}^n x\| < 1 + \|u\|$ and $|\langle e_0^*, B_{\boldsymbol{w}}^n x\rangle| > \alpha$ for all $n \in \mathcal{N}$. By Lemma 2.2, this concludes the proof. □

Remark 2.3. — Proposition 2.1 is valid in a more general setting: the same proof shows that if X is a Banach sequence space for which $(e_n)_{n\geqslant 0}$ is a boundedly complete unconditional basis, then a weighted shift $B_{\boldsymbol{w}}$ acting on X admits a non-trivial invariant measure if and only if the series $\sum \frac{1}{w_1\cdots w_n}\, e_n$ is convergent in X. We do not know if the analogous result holds for weighted shifts acting on $c_0(\mathbb{Z}_+)$.

3. A general criterion for orthogonality

Although the definition of orthogonality was given in the introduction for pairs of weighted shifts only, it makes sense in a much more general context. If T is any continuous self-map of a Polish (*i.e.* separable and completely metrizable) space X, let us denote by $\mathcal{P}_T(X)$ the set of all T-invariant (Borel probability) measures on X.

Definition 3.1. — Let X be a Polish space, and let (P) be any property of measures on X. We say that two continuous self-maps T_1, T_2 of X are *orthogonal with respect to measures satisfying* (P) if the following holds true: whenever $m_1 \in \mathcal{P}_{T_1}(X)$ and $m_2 \in \mathcal{P}_{T_2}(X)$ satisfy (P), it follows that $m_1 \perp m_2$.

In accordance with Definition 1.1, if X is a Polish topological vector space and T_1, T_2 are continuous linear operators on X, we will say that T_1 and T_2 are *orthogonal* if they are orthogonal with respect to measures not charging the singleton $\{0\}$.

We now present a general sufficient condition for orthogonality. Recall that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, then $\overline{\lim}\, A_n$ denotes the set of all $x \in X$ that belong to infinitely many A_n's.

Theorem 3.2. — *Let X be a Polish space, and let Λ be a Borel subset of X. Let also T_1 and T_2 be two continuous self-maps of X. Assume that for any compact set $K \subseteq X$ with $K \cap \Lambda = \varnothing$, there exists an infinite set $I \subseteq \mathbb{N}$ such that*

$$\overline{\lim_{n\in I}}\; T_1^{-n}(K) \cap T_2^{-n}(K) = \varnothing. \tag{3.1}$$

Then, T_1 and T_2 are orthogonal with respect to measures not charging Λ.

Proof. — Let $m_1 \in \mathcal{P}_{T_1}(X)$ and $m_2 \in \mathcal{P}_{T_2}(X)$ be such that $m_1(\Lambda) = 0 = m_2(\Lambda)$. We have to show that $m_1 \perp m_2$, and this will be an easy consequence of the following fact.

Fact 3.3. — For any $\varepsilon > 0$, one can find a Borel set $E_\varepsilon \subseteq X$ such that $m_1(E_\varepsilon) < \varepsilon$ and $m_2(E_\varepsilon) > 1 - \varepsilon$.

Proof of Fact 3.3. — By regularity of the measures m_1 and m_2 and since m_1 and m_2 do not charge Λ, there exists a compact set $K \subseteq X\backslash\Lambda$ such that $m_1(K) > 1 - \varepsilon/2$ and $m_2(K) > 1 - \varepsilon/2$. Let I be an infinite subset of $\mathbb{N}$ such that (3.1) holds true. Then $m_1\big(T_1^{-n}(K) \cap T_2^{-n}(K)\big) \to 0$ as $n \to \infty$ along I by Fatou's Lemma; so one can find an integer $n \geqslant 1$ such that $m_1\big(T_1^{-n}(K) \cap T_2^{-n}(K)\big) < \varepsilon/2$. Since $m_1\big(T_1^{-n}(K)\big) = m_1(K) > 1 - \varepsilon/2$, it follows that $m_1\big(T_2^{-n}(K)\big) < \varepsilon$. On the other hand, $m_2\big(T_2^{-n}(K)\big) = m_2(K) > 1 - \varepsilon$; so we may take $E := T_2^{-n}(K)$. □

If we now define $E := \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} E_{\frac{2^{-n}}{k}}$, we see that $m_1(E) = 0$ and $m_2(E) = 1$, which shows that $m_1 \perp m_2$. □

Remark 3.4. — The assumption in Theorem 3.2 is here to ensure that for any compact set $K \subseteq X\backslash\Lambda$ and any measure m on X, it holds that $\inf_{n\geqslant 1} m\big(T_1^{-n}(K) \cap T_2^{-n}(K)\big) = 0$. However, it is very likely that (3.1) is a too strong assumption. More generally, it would be nice to have a "measure-free" characterization of the sequences of Borel sets $(A_n)_{n\geqslant 1} \subseteq X$ such that $\inf_{n\geqslant 1} m(A_n) = 0$ for every measure m on X. Note that there is a very simple characterization if "inf" is replaced by "lim": we have $\lim_{n\to\infty} m(A_n) = 0$ for every measure m on X if and only if $\overline{\lim}\, A_n = \varnothing$. On the other hand, for a sequence $(A_n)_{n\geqslant 1}$ of non-empty Borel subsets of X, consider the following statements.

(i) There exists an infinite set $I \subseteq \mathbb{N}$ such that $\overline{\lim}_{n\in I} A_n = \varnothing$.
(ii) $\inf_{n\geqslant 1} m(A_n) = 0$ for every measure m on X.
(iii) $\inf_{n\geqslant 1} m(A_n) = 0$ for every discrete measure m on X.
(iv) $\inf_{n\geqslant 1} m(A_n) = 0$ for every finitely supported measure m on X.
(v) For any finite set $F \subseteq X$, there exist infinitely many n such that $A_n \cap F = \varnothing$.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v), but (v) does not imply (ii) and (ii) does not imply (i).

Proof. — (i) $\Longrightarrow$ (ii) by Fatou's Lemma, and obviously (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). Moreover, it is rather clear that in fact (iii) $\Longleftrightarrow$ (iv), and that (v) $\Longrightarrow$ (iv).

To prove that (iv) $\Longrightarrow$ (v), assume that for some finite set $F \subseteq X$ and some integer N, we have $A_n \cap F \neq \varnothing$ for all $n > N$. Choose a point $a_i \in A_i$ for $i = 1, \dots, N$. Then, the finitely supported measure $m := \frac{1}{2}\left(\frac{1}{N}\sum_{i=1}^{N} \delta_{a_i} + \frac{1}{\#F}\sum_{x\in F} \delta_x\right)$ is such that $\inf_{n\geqslant 1} m(A_n) > 0$.

To show that (v) does not imply (ii), let $X := [0,1]$, and define

$$A_n := \{t \in [0,1];\ |\sin(2\pi n t)| \geqslant 1/\pi\}.$$

By Dirichlet's theorem, for any finite set $F \subseteq [0,1]$ one can find an increasing sequence of integers (n_k) such that $\sin(2\pi n_k t) \to 0$ on F; so (v) is satisfied. However, we have $\int_0^1 |\sin(2\pi n t)|\, dt = 2/\pi$ for all $n \geqslant 1$. Hence $m(A_n) \geqslant 1/\pi$ for all $n \geqslant 1$, where m is Lebesgue measure on $[0,1]$.

Let us now show that (ii) does not imply (i). This example is due to N. de Rancourt. Consider the Cantor space $\Delta = \{0,1\}^{\mathbb{N}}$ identified with the power set of $\mathbb{N}$. Let

$$X := \left\{J \subseteq \mathbb{N};\ \sum_{n\in J} \frac{1}{n} \leqslant 1\right\},$$

which is a closed subset of Δ and hence a (compact) Polish space. For each $n \in \mathbb{N}$, let

$$A_n := \{J \in X;\ n \in J\}.$$

It is clear that the sequence (A_n) does not satisfy (i): indeed, for any infinite set $I \subseteq \mathbb{N}$, one can find an infinite set $J \subseteq I$ such that $\sum_{n\in J} \frac{1}{n} \leqslant 1$, and this J belongs to $\overline{\lim}\, A_n$ by definition.

However, (A_n) satisfies (ii); in fact, for any measure m on X, we have $\sum_{n=1}^{\infty} \frac{m(A_n)}{n} < \infty$. Indeed, we have

$$\sum_{n=1}^{\infty} \frac{m(A_n)}{n} = \int_X \left(\sum_{n=1}^{\infty} \frac{1}{n}\, \mathbf{1}_{A_n}(J)\right) dm(J) = \int_X \left(\sum_{n\in J} \frac{1}{n}\right) dm(J) \leqslant 1.$$

Here is yet another example showing that (ii) does not imply (i), which is a kind of twin of the previous one. Let

$$X := \left\{ x \in [0,\pi];\ \sum_{n=1}^{\infty} \frac{|\sin(2^n x)|}{n} \leqslant 1 \right\},$$

which is a closed subset of $[0,\pi]$. For each $n \geqslant 1$, let

$$A_n := \{x \in X;\ |\sin(2^n x)| \geqslant 1/2\}.$$

With some effort, one can prove the following: *if $(m_i)_{i\geqslant 1}$ is a sufficiently fast increasing sequence of integers and if we define*

$$x := \frac{\pi}{2} \sum_{i=1}^{\infty} \frac{1}{2^{m_i}},$$

then $x \in A_{m_k}$ for every $k \geqslant 1$. It follows imediately that the sequence (A_n) does not satisfy (i). However, one shows in the same way as above that (A_n) satisfies (ii). □

Remark 3.5. — With the notation of Theorem 3.2, consider the following statements.

(1) For any compact set $K \subseteq X\backslash\Lambda$, it holds that $\overline{\lim}\, T_1^{-n}(K) \cap T_2^{-n}(K) = \varnothing$.
(2) It is not possible to find $x \in X$ and an increasing sequence of integers (n_k) such that the sequences $(T_1^{n_k}x)$ and $(T_2^{n_k}x)$ both converge to a limit not belonging to Λ.

Then (2) $\Longrightarrow$ (1), and (1) $\Longleftrightarrow$ (2) if Λ is a *closed* subset of X.

Proof. — It should be rather clear that (2) implies (1). Indeed, let K be a compact subset of $X\backslash\Lambda$, and assume that $\overline{\lim}\, T_1^{-n}(K) \cap T_2^{-n}(K) \neq \varnothing$. Then, there exists $x \in X$ and an increasing sequence of integers $(n_k)_{k\geqslant 0}$ such that $T_1^{n_k}x \in K$ and $T_2^{n_k}x \in K$ for all $k \geqslant 0$. Since K is compact, we may assume, upon extracting subsequences, that both sequences $(T_1^{n_k}x)$ and $(T_2^{n_k}x)$ converge, $T_1^{n_k}x \to u \in K$ and $T_2^{n_k}x \to v \in K$; and since $K \subseteq X\backslash\Lambda$, this shows that (2) is not satisfied.

Conversely, assume that Λ is closed and that (2) is not satisfied, *i.e.* there exist $x \in X$ and an increasing sequence of integers $(n_k)_{k\geqslant 0}$ such that $T_1^{n_k}x \to u$ and $T_2^{n_k}x \to v$, where $u, v \in X\backslash\Lambda$. Since $X\backslash\Lambda$ is open, one can find k_0 such that $T_1^{n_k}x, T_2^{n_k}x \in X\backslash\Lambda$ for all $k \geqslant k_0$. Then the set $K := \{T_1^{n_k}x;\ k \geqslant k_0\} \cup \{T_2^{n_k}x;\ k \geqslant k_0\} \cup \{u, v\}$ is a compact subset of $X\backslash\Lambda$ and $x \in T_1^{-n_k}(K) \cap T_2^{-n_k}(K)$ for all $k \geqslant k_0$, so that (1) is not satisfied. □

Here is a first application of Theorem 3.2.

Corollary 3.6. — *Let X be a Polish topological vector space, and let $T \in \mathcal{L}(X)$. Let also $a, b \in \mathbb{K}$, and assume that aT and bT admit non-trivial invariant measures. Then, aT and bT are orthogonal if and only if $|a| \neq |b|$. More precisely: aT and bT are orthogonal if $|a| \neq |b|$; and they share a non-trivial invariant measure if $|a| = |b|$.*

Proof. — To prove that aT and bT are orthogonal if $|a| \neq |b|$, we use Theorem 3.2 and Remark 3.5 with $\Lambda = \{0\}$. Assume that there exists $x \in X$ and an increasing sequence of integers (n_k) such that $a^{n_k}T^{n_k}x \to u \in X$ and $b^{n_k}T^{n_k}x \to v \in X$. We must show that $u = 0$ or $v = 0$; and this follows from the continuity of the map $(\lambda, z) \mapsto \lambda z$: indeed, assuming for example that $|a| > |b|$ and writing $b^{n_k}T^{n_k}x = (b/a)^{n_k}\, a^{n_k}T^{n_k}x$, we see that $b^{n_k}T^{n_k}x \to 0$.

The converse will follow from Proposition 6.1 below. □

Remark 3.7. — If $B_{\boldsymbol{w}}$ is a weighted shift acting on $X = \ell_p(\mathbb{Z}_+)$ and $a, b \in \mathbb{K}$ are such that $|a| = |b| =: r$ with $\sum_{n=1}^{\infty} \frac{1}{(r^n|w_1\cdots w_n|)^p} < \infty$, then the operators $aB_{\boldsymbol{w}}$ and $bB_{\boldsymbol{w}}$ admit equivalent ergodic Gaussian measures with full support. This will follow from Theorem 5.7 below.

Corollary 3.6 leads to naive speculations regarding orthogonality of weighted shifts. Recall that two operators T_1, T_2 acting on X are said to be *similar* if there exists an invertible operator J such that $T_1 = JT_2J^{-1}$. By [22, []p. 54, Th. 2(a)], two weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on ℓ_p are similar if and only if

$$\underline{\lim} \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| > 0 \qquad \text{and} \qquad \overline{\lim} \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| < \infty. \tag{3.2}$$

When $\boldsymbol{u}$ and $\boldsymbol{v}$ have the form $\boldsymbol{u} = a\boldsymbol{w}$ and $\boldsymbol{v} = b\boldsymbol{w}$ for some weight sequence $\boldsymbol{w}$ and $a, b \in \mathbb{K}$, then $\frac{u_1\cdots u_n}{v_1\cdots v_n} = \left(\frac{a}{b}\right)^n$, and hence (3.2) holds true if and only if $|a| = |b|$. So it is tempting to "conjecture" that two weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal if and only if they are not similar. This is however not true, as will be seen in Section 4 below. Nevertheless, we now present two results showing that this is not *that far* from being true.

In what follows we denote by e_j^*, $j \geqslant 0$ the coordinate functionals on $X = \ell_p(\mathbb{Z}_+)$. And if $x \in X$, we write $x = \sum_{j=0}^{\infty} x_j e_j$ where $x_j = \langle e_j^*, x\rangle$.

Proposition 3.8. — *Let $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ be two weighted shifts acting on $X = \ell_p$. Assume that $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not similar, i.e. that*

$$\textit{either} \quad \underline{\lim} \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| = 0 \qquad \textit{or} \quad \overline{\lim} \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| = \infty. \tag{3.3}$$

Then, $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal with respect to measures not charging $\Lambda_{e_0^*} := \ker(e_0^*)$.

Proof. — We apply Theorem 3.2 with $\Lambda := \Lambda_{e_0^*}$. So let K be a compact subset of $X\setminus \ker(e_0^*)$. Choose some constants $M < \infty$ and $\gamma > 0$ such that $\|z\| \leqslant M$ and $|\langle e_0^*, z\rangle| \geqslant \gamma$ for all $z \in K$. If $x = \sum_{j=0}^{\infty} x_j e_j \in X$ then, for every $n \geqslant 1$, we have

$$\langle e_0^*, B_{\boldsymbol{u}}^n x\rangle = u_1\cdots u_n\, x_n \qquad \text{and} \qquad \langle e_0^*, B_{\boldsymbol{v}}^n x\rangle = v_1\cdots v_n\, x_n;$$

so we see that

$$B_{\boldsymbol{u}}^{-n}(K) \subseteq \big\{x \in X;\ |x_n|\cdot|u_1\cdots u_n| \geqslant \gamma\big\}$$

and

$$B_{\boldsymbol{v}}^{-n}(K) \subseteq \big\{x \in X;\ |x_n|\cdot|v_1\cdots v_n| \leqslant M\big\}.$$

Hence, the following implication holds :

$$B_{\boldsymbol{u}}^{-n}(K) \cap B_{\boldsymbol{v}}^{-n}(K) \neq \varnothing \implies \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| \geqslant \frac{\gamma}{M}.$$

It follows that if $\underline{\lim} \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| = 0$, then $B_{\boldsymbol{u}}^{-n}(K) \cap B_{\boldsymbol{u}}^{-n}(K) = \varnothing$ for infinitely many n; and symmetrically, if $\overline{\lim} \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| = \infty$ then $B_{\boldsymbol{v}}^{-n}(K) \cap B_{\boldsymbol{u}}^{-n}(K) = \varnothing$ for infinitely many n. By Theorem 3.2, this concludes the proof. □

Remark 3.9. — One may observe that for any weight sequence $\boldsymbol{w}$ and every $n \geqslant 1$, we have $B_{\boldsymbol{w}}^{-n}(\Lambda_{e_0^*}) = \Lambda_{e_n^*}$. Hence, a measure $m \in \mathcal{P}_{\boldsymbol{w}}(X)$ does not charge $\Lambda_{e_0^*}$ if and only if it does not charge $\Lambda_{e_n^*}$. Therefore, m does not charge $\Lambda_{e_0^*}$ if and only if it is supported on the set $\{x \in X;\ \langle e_n^*, x\rangle \neq 0 \text{ for all } n \geqslant 0\}$.

The next theorem shows that "full" orthogonality of $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ can be deduced from stronger assumptions on the weights $\boldsymbol{u}$ and $\boldsymbol{v}$.

Theorem 3.10. — *Let $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ be two weighted shifts acting on $X = \ell_p$. Assume that either*

$$\varlimsup_{n\to\infty} \max_{0\leqslant d\leqslant N} \left|\frac{u_1\cdots u_{n+d}}{v_1\cdots v_{n+d}}\right| = 0 \qquad \text{for all } N \geqslant 0 \tag{3.4}$$

or

$$\varliminf_{n\to\infty} \min_{0\leqslant d\leqslant N} \left|\frac{u_1\cdots u_{n+d}}{v_1\cdots v_{n+d}}\right| = \infty \qquad \text{for all } N \geqslant 0. \tag{3.5}$$

Then, $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal.

