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# Generation of value function data for bilevel optimal control method

Olivier Cots\*      Rémy Dutto<sup>†‡§</sup>      Sophie Jan<sup>‡</sup>      Serge Laporte<sup>‡</sup>

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## Abstract

In this article, we present two numerical methods to create a database for the approximation of the value function of a bilevel optimal control problem. The first method is based on the computation of the value function via indirect simple shooting, which implies to find the roots of a function. The second one consists in solving a Cauchy problem. These two techniques are compared, in terms of prior information, computation cost and data distribution, on an industrial application: the torque split and gear shift optimal control problem on hybrid electric vehicles.

**Keywords.** Database generation, Bilevel optimal control, Pontryagin maximum principle, Indirect method, Value function, Hybrid electric vehicle.

## 1 Introduction

The value function is a central object in optimal control theory, that describes how the optimal cost depends on the initial conditions. This function was studied in the 1950's by Richard Bellman and leads to the Hamilton-Jacobi-Bellman (HJB) partial differential equations. These equations provide necessary and sufficient conditions for an optimal control problem, as well as the optimal control in a feedback form. They are the baseline of dynamic programming [3] and reinforcement learning [11], which are two of the main optimal control methods used in industrial applications. However, these methods are subject to the curse of dimensionality, which is a key numerical issue for embedded solutions.

The Pontryagin Maximum Principle [8] also introduced in the 1950's gives necessary optimality conditions for optimal control problems, and leads to the indirect numerical methods, which promise to be accurate and fast enough to be used for embedded solutions. For a description of classical numerical methods for optimal control problems, see [5].

A new method based on a bilevel decomposition of the optimal control problem is presented in [6], promises to reduce the number of computations and to be fast enough for embedded solution. This method uses some value functions between fixed times, which can rarely be calculated explicitly, and the strategy is to approximate them from a set of precomputed data. The main objective of this paper is to propose and compare two methods to create the associated database. The first method is based on the effective computation of the value function while the second one consists in integrating the Hamiltonian flow.

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In this paper, we first introduce the bilevel optimal control method in Section 2. Then we present two methods to create the value function database in Section 3. We compare in Section 4 the numerical results we have obtained when applying these two methods to an industrial application: the torque split and gear shift optimal control of a hybrid electric vehicle. Finally Section 5 concludes the article.

## 2 Motivation

### 2.1 Optimal control problem

We consider the following Optimal Control Problem in Bolza form:

$$\begin{aligned}
 \text{(OCP)} \quad & \begin{cases} \min_{x,u} \int_{t_0}^{t_f} f^0(t, x(t), u(t)) dt + g(x(t_f)), & (1) \\ \text{s.t. } \dot{x}(t) = f(t, x(t), u(t)), & t \in [t_0, t_f] \text{ a.e.}, & (2) \\ u(t) \in U, & t \in [t_0, t_f], & (3) \\ x(t_0) = x_0, & & (4) \end{cases}
 \end{aligned}$$

where the function  $f^0: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ , the terminal cost  $g: \mathbb{R}^n \rightarrow \mathbb{R}$ , and the state dynamic function  $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  are of class  $\mathcal{C}^1$ . The initial and final times  $t_0 < t_f$  are fixed, as well as the initial state  $x_0 \in \mathbb{R}^n$ . The control domain  $U \subset \mathbb{R}^m$  is a nonempty set. Solving (OCP) consists in finding an absolutely continuous *state*  $x \in AC([t_0, t_f], \mathbb{R}^n)$  and an essentially bounded *control*  $u \in L^\infty([t_0, t_f], \mathbb{R}^m)$ , which minimise the cost (1) and satisfy the constraints (2), (3) and (4).