Proof. — We apply Theorem 3.2 once again, with $\Lambda := \{0\}$. So, let K be a compact subset of $X\backslash\{0\}$.

We first note that there exists an integer N such that $\max_{0\leqslant d\leqslant N} |z_d| > 0$ for every $z \in K$. Indeed, the compact set K is covered by the open sets $O_d := \{z \in X;\ |z_d| > 0\}$.

Choose $\gamma > 0$ such that $\max_{0\leqslant d\leqslant N} |z_d| \geqslant \gamma$ for every $z \in K$, and $M < \infty$ such that $\|z\| \leqslant M$ for every $z \in K$. As in the proof of Proposition 3.8, we get that for every $n \geqslant 1$,

$$B_{\boldsymbol{u}}^{-n}(K) \subseteq \left\{x \in X;\ \max_{0\leqslant d\leqslant N} |u_{1+d}\cdots u_{n+d}|\cdot|x_{n+d}| \geqslant \gamma\right\}$$

and

$$B_{\boldsymbol{v}}^{-n}(K) \subseteq \left\{x \in X;\ \max_{0\leqslant d\leqslant N} |v_{1+d}\cdots v_{n+d}|\cdot|x_{n+d}| \leqslant M\right\};$$

so that

$$B_{\boldsymbol{u}}^{-n}(K) \cap B_{\boldsymbol{v}}^{-n}(K) \neq \varnothing \implies \max_{0\leqslant d\leqslant N} \left|\frac{u_{1+d}\cdots u_{n+d}}{v_{1+d}\cdots v_{n+d}}\right| \geqslant \frac{\gamma}{M}.$$

Now, we have $\max_{0\leqslant d\leqslant N} \left|\frac{u_{1+d}\cdots u_{n+d}}{v_{1+d}\cdots v_{n+d}}\right| \leqslant C_N \max_{0\leqslant d\leqslant N} \left|\frac{u_1\cdots u_{n+d}}{v_1\cdots v_{n+d}}\right|$ for some constant C_N. Hence, if (3.4) is satisfied, then one can find an infinite set $I \subseteq \mathbb{N}$ such that

$$\max_{0\leqslant d\leqslant N} \left|\frac{u_{1+d}\cdots u_{n+d}}{v_{1+d}\cdots v_{n+d}}\right| \to 0 \qquad \text{as} \quad n \to \infty \ \text{ along } I;$$

and it follows that $B_{\boldsymbol{u}}^{-n}(K) \cap B_{\boldsymbol{v}}^{-n}(K) = \varnothing$ for all $n \in I$ sufficiently large. Symmetrically, if (3.5) is satisfied, then one can find an infinite set $I \subseteq \mathbb{N}$ such that $B_{\boldsymbol{v}}^{-n}(K) \cap B_{\boldsymbol{u}}^{-n}(K) = \varnothing$ for all $n \in I$ sufficiently large. So Theorem 3.2 applies. □

Corollary 3.11. — *Assume that the weight sequences $\boldsymbol{u}$ ad $\boldsymbol{v}$ are bounded below, i.e. $\inf_{n\geqslant 1} |u_n| > 0$ and $\inf_{n\geqslant 1} |u_n| > 0$. If $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not similar, then they are orthogonal.*

Proof. — The assumption on $\boldsymbol{u}$ and $\boldsymbol{v}$ implies that for every $N \geqslant 0$, there are constants $c_N > 0$ and $C_N < \infty$ such that $\max_{0\leqslant d\leqslant N} \left|\frac{u_1\cdots u_{n+d}}{v_1\cdots v_{n+d}}\right| \leqslant C_N \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right|$ and $\min_{0\leqslant d\leqslant N} \left|\frac{u_1\cdots u_{n+d}}{v_1\cdots v_{n+d}}\right| \geqslant c_N \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right|$. Hence, if $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not similar then (3.4) or (3.5) is satisfied. □

Remark 3.12. — Theorem 3.10 can be reformulated in the following way: if

- either, for every $\varepsilon > 0$, the set $A_{\boldsymbol{u},\boldsymbol{v},\varepsilon} := \left\{n \in \mathbb{Z}_+;\ \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| < \varepsilon\right\}$ contains arbitrarily long intervals,
- or, for every $M < \infty$, the set $A'_{\boldsymbol{u},\boldsymbol{v},M} := \left\{n \in \mathbb{Z}_+;\ \left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| > M\right\}$ contains arbitrarily long intervals,

then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal.

4. Non-orthogonality *via* periodic points

In this section, our aim is to present some examples of non-orthogonal weighted shifts using measures supported by periodic orbits. We first note the following easy fact.

Fact 4.1. — Let X be a Polish topological vector space, and let $T_1, T_2 \in \mathcal{L}(X)$. If T_1 and T_2 share a non-zero periodic point, then they are not orthogonal.

Proof. — Let $x \neq 0$ be a common periodic point for T_1 and T_2; so $T^{d_1}x = x = T^{d_2}x$ for some $d_1, d_2 \in \mathbb{N}$. For $i = 1, 2$, the measure $m_i := \frac{1}{d_i}\sum_{n=0}^{d_i-1} \delta_{T_i^n x}$ is T_i-invariant, and $m_i(\{0\}) = 0$ since $x \neq 0$. Moreover, m_1 and m_2 are not orthogonal since $m_1(\{x\}) > 0$ and $m_2(\{x\}) > 0$. □

Theorem 4.2. — *Let $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ be two weighted shifts on $X = \ell_p(\mathbb{Z}_+)$. Assume that $\sum_{n=1}^{\infty} \frac{1}{|u_1\cdots u_n|^p} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|v_1\cdots v_n|^p} < \infty$. The following are equivalent.*

(1) *$B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ share a non-zero periodic point;*
(2) *there exist $d \in \mathbb{N}$ and $0 \leqslant j \leqslant d-1$ such that $u_{1+j}\cdots u_{dm+j} = v_{1+j}\cdots v_{dm+j}$ for all $m \geqslant 1$;*
(3) *there exist $d \in \mathbb{N}$ and $0 \leqslant j \leqslant d-1$, and a non-zero scalar C such that $u_1\cdots u_{dm+j} = C\, v_1\cdots v_{dm+j}$ for all $m \geqslant 0$ (where an empty product is declared to be equal to 1).*

Proof. — (1) $\Longrightarrow$ (2) Assume that $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ share a non-zero periodic point x. Let d_1 be the period of x as a periodic point of T_1, and let d_2 be the period of x as a periodic point of T_2. Setting $d := d_1 d_2$, we then have $B_{\boldsymbol{u}}^d x = x = B_{\boldsymbol{v}}^d x$ (and hence $B_{\boldsymbol{u}}^{md} x = x = B_{\boldsymbol{v}}^{md} x$ for all $m \geqslant 1$.) Writing $x = \sum_{j\geqslant 0} x_j e_j$, we have

$$x_{j+md} = \frac{1}{u_{j+1}\cdots u_{j+md}}\, x_j = \frac{1}{v_{j+1}\cdots v_{j+md}}\, x_j \qquad \text{for all } j \geqslant 0.$$

Since $x \neq 0$, one can find $0 \leqslant j \leqslant d-1$ such that $x_j \neq 0$, and then $u_{j+md}\cdots u_{j+1} = v_{j+md}\cdots v_{j+1}$ for all $m \geqslant 1$.

(2) $\Longrightarrow$ (3) This is clear: if (2) is satisfied for some d and $0 \leqslant j \leqslant d-1$, then (3) is satisfied with the same d and j and $C := \frac{u_1\cdots u_j}{v_1\cdots v_j}$ (so $C = 1$ if $j = 0$).

(3) $\Longrightarrow$ (1) Assume that (3) holds true for some d, j, C. Since $\sum_{n=1}^{\infty} \frac{1}{|u_1\cdots u_n|^p} < \infty$, the vector

$$x := \sum_{m=0}^{\infty} \frac{1}{u_1\cdots u_{md+j}}\, e_{md+j} = \frac{1}{C}\sum_{m=0}^{\infty} \frac{1}{v_1\cdots v_{md+j}}\, e_{md+j} \in \ell_p$$

is well-defined, and clearly $B_{\boldsymbol{u}}^d x = x = B_{\boldsymbol{v}}^d x$. Thus, x is a non-zero common periodic point for $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$. □

Corollary 4.3. — *Let $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ be two weighted shifts on $\ell_p(\mathbb{Z}_+)$, with $\sum_{n=1}^{\infty} \frac{1}{|u_1 \cdots u_n|^p} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|v_1 \cdots v_n|^p} < \infty$. Assume that there exist $d \in \mathbb{N}$, $0 \leqslant j \leqslant d-1$ and a scalar $C \neq 0$ such that $u_1 \cdots u_{dm+j} = C\, v_1 \cdots v_{dm+j}$ for all $m \geqslant 0$. Then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not orthogonal.*

Example 4.4. — Assume that $u_j = 2$ for all $j \geqslant 1$, $v_1 = 3$ and $v_j = 2$ for all $j \geqslant 2$. Then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on ℓ_p share a non-zero fixed point, and hence they are not orthogonal.

In this example, the operators $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are obviously similar. We now give an example of non-orthogonal weighted shifts which are not similar.

Example 4.5. — Let $(r_k)_{k \geqslant 1}$ be an increasing sequence of integers, and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the two weight sequences defined as follows:

$$u_j := \begin{cases} 2 & \text{if} \quad j \notin \{5r_k + 1, 5r_k + 4;\ k \geqslant 1\} \\ \frac{1}{k} & \text{if} \quad j = 5r_k + 1 \text{ or } j = 5r_k + 4 \ \text{ for some } k \geqslant 1 \end{cases}$$

and

$$v_j := \begin{cases} 2 & \text{if} \quad j \notin \{5r_k + 2, 5r_k + 3;\ k \geqslant 1\} \\ \frac{1}{k} & \text{if} \quad j = 5r_k + 2 \text{ or } j = 5r_k + 3 \ \text{ for some } k \geqslant 1. \end{cases}$$

If (r_k) is sufficiently fast increasing, then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on ℓ_p share a non-zero periodic point and hence they are not orthogonal. However, we have

$$\underline{\lim}\ \frac{u_1 \cdots u_n}{v_1 \cdots v_n} = 0 \qquad \text{and} \qquad \overline{\lim}\ \frac{u_1 \cdots u_n}{v_1 \cdots v_n} = \infty,$$

so that, in particular, $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not similar.

Proof. — It is clear that if (r_k) is sufficiently fast increasing, then $\sum_{n=0}^{\infty} \frac{1}{|u_1 \cdots u_n|^p} < \infty$ and $\sum_{n=0}^{\infty} \frac{1}{|v_1 \cdots v_n|^p} < \infty$.

Observe that if m is a positive integer and if we consider the largest k such that $r_k \leqslant m$, then either $r_k = m$ or $5r_k + 4 < 5m$. Since $u_1 \cdots u_{5r_k} = v_1 \cdots v_{5r_k}$ by definition of the weights, it follows that

$$u_1 \cdots u_{5m} = v_1 \cdots v_{5m} \qquad \text{for all } m \geqslant 1.$$

Hence, $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ share a non-zero periodic point by Theorem 4.2.

However, for any $k \geqslant 1$ we have

$$\frac{u_1 \cdots u_{5r_k+1}}{v_1 \cdots v_{5r_k+1}} = \frac{1}{2k} \qquad \text{and} \qquad \frac{u_1 \cdots u_{5r_k+3}}{v_1 \cdots v_{5r_k+3}} = 2k,$$

so that $\underline{\lim}\ \frac{u_1 \cdots u_n}{v_1 \cdots v_n} = 0$ and $\overline{\lim}\ \frac{u_1 \cdots u_n}{v_1 \cdots v_n} = \infty$.

□

Remark 4.6. — In the above example, the weights are not bounded below. This is not accidental: indeed, if two weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ share a non-zero periodic point and if $\inf_{n \geqslant 1} |u_n| > 0$ and $\inf_{n \geqslant 1} |v_n| > 0$, then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are necessarily similar by Corollary 3.11. (Alternatively, this follows easily from Theorem 4.2.)

5. Non-orthogonality *via* product measures

5.1. Invariant product measures for weighted shifts. — The question we consider in this section is the following: given two weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ on $X = \ell_p(\mathbb{Z}_+)$, when is it possible to find invariant measures $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ (for $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ respectively) not charging $\{0\}$ which are not orthogonal and are also *product measures*?

The terminology requires some explanation since product measures should be defined on products of measurable spaces and $X = \ell_p$ is not such a product space. However, ℓ_p is contained in the product space $\Omega := \mathbb{K}^{\mathbb{Z}_+}$. It is well-known that the Borel σ-algebra of Ω (induced by the product topology) is identical with the product σ-algebra $\otimes_{n\geqslant 0}\mathcal{B}_{\mathbb{K}}$. Moreover, ℓ_p is a Borel subset of Ω when Ω is endowed with the product topology, and a subset of ℓ_p is Borel in $(\ell_p, \|\cdot\|_p)$ if and only if is Borel in Ω. It follows that if μ is a Borel probability measure on Ω, then the restriction of μ to ℓ_p is a Borel measure on ℓ_p endowed with its usual topology (but not a probability measure unless $\mu(\ell_p) = 1$); and conversely, any Borel probability measure on ℓ_p can be considered as a Borel probability measure on Ω (supported on ℓ_p). We will say that a Borel probability measure m on ℓ_p is a *product measure on* ℓ_p if m is the restriction to ℓ_p of a (probability) product measure μ on Ω, *i.e.* $\mu = \bigotimes_{n\geqslant 0}\mu_n$ where each μ_n is a Borel probability measure on $\mathbb{K}$, such that $\mu(\ell_p) = 1$. In this case, the measure μ is uniquely determined by m since $\mu(B) = m(B \cap \ell_p)$ for every Borel set $B \subseteq \Omega$, so we identify m and μ and simply write $m = \otimes_{n\geqslant 0}\mu_n$.

The following essentially obvious remark will be used repeatedly.

Fact 5.1. — Let X be a Borel subset of $\Omega = \mathbb{K}^{\mathbb{Z}_+}$, and let μ be a Borel (probability) measure on Ω such that $\mu(X) = 1$. Set $m := \mu_{|X}$. Let also $T : \Omega \to \Omega$ be a Borel map such that $T(X) \subseteq X$.Then μ is T - invariant if and only if m is $(T_{|X})$ - invariant.

Proof. — The measure μ is T - invariant if and only if $\mu(T^{-1}(B)) = \mu(B)$ for every Borel set $B \subseteq \Omega$. Now, we have $\mu(B) = m(X \cap B)$ and $\mu(T^{-1}(B)) = m(X \cap T^{-1}(B)) = m\big((T_{|X})^{-1}(X \cap B)\big)$ because $T(X) \subseteq X$. So μ is T - invariant if and only if $m(X \cap B) = m\big((T_{|X})^{-1}(X \cap B)\big)$ for every Borel set $B \subseteq \Omega$, which means exactly that m is $(T_{|X})$ - invariant. □

From this observation, it follows that if m is a product measure on $X = \ell_p$ and if we denote by μ the measure m considered as a measure on $\Omega = \mathbb{K}^{\mathbb{Z}_+}$, then m is invariant under some weighted shift $B_{\boldsymbol{w}}$ if and only if μ is invariant under the natural extension of $B_{\boldsymbol{w}}$ to Ω (defined by the same formula and also denoted by $B_{\boldsymbol{w}}$). The next lemma says precisely when this happens.

Lemma 5.2. — *Let $\mu = \otimes_{n\geqslant 0}\mu_n$ be a product measure on $\Omega = \mathbb{K}^{\mathbb{Z}_+}$, and let $\boldsymbol{w}$ be a weight sequence. Then, μ is $B_{\boldsymbol{w}}$ - invariant if and only if, for each $n \geqslant 1$, the measure μ_n is the image of μ_0 under the map $t \mapsto \frac{1}{w_1\cdots w_n}\, t$, i.e. $\mu_n(A) = \mu_0(w_1 \cdots w_n A)$ for every Borel set $A \subseteq \mathbb{K}$.*

Proof. — For any Borel sets $A_0 \ldots, A_N \subseteq \mathbb{K}$, denote by $[A_0, \ldots, A_N] \subseteq \Omega$ the "cylinder set" defined as follows:

$$[A_0, \ldots, A_N] = \big\{t = (t_j)_{j\geqslant 0} \in \Omega;\ t_j \in A_j \quad \text{for } j = 0, \ldots, N\big\}.$$

Since the cylinder sets generate the Borel σ-algebra of Ω, the measure μ is $B_{\boldsymbol{w}}$ - invariant if and only if

$$\mu\big(B_{\boldsymbol{w}}^{-1}([A_0, \ldots, A_N])\big) = \mu\big([A_0, \ldots, A_N]\big)$$

for all Borel sets $A_0, \dots, A_N \subseteq \mathbb{K}$. Now, by definition of $B_{\boldsymbol{w}}$ we have

$$B_{\boldsymbol{w}}^{-1}([A_0, \dots, A_N]) = \big\{t = (t_j)_{j\geqslant 0} \in \Omega;\ w_{j+1}t_{j+1} \in A_j \quad \text{for } j = 0, \dots, N\big\},$$

so that

$$B_{\boldsymbol{w}}^{-1}([A_0, \dots, A_N]) = [\mathbb{K}, (1/w_1)A_0, \dots, (1/w_{N+1})A_N]\,.$$

Since $\mu = \otimes_{j\geqslant 0}\mu_j$, it follows that μ is $B_{\boldsymbol{w}}$ - invariant if and only if

$$\prod_{j=0}^{N} \mu_j(A_j) = \prod_{j=0}^{N} \mu_{j+1}\big((1/w_{j+1})A_j\big)$$

for every $N \geqslant 0$ and all Borel sets $A_0, \dots, A_N \subseteq \mathbb{K}$. This is clearly equivalent to the fact that

$$\mu_{j+1}((1/w_{j+1})A) = \mu_j(A) \qquad \text{for all } j \geqslant 0 \text{ and every Borel set } A \subseteq \mathbb{K},$$

which proves the lemma. □

Corollary 5.3. — *Let $\mu = \otimes_{n\geqslant 0}\mu_n$ be a product measure on $\Omega = \mathbb{K}^{\mathbb{Z}_+}$, and let $\boldsymbol{w}$ be a weight sequence. Assume that μ_0 has a density $\mathtt{p}$ with respect to Lebesgue measure on $\mathbb{K}$. Then μ is $B_{\boldsymbol{w}}$ - invariant if and only if, for each $n \geqslant 1$, the measure μ_n has a density $\mathtt{p}_n$ given by $\mathtt{p}_n(t) := |w_1 \cdots w_n|^d\, \mathtt{p}(w_1 \cdots w_n t)$, where $d = 1$ if $\mathbb{K} = \mathbb{R}$ and $d = 2$ if $\mathbb{K} = \mathbb{C}$.*

Proof. — This follows immediately from Lemma 5.2 □

Corollary 5.4. — *Let $\mu = \otimes_{n\geqslant 0}\mu_n$ be a product measure on $\Omega = \mathbb{K}^{\mathbb{Z}_+}$, and let $\boldsymbol{w}$ be a weight sequence such that $\sum_{n=0}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$. If μ is $B_{\boldsymbol{w}}$ - invariant and if the measure μ_0 is such that $\int_{\mathbb{K}} |t|^p d\mu_0(t) < \infty$, then $\mu(\ell_p) = 1$.*

Proof. — By Lemma 5.2, we have

$$\begin{aligned}\int_\Omega \left(\sum_{n=0}^{\infty} |t_n|^p\right) d\mu(t) &= \sum_{n=0}^{\infty} \int_{\mathbb{K}} |t_n|^p d\mu_n(t_n) \\ &= \left(\int_{\mathbb{K}} |s|^p d\mu_0(s)\right) \times \left(1 + \sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p}\right) < \infty.\end{aligned}$$

In particular, we see that $\sum_{n=0}^{\infty} |t_n|^p < \infty$ for μ - almost every $t \in \Omega$, *i.e.* $\mu(\ell_p) = 1$. □

To conclude this section, we point out the following general fact.

Fact 5.5. — Let $B_{\boldsymbol{w}}$ be a weighted shift on ℓ_p where $\boldsymbol{w}$ satisfies $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$, and let $m = \otimes_{n\geqslant 0}\mu_n$ be a $B_{\boldsymbol{w}}$ - invariant product measure on ℓ_p. If the measure μ_0 has full support, then m is an ergodic measure with full support for $B_{\boldsymbol{w}}$.

Proof. — Let $(\xi_n)_{n\geqslant 0}$ be a sequence of independent $\mathbb{K}$ - valued random variables with law μ_0. As a measure on $\mathbb{K}^{\mathbb{Z}_+}$, m is the distribution of the $\mathbb{K}^{\mathbb{Z}_+}$ - valued random variable $\xi := \xi_0 e_0 + \sum_{n=1}^{\infty} \frac{\xi_n}{w_1 \cdots w_n} e_n$. Since $m(\ell_p) = 1$, we have $\xi \in \ell_p$ almost surely, which means that the series defining ξ is almost surely convergent with respect to the ℓ_p - norm. Moreover, since μ_0 has full support, the measure m has full support. By [1, []Proposition 2.5], it follows that m is an ergodic measure (in fact, a strongly mixing measure) for $B_{\boldsymbol{w}}$. □

5.2. Non-orthogonality of product measures. — A basic tool in the study of product measures is a famous classical result of Kakutani [17] giving a necessary and sufficient condition for the equivalence of two infinite product probability measures $\nu = \otimes_{n\geqslant 0}\nu_n$ and $\nu' = \otimes_{n\geqslant 0}\nu'_n$ defined on a product of measurable spaces $(\Omega, \mathcal{B}) = \otimes_{n\geqslant 0}(\Omega_n, \mathcal{B}_n)$.

Let us first recall the definition of the so-called *Hellinger integral* $H(\alpha, \beta)$ of two probability measures α, β on some measurable space $(T, \mathcal{T})$:

$$H(\alpha,\beta) = \int_T \sqrt{\frac{d\alpha}{d\tau}}\sqrt{\frac{d\beta}{d\tau}}\, d\tau,$$

where τ is any sigma-finite positive measure on $(T, \mathcal{T})$ such that α and β are absolutely continuous with respect to τ. If we write $\alpha = \frac{d\alpha}{d\beta}\,\beta + \nu$ the Lebesgue-Nikodym decomposition of α with respect to β, then $\left(\frac{d\alpha}{d\tau} - \frac{d\alpha}{d\beta}\frac{d\beta}{d\tau}\right)\frac{d\beta}{d\tau} = 0$ τ-a.e. since $\nu \perp \beta$, so $\frac{d\alpha}{d\tau}\frac{d\beta}{d\tau} = \frac{d\alpha}{d\beta}\left(\frac{d\beta}{d\tau}\right)^2$ and hence

$$H(\alpha,\beta) = \int_T \sqrt{\frac{d\alpha}{d\beta}}\, d\beta;$$

which shows that $H(\alpha, \beta)$ is indeed independent of the choice of τ. Note that we always have $0 \leqslant H(\alpha, \beta) \leqslant 1$ by the Cauchy-Schwarz inequality; and that $H(\alpha, \beta) = 1$ if and only if $\alpha = \beta$, whereas $H(\alpha, \beta) = 0$ if and only if $\alpha \perp \beta$.

Kakutani's Theorem as we will need it can now be stated as follows.

Theorem 5.6. — *Let $\nu = \otimes_{n\geqslant 0}\nu_n$ and $\nu' = \otimes_{n\geqslant 0}\nu'_n$ be two product probability measures on $(\Omega, \mathcal{B}) = \otimes_{n\geqslant 0}(\Omega_n, \mathcal{B}_n)$. The measures ν and ν' are non-orthogonal if and only if*

$$\prod_{n=0}^{\infty} H(\nu_n, \nu'_n) > 0.$$

Moreover, under the assumption that $\nu_n \sim \nu'_n$ for each $n \geqslant 0$, the measures ν and ν' are either orthogonal or equivalent.

Note that this is not exactly what is proved in [17]: the main result of [17] states that under the assumption that $\nu_n \sim \nu'_n$ for all n, the measures ν and ν' are equivalent if $\prod_{n=0}^{\infty} H(\nu_n, \nu'_n) > 0$ and orthogonal otherwise. However, the above version of the theorem is certainly well-known (it is stated for example in [21]), and it can be obtained by slight modifications of Kakutani's original proof; see Section 6 for more details.

5.3. Gaussian product measures. — A particularly interesting class of product measures is that of *Gaussian product measures*. Our definition will be very restrictive: we will say that a product measure $\mu = \otimes_{n\geqslant 0}\mu_n$ on $\Omega = \mathbb{K}^{\mathbb{Z}_+}$ is a Gaussian product measure if for each $n \geqslant 0$, the measure μ_n is a Gaussian measure on $\mathbb{K}$ of the form $\mathcal{N}(0, \sigma_n^2)$ for some $\sigma_n > 0$, *i.e.* μ_n is a centered Gaussian measure with covariance matrix $\sigma_n^2 I_{\mathbb{K}}$. Accordingly, we say that a measure m on $X = \ell_p$ is a Gaussian product measure if m is the restriction to ℓ_p of a Gaussian product measure μ on Ω such that $\mu(\ell_p) = 1$.

By Fernique's integrability theorem (see e.g. [9]), if m is a Gaussian measure on $X = \ell_p$, then m admits moments of all orders and in particular $\int_X \|x\|^p dm < \infty$. It follows that if $\mu = \otimes_{n\geqslant 0}\mu_n$ is a Gaussian product measure on $\Omega = \mathbb{K}^{\mathbb{Z}_+}$, then μ is supported on ℓ_p if and only if $\int_\Omega \left(\sum_{n=0}^{\infty} |t_n|^p\right) d\mu(t) < \infty$. Moreover, if μ is invariant under some weighted shift $B_{\boldsymbol{w}}$, then $\int_\Omega |t_n|^p d\mu(t) = \frac{1}{|w_1\cdots w_n|^p}\int_{\mathbb{K}} |t|^p d\mu_0(t)$ for all $n \geqslant 1$ by Lemma 5.2. Altogether, we

see that a weighted shift $B_{\boldsymbol{w}}$ acting on ℓ_p admits an invariant Gaussian product measure if and only if $\sum_{n=0}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$, and that if this holds, then any $B_{\boldsymbol{w}}$-invariant Gaussian product measure on $\mathbb{K}^{\mathbb{Z}_+}$ is supported on ℓ_p. (For the "only if" part, one could also have used Proposition 2.1 rather than Fernique's integrability theorem.) We refer the reader to [2], [3], [5] and [4, Chapter 5] for an in-depth study of invariant Gaussian measures for operators on Hilbert or Banach spaces. We just point out here that if $\boldsymbol{w}$ is a weight sequence such that $\sum_{n=0}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$ and if m is a $B_{\boldsymbol{w}}$-invariant Gaussian product measure on ℓ_p, then m is an ergodic measure with full support for $B_{\boldsymbol{w}}$ (see Fact 5.5).

The next theorem will provide a necessary and sufficient condition for two weighted backward shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ on $X = \ell_p$ to admit equivalent invariant Gaussian product measures.

Theorem 5.7. — *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two weight sequences, and let $\mu_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{v},n}$ be two Gaussian product measures on $\mathbb{K}^{\mathbb{Z}_+}$ invariant under $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ respectively, with $\mu_{\boldsymbol{u},0} = \mathcal{N}(0,\sigma^2)$ and $\mu_{\boldsymbol{v},0} = \mathcal{N}(0,\sigma'^2)$. Then $\mu_{\boldsymbol{u}}$ and $\mu_{\boldsymbol{v}}$ are either equivalent or orthogonal, and they are equivalent if and only if*

$$\sum_{n=1}^{\infty} \left(1 - \frac{\sigma'}{\sigma}\,\frac{|u_1 \cdots u_n|}{|v_1 \cdots v_n|}\right)^2 < \infty. \tag{5.1}$$

From this result, we immediately deduce

Corollary 5.8. — *Let $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ be two weighted shifts on $X = \ell_p$, where the weight sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ are such that $\sum_{n=0}^{\infty} \frac{1}{|u_1 \cdots u_n|^p} < \infty$ and $\sum_{n=0}^{\infty} \frac{1}{|v_1 \cdots v_n|^p} < \infty$. The following are equivalent.*

(a) *$B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit equivalent invariant Gaussian product measures.*

(b) *There exists a constant $\kappa > 0$ such that $\sum_{n=1}^{\infty} \left(1 - \kappa\,\frac{|u_1 \cdots u_n|}{|v_1 \cdots v_n|}\right)^2 < \infty$.*

Here is another consequence of Theorem 5.7.

Corollary 5.9. — *Let $\boldsymbol{\Lambda}$ be a countable family of weight sequences. Assume that $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$ for all $\boldsymbol{w} \in \boldsymbol{\Lambda}$ and $\sum_{n=1}^{\infty} \left(1 - \frac{|u_1 \cdots u_n|}{|v_1 \cdots v_n|}\right)^2 < \infty$ for any $\mathbf{u}, \mathbf{v} \in \boldsymbol{\Lambda}$. Then, there exists a Gaussian measure with full support m on $X = \ell_p$ with the following property: given any sequence of Borel sets $(A_i)_{i\geqslant 0} \subseteq X$ with $m(A_i) > 0$, there exists $x \in X$ such that for every $\boldsymbol{w} \in \boldsymbol{\Lambda}$ and all $i \geqslant 0$, the set $\mathcal{N}_{B_{\boldsymbol{w}}}(x, A_i) := \{n \in \mathbb{N};\ B_{\boldsymbol{w}}^n x \in A_i\}$ has a positive density.*

Proof. — For each $\boldsymbol{w} \in \boldsymbol{\Lambda}$, let $m_{\boldsymbol{w}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{w},n}$ be the $B_{\boldsymbol{w}}$-invariant Gaussian product measure on ℓ_p defined by $\mu_{\boldsymbol{w},0} := \mathcal{N}(0,1)$. This is an ergodic measure with full support for $B_{\boldsymbol{w}}$. Since the measures $m_{\boldsymbol{w}}$, $\boldsymbol{w} \in \boldsymbol{\Lambda}$ are pairwise equivalent by Theorem 5.7 and since $\boldsymbol{\Lambda}$ is countable, the result follows from the pointwise ergodic theorem, taking $m := m_{\boldsymbol{w}_0}$ for any $\boldsymbol{w}_0 \in \boldsymbol{\Lambda}$. □

Remark 5.10. — Taking as $(A_i)_{i\geqslant 0}$ a countable basis of open sets for X in Corollary 5.9, we get a vector $x \in X$ which is frequently hypercyclic for all operators $B_{\boldsymbol{w}}$, $\boldsymbol{w} \in \boldsymbol{\Lambda}$. However, it turns out that something much stronger holds true: if $\boldsymbol{u}$ and $\boldsymbol{v}$ are two weight sequences such that $\frac{u_1 \cdots u_n}{v_1 \cdots v_n}$ has a non-zero limit as $n \to \infty$, then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ have in fact

the same frequently hypercyclic vectors. This result, which is due to S. Charpentier and the third author, will be proved in Section 6.

Before proving Theorem 5.7, we use Corollary 5.8 to give an example of two non-orthogonal weighted shifts sharing no non-zero periodic points.

Example 5.11. — Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be the weight sequences defined as follows:

$$\boldsymbol{u}_n := 2 \qquad \text{and} \qquad \boldsymbol{v}_n := 2\,\frac{1+\varepsilon_n}{1+\varepsilon_{n-1}},$$

where $\varepsilon_0 = 0$ and $(\varepsilon_n)_{n\geqslant 1}$ is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n^2 < \infty$. Then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on $X = \ell_p$ share no non-zero periodic point, but they admit equivalent invariant Gaussian product measures (and hence they are not orthogonal).

Proof. — We have $u_1 \cdots u_n = 2^n$ and $v_1 \cdots v_n = 2^n(1+\varepsilon_n)$ for all $n \geqslant 1$. In particular, $\sum_{n=1}^{\infty} \frac{1}{(u_1\cdots u_n)^p} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{(v_1\cdots v_n)^p} < \infty$. Since $\frac{u_1\cdots u_n}{v_1\cdots v_n} = \frac{1}{(1+\varepsilon_n)}$ is increasing, the conditions of Theorem 4.2 cannot be satisfied, so $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ do not share any non-zero periodic point. However

$$1 - \frac{u_1\cdots u_n}{v_1\cdots v_n} = 1 - \frac{1}{1+\varepsilon_n} \sim \varepsilon_n \qquad \text{as } n \to \infty,$$

so $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit equivalent invariant Gaussian product measures by Corollary 5.8. □

It is now time to prove Theorem 5.7.

Proof of Theorem 5.7. — In what follows, we set $d := 1$ if $\mathbb{K} = \mathbb{R}$ and $d := 2$ if $\mathbb{K} = \mathbb{C}$.

By Kakutani's Theorem, we just have to show that $\prod_{n=1}^{\infty} H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) > 0$ if and only if (5.1) is satisfied.

By Lemma 5.2, the measures $\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v},n}$ are uniquely determined by $\mu_{\boldsymbol{u},0} = \mathcal{N}(0,\sigma^2)$ and $\mu_{\boldsymbol{v},0} = \mathcal{N}(0,\sigma'^2)$. Explicitely,

$$\mu_{\boldsymbol{u},n} = \mathcal{N}(0,\sigma_n^2) \qquad \text{with} \quad \sigma_n^2 = \frac{\sigma^2}{|u_1\cdots u_n|^2}$$

and

$$\mu_{\boldsymbol{v},n} = \mathcal{N}(0,\sigma_n'^2) \qquad \text{with} \quad \sigma_n'^2 = \frac{\sigma'^2}{|v_1\cdots v_n|^2}.$$

The computation of $H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n})$ now relies on the following fact.

Fact 5.12. — If μ and μ' are two Gaussian measures on $\mathbb{K}$ of the form $\mu = \mathcal{N}(0,\sigma^2)$ and $\mu' = \mathcal{N}(0,\sigma'^2)$, then

$$H(\mu,\mu') = \left(\frac{2\sigma\sigma'}{\sigma^2+\sigma'^2}\right)^{d/2} = \left(\frac{2(\sigma'/\sigma)}{1+(\sigma'/\sigma)^2}\right)^{d/2}.$$

Proof of Fact 5.12. — The measures μ and μ' are absolutely continuous with respect to Lebesgue measure on $\mathbb{K}$ with densities $\frac{1}{(2\pi\sigma^2)^{d/2}} e^{-|t|^2/2\sigma^2}$ and $\frac{1}{(2\pi\sigma'^2)^{d/2}} e^{-|t|^2/2\sigma'^2}$. Hence,

$$H(\mu,\mu') = \frac{1}{(2\pi\sigma\sigma')^{d/2}} \int_{\mathbb{R}} e^{-\frac{1}{4}\left(\frac{1}{\sigma^2}+\frac{1}{\sigma'^2}\right)|t|^2} dt = \frac{1}{(2\pi\sigma\sigma')^{d/2}} \times \left(\frac{4\pi}{\frac{1}{\sigma^2}+\frac{1}{\sigma'^2}}\right)^{d/2}.$$

□

Going back to the measures $m_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{v},n}$, let us set for each $n \geqslant 1$:

$$\lambda_n := \frac{\sigma'_n}{\sigma_n} = \frac{\sigma'}{\sigma}\,\frac{|u_1\cdots u_n|}{|v_1\cdots v_n|}.$$

By Fact 5.12, we have

$$H(\mu_{\boldsymbol{u},n},\mu_{\boldsymbol{v},n}) = \left(\frac{2\,\lambda_n}{1+\lambda_n^2}\right)^{d/2}.$$

Since $0 \leqslant H(\mu_{\boldsymbol{u},n},\mu_{\boldsymbol{v},n}) \leqslant 1$ for all $n \geqslant 1$, it follows that

$$\prod_{n=1}^{\infty} H(\mu_{\boldsymbol{u},n},\mu_{\boldsymbol{v},n}) > 0 \quad \text{if and only if} \quad \sum_{n=1}^{\infty}\left(1-\left(\frac{2\,\lambda_n}{1+\lambda_n^2}\right)^{d/2}\right) < \infty.$$

This can be satisfied only if $\frac{2\,\lambda_n}{1+\lambda_n^2} \to 1$ as $n\to\infty$; and an examination of the function $\Psi(\lambda) := \frac{2\,\lambda}{1+\lambda^2}$ shows that this is equivalent to the fact that $\lambda_n \to 1$. In this case, writing $\lambda_n = 1+u_n$ and using Taylor's formula, we see that

$$1-\left(\frac{2\,\lambda_n}{1+\lambda_n^2}\right)^{d/2} \sim \frac{d}{4}\,u_n^2 = \frac{d}{4}\,(1-\lambda_n)^2.$$

So we get that

$$\prod_{n=1}^{\infty} H(\mu_{\boldsymbol{u},n},\mu_{\boldsymbol{v},n}) > 0 \quad \text{if and only if} \quad \sum_{n=1}^{\infty}(1-\lambda_n)^2 < \infty,$$

which is (5.1). Theorem 5.7 is proved. □

5.4. Products of absolutely continuous measures. — It is natural to believe that some condition strictly weaker than (b) of Corollary 5.8 might still yield the existence of (non-Gaussian) equivalent non-trivial invariant product measures for the backward shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$. The next theorem shows that this is in fact not the case, at least for product measures whose marginals are absolutely continuous with respect to Lebesgue measure.