**Remark 1.** *For conciseness and clarity, we consider an optimal control in this simple Bolza form but our approach and results can be easily extended to a more general Bolza problem with fixed final state, or mixed limit conditions for instance.*

### 2.2 Bilevel optimal control method

Considering intermediate times  $t_0 < t_1 < \dots < t_N < t_{N+1} = t_f$ , the optimal control problem (OCP) can be decomposed into the following Bilevel Optimal Control Problem

$$\text{(BOCP)} \quad \begin{cases} \min_X \sum_{i=0}^N V_i(X_i, X_{i+1}) + g(X_{N+1}), \\ \text{s.t. } X = (X_0, \dots, X_{N+1}) \in \mathcal{X}, \quad X_0 = x_0, \end{cases}$$

where for all  $i \in \mathbb{N}_N := \{0, \dots, N\}$ , the functions  $V_i$  are the *intermediate value functions*: for all admissible<sup>1</sup> pair  $(a, b)$ ,  $V_i(a, b)$  is the optimal cost of

$$\text{(OCP}_{i,a,b}) \quad \begin{cases} V_i(a, b) = \min_{x,u} \int_{t_i}^{t_{i+1}} f^0(t, x(t), u(t)) dt, \\ \text{s.t. } \dot{x}(t) = f(t, x(t), u(t)), & t \in [t_i, t_{i+1}] \text{ a.e.}, \\ u(t) \in U, & t \in [t_i, t_{i+1}], \\ x(t_i) = a, x(t_{i+1}) = b. \end{cases}$$

The set  $\mathcal{X}$  is defined as follows:  $X \in \mathcal{X} \subset (\mathbb{R}^n)^{N+2}$  if for all  $i \in \mathbb{N}_N$ , the pair  $(X_i, X_{i+1})$  is admissible for (OCP <sub>$i, X_i, X_{i+1}$</sub> ). Noting that for all  $i \in \mathbb{N}_N$ , for all admissible  $(a, b)$  and for all  $(x_i, u_i)$  solution

<sup>1</sup>The pair  $(a, b)$  is admissible for (OCP <sub>$i,a,b$</sub> ) if  $b$  is reachable at time  $t_{i+1}$  from  $a$  at  $t_i$ .

of  $(\text{OCP}_{i,a,b})$  we have  $V_i(a,b) = \int_{t_i}^{t_{i+1}} f^0(t, x_i(t), u_i(t)) dt$ ,  $(\text{BOCP})$  is equivalent to

$$\begin{cases} \min_X \max_{x,u} \sum_{i=0}^N \int_{t_i}^{t_{i+1}} f^0(t, x_i(t), u_i(t)) dt + g(X_{N+1}) \\ \text{s.t. } X \in \mathcal{X}, \quad X_0 = x_0, \\ \forall i \in \mathbb{N}_N, \quad (x_i, u_i) \text{ solution of } (\text{OCP}_{i,X_i,X_{i+1}}), \end{cases}$$

which is a bilevel optimal control problem [2].

In most applications, the intermediate value functions  $V_i$  cannot be computed easily. Thus we propose to replace them by approximations denoted  $C_i$ . This leads to a hierarchical method in two steps, called  $(\text{Macro})$ - $(\text{Micro})$ . The first step consists in solving the finite dimensional optimization problem

$$(\text{Macro}) \quad \begin{cases} \min_X \sum_{i=0}^N C_i(X_i, X_{i+1}) + g(X_{N+1}), \\ \text{s.t. } X \in \mathcal{X}, \quad X_0 = x_0, \end{cases}$$

to get the intermediate state  $\hat{X} = (\hat{X}_0, \dots, \hat{X}_{N+1})$ . The second step is to solve  $N+1$  optimal control problems of the following Lagrange form

$$(\text{Micro}) \quad \begin{cases} \min_{x,u} \int_{t_i}^{t_{i+1}} f^0(t, x(t), u(t)) dt, \\ \text{s.t. } \dot{x}(t) = f(t, x(t), u(t)), & t \in [t_i, t_{i+1}] \text{ a.e.}, \\ u(t) \in U, & t \in [t_i, t_{i+1}], \\ x(t_i) = \hat{X}_i, \quad x(t_{i+1}) = \hat{X}_{i+1}, \end{cases}$$

to get the state and command trajectory on each sub-interval  $[t_i, t_{i+1}]$ . This method is sub-optimal, due to the approximation error between  $C_i$  and  $V_i$ , but it promises to be faster and to require less computations than classical methods, which is particularly interesting for embedded applications.