We point out that it is possible to deduce parts (1ii) and (2) of this theorem from a Theorem of Shepp [21] concerning (non-)orthogonality of translates of a product measure on $\mathbb{R}^{\mathbb{Z}_+}$, namely [21, Theorem 1]. However, we are going to give here a self-contained proof which looks rather different from what is done in [21] (except, of course, for the use of Kakutani's Theorem), and we will indicate in Section 6 how Shepp's result can be used in our context; see Theorem 6.11.

For simplicity, we will assume that $\mathbb{K} = \mathbb{R}$ and that all the weights are positive.

Theorem 5.13. — *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two positive weight sequences. Assume that the backward shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on the real space $X = \ell_p$ admit invariant product measures $m_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{v},n}$ such that $\mu_{\boldsymbol{u},0}$ and $\mu_{\boldsymbol{v},0}$ are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, $\mu_{\boldsymbol{u},0} = \mathtt{p}(t)dt$ and $\mu_{\boldsymbol{v},0} = \mathtt{q}(t)dt$.*

(1) *If $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ are not orthogonal, then:*

  (i) *the quotient $\frac{u_1\cdots u_n}{v_1\cdots v_n}$ has a limit $a \in (0,\infty)$ as $n\to\infty$, and $\mathtt{q}(t) = a\mathtt{p}(at)$ almost everywhere;*

  (ii) *we have $\sum_{n=1}^{\infty}\left(1-\frac{1}{a}\,\frac{u_1\cdots u_n}{v_1\cdots v_n}\right)^2 < \infty$.*

(2) *Assume that the function* $f := \sqrt{\mathtt{p}}$ *is continuous on* $\mathbb{R}\backslash\{0\}$ *and* $\mathcal{C}^1$*-smooth except at a finite number of points with* $tf'(t) \in L_2(\mathbb{R})$*, and that* $\mathtt{q}(t) = a\mathtt{p}(at)$ *for some* $a \in (0,\infty)$*. Then,* $m_{\boldsymbol{u}}$ *and* $m_{\boldsymbol{v}}$ *are non-orthogonal* if and only if $\sum_{n=1}^{\infty} \left(1 - \frac{1}{a}\frac{u_1\cdots u_n}{v_1\cdots v_n}\right)^2 < \infty$.

(3) *Assume that* $f = \sqrt{\mathtt{p}}$ *is* $\mathcal{C}^1$*-smooth except at a finite number of points with* $tf'(t) \in L_2(\mathbb{R})$*, that* $\mathtt{p}$ *has at least one discontinuity point in* $\mathbb{R}\backslash\{0\}$*, and that* $\mathtt{q}(t) = a\mathtt{p}(at)$ *for some* $a \in (0,\infty)$*. Then,* $m_{\boldsymbol{u}}$ *and* $m_{\boldsymbol{v}}$ *are non-orthogonal if and only if* $\sum_{n=1}^{\infty} \left|1 - \frac{1}{a}\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| < \infty$.

From part (1) of this theorem and Theorem 5.7, we immediately deduce

Corollary 5.14. — *Let* $B_{\boldsymbol{u}}$ *and* $B_{\boldsymbol{v}}$ *be two shifts with positive weights acting on the real space* ℓ_p*. The following are equivalent.*

(a) $B_{\boldsymbol{u}}$ *and* $B_{\boldsymbol{v}}$ *are not orthogonal with respect to product measures whose marginals are absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}$*.*

(b) *There exists* $\kappa > 0$ *such that* $\sum_{n=1}^{\infty} \left(1 - \kappa\frac{u_1\cdots u_n}{v_1\cdots v_n}\right)^2 < \infty$.

(c) $B_{\boldsymbol{u}}$ *and* $B_{\boldsymbol{v}}$ *admit equivalent invariant Gaussian product measures.*

We point out that Corollary 5.14 is also valid in the complex case and for not necessarily positive weights; see Corollary 6.12.

For the proof of Theorem 5.13, we will need the following lemma.

Lemma 5.15. — *Let* $h \in L_2(\mathbb{R})$ *be non-zero and real-valued, and let* $\mathcal{P}h : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$\mathcal{P}h(\alpha) := \int_{\mathbb{R}} h(x)h(x+\alpha)\,dx.$$

(a) *The function* $\mathcal{P}h$ *is continuous and even, with* $\mathcal{P}h(\alpha) \leqslant \|h\|_2^2$ *for avery* $\alpha \in \mathbb{R}$ *and* $\mathcal{P}h(0) = \|h\|_2^2$*.*

(b) *If* h *belongs to the Sobolev space* $W^{1,2}(\mathbb{R})$*, then* $\mathcal{P}h$ *is* $\mathcal{C}^2$*-smooth on* $\mathbb{R}$*, with* $(\mathcal{P}h)'(0) = 0$ *and* $(\mathcal{P}h)''(0) < 0$*. In particular,* $\|h\|_2^2 - \mathcal{P}h(\alpha) \sim c\alpha^2$ *as* $\alpha \to 0$*, for some constant* $c > 0$*.*

(c) *If* $h \notin W^{1,2}(\mathbb{R})$*, then* $(\|h\|_2^2 - \mathcal{P}h(\alpha))/\alpha^2 \to \infty$ *as* $\alpha \to 0$*.*

(d) *Assume that* h *is* $\mathcal{C}^1$*-smooth except at finitely many points, with* $h' \in L_2(\mathbb{R})$*, and that* h *has at least one discontinuity point. Then* $\mathcal{P}h$ *is left-differentiable and right-differentiable at* 0*, with* $(\mathcal{P}h)'(0^-) > 0$ *and* $(\mathcal{P}h)'(0^+) < 0$*.*

This lemma has the following immediate consequence.

Corollary 5.16. — *For any non-zero real-valued* $h \in L_2(\mathbb{R})$*, there exists a constant* $c > 0$ *such that* $\|h\|_2^2 - \mathcal{P}h(\alpha) \geqslant c\alpha^2$ *for* α *sufficiently close to* 0*.*

Note that using Plancherel's formula, it can be seen that this can be stated equivalently as follows: for any $g \in L_2(\mathbb{R})\backslash\{0\}$ with real-valued Fourier transform, there exists a constant c such that $\int_{\mathbb{R}} |g(t)|^2\big(1 - \cos(\alpha t)\big)\,dt \geqslant c\alpha^2$ for α sufficiently close to 0.

Proof of Lemma 5.15. — For notational brevity, let us set $F := \mathcal{P}h$.

(a) We have $F(\alpha) = \langle h, \tau_\alpha h\rangle_{L_2(\mathbb{R})}$, where $\tau_\alpha h(x) := h(x+\alpha)$, so F is continuous by continuity of the map $\alpha \mapsto \tau_\alpha h$ from $\mathbb{R}$ into $L_2(\mathbb{R})$. It is clear that F is even and that $F(0) = \|h\|_2^2$. Finally, $F(\alpha) \leqslant \|h\|_2^2$ by the Cauchy-Schwarz inequality.

(b) Since $h \in W^{1,2}(\mathbb{R})$, the map $\alpha \mapsto \tau_\alpha h$ is $\mathcal{C}^1$-smooth from $\mathbb{R}$ into $L^2(\mathbb{R})$, with derivative $\alpha \mapsto \tau_\alpha h'$. Since $F(\alpha) = \langle h, \tau_\alpha h\rangle_{L_2(\mathbb{R})}$, it follows that F is $\mathcal{C}^1$-smooth on $\mathbb{R}$, with

$$F'(\alpha) = \langle h, \tau_\alpha h'\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} h(u)h'(u+\alpha)\,du = \int_{\mathbb{R}} h(u-\alpha)h'(u)\,du.$$

Hence, the same argument shows that F is in fact $\mathcal{C}^2$-smooth on $\mathbb{R}$, with

$$F''(\alpha) = -\int_{\mathbb{R}} h'(u-\alpha)h'(u)\,du.$$

We have $F'(0) = 0$ because F is even; and

$$F''(0) = -\int_{\mathbb{R}} h'^2 < 0.$$

The strict inequality holds because if we had $F''(0) = 0$, then h' would be equal to 0 almost everywhere, so h would be 0 since it belongs to $W^{1,2}(\mathbb{R})$.

(c) Note that

$$F(\alpha) = \langle h, \tau_\alpha h\rangle_{L_2(\mathbb{R})} = \frac{1}{2}\left(\|h\|_2^2 + \|\tau_\alpha h\|^2 - \|h - \tau_\alpha h\|_2^2\right) = \|h\|_2^2 - \frac{1}{2}\|h - \tau_\alpha h\|_2^2,$$

so that $\|h\|_2^2 - F(\alpha) = \frac{1}{2}\|h - \tau_\alpha h\|_2^2$. Now since $h \notin W^{1,2}(\mathbb{R})$, we have $\lim_{\alpha\to 0}\left\|\frac{\tau_\alpha h - h}{\alpha}\right\| = \infty$. Indeed, otherwise one could find a sequence (α_n) tending to 0 such that $\frac{\tau_{\alpha_n} h - h}{\alpha_n}$ has a weak limit $u \in L_2(\mathbb{R})$, and this would give that $h \in W^{1,2}(\mathbb{R})$ with $h' = u$. This proves (c).

(d) Since F is an even function, it is enough to show that F is right-differentiable at 0 with $F'(0^+) < 0$.

Let $u_0 < \cdots < u_N$ be the discontinuity points of h. Also, let $I_0 := (-\infty, u_0)$, $I_{N+1} := (u_n, \infty)$ and $I_k := (u_{k-1}, u_k)$ for $1 \leqslant k \leqslant N$. Note that by assumption on h, the restriction of h to each interval I_k belongs to the Sobolev space $W^{1,2}(I_k)$. In particular, h has a left limit $h(u_k^-)$ and a right limit $h(u_k^+)$ at each point u_k.

For every $\alpha > 0$, we have

$$F(\alpha) = \sum_{k=0}^{N}\int_{I_k} h(u)h(u+\alpha)\,du =: \sum_{k=0}^{N+1} F_k(\alpha).$$

We consider separately the functions F_0, F_{N+1} and F_k for $1 \leqslant k \leqslant N$.

Let us start with F_k, $1 \leqslant k \leqslant N$. For $0 < \alpha < u_k - u_{k-1}$, we write

$$F_k(\alpha) = \int_{u_{k-1}}^{u_k-\alpha} h(u)h(u+\alpha)\,du + \int_{u_k-\alpha}^{u_k} h(u)h(u+\alpha)\,du =: F_{k,1}(\alpha) + F_{k,2}(\alpha).$$

Consider the open triangle $\Omega := \{(\alpha,\beta);\ u_{k-1} < \beta < u_k \text{ and } 0 < \alpha < u_k - \beta\}$, and the function $G : \overline{\Omega} \to \mathbb{R}$ defined by $G(\alpha,\beta) := \int_{u_{k-1}}^{\beta} h(u)h(u+\alpha)\,du$. Since the restriction of h to $I_k = (u_{k-1}, u_k)$ belongs to the Sobolev space $W^{1,2}(I_k)$, the map G is $\mathcal{C}^1$-smooth on Ω, with $\partial_\alpha G(\alpha,\beta) = \int_{u_{k-1}}^{\beta} h(u)h'(u+\alpha)\,du$ and $\partial_\beta G(\alpha,\beta) = h(\beta)h(\beta+\alpha)$. Moreover, G is continuous on $\overline{\Omega}$ and its partial derivatives extend continuously to $\overline{\Omega}$, with $\partial_\alpha G(\alpha, u_k - \alpha) = \int_{u_{k-1}}^{u_k-\alpha} h(u)h'(u+\alpha)\,du$ and $\partial_\beta G(\alpha, u_k - \alpha) = h(u_k - \alpha)h(u_k^-)$ for $0 < \alpha < u_k - u_{k-1}$. It follows that $F_{k,1}(\alpha) = G(\alpha, u_k - \alpha)$ is $\mathcal{C}^1$-smooth on $(0, u_k - u_{k-1})$, with $F'_{k,1}(\alpha) = \int_{u_{k-1}}^{u_k-\alpha} h(u)h'(u+\alpha)\,du - h(u_k-\alpha)h(u_k^-)$. To see this, observe that for any $\alpha_0 \in (0, u_k - u_{k-1})$, the function $G_n(\alpha) := G(\alpha, u_k - \alpha - \frac{1}{n})$ is well-defined in a neighbourhood

of α_0 if n is large enough, that $G_n(\alpha) \to F_{k,1}(\alpha)$ uniformly in a neighbourhood of α_0 as $n \to \infty$, and that $G_n'(\alpha) \to \int_{u_{k-1}}^{u_k-\alpha} h(u)h'(u+\alpha)\,du - h(u_k-\alpha)h(u_k^-)$ uniformly in a neighbourhood of α_0. Since $F_{k,1}'(\alpha) \to \int_{u_{k-1}}^{u_k} h(u)h'(u)\,du - h(u_k^-)^2$ as 0^+, we deduce that $F_{k,1}$ is right-differentiable at 0 with

$$\begin{aligned} F_{k,1}'(0^+) &= \int_{u_{k-1}}^{u_k} h(u)h'(u)\,du - h(u_k^-)^2 \\ &= \left[\frac{1}{2}h(u)^2\right]_{u_{k-1}}^{u_k} - h(u_k^-)^2 \\ &= -\frac{1}{2}\big(h(u_k^-)^2 + h(u_{k-1}^+)^2\big). \end{aligned}$$

Similar arguments show that $F_{k,2}(\alpha) = \int_{u_k-\alpha}^{u_k} h(u)h(u+\alpha)\,du$ is $\mathcal{C}^1$- smooth on $(0,c)$ for some $c>0$, with $F_{k,2}'(\alpha) = \int_{u_k-\alpha}^{u_k} h(u)h'(u+\alpha)\,du + h(u_k-\alpha)h(u_k^+)$; and it follows that $F_{k,2}$ is right-differentiable at 0 with $F_{k,2}'(0^+) = h(u_k^-)h(u_k^+)$. Hence, F_k is right-differentiable at 0, with

$$F_k'(0^+) = h(u_k^-)h(u_k^+) - \frac{1}{2}\big(h(u_k^-)^2 + h(u_{k-1}^+)^2\big).$$

One shows in the same way that $F_0(\alpha) = \int_{-\infty}^{u_0} h(u)h(u+\alpha)\,du$ and $F_{N+1}(\alpha) = \int_{u_N}^{\infty} h(u)h(u+\alpha)\,du$ are right-differentiable at 0, with

$$F_0'(0^+) = h(u_0^-)h(u_0^+) - \frac{1}{2}h(u_0^-)^2 \qquad \text{and} \qquad F_{N+1}'(0^+) = -\frac{1}{2}h(u_N^+)^2.$$

Altogether, we see that $F = \sum_{k=0}^{N+1} F_k$ is right-differentiable at 0, with

$$\begin{aligned} F'(0^+) &= h(u_0^-)h(u_0^+) - \frac{1}{2}h(u_0^-)^2 \\ &\qquad + \sum_{k=1}^{N}\left(h(u_k^-)h(u_k^+) - \frac{1}{2}\big(h(u_k^-)^2 + h(u_{k-1}^+)^2\big)\right) - \frac{1}{2}h(u_N^+)^2 \\ &= \sum_{k=0}^{N} h(u_k^-)h(u_k^+) - \frac{1}{2}\sum_{k=0}^{N}\big(h(u_k^-)^2 + h(u_k^+)^2\big) \\ &= -\frac{1}{2}\sum_{k=0}^{N}\big(h(u_k^+) - h(u_k^-)\big)^2 < 0. \end{aligned}$$

□

We can now give the proof of Theorem 5.13.

Proof of Theorem 5.13. — In what follows, we set $f := \sqrt{\mathtt{p}}$ and $g := \sqrt{\mathtt{q}}$. Also, for each $n \geqslant 1$, let

$$\lambda_n := \frac{u_1 \cdots u_n}{v_1 \cdots v_n}.$$

The following simple computation will be essential for the proof.

Fact 5.17. — For every $n \geqslant 1$, we have

$$H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) = \Psi(\lambda_n),$$

where $\Psi : (0,\infty) \to \mathbb{R}$ is the function defined by

$$\Psi(\lambda) := \sqrt{\lambda} \int_{\mathbb{R}} f(\lambda t) g(t)\, dt.$$

Proof of Fact 5.17. — Denoting by τ the Lebesgue measure on $\mathbb{R}$, we have

$$\frac{d\mu_{\boldsymbol{u},n}}{d\tau}(t) = u_1 \cdots u_n\, \mathtt{p}(u_1 \cdots u_n t) \qquad \text{and} \qquad \frac{d\mu_{\boldsymbol{v},n}}{d\tau}(t) = v_1 \cdots v_n\, \mathtt{q}(v_1 \cdots v_n t).$$

Hence

$$\begin{aligned} H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) &= \int_{\mathbb{R}} \sqrt{u_1 \cdots u_n}\, f(u_1 \cdots u_n t) \times \sqrt{v_1 \cdots v_n}\, g(v_1 \cdots v_n t)\, dt \\ &= \sqrt{\frac{u_1 \cdots u_n}{v_1 \cdots v_n}} \int_{\mathbb{R}} f\left(\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\, s\right) g(s)\, ds \\ &= \Psi(\lambda_n). \end{aligned}$$

□

By Fact 5.17 and Kakutani's Theorem, we know that $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ are not orthogonal if and only if

$$\sum_{n=1}^{\infty} (1 - \Psi(\lambda_n)) < \infty; \tag{5.2}$$

and in that case, we have in particular that

$$\Psi(\lambda_n) \to 1 \quad \text{as } n \to \infty. \tag{5.3}$$

Proof of (1i) *in Theorem 5.13.* — We first note that we must have $\underline{\lim}\, \lambda_n > 0$ and $\overline{\lim}\, \lambda_n < \infty$. Indeed, otherwise $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are orthogonal with respect to measures not charging $\ker(e_0^*)$ by Proposition 3.8. The measures $\mu_{\boldsymbol{u}}$ and $\mu_{\boldsymbol{v}}$ have this property since $\mu_{\boldsymbol{u},0}$ and $\mu_{\boldsymbol{v},0}$ do not charge $\{0\}$, so this cannot happen.

Now, the key point is the following fact.

Fact 5.18. — The function Ψ is continuous on $(0,\infty)$ with $0 \leqslant \Psi(\lambda) \leqslant 1$, there is at most one $\lambda \in (0,\infty)$ such that $\Psi(\lambda) = 1$, and if $\Psi(\lambda) = 1$ then $\mathtt{q}(t) = \lambda \mathtt{p}(\lambda t)$ almost everywhere..

Proof of Fact 5.18. — It is clear that $\Psi(\lambda) \geqslant 0$. Moreover, by the Cauchy-Schwarz inequality, we have

$$\Psi(\lambda) \leqslant \left(\int_{\mathbb{R}} \lambda \mathtt{p}(\lambda t)\, dt\right)^{1/2} \left(\int_{\mathbb{R}} \mathtt{q}(t)\, dt\right)^{1/2} = 1.$$

If $\Psi(\lambda) = 1$ then, by the equality case in Cauchy-Schwarz's inequality and since $\int_{\mathbb{R}} \mathtt{p} = \int_{\mathbb{R}} \mathtt{q} = 1$, we must have $\mathtt{q}(t) = \lambda \mathtt{p}(\lambda t)$ almost everywhere. It follows easily that there can be at most one λ such that $\Psi(\lambda) = 1$. Indeed, assume that $\Psi(\lambda) = 1 = \Psi(\lambda')$ for some $0 < \lambda < \lambda'$. Then $\lambda \mathtt{p}(\lambda t) = \lambda' \mathtt{p}(\lambda' t)$ almost everywhere. Setting $c := \lambda'/\lambda > 1$, it follows that $\mathtt{p}(x) = c\, \mathtt{p}(cx)$ almost everywhere. Hence $\mathtt{p}(x) = c^n \mathtt{p}(c^n x)$ almost everywhere for every $n \in \mathbb{Z}$. So, with $I_n := (c^n, c^{n+1}]$, we obtain that

$$\int_{I_n} \mathtt{p}(x)\, dx = \int_{I_0} c^n \mathtt{p}(c^n x)\, dx = \int_{I_0} \mathtt{p}(x)\, dx \qquad \text{for all } n \in \mathbb{Z}.$$

Since the intervals I_n form a partition of $(0,\infty)$ and $\int_0^\infty \mathtt{p}(x)\,dx < \infty$, this implies that $\mathtt{p}(x) = 0$ almost everywhere on $(0,\infty)$. Similarly $\mathtt{p}(x) = 0$ almost everywhere on $(-\infty, 0)$, hence $\mathtt{p}(x) = 0$ almost everywhere on $\mathbb{R}$, which is a contradiction since $\int_{\mathbb{R}} \mathtt{p} = 1$.

Finally, to prove that Ψ is continuous, we note that $\Psi(\lambda) = \sqrt{\lambda}\langle f_\lambda, g\rangle_{L^2(\mathbb{R})}$, where $f_\lambda(t) := f(\lambda t)$. Since $f \in L^2(\mathbb{R})$, the map $\lambda \mapsto f_\lambda$ is continuous from $(0,\infty)$ into $L^2(\mathbb{R})$; so Ψ is indeed continuous. □

It is now easy to conclude the proof of (1i). Since $\Psi(\lambda_n) \to 1$ and Ψ is continuous on $(0,\infty)$, any cluster point $\lambda \in (0,\infty)$ of the sequence (λ_n) must be such that $\Psi(\lambda) = 1$; hence, (λ_n) has at most one cluster point in $(0,\infty)$ by Fact 5.18. Since $0 < \underline{\lim}\,\lambda_n \leqslant \overline{\lim}\,\lambda_n < \infty$, it follows that (λ_n) has a limit $a \in (0,\infty)$; and by Fact 5.18 again, we have ($\Psi(a) = 1$ and) $\mathtt{q}(t) = a\mathtt{p}(at)$ almost everywhere. □

Proof of (1ii) *in Theorem 5.13.* — Let $\Phi : (0,\infty) \to \mathbb{R}$ and $\Theta : (0,\infty) \to \mathbb{R}$ be defined by

$$\Phi(\lambda) := \int_{\mathbb{R}} f(t)f(\lambda t)\,dt \qquad \text{and} \qquad \Theta(\lambda) := \sqrt{\lambda}\,\Phi(\lambda).$$

Since $g(t) = \sqrt{a}f(at)$ almost everywhere by (1i), we have

$$\Psi(\lambda) = \sqrt{\lambda}\int_{\mathbb{R}} f(\lambda t) \times \sqrt{a}f(at)\,dt = \sqrt{\lambda/a}\,\Phi(\lambda/a) = \Theta(\lambda/a).$$

So, we see that $0 \leqslant \Theta(\lambda) \leqslant 1$ for every $\lambda \in (0,\infty)$, and by (5.2):

$$\sum_{n=1}^{\infty} \big(1 - \Theta(\lambda_n/a)\big) < \infty. \tag{5.4}$$

From that, we have to deduce that

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{a}\,\lambda_n\right)^2 < \infty. \tag{5.5}$$

To do this, we perform a change of variable in order to apply Lemma 5.15. If we set $\lambda = e^\alpha$, then

$$\begin{aligned}
\Theta(\lambda) &= e^{\alpha/2}\left(\int_0^{+\infty} f(t)f(e^\alpha t)\,dt + \int_{-\infty}^0 f(t)f(e^\alpha t)\,dt\right)\\
&= e^{\alpha/2}\int_{\mathbb{R}} f(e^u)f(e^{\alpha+u})\,e^u du + e^{\alpha/2}\int_{\mathbb{R}} f(-e^u)f(-e^{\alpha+u})\,e^u du\\
&= \int_{\mathbb{R}} h_+(u)h_+(u+\alpha)\,du + \int_{\mathbb{R}} h_-(u)h_-(u+\alpha)\,du,
\end{aligned}$$

where h_+ and h_- are the functions defined on $\mathbb{R}$ by

$$h_+(x) := f(e^x)e^{x/2} \qquad \text{and} \qquad h_-(x) := f(-e^x)e^{x/2}.$$

Note that the functions h_+ and h_- belong to $L_2(\mathbb{R})$, and that

$$\|h_+\|_2^2 + \|h_-\|_2^2 = \int_{\mathbb{R}} f^2 = 1.$$

So, with the notation of Lemma 5.15, we have

$$\Theta(\lambda) = \mathcal{P}h_+\big(\log(\lambda)\big) + \mathcal{P}h_-\big(\log(\lambda)\big). \tag{5.6}$$

Since at least one of $h_\pm$ is non-zero, it follows that there exists a constant $c > 0$ such that $1 - \Theta(\lambda) \geqslant c \log(\lambda)^2$ for λ sufficiently close to 1. Hence $1 - \Theta(\lambda) \geqslant c(\lambda - 1)^2$ for λ sufficiently close to 1 (and some other constant $c > 0$), so that (5.5) is indeed a direct consequence of (5.4) since we already know that $\lambda_n \to a$. This concludes the proof of (1ii). □

Proof of (2) *in Theorem 5.13.* — By Kakutani's Theorem and with the notation of the proof of (1ii), we have to show that

$$\sum_{n=1}^{\infty} \big(1 - \Theta(\lambda_n/a)\big) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \left(1 - \frac{1}{a}\,\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right)^2 < \infty.$$

By (1ii), the "only if" implication is already known. To prove the converse, we keep the notation of the proof of (1ii). So we have as above

$$\Theta(\lambda) = \mathcal{P}h_+\big(\log(\lambda)\big) + \mathcal{P}h_-\big(\log(\lambda)\big),$$

where the functions h_+ and h_- are defined by

$$h_\pm(x) = f(\pm e^x)e^{x/2}.$$

Note that by assumption on f, the functions h_+ and h_- are continuous on $\mathbb{R}$ and $\mathcal{C}^1$-smooth except at a finite number of points. Moreover, $h'_\pm \in L_2(\mathbb{R})$. Indeed, we have $h'_\pm(x) = \pm f'(\pm e^x)e^{3x/2} + \frac{1}{2} f(\pm e^x)e^{x/2}$ almost everywhere, and both terms belong to $L_2(\mathbb{R})$ since

$$\int_{\mathbb{R}} f'(\pm e^x)^2 e^{3x} dx = \int_{\pm(0,\infty)} t^2 f'(t)^2 dt < \infty \qquad \text{and}$$

$$\int_{\mathbb{R}} f(\pm e^x)^2 e^x \, dx = \int_{\pm(0,\infty)} f(t)^2 dt < \infty.$$

So $h_\pm$ belongs to the Sobolev space $W^{1,2}(\mathbb{R})$. Hence, by Lemma 5.15 and since $\|h_+\|_2^2 + \|h_-^2\|_2 = 1$, we know that the function $F := \mathcal{P}h_+ + \mathcal{P}h_-$ is $\mathcal{C}^2$-smooth on $\mathbb{R}$, with $F(0) = 1$, $F'(0) = 0$ and $F''(0) < 0$. It follows that there is a constant $c > 0$ such that $1 - \Theta(\lambda) \sim c(\lambda - 1)^2$ as $\lambda \to 1$; and this concludes the proof of the "if" implication in (2). □

Proof of (3) *in Theorem 5.13.* — We keep the notation of the proofs of (1ii) and (2). This time, we have to show that

$$\sum_{n=1}^{\infty} \big(1 - \Theta(\lambda_n/a)\big) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \left|1 - \frac{1}{a}\,\lambda_n\right| < \infty. \tag{5.7}$$

As above, let $F := \mathcal{P}h_+ + \mathcal{P}h_-$, so that $\Theta(\lambda) = F\big(\log(\lambda)\big)$. By assumption on $\mathtt{p}$, the functions $h_+(x) = f(e^x)e^{x/2}$ and $h_-(x) = f(-e^x)e^{x/2}$ are $\mathcal{C}^1$-smooth except at finitely many points, with $h'_\pm \in L_2(\mathbb{R})$, and at least one of them has a discontinuity point. By Lemma 5.15, it follows that there exists two constants $c > 0$ and $c' < \infty$ such that $c\,|\alpha| \leqslant 1 - F(\alpha) \leqslant c'\,|\alpha|$ for α sufficiently close to 0. Hence $c\,|1 - \lambda| \leqslant 1 - \Theta(\lambda) \leqslant c'\,|1 - \lambda|$ for λ sufficiently close to 1, and (5.7) follows. □

The proof of Theorem 5.13 is now complete. □

Remark 5.19. — The proofs of parts (2) and (3) of Theorem 5.13 rely on the local properties of the function $\Theta(\lambda) = \sqrt{\lambda} \int_{\mathbb{R}} f(t) f(\lambda t)\, dt$ at $\lambda = 1$. It is perhaps worth noticing that if Θ happens to be differentiable at $\lambda = 1$, then we necessarily have $\Theta'(1) = 0$. This is because Θ is "symmetric with respect to 1", *i.e.* $\Theta(1/\lambda) = \Theta(\lambda)$.

Remark 5.20. — Part (3) of Theorem 5.13 may be applied for example if $\mu_{\boldsymbol{u},0}$ is the uniform distribution over some bounded interval $I \subseteq \mathbb{R}$ and $\mu_{\boldsymbol{v},0}$ is the uniform distribution over $\frac{1}{a}\, I$.

Remark 5.21. — Part (2) of Theorem 5.7 remains valid under the following assumption on $f = \sqrt{p}$: the restrictions of f to the intervals $(0,\infty)$ and $(-\infty,0)$ have locally integrable derivatives (in the distribution sense), and $tf'(t) \in L_2(\mathbb{R})$.

Proof. — Looking back at the proof of (2), we see that the only thing to check is that the functions h_+ and h_- belong to the Sobolev space $W^{1,2}(\mathbb{R})$. We do this for h_+ and, for notational simplicity, we set $h := h_+$ and we denote by f the restriction of f to $(0,\infty)$. So we have to show that if $f \in L_2(0,\infty)$ has a distributional derivative $f' \in L_1^{\mathrm{loc}}(0,\infty)$ such that $tf(t) \in L_2(0,\infty)$, then $h(x) = f(e^x)e^{x/2} \in W^{1,2}(\mathbb{R})$.

Since $f \in L_2(0,\infty)$, it is clear that $h \in L_2(\mathbb{R})$. To find the distributional derivative of h, let us fix a test function $\varphi \in \mathcal{D}(\mathbb{R})$. Let $\psi \in \mathcal{D}(0,\infty)$ be the function defined by the relation $\varphi(x) = \psi(e^x)e^{x/2}$, *i.e.* $\psi(t) = \frac{1}{\sqrt{t}}\,\varphi(\log(t))$. Writing $\varphi'(x) = e^{3x/2}\psi'(e^x) + \frac{1}{2}e^{x/2}\psi(e^x)$ and using the change of variable $x = \log(t)$, an elementary computation reveals that

$$\int_{\mathbb{R}} h(x)\,\varphi'(x)\,dx = -\int_0^\infty \left(e^{3x/2}f'(e^x) + \frac{1}{2}\,e^{x/2}f(e^x)\right)\varphi(x)\,dx.$$

This means that h has a distributional derivative $h' \in L_1^{\mathrm{loc}}(\mathbb{R})$ given (as expected) by $h'(x) = e^{3x/2}f'(e^x) + \frac{1}{2}e^{x/2}f(e^x)$. Since $f \in L_2(0,\infty)$ and $tf'(t) \in L_2(0,\infty)$, we see that $h' \in L_2(\mathbb{R})$, hence $h \in W^{1,2}(\mathbb{R})$.

□

5.5. Products of discrete measures. — Theorem 5.13 (3) shows that for weighted shifts acting on ℓ_p, to admit non-orthogonal invariant measures which are products of absolutely continuous measures having some "singularities" is in fact a strictly stronger requirement on the weights than to admit equivalent invariant Gaussian product measures. The next proposition goes along the same lines, in an even more dramatic way.

Proposition 5.22. — *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two weight sequences.*

(a) *If $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit non-orthogonal invariant product measures $m_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{v},n}$ such that $\mu_{\boldsymbol{u},0}$ and $\mu_{\boldsymbol{v},0}$ have non-zero discrete parts and do not charge $\{0\}$, then $\left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right|$ is eventually constant (and hence $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit equivalent invariant Gaussian product measures).*

(b) *Assume that $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit equivalent non-trivial invariant product measures or, more generally, non-trivial invariant product measures $m_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{v},n}$ such that $\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v},n}$ have the same support for each $n \geqslant 0$. Let $S := \mathrm{supp}(\mu_{\boldsymbol{u},0}) = \mathrm{supp}(\mu_{\boldsymbol{v},0})$. If either S is compact or $0 \notin S$, then $\left|\frac{u_1\cdots u_n}{v_1\cdots v_n}\right| = 1$ for all $n \geqslant 1$; and hence $\boldsymbol{u} = \boldsymbol{v}$ if $\boldsymbol{u}$ and $\boldsymbol{v}$ are positive weight sequences.*

Proof. — For each $n \geqslant 1$, let us set as usual $\lambda_n := \frac{u_1\cdots u_n}{v_1\cdots v_n}$.

(a) Since the measures $\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v},n}$ do not charge $\{0\}$, one can find a countable multiplicative subgroup $\mathcal{S}$ of $\mathbb{K}\backslash\{0\}$ such that the discrete parts of all measures $\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v},n}$ are supported on $\mathcal{S}$ and $\mathcal{S}$ contains the set $\{u_n;\ n \geqslant 1\} \cup \{v_n;\ n \geqslant 1\}$. Denote by $\tau_{\mathcal{S}}$ the counting measure on $\mathcal{S}$. Let also τ_c be a continuous measure such that the continuous

parts of all measures $\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v},n}$ are absolutely continuous with respect to τ_c. Then, one can take $\tau := \tau_c + \tau_{\mathcal{S}}$ to compute $H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n})$, for every $n \geqslant 0$.

For each $n \geqslant 0$, write

$$\mu_{\boldsymbol{u},n} = \alpha_n \tau_c + \mathtt{p}_n \tau_{\mathcal{S}} \qquad \text{and} \qquad \mu_{\boldsymbol{v},n} = \beta_n \tau_c + \mathtt{q}_n \tau_{\mathcal{S}}$$

for some non-negative measurable functions $\alpha_n, \mathtt{p}_n, \beta_n, \mathtt{q}_n$. Without loss of generality, we may assume that $\alpha_n \mathbf{1}_{\mathcal{S}} = \beta_n \mathbf{1}_{\mathcal{S}} = 0$ and $\mathtt{p}_n \mathbf{1}_{\mathbb{K}\setminus\mathcal{S}} = \mathtt{q}_n \mathbf{1}_{\mathbb{K}\setminus\mathcal{S}} = 0$, so that we also have

$$\mu_{\boldsymbol{u},n} = (\alpha_n + \mathtt{p}_n)\tau \qquad \text{and} \qquad \mu_{\boldsymbol{v},n} = (\beta_n + \mathtt{q}_n)\tau.$$

Hence,

$$H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) = \int_{\mathbb{K}} \sqrt{\alpha_n + \mathtt{p}_n}\, \sqrt{\beta_n + \mathtt{q}_n}\, d\tau = \big\langle \sqrt{\alpha_n + \mathtt{p}_n}, \sqrt{\beta_n + \mathtt{q}_n} \big\rangle_{L_2(\tau)}.$$

Now, by Kakutani's Theorem we know that $H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) \to 1$ as $n \to \infty$. Since $\|\sqrt{\alpha_n + \mathtt{p}_n}\|_{L_2(\tau)} = \|\sqrt{\beta_n + \mathtt{q}_n}\|_{L_2(\tau)} = 1$, it follows that

$$\big\|\sqrt{\alpha_n + \mathtt{p}_n} - \sqrt{\beta_n + \mathtt{q}_n}\,\big\|_{L_2(\tau)} \to 0.$$

Moreover, by assumption on $\alpha_n, \beta_n, \mathtt{p}_n, \mathtt{q}_n$, we have

$$\big\|\sqrt{\alpha_n + \mathtt{p}_n} - \sqrt{\beta_n + \mathtt{q}_n}\big\|^2_{L_2(\tau)} = \big\|\sqrt{\alpha_n} - \sqrt{\beta_n}\big\|^2_{L_2(\tau_c)} + \big\|\sqrt{\mathtt{p}_n} - \sqrt{\mathtt{q}_n}\big\|^2_{L_2(\tau_{\mathcal{S}})}.$$

Hence, we have in particular that $\|\sqrt{\mathtt{p}_n} - \sqrt{\mathtt{q}_n}\|^2_{L_2(\tau_{\mathcal{S}})} \to 0$ as $n \to \infty$; and since $\tau_{\mathcal{S}}$ is the counting measure on $\mathcal{S}$, this means that

$$\sum_{s\in\mathcal{S}} \Big(\sqrt{\mathtt{p}_n(s)} - \sqrt{\mathtt{q}_n(s)} \Big)^2 \to 0 \qquad \text{as } n \to \infty. \tag{5.8}$$

So far, we have not used the fact that the measures $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ are invariant under $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$. Set $\mathtt{p} := \mathtt{p}_0$ and $\mathtt{q} := \mathtt{q}_0$. Then, the discrete parts of $\mu_{\boldsymbol{u},0}$ and $\mu_{\boldsymbol{v},0}$ are respectively $\sum_{s\in\mathcal{S}} \mathtt{p}(s)\, \delta_s$ and $\sum_{s\in\mathcal{S}} \mathtt{q}(s)\, \delta_s$. So, by the invariance properties and since $\mathcal{S}$ is a multiplicative group containing $u_1 \cdots u_n$ and $v_1 \cdots v_n$, the discrete parts of $\mu_{\boldsymbol{u},n}$ and $\mu_{\boldsymbol{v},n}$ (for $n \geqslant 1$) are respectively $\sum_{s\in\mathcal{S}} \mathtt{p}(u_1 \cdots u_n s)\, \delta_s$ and $\sum_{s\in\mathcal{S}} \mathtt{q}(v_1 \cdots v_n s)\, \delta_s$; in other words, we have $\mathtt{p}_n(s) = \mathtt{p}(u_1 \cdots u_n s)$ and $\mathtt{q}_n(s) = \mathtt{q}(v_1 \cdots v_n s)$ for all $s \in \mathcal{S}$. Hence, if we set $f := \sqrt{\mathtt{p}}$ and $g := \sqrt{\mathtt{q}}$, (5.8) can be re-written as follows:

$$\sum_{s\in\mathcal{S}} \big(f(u_1 \cdots u_n s) - g(v_1 \cdots v_n s) \big)^2 \to 0 \qquad \text{as } n \to \infty$$

or, equivalently,

$$\sum_{s\in\mathcal{S}} \big(f(\lambda_n s) - g(s) \big)^2 \to 0.$$

It follows in particular, $f(\lambda_n s) \to g(s)$ for all $s \in \mathcal{S}$. So, taking any s such that $g(s) > 0$ (such an s exists since the discrete part of $\mu_{\boldsymbol{v},0}$ is non-zero), we see that one can find $\varepsilon > 0$, e.g. $\varepsilon := g(s)/2$, such that $f(\lambda_n s) \geqslant \varepsilon$ for all n sufficiently large. It follows that the sequence (λ_n) can take only finitely many distinct values: indeed, otherwise $f(s') \geqslant \varepsilon$ for infinitely many $s' \in \mathcal{S}$, a contradiction since $f \in \ell_2(\mathcal{S})$. Now, assume that $|\lambda_n|$ is not eventually constant. Then, since λ_n takes only finitely many values, $\lambda_n \in \mathcal{S}$ and $f(\lambda_n s) \to g(s)$ for all $s \in \mathcal{S}$, we see that one can find $\lambda, \lambda' \in \mathcal{S}$ with $|\lambda| \neq |\lambda'|$ such that $f(\lambda s) = f(\lambda' s)$ for all $s \in \mathcal{S}$. Setting $\alpha := \lambda/\lambda'$, we then have $f(\alpha^k s) = f(s)$ for any $s \in \mathcal{S}$ and all $k \in \mathbb{N}$; and taking any s such that $f(s) > 0$ (again, such an s exists), we obtain a contradiction since $f \in \ell_2(\mathcal{S})$ and the α^k, $k \in \mathbb{N}$ are pairwise distinct. This concludes the proof of (a).

(b) Let us fix $n \geqslant 1$ and, towards a contradiction, assume that $|\lambda_n| \neq 1$, say $|\lambda_n| < 1$. By the invariance properties, we have $\mathrm{supp}(\mu_{\boldsymbol{u},n}) = \frac{1}{u_1 \cdots u_n} S$ and $\mathrm{supp}(\mu_{\boldsymbol{v},n}) = \frac{1}{v_1 \cdots v_n} S$, so that $\lambda_n S = S$. Hence $\lambda_n^r S = S$ for all $r \in \mathbb{Z}$. If S is compact, it follows that $S = \{0\}$ by letting $r \to +\infty$; hence $\mu_{\boldsymbol{u},0} = \delta_0 = \mu_{\boldsymbol{v},0}$, which is the required contradiction. If $0 \notin S$, then we get $S = \varnothing$ by letting $r \to -\infty$, which is another contradiction. □

Corollary 5.23. — *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be positive weight sequences such that $\sum_{n=1}^{\infty} \frac{1}{(u_1 \cdots u_n)^p} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{(v_1 \cdots v_n)^p} < \infty$. The weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on ℓ_p are not orthogonal with respect to product measures whose marginals have non-zero discrete parts and do not charge $\{0\}$ if and only if $\frac{u_1 \cdots u_n}{v_1 \cdots v_n}$ is eventually constant; and in that case they admit equivalent invariant product measures with purely discrete marginals not charging $\{0\}$.*

Proof. — The "only if" part follows from Proposition 5.22 (a). Conversely, assume that $\lambda_n := \frac{u_1 \cdots u_n}{v_1 \cdots v_n}$ is eventually constant, say $\lambda_n = \lambda$ for $n > n_0$. Let $\mathcal{S}$ be the multiplicative subgroup of $(0, \infty)$ generated by the set $\{u_n;\ n \geqslant 1\} \cup \{v_n;\ \geqslant 1\}$. Let $\mathtt{p} : \mathcal{S} \to \mathbb{R}_+$ be a strictly positive probability density function such that $\sum_{s \in \mathcal{S}} \mathtt{p}(s)\, s^p < \infty$, and let $\mathtt{q} : \mathcal{S} \to \mathbb{R}_+$ be the probability density function defined by $\mathtt{q}(s) := \mathtt{p}(\lambda s)$. Finally, let $\mu_{\boldsymbol{u},0} := \sum_{s \in \mathcal{S}} \mathtt{p}(s) \delta_s$ and $\mu_{\boldsymbol{v},0} := \sum_{s \in \mathcal{S}} \mathtt{q}(s) \delta_s$, and denote by $m_{\boldsymbol{u}} = \oplus_{n \geqslant 0} \mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \oplus_{n \geqslant 0} \mu_{\boldsymbol{v},n}$ the associated $B_{\boldsymbol{u}}$ - invariant and $B_{\boldsymbol{v}}$ - invariant measures on $\mathbb{K}^{\mathbb{Z}_+}$. By Corollary 5.4 and the assumption on $\boldsymbol{u}$, $\boldsymbol{v}$ and $\mathtt{p}$, the measures $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ are supported on ℓ_p, and they have purely discrete marginals not charging $\{0\}$. Finally, by the proof of Proposition 5.22 (a), we have $H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) = \sum_{s \in \mathcal{S}} \sqrt{\mathtt{p}(\lambda_n s) \mathtt{q}(s)}$ for all $n \geqslant 0$, so that $H(\mu_{\boldsymbol{u},n}, \mu_{\boldsymbol{v},n}) = \sum_{s \in \mathcal{S}} \mathtt{q}(s) = 1$ for all $n > n_0$. Since $\mu_{\boldsymbol{u},n} \sim \mu_{\boldsymbol{v},n}$ for all $n \geqslant 0$, it follows that $m_{\boldsymbol{u}} \sim m_{\boldsymbol{v}}$. This concludes the proof. □

Remark 5.24. — In Proposition 5.22 (b), one cannot replace "equivalent" by "non-orthogonal". For example, let $\boldsymbol{u}$ and $\boldsymbol{v}$ be defined as follows: $u_n = 2$ for all $n \geqslant 1$, $v_1 = 1$, $v_2 = 4$ and $v_n = 2$ for all $n \geqslant 3$. Then $\boldsymbol{u} \neq \boldsymbol{v}$ (!), and yet $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit non-orthogonal product measure $m_{\boldsymbol{u}} = \otimes_{n \geqslant 0} \mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \otimes_{n \geqslant 0} \mu_{\boldsymbol{v},n}$ for which $\mu_{\boldsymbol{u},0} = \mu_{\boldsymbol{v},0}$ has a compact support not containing 0; for example, $\mu_{\boldsymbol{u},0} = \mu_{\boldsymbol{v},0}$ could be the uniform distribution on the interval $[1, 3]$.

6. Additional facts

6.1. Invariant measures with symmetry properties. — It is clear that if an operator T acting on a Polish topological vector space X admits a non-trivial invariant measure m, then the measure $\widetilde{m}$ defined by $\widetilde{m}(A) := \frac{1}{2}(m(A) + m(-A))$ is a non-trivial invariant measure for both T and $-T$, and that $\widetilde{m}$ is additionally *symmetric*, *i.e.* $\widetilde{m}(-A) = \widetilde{m}(A)$ for every Borel set $A \subseteq X$. The next proposition goes along the same lines.

Proposition 6.1. — *Let X be a Polish space, and let T be a continuous self-map of X admitting an invariant measure m. Let also $\mathbf{G}$ be a compact abelian group acting continuously on X, and assume that $Tg = \beta(g)T$ for all $g \in \mathbf{G}$, where $\beta : \mathbf{G} \to \mathbf{G}$ is a continuous map invariant under the Haar measure of $\mathbf{G}$. Let $\widetilde{m}$ be the measure on X defined by*

$$\widetilde{m}(A) := \int_{\mathbf{G}} m(g^{-1}A)\, dg,$$

where dg denotes integration with respect to the Haar measure of $\mathbf{G}$. Then $\widetilde{m}$ is $\mathbf{G}$ - invariant and both gT - invariant and Tg - invariant for every $g \in \mathbf{G}$.

Proof. — It is clear that $\widetilde{m}$ is $\mathbf{G}$-invariant. Moreover, if $h \in \mathbf{G}$ then, for any bounded Borel function $f : X \to \mathbb{R}^+$, we have

$$\begin{aligned}\int_X f \circ (\beta(h)T)\, d\widetilde{m} &= \int_X \left(\int_{\mathbf{G}} f(\beta(h)T(gx))\, dg \right) dm(x) \\ &= \int_X \left(\int_{\mathbf{G}} f(\beta(h)\beta(g)T(x))\, dg \right) dm(x) \\ &= \int_X \left(\int_{\mathbf{G}} f(\beta(h)gT(x))\, dg \right) dm(x) \\ &= \int_X \left(\int_{\mathbf{G}} f(gT(x))\, dg \right) dm(x) \\ &= \int_{\mathbf{G}} \left(\int_X f(gu)\, dm(u) \right) dg \\ &= \int_X f\, d\widetilde{m}.\end{aligned}$$

Since β is necessarily onto (because the Haar measure of $\mathbf{G}$ has full support), it follows that $\widetilde{m}$ if gT-invariant for every $g \in \mathbf{G}$; and hence also Tg-invariant since $Tg = \beta(g)T$. □

Corollary 6.2. — *Let X be a Polish space, and let T_1, T_2 be two continuous self-maps of X. Let also $\mathbf{G}$ be a compact abelian group acting continuously on X, and assume that $T_i g = gT_i$ for all $g \in \mathbf{G}$ and $i = 1, 2$. Finally, let $x_0 \in X$ be a fixed point for the action of $\mathbf{G}$. If T_1 and T_2 admit equivalent (resp. non-orthogonal) invariant measures m_1, m_2 not charging $\{x_0\}$, then they also admit equivalent (resp. non-orthogonal) invariant measures not charging $\{x_0\}$ which are additionally $\mathbf{G}$-invariant.*

Proof. — Let $\widetilde{m_1}$ and $\widetilde{m_2}$ be defined as in Proposition 6.1. It is clear that $\widetilde{m_1}$, $\widetilde{m_2}$ do not charge $\{x_0\}$. So we just have to check that if $m_1 \sim m_2$ then $\widetilde{m_1} \sim \widetilde{m_2}$, and that if $\widetilde{m_1} \perp \widetilde{m_2}$ then $m_1 \perp m_2$.

Assume that $m_1 \sim m_2$. Let $A \subseteq X$ be a Borel set such that $\widetilde{m_1}(A) = 0$. Then $m_1(gA) = 0$ for almost every $g \in \mathbf{G}$; hence $m_2(gA) = 0$ for almost every $g \in \mathbf{G}$ since $m_2 \ll m_1$, and hence $\widetilde{m_2}(A) = 0$. This shows that $\widetilde{m_2} \ll \widetilde{m_1}$; and similarly $\widetilde{m_1} \ll \widetilde{m_2}$.

Assume that $\widetilde{m_2} \perp \widetilde{m_1}$. Let $A \subseteq X$ be a Borel set such that $\widetilde{m_1}(A) = 0$ and $\widetilde{m_2}(X \backslash A) = 0$. Then $m_1(gA) = 0$ for almost every $g \in \mathbf{G}$ and $m_2(X \backslash gA) = m_2(g(X \backslash A)) = 0$ for almost every $g \in \mathbf{G}$. So one can find at least one g such that $m_1(gA) = 0 = m_2(X \backslash gA)$, which shows that $m_1 \perp m_2$. □

Corollary 6.3. — *Let X be a Polish topological vector space, and let $T \in \mathcal{L}(X)$. Assume that κT admits a non-trivial invariant measure for some $\kappa \in \mathbb{K}$. Then, there exists a non-trivial measure which is aT-invariant for all $a \in \mathbb{K}$ such that $|a| = |\kappa|$.*

Proof. — Apply Proposition 6.1 with the group $\mathbf{G} := \{\omega \in \mathbb{K};\ |\omega| = 1\}$ acting by multiplication on X. □

Corollary 6.4. — *Let $B_{\boldsymbol{u}}$ be a backward shift acting on ℓ_p. If $B_{\boldsymbol{u}}$ admits a non-trivial invariant measure, then there exists a measure m on ℓ_p with full support which is invariant for all backward shifts $B_{\boldsymbol{v}}$ such that $|v_n| = |u_n|$ for all $n \geqslant 1$.*

Proof. — Since $\sum_{n=1}^{\infty} \frac{1}{|u_1 \cdots u_n|^p} < \infty$, we know that $B_{\boldsymbol{u}}$ admit an invariant measure m with full support (which can even be taken to be Gaussian and ergodic for $B_{\boldsymbol{u}}$). Let

$$\mathbf{G} := \Big\{ (\omega_n)_{n \geqslant 0} \in \mathbb{K}^{\mathbb{Z}_+};\ |\omega_n| = 1 \ \text{ for all } n \geqslant 0 \Big\}.$$

This is a compact abelian group acting continuously on $X := \mathbb{K}^{\mathbb{Z}_+}$ by coordinatewise multiplication. Moreover, if we set $T := B_{\boldsymbol{u}}$ acting on X, we have $B_{\boldsymbol{u}} g = \sigma(g) B_{\boldsymbol{u}}$ for every $g \in \mathbf{G}$, where $\sigma : \mathbf{G} \to \mathbf{G}$ is (the restriction to $\mathbf{G}$ of) the canonical backward shift. Hence, considering m as a Borel measure on X, we may apply Proposition 6.1. This gives a measure $\widetilde{m}$ on $\mathbb{K}^{\mathbb{Z}_+}$ which is $B_{\boldsymbol{v}}$ - invariant for every weight sequence $\boldsymbol{v}$ such that $|v_n| = |u_n|$ for all $n \geqslant 1$. Moreover, since $B_{\boldsymbol{u}}$ is supported on ℓ_p and ℓ_p is $\mathbf{G}$ - invariant, $\widetilde{m}$ is supported on ℓ_p; and since m has full support, it is readily checked that $\widetilde{m}$ has full support. □

Remark 6.5. — It is well-known (see [22]) that two weight sequences $\boldsymbol{u}$ and $\boldsymbol{v}$ are such that $|u_n| = |v_n|$ for all $n \geqslant 1$ if and only if the backward shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ (acting on any ℓ_p) are unitarily similar, *i.e.* there exists an isometry J of ℓ_p such that $B_{\boldsymbol{v}} = J B_{\boldsymbol{u}} J^{-1}$.

6.2. Product measures charging small subspaces. — The following proposition says in essence that whether or not a product measure $\mathbb{K}^{\mathbb{Z}_+}$ invariant under a weighted shift $B_{\boldsymbol{w}}$ is supported on some "small" subspace of $\mathbb{K}^{\mathbb{Z}_+}$ depends on the rate of growth of the products $w_1 \cdots w_n$.

Proposition 6.6. — *Let μ_0 be a Borel probability measure on $\mathbb{K}$, let $\boldsymbol{w}$ be a sequence of non-zero scalars, and let $\mu_{\boldsymbol{w}} = \otimes_{n \geqslant 0} \mu_n$ be the $B_{\boldsymbol{w}}$ - invariant measure on $\mathbb{K}^{\mathbb{Z}_+}$ defined by μ_0, i.e. $\mu_n(A) = \mu_0(w_1 \cdots w_n A)$ for each $n \geqslant 1$ and every Borel set $A \subseteq \mathbb{K}$.*

(a) *Let $p \in [1, \infty)$. If $\int_{\mathbb{K}} |t|^p d\mu_0(t) < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{|w_1 \cdots w_n|^p} < \infty$, then $\mu_{\boldsymbol{w}}(\ell_p) = 1$.*

(b) *If there exists a summable sequence of positive real numbers (ε_n) such that*

$$\sum_{n=1}^{\infty} \mu_0\big(|t| > |w_1 \cdots w_n|\, \varepsilon_n\big) < \infty,$$

then $\mu_{\boldsymbol{w}}(\ell_1) = 1$.

(c) *If $|w_1 \cdots w_n| \not\to \infty$ and $\mu_0 \neq \delta_0$, then $\mu_{\boldsymbol{w}}(c_0) = 0$. If $|w_1 \cdots w_n| \not\to \infty$ and μ_0 does not have compact support, then $\mu_{\boldsymbol{w}}(\ell_\infty) = 0$.*

Proof. — (a) This is Corollary 5.4.

(b) Since ℓ_1 is a tail subset of $\mathbb{K}^{\mathbb{Z}_+}$, we have $\mu_{\boldsymbol{w}}(\ell_1) = 0$ or 1, by Kolmogorov's 0 - 1 law. Let

$$A := \big\{ x \in \mathbb{K}^{\mathbb{Z}_+};\ |x_n| \leqslant \varepsilon_n \quad \text{for all } n \geqslant 1 \big\}.$$

Obviously $A \subseteq \ell_1$, and we have

$$\begin{aligned} \mu_{\boldsymbol{w}}(A) &= \prod_{n=1}^{\infty} \mu_n\big(|t| \leqslant \varepsilon_n\big) \\ &= \prod_{n=1}^{\infty} \mu_0\big(|t| \leqslant |w_1 \cdots w_n|\, \varepsilon_n\big). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \big(1 - \mu_0(|t| \leqslant |w_1 \cdots w_n|\, \varepsilon_n)\big) < \infty$ by assumption, it follows that $\mu_{\boldsymbol{w}}(A) > 0$, and hence $\mu_{\boldsymbol{w}}(\ell_1) = 1$.

(c) Assume that $\mu_{\boldsymbol{w}}(c_0) > 0$. Then, there exists a compact set $K \subseteq c_0$ such that $\mu_{\boldsymbol{w}}(K) > 0$. By the well-known description of the compact subsets of c_0, one can find a sequence of positive real numbers (ε_n) tending to 0 such that

$$K \subseteq \big\{x \in \mathbb{K}^{\mathbb{Z}_+};\ |x_n| \leqslant \varepsilon_n \quad \text{for all } n \geqslant 0\big\}.$$

As in the proof of (b), it follows that

$$\sum_{n=1}^{\infty} \mu_0\big(|t| > |w_1 \cdots w_n|\, \varepsilon_n\big) < \infty.$$

In particular, $\mu_0(|t| > |w_1 \cdots w_n|\, \varepsilon_n) \to 0$ as $n \to \infty$. If $\mu_0 \neq \delta_0$, this implies that no subsequence of $(|w_1 \cdots w_n|\varepsilon_n)$ can tend to 0; and hence $|w_1 \cdots w_n| \to \infty$.

The second part of (c) is proved in a similar way. □

Corollary 6.7. — *For any probability measure μ_0 on $\mathbb{K}$, one can find a sequence of positive real numbers $\boldsymbol{w}$ such that $\mu_{\boldsymbol{w}}(\ell_1) = 1$.*

Proof. — This is clear by (b): choose a sequence of positive numbers (X_n) such that $\sum_{n=1}^{\infty} \mu_0(|t| > X_n) < \infty$, and then take $\boldsymbol{w}$ such that $2^{-n} w_1 \cdots w_n \geqslant X_n$ for all $n \geqslant 1$. □

Corollary 6.8. — *If the measure μ_0 is such that $\sum_{n=1}^{\infty} \mu_0(|t| > C^n) < \infty$ for some constant C, then $\mu_{\boldsymbol{w}}(\ell_1) = 1$ for any weight sequence $\boldsymbol{w}$ such that $\varliminf |w_1 \cdots w_n|^{1/n} > C$. In particular, if $\mu_0(|t| > X) = O\big(\log(X)^{-\alpha}\big)$ as $X \to \infty$ for some constant $\alpha > 1$, then $\mu_{\boldsymbol{w}}(\ell_1) = 1$ for any weight sequence $\boldsymbol{w}$ such that $\varliminf |w_n|^{1/n} > 1$.*

Proof. — Take $\varepsilon_n := (C/\rho)^n$ in (b), where $\rho := \varliminf |w_n|^{1/n}$. □

6.3. When frequently hypercyclic vectors are the same. — As mentioned in Remark 5.10, the following result is due to S. Charpentier and the third author. We thank S. Charpentier for allowing us to include it here.

Proposition 6.9. — *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two weight sequences such that $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are frequently hypercyclic on ℓ_p. If $\frac{u_1 \cdots u_n}{v_1 \cdots v_n}$ has a non-zero limit as $n \to \infty$, then $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ have the same frequently hypercyclic vectors.*

Proof. — In what follows, we set $a := \lim\limits_{n\to\infty} \frac{u_1 \cdots u_n}{v_1 \cdots v_n} \in \mathbb{K}\backslash\{0\}$.

Let x be a frequently vector for $B_{\boldsymbol{u}}$; we want to show that x is also a frequenty hypercyclic vector for $B_{\boldsymbol{v}}$. So, we fix $y = \sum_{k=0}^{d} y_k e_k \in c_{00}$ and $\varepsilon > 0$, and our task is to show that the set $\mathcal{N}_{B_{\boldsymbol{v}}}\big(x, B(y,\varepsilon)\big)$ has positive lower density.

Since x is a frequently hypercyclic vector for $B_{\boldsymbol{u}}$, it is enough to find a vector $z \in \ell_p$ and $\alpha > 0$ such that any sufficiently large $n \in \mathcal{N}_{B_{\boldsymbol{u}}}\big(x, B(z,\alpha)\big)$ belongs to $\mathcal{N}_{B_{\boldsymbol{v}}}\big(x, B(y,\varepsilon)\big)$. We consider

$$z := \sum_{k=0}^{d} a\, \frac{v_1 \cdots v_k}{u_1 \cdots u_k}\, y_k\, e_k \in c_{00}$$

and

$$\alpha := \varepsilon/2K \qquad \text{where} \quad K := \sup_{n,k\geqslant 0} \left|\frac{v_{k+1} \cdots v_{k+n}}{u_{k+1} \cdots u_{k+n}}\right|.$$

Note that $K < \infty$ since for every $n, k \geqslant 0$, we have

$$\left|\frac{v_{k+1} \cdots v_{k+n}}{u_{k+1} \cdots u_{k+n}}\right| = \left|\frac{u_1 \cdots u_k}{v_1 \cdots v_k}\right| \left|\frac{v_1 \cdots v_{k+n}}{u_1 \cdots u_{k+n}}\right| \leqslant \left(\sup_{m\geqslant 1} \left|\frac{u_1 \cdots u_m}{v_1 \cdots v_m}\right|\right) \left(\sup_{m\geqslant 1} \left|\frac{v_1 \cdots v_m}{u_1 \cdots u_m}\right|\right).$$

Let us show that if $n \in \mathcal{N}_{B_{\boldsymbol{u}}}\big(x, B(z, \frac{\varepsilon}{2K})\big)$ is large enough, then $n \in \mathcal{N}_{B_{\boldsymbol{v}}}\big(x, B(y,\varepsilon)\big)$. We have

$$\begin{aligned}
\|B_{\boldsymbol{v}}^n x - y\|^p &= \sum_{k=0}^{d} |(v_{k+1}\cdots v_{k+n})x_{k+n} - y_k|^p + \sum_{k=d+1}^{\infty} |(v_{k+1}\cdots v_{k+n})x_{k+n}|^p \\
&\leqslant \sum_{k=0}^{d} \left|\frac{v_{k+1}\cdots v_{k+n}}{u_{k+1}\cdots u_{k+n}}\right|^p \left|(u_{k+1}\cdots u_{k+n})x_{k+n} - \frac{u_{k+1}\cdots u_{k+n}}{v_{k+1}\cdots v_{k+n}}\, y_k\right|^p \\
&\qquad + \sum_{k=d+1}^{\infty} \left|\frac{v_{k+1}\cdots v_{k+n}}{u_{k+1}\cdots u_{k+n}}\right|^p |(u_{k+1}\cdots u_{k+n})x_{k+n}|^p \\
&\leqslant K^p \|B_{\boldsymbol{u}}^n x - \tilde{z}^{(n)}\|^p
\end{aligned}$$

where

$$\tilde{z}^{(n)} := \sum_{k=0}^{d} \frac{u_{k+1}\cdots u_{k+n}}{v_{k+1}\cdots v_{k+n}} y_k\, e_k.$$

Therefore, if $n \in \mathcal{N}_{B_{\boldsymbol{u}}}\big(x, B(z, \frac{\varepsilon}{2K})\big)$, we get

$$\|B_{\boldsymbol{v}}^n x - y\| \leqslant K\|B_{\boldsymbol{u}}^n x - z\| + K\|z - \tilde{z}^{(n)}\| \leqslant \frac{\varepsilon}{2} + K\|z - \tilde{z}^{(n)}\|;$$

and the desired result follows because

$$\begin{aligned}
\|z - \tilde{z}^{(n)}\| &\leqslant \sum_{k\leqslant d} \left|z_k - \frac{u_{k+1}\cdots u_{k+n}}{v_{k+1}\cdots v_{k+n}}\, y_k\right| \\
&\leqslant \|y\|_1 \max_{k\leqslant d} \left|a\,\frac{v_1\cdots v_k}{u_1\cdots u_k} - \frac{u_{k+1}\cdots u_{k+n}}{v_{k+1}\cdots v_{k+n}}\right| \\
&\leqslant \|y\|_1 \left(\max_{k\leqslant d}\left|\frac{v_1\cdots v_k}{u_1\cdots u_k}\right|\right)\left(\sup_{m\geqslant n}\left|a - \frac{u_1\cdots u_m}{v_1\cdots v_m}\right|\right) \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

□

6.4. Using Shepp's Theorem. — In this section, we show how results like Shepp's theorem from [21] mentioned before Theorem 5.13 can be used in the context of weighted shifts. Specifically, we will make use of the following theorem. For any measure m on $X = \mathbb{R}^{\mathbb{Z}_+}$ and $\boldsymbol{\alpha} \in X$, let us denote by $m_{\boldsymbol{\alpha}}$ the translate of m by $\boldsymbol{\alpha}$, which is the measure on X defined by

$$m_{\boldsymbol{\alpha}}(A) := m(A + \boldsymbol{\alpha}).$$

Theorem 6.10. — *Let $\tilde{\mu}$ and $\tilde{\mu}'$ be two measures on $\mathbb{R}$, and let $\tilde{m}$ and $\tilde{m}'$ be the product measures on $\mathbb{R}^{\mathbb{Z}_+}$ with marginals $\tilde{\mu}_n := \tilde{\mu}$ and $\tilde{\mu}'_n := \tilde{\mu}'$. Let also $\boldsymbol{\alpha} = (\alpha_n)_{n\geqslant 0} \in \mathbb{R}^{\mathbb{Z}_+}$, and assume that the measures $\tilde{m}$ and $\tilde{m}'_{\boldsymbol{\alpha}}$ are not orthogonal.*

(1) *If $\tilde{\mu} = \tilde{\mu}'$, then $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.*
(2) *If $\tilde{\mu}$ and $\tilde{\mu}'$ have a moment of order 2, then $\sum_{n=0}^{\infty} (\alpha_n - \alpha)^2 < \infty$ for some $\alpha \in \mathbb{R}$.*

Proof. — (1) is (the first third of) [21, Theorem 1].

To prove (2), we use a method devised by Dudley [13] in order to generalize Shepp's theorem. Denote by e_n^*, $n \geqslant 0$ the coordinate linear functionals on $\Omega := \mathbb{R}^{\mathbb{Z}_+}$. The assumption on $\tilde{\mu}$ and $\tilde{\mu}'$ implies that e_n^* belongs to $L_2(\Omega, \tilde{m}) \cap L_2(\Omega, \tilde{m}')$ with L_2-norms

respectively equal to $\int_{\mathbb{R}} x^2\, d\widetilde{\mu}(x)$ and $\int_{\mathbb{R}} x^2\, d\widetilde{\mu}'(x)$, hence independent of n. Likewise, $\int_\Omega e_n^*\, d\widetilde{m} = \int_{\mathbb{R}} x\, d\widetilde{\mu}(x)$ and $\int_\Omega e_n^*\, d\widetilde{m}' = \int_{\mathbb{R}} x\, d\widetilde{\mu}'(x)$ do not depend on n.

Assume first that $\int_\Omega e_n^*\, d\widetilde{m} = 0$ and $\int_\Omega e_n^*\, d\widetilde{m}' = 0$ for all $n \geqslant 0$. We claim that in this case, we have $\sum_{n=0}^\infty \alpha_n^2 < \infty$. Towards a contradiction, assume that $\sum_{n=0}^\infty \alpha_n^2 = \infty$. Then, one can find a sequence of real numbers (β_n) such that $\sum_{n=0}^\infty \beta_n^2 < \infty$, $\beta_n\alpha_n \geqslant 0$ for all n and $\sum_{n=0}^\infty \beta_n\alpha_n = \infty$. Now, consider the measure $\tau := \frac{1}{2}(\widetilde{m} + \widetilde{m}')$. By our assumption, the sequence $(e_n^*)_{n\geqslant 0}$ is orthogonal and bounded in $L_2(\Omega, \tau)$. Since $\sum_{n=0}^\infty \beta_n^2 < \infty$, it follows that the series $\sum \beta_n e_n^*$ is convergent in $L_2(\Omega, \tau)$. Hence, there is an increasing sequence of integers $(N_k)_{k\geqslant 0}$ such that if we define $f_k := \sum_{n=0}^{N_k} e_n^*$, then the sequence of linear functionals (f_k) converges τ-a.e. on Ω. So, the linear subspace

$$E := \big\{x \in \Omega;\ \text{the sequence } (f_k(x)) \text{ is convergent}\big\} \subseteq \Omega$$

is such that $\tau(E) = 1$ and hence $\widetilde{m}(E) = 1$. On the other hand, we have $f_k(\boldsymbol{\alpha}) = \sum_{n=0}^{N_k} \beta_n\alpha_n$, so the sequence $(f_k(\boldsymbol{\alpha}))$ is *not* convergent, *i.e.* $\boldsymbol{\alpha} \notin E$; and since E is a linear subspace of Ω, it follows that $E \cap (E + \boldsymbol{\alpha}) = \varnothing$. So we have $\tau(E + \boldsymbol{\alpha}) = 0$, and hence $\widetilde{m}'_{\boldsymbol{\alpha}}(E) = \widetilde{m}'(E + \boldsymbol{\alpha}) = 0$. Since $\widetilde{m}$ and $\widetilde{m}'_{\boldsymbol{\alpha}}$ are not orthogonal, this is a contradiction.

Now, let us consider the general case. Let $c := \int_{\mathbb{R}} x\, d\widetilde{\mu}(x)$ and $c' := \int_{\mathbb{R}} x\, d\widetilde{\mu}'(x)$, so that $\int_\Omega e_n^*\, d\widetilde{m} = c$ and $\int_\Omega e_n^*\, d\widetilde{m}' = c'$ for all $n \geqslant 0$. Then $\int_\Omega e_n^*\, d\widetilde{m}_{\mathbf{c}} = 0$ and $\int_\Omega e_n^*\, d\widetilde{m}'_{\mathbf{c}'} = 0$ for all $n \geqslant 0$, where $\mathbf{c} := (c, c, \dots)$ and $\mathbf{c}' := (c', c', \dots)$. Since the measures $\widetilde{m}_{\mathbf{c}}$ and $\big(\widetilde{m}'_{\mathbf{c}'}\big)_{\boldsymbol{\alpha}+\mathbf{c}-\mathbf{c}'} = \big(\widetilde{m}'_{\boldsymbol{\alpha}}\big)_{\mathbf{c}}$ are not orthogonal, it follows that $\sum_{n=0}^\infty (\alpha_n - (c' - c))^2 < \infty$. □

From Theorem 6.10, it is essentially a formal matter to deduce the following theorem, which is not far from saying that if two weighted shifts are not orthogonal with respect to product measures whose marginals do not charge $\{0\}$, then they admit equivalent invariant Gaussian product measures.

Theorem 6.11. — *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two sequences of non-zero scalars, and let $m_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{n,\boldsymbol{u}}$ and $m_{\boldsymbol{v}} = \otimes_{n\geqslant 0}\mu_{n,\boldsymbol{v}}$ be two product measures on $\mathbb{K}^{\mathbb{Z}_+}$, respectively $B_{\boldsymbol{u}}$-invariant and $B_{\boldsymbol{v}}$-invariant, with $\mu_{0,\boldsymbol{u}}(\{0\}) = 0 = \mu_{0,\boldsymbol{v}}(\{0\})$. Assume that the measures $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ are not orthogonal. Finally, let $\lambda_n := \frac{u_1\cdots u_n}{v_1\cdots v_n}$ for all $n \geqslant 1$.*

(1) *If there exists $a \in \mathbb{K}\backslash\{0\}$ such that $\mu_{0,\boldsymbol{v}}(A) = \mu_{0,\boldsymbol{u}}(aA)$ for every Borel set $A \subseteq \mathbb{K}$, then $\sum_{n=1}^\infty \big(1 - |a^{-1}\lambda_n|\big)^2 < \infty$.*

(2) *If $\int_{\mathbb{K}\backslash\{0\}} \big(\log|t|\big)^2 d\mu_{0,\boldsymbol{u}}(t) < \infty$ and $\int_{\mathbb{K}\backslash\{0\}} \big(\log|t|\big)^2 d\mu_{0,\boldsymbol{v}}(t) < \infty$, then there exists $a \in \mathbb{K}\backslash\{0\}$ such that $\sum_{n=1}^\infty \big(1 - |a^{-1}\lambda_n|\big)^2 < \infty$.*

Proof. — (1) Since $\mu_{0,\boldsymbol{u}}(\{0\}) = 0 = \mu_{0,\boldsymbol{v}}(\{0\})$, we view $\mu_{0,\boldsymbol{u}}$ and $\mu_{0,\boldsymbol{v}}$ as measures on $G := \mathbb{K}\backslash\{0\}$, and hence we view $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ as measures on $G^{\mathbb{Z}_+}$. Note that $G^{\mathbb{Z}_+}$ is an abelian group under entry-wise multiplication.

Consider the measures $m = \otimes_{n\geqslant 0}\mu_n$ and $m' = \otimes_{n\geqslant 0}\mu'_n$ where $\mu_n := \mu_{0,\boldsymbol{u}}$ and $\mu'_n := \mu_{0,\boldsymbol{v}}$ for all $n \geqslant 0$. If we set $\mathbf{U} := (1, u_1, u_1u_2, \dots)$, $\mathbf{V} := (1, v_1, v_1v_2, \dots)$ and $\mathbf{a} := (a, a, a, \dots)$, then we have for every Borel set $B \subseteq G^{\mathbb{Z}_+}$:

$$m_{\boldsymbol{u}}(B) = m(\mathbf{U}B) \qquad \text{and} \qquad m_{\boldsymbol{v}}(B) = m'(\mathbf{V}B) = m(\mathbf{a}\mathbf{V}B).$$

Now, consider the map $L : G^{\mathbb{Z}_+} \to \mathbb{R}^{\mathbb{Z}_+}$ defined by $L(t_0, t_1, \dots) := \big(\log|t_0|, \log|t_1|, \dots\big)$. Denote by $\widetilde{m}_{\boldsymbol{u}}$, $\widetilde{m}_{\boldsymbol{v}}$, $\widetilde{m}$ and $\widetilde{m}'$ the images of $m_{\boldsymbol{u}}$, $m_{\boldsymbol{v}}$, m and m' under this map L. Then $\widetilde{m} = \otimes_{n\geqslant 0}\widetilde{\mu}_n$ and $\widetilde{m}' = \otimes_{n\geqslant 0}\widetilde{\mu}'_n$, where $\widetilde{\mu}_n$ and $\widetilde{\mu}'_n$ are the images of $\mu_{0,\boldsymbol{u}}$ and $\mu_{0,\boldsymbol{v}}$ under

the map $t \mapsto \log|t|$, for every $n \geqslant 0$. Moreover, since L is a group homomorphism, we have for every Borel set $B \subseteq \mathbb{R}^{\mathbb{Z}_+}$:

$$\widetilde{m}_{\boldsymbol{u}}(B) = \widetilde{m}\big(B + L(\mathbf{U})\big) \qquad \text{and} \qquad \widetilde{m}_{\boldsymbol{u}}(B) = \widetilde{m}\big(B + L(\mathbf{aV})\big).$$

Now, the measures $\widetilde{m}_{\boldsymbol{u}} = m_{\boldsymbol{u}} \circ L^{-1}$ and $\widetilde{m}_{\boldsymbol{v}} = m_{\boldsymbol{v}} \circ L^{-1}$ are not orthogonal since $m_{\boldsymbol{u}}$ and $m_{\boldsymbol{v}}$ are not orthogonal. So their translates by $-L(\mathbf{aV})$ are not orthogonal either. In other words, if we set

$$\boldsymbol{\alpha} := L(\mathbf{UV}^{-1}\mathbf{a}^{-1}) = (0, \log|a^{-1}\lambda_1|, \log|a^{-1}\lambda_2|, \dots),$$

then the measure $\widetilde{m}$ and $\widetilde{m}_{\boldsymbol{\alpha}}$ are not orthogonal. By Theorem 6.10 (1), it follows that $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$, *i.e.*

$$\sum_{n=1}^{\infty} (\log|a^{-1}\lambda_n|)^2 < \infty;$$

which concludes the proof.

(2) The proof is similar, using Theorem 6.10 (2). □

Corollary 6.12. — *Let $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ be two weighted shifts acting on ℓ_p. The following are equivalent.*

(a) *$B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not orthogonal with respect to product measures whose marginals are absolutely continuous with respect to Lebesgue measure on $\mathbb{K}$.*
(b) *$B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ are not orthogonal with respect to product measures whose marginals do not charge $\{0\}$ and are such that $\log|t| \in L_2$.*
(c) *There exists $\kappa > 0$ such that $\sum_{n=1}^{\infty} \left(1 - \kappa \left|\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right|\right)^2 < \infty$.*
(d) *$B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit equivalent invariant Gaussian product measures.*

Proof. — By Theorem 6.11 and keeping the same notation, we just have to check that if $\mu_{0,\boldsymbol{u}}$ and $\mu_{0,\boldsymbol{v}}$ are absolutely continuous with respect to Lebesgue measure on $\mathbb{K}$, then there exists $a \in \mathbb{K}\backslash\{0\}$ such that $\mu_{0,\boldsymbol{v}}(A) = \mu_{0,\boldsymbol{u}}(aA)$ for every Borel set $A \subseteq \mathbb{K}$. When $\mathbb{K} = \mathbb{R}$ this is done in the proof of (1i) in Theorem 5.13. We repeat the argument here, in a way that makes it work when $\mathbb{K} = \mathbb{C}$ as well. Write $\mu_{0,\boldsymbol{u}} = \mathtt{p}(t)dt$ and $\mu_{0,\boldsymbol{v}} = \mathtt{q}(t)dt$. Then $\Psi(\lambda_n) \to 1$ as $n \to \infty$, where $\Psi : \mathbb{K}\backslash\{0\} \to [0,1]$ is the function defined by $\Psi(\lambda) := |\lambda|^{d/2} \int_{\mathbb{K}} f(\lambda t) g(t)\, dt$ and $d = 1$ or 2 depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Moreover, since $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ must be similar by Proposition 3.8, the sequence (λ_n) has a cluster point $a \in \mathbb{K}\backslash\{0\}$. Then $\Psi(a) = 1$ by continuity of Ψ, and it follows that $\mathtt{q}(t) = |a|^d\, \mathtt{p}(at)$ almost everywhere. □

6.5. Kakutani's Theorem. — In this section, our aim is to prove the following variant of Kakutani's Theorem from [17].

Theorem 6.13. — *Let $\nu = \otimes_{n\geqslant 0}\nu_n$ and $\nu' = \otimes_{n\geqslant 0}\nu'_n$ be two product probability measures on $\Omega = \prod_{n\geqslant 0}\Omega_n$. The measures ν and ν' are orthogonal if and only if $\prod_{n=0}^{\infty} H(\nu_n, \nu'_n) = 0$. Moreover, if $\prod_{n=0}^{\infty} H(\nu_n, \nu'_n) > 0$ and $\nu_n \ll \nu'_n$ for all n, then $\nu \ll \nu'$.*

Note that this variant was used in the proofs of Theorem 5.13 and Proposition 5.22. We give a rather detailed proof since we were not able to locate one in the literature. However, this is essentially a "copy and paste" of Kakutani's original proof of his theorem.

Proof of Theorem 6.13. — For each $n \geqslant 0$, let us choose a Radon-Nikodym derivative of ν_n with respect to ν'_n, *i.e.* a measurable function $\mathtt{p}_n : \Omega_n \to \mathbb{R}^+$ such that

$$\nu_n = \mathtt{p}_n \, \nu'_n + \alpha_n,$$

where α_n is a positive measure orthogonal to ν'_n. Let also $f_n := \sqrt{\mathtt{p}_n}$. With this notation, we have

$$H(\nu_n, \nu'_n) = \int_{\Omega_n} \sqrt{\mathtt{p}_n} \, d\nu'_n = \int_{\Omega_n} f_n \, d\nu'_n.$$

(i) Assume that $\prod_{n=0}^{\infty} H(\nu_n, \nu'_n) > 0$, and let us show that the measures ν and ν' are not orthogonal, with $\nu \ll \nu'$ if $\nu_n \ll \nu'_n$ for all n.

For each $k \geqslant 0$, let $F_k := \otimes_{n=0}^{k} f_n$, *i.e.* $F_k : \Omega \to \mathbb{R}^+$ is the function defined by

$$F_k(\omega) := \prod_{n=0}^{k} f_n(\omega_n).$$

We claim that F_k converges in $L_2(\Omega, \nu')$ to some $F : \Omega \to \mathbb{R}^+$. Indeed, if $0 \leqslant p < q$, then

$$\begin{aligned} \|F_q - F_p\|^2_{L_2(\nu')} &= \|F_p\|^2_{L_2(\nu')} \left\| 1 - \otimes_{n=p+1}^{q} f_n \right\|^2_{L_2(\nu')} \\ &\leqslant \left\| 1 - \otimes_{n=p+1}^{q} f_n \right\|^2_{L_2(\nu')} \\ &\leqslant 2 \left(1 - \prod_{n=p+1}^{q} H(\nu_n, \nu'_n) \right); \end{aligned}$$

so we see that $(F_k)_{k \geqslant 0}$ is a Cauchy sequence in $L_2(\Omega, \nu)$.

By L_2 - convergence and since $\int_{\Omega_n} f_n^2 \, d\nu'_n \geqslant \left(\int_{\Omega_n} f_n \, d\nu'_n \right)^2$ for all $n \geqslant 0$, we have

$$\int_{\Omega} F^2 \, d\nu' = \lim_{k \to \infty} \int_{\Omega} F_k^2 \, d\nu' \geqslant \prod_{n=0}^{\infty} H(\nu_n, \nu'_n)^2 > 0.$$

Moreover, for any cylinder set $A = [A_0, \ldots, A_N] \subseteq \Omega$, we have

$$\begin{aligned} \nu(A) &= \prod_{n=0}^{N} \nu_n(A_n) \\ &\geqslant \prod_{n=0}^{N} \int_{A_n} \mathtt{p}_n \, d\nu'_n \\ &= \prod_{n=0}^{N} \int_{A_n} f_n^2 \, d\nu'_n \geqslant \int_A F^2 \, d\nu', \end{aligned}$$

where the last inequality follows from the fact that $\int_{\Omega_n} f_n^2 \, d\nu'_n \leqslant 1$ for every $n > N$.

Therefore, the positive measure $F^2 \, \nu'$ is non-zero and absolutely continuous with respect to both ν' and ν, and hence ν and ν' are not orthogonal. Finally, if $\nu_n \ll \nu'_n$ for all n, *i.e.* $\alpha_n = 0$, then we see that $\nu(A) = \int_A F^2 \, d\nu'$ for all cylinder sets A, so $\nu \ll \nu'$.

(ii) Now, assume that $\prod_{n=0}^{\infty} H(\nu_n, \nu'_n) = 0$, and let us show that $\nu \perp \nu'$. It is enough to show that for any $\varepsilon > 0$, one can find a measurable set $E \subseteq \Omega$ such that $\nu'(E) < \varepsilon$ and $\nu(\Omega \backslash E) < \varepsilon$.

Let $\varepsilon > 0$, and let us choose $N \geqslant 0$ such that $\prod_{n=0}^{N} H(\nu_n, \nu'_n) < \varepsilon$, *i.e.*

$$\int_\Omega F_N \, d\nu' < \varepsilon.$$

Let us also choose, for each $n \in [\![0, N]\!]$, a measurable set $A_n \subseteq \Omega_n$ such that $\nu'_n(A_n) = 0$ and $\alpha_n(\Omega_n \backslash A_n) = 0$, and let

$$E := \{\omega \in \Omega;\ F_N(\omega) \geqslant 1\} \cup \{\omega \in \Omega;\ \omega_n \in A_n \text{ for some } n \in [\![0, N]\!]\}.$$

We have on the one hand $\nu'(E) = \nu'(F_N \geqslant 1) < \varepsilon$; and, on the other hand,

$$\begin{aligned}
\nu(\Omega\backslash E) &= \otimes_{n=0}^{\infty}(\mathtt{p}_n \, \nu'_n + \alpha_n)\left(\{F_N < 1\} \cap [\Omega_0\backslash A_0, \dots, \Omega_N \backslash A_N]\right) \\
&= \otimes_{n=0}^{N}(\mathtt{p}_n \, \nu'_n)\left(\{(\omega_0, \dots, \omega_N);\ \sqrt{\mathtt{p}_0(\omega_0) \cdots \mathtt{p}_N(\omega_N)} < 1\}\right) \\
&= \int_{\sqrt{\mathtt{p}_0 \otimes \cdots \otimes \mathtt{p}_N} < 1} \mathtt{p}_0 \otimes \cdots \otimes \mathtt{p}_N \, d\nu' \\
&\leqslant \int_\Omega \sqrt{\mathtt{p}_0 \otimes \cdots \otimes \mathtt{p}_N} \, d\nu' = \int_\Omega F_N \, d\nu' < \varepsilon.
\end{aligned}$$

This concludes the proof. □

7. Some questions

Of course, the main open question that remains at the end of this work is the one from which we started.

Question 7.1. — Find a characterization of the pairs of weight sequences $(\boldsymbol{u}, \boldsymbol{v})$ such that the weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ acting on ℓ_p are orthogonal.

In Section 3, we saw that orthogonality might be related to non-similarity, in a way which is not yet clear. In particular, in view of Theorem 3.10, the following question is natural.

Question 7.2. — Is it true that if $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit non-trivial invariant measures and are orthogonal, then either

$$\underline{\lim}_{n\to\infty} \max_{0\leqslant d\leqslant N} \left|\frac{u_1 \cdots u_{n+d}}{v_1 \cdots v_{n+d}}\right| = 0 \qquad \text{for all } N \geqslant 0$$

or

$$\overline{\lim}_{n\to\infty} \min_{0\leqslant d\leqslant N} \left|\frac{u_1 \cdots u_{n+d}}{v_1 \cdots v_{n+d}}\right| = \infty \qquad \text{for all } N \geqslant 0?$$

In the same spirit, Example 4.5 shows that non-similarity does not imply orthogonality. Conversely, it would be quite interesting to know whether similarity implies non-orthogonality.

Question 7.3. — Suppose that $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit non-trivial invariant measures and are similar, *i.e.*

$$0 < \underline{\lim} \left|\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right| \leqslant \overline{\lim} \left|\frac{u_1 \cdots u_n}{v_1 \cdots v_n}\right| < \infty.$$

Does it follow that $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{w}}$ are non-orthogonal?

Note that a positive answer to Question 7.2 would imply a positive answer to Question 7.3. A very special case of the latter has a positive answer by Corollary 6.4: if $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit non-trivial invariant measures and are *unitarily* similar, then they are non-orthogonal.

Next, in view of Theorem 5.13, Proposition 5.22 and Theorem 6.11, it is natural to ask whether the existence of equivalent non-trivial invariant product measures always implies the existence of equivalent invariant *Gaussian* product measures.

Question 7.4. — Is it true in general that if $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ admit equivalent non-trivial invariant product measures $m_{\boldsymbol{u}} = \otimes_{n\geqslant 0}\mu_{\boldsymbol{u},n}$ and $m_{\boldsymbol{v}} = \otimes\mu_{\boldsymbol{v},n}$, then they admit equivalent invariant Gaussian product measures?

From a more general point of view, the following question seems also natural.

Question 7.5. — If T_1 and T_2 are non-orthogonal operators, does it follow that they admit equivalent non-trivial invariant measures?

In the same spirit, one may ask

Question 7.6. — When do two weighted shifts $B_{\boldsymbol{u}}$ and $B_{\boldsymbol{v}}$ share a common non-trivial invariant measure?

Finally, the following seems to be unknown.

Question 7.7. — Let $B_{\boldsymbol{w}}$ be a weighted shift acting on $c_0(\mathbb{Z}_+)$. Is it true that $B_{\boldsymbol{w}}$ admits a non-trivial invariant measure if and only if $|w_1 \cdots w_n| \to \infty$ as $n \to \infty$?

References

[1] K. Agneessens, *Frequently hypercyclic random vectors*, Proc. Amer. Math. Soc. **151** (2023), no. 3, 1103–1117. ↑13

[2] F. Bayart and S. Grivaux, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 5083–5117. ↑2, 15

[3] F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 1, 181–210. ↑15

[4] F. Bayart and É. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009. ↑2, 15

[5] F. Bayart and É. Matheron, *Mixing operators and small subsets of the circle*, J. Reine Angew. Math. **715** (2016), 75–123. ↑15

[6] F. Bayart and I. Z. Ruzsa, *Difference sets and frequently hypercyclic weighted shifts*, Ergodic Theory Dynam. Systems **35** (2015), no. 3, 691–709. ↑2, 4

[7] N. C. Bernardes Jr., P. R. Cirilo, U. B. Darji, A. Messaoudi, and E. R. Pujals, *Expansivity and shadowing in linear dynamics*, J. Math. Anal. Appl. **461** (2018), no. 1, 796–816. ↑2

[8] N. C. Bernardes Jr. and A. Messaoudi, *Shadowing and structural stability for operators*, Ergodic Theory Dynam. Systems **41** (2021), no. 4, 961–980. ↑2

[9] V. I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. ↑14

[10] S. Charpentier, R. Ernst, M. Mestiri, and A. Mouze, *Common frequent hypercyclicity*, J. Funct. Anal. **283** (2022), no. 3, Paper No. 109526, 51. ↑3

[11] S. Charpentier, K. Grosse-Erdmann, and Q. Menet, *Chaos and frequent hypercyclicity for weighted shifts*, Ergodic Theory Dynam. Systems **41** (2021), no. 12, 3634–3670. ↑2, 4

[12] G. Costakis and M. Sambarino, *Topologically mixing hypercyclic operators*, Proc. Amer. Math. Soc. **132** (2003), no. 2, 385–389. ↑2

[13] R. M. Dudley, *Singularity of measures on linear spaces*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **6** (1966), 129–132. ↑30

[14] E. Glasner and B. Weiss, *A universal hypercyclic representation*, J. Funct. Anal. **268** (2015), no. 11, 3478–3491. ↑2
[15] S. Grivaux, *Some new examples of universal hypercyclic operators in the sense of Glasner and Weiss*, Trans. Amer. Math. Soc. **369** (2017), no. 11, 7589–7629. ↑2
[16] K.-G. Grosse-Erdmann and A. Peris, *Linear chaos*, Universitext, Springer, London, 2011. ↑2
[17] S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. (2) **49** (1948), 214–224. ↑14, 32
[18] N. K. Nikolski, *Treatise on the shift operator*, Grundlehren der mathematischen Wissenschaften, vol. 273, Springer-Verlag, Berlin, 1986. ↑1
[19] H. N. Salas, *Hypercyclic weighted shifts*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 993–1004. ↑1
[20] H. N. Salas, *Supercyclicity and weighted shifts*, Studia Math. **135** (1999), no. 1, 55–74. ↑1
[21] L. A. Shepp, *Distinguishing a sequence of random variables from a translate of itself*, Ann. Math. Statist. **36** (1965), 1107–1112. ↑14, 17, 30
[22] A. L. Shields, *Weighted shift operators and analytic function theory*, Topics in operator theory, Math. Surveys, No. 1, Amer. Math. Soc., Providence, RI, 1974, pp. 49–128. ↑1, 8, 28

S. Grivaux, CNRS, Univ. Lille, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
E-mail : sophie.grivaux@univ-lille.fr

É. Matheron, Univ. Artois, UR 2462 - Laboratoire de Mathématiques de Lens (LML), F-62300 Lens, France • *E-mail :* etienne.matheron@univ-artois.fr

Q. Menet, Service de Probabilité et Statistique, Département de Mathématique, Université de Mons, Place du Parc 20, 7000 Mons, Belgium • *E-mail :* quentin.menet@umons.ac.be