However, before applying the  $(\text{Macro})$ - $(\text{Micro})$  method, we need to build the approximation functions  $C_i$ . We postpone to Section 3 the creation of the optimal values database which will be used for that purpose and which consists in the main contribution of this article. From now on, we consider one  $(\text{Micro})$  problem, for a given  $i \in \mathbb{N}_N$  fixed and we give in the next section some details about the so-called simple indirect shooting method to solve it.

## 2.3 Indirect method

The considered  $(\text{Micro})$  problem is solved using the indirect simple shooting method which is based on the Pontryagin Maximum Principle (PMP) [8]. The core function of this principle is the *pseudo-Hamiltonian*

$$h(t, x, p, u) = -f^0(t, x, u) + (p \mid f(t, x, u)),$$

where  $(\cdot \mid \cdot)$  stands for the usual scalar product in  $\mathbb{R}^n$ . If we assume that the *Hamiltonian*

$$H(t, x, p) = \max_{u \in U} h(t, x, p, u)$$

is defined and smooth, the PMP leads to the resolution of the two points boundary value problem

$$(\text{TPBVP}) \quad \begin{cases} (x_{i+1}, p_{i+1}) = \exp_{\vec{H}}(t_{i+1}, t_i, (x_i, p_i)), \\ x_i = \hat{X}_i, \quad x_{i+1} = \hat{X}_{i+1}, \end{cases}$$

where  $\vec{H}$  is the *Hamiltonian vector field* defined by

$$\vec{H}(t, (x, p)) = \left( \nabla_p H(t, (x, p)), -\nabla_x H(t, (x, p)) \right),$$

and  $\exp_{\vec{H}}(t_{i+1}, t_i, z_i)$  is the *exponential map* of  $\vec{H}$ : it corresponds to the solution at time  $t_{i+1}$  of the Cauchy problem

$$\forall t \in [t_i, t_{i+1}], \quad \dot{z}(t) = \vec{H}(t, z(t)), \quad z(t_i) = z_i.$$

Let us denote by  $\bar{x}_{i+1}(z_i)$  the state at time  $t_{i+1}$  obtained by the integration of the Hamiltonian flow from  $z_i$  at time  $t_i$ , defined by

$$\bar{x}_{i+1}(z_i) = \pi_x \left( \exp_{\vec{H}}(t_{i+1}, t_i, z_i) \right) \quad (5)$$

where  $\pi_x(x, p) = x$  is the projection on the state space.

In the following, we assume that for all initial and final admissible state  $(a, b)$ , if  $(x, u)$  is a solution of  $(\text{OCP}_{i,a,b})$ , there exists  $p \in \text{AC}([t_i, t_{i+1}], \mathbb{R}^n)$  such that the pair  $(x, p)$  is a solution of  $(\text{TPBVP})$ . Such a pair that satisfies the Hamiltonian vector field is called an *extremal*. A solution of  $(\text{TPBVP})$  is an extremal that satisfies the boundary conditions, and it is called a *BC-extremal*. This assumption is linked to the normality of the BC-extremals associated to the solution (cf. [1] for more details). For the sake of simplicity, we denote by  $\bar{c}_i(z_i)$  the cost of the state and command trajectory, given by the integration of the Hamiltonian vector field on the time interval  $[t_i, t_{i+1}]$ , with the initial state and costate conditions  $z_i = (x_i, p_i)$ . We also consider the following hypothesis, that will be numerically verified in our application.

**Hypothesis 1.** *Let  $(x_i, x_{i+1})$  be a given initial and final admissible state. Then, all the zeros of*

$$p_i \mapsto \bar{x}_{i+1}(x_i, p_i) - x_{i+1}$$

*are associated to the same state and command trajectory  $(x, u)$ , depending on the pair  $(x_i, x_{i+1})$ .*

**Remark 2.** *Under the previous assumptions and Hypothesis 1, for all initial and final admissible state  $(a, b)$ , there exists at most a unique solution of  $(\text{OCP}_{i,a,b})$ .*

The indirect simple shooting method aims to solve  $(\text{TPBVP})$  by finding a zero of the *shooting function*

$$S_{i, \hat{X}_i, \hat{X}_{i+1}}(p_i) = \bar{x}_{i+1}(\hat{X}_i, p_i) - \hat{X}_{i+1}. \quad (6)$$

Thanks to Hypothesis 1, all the zeros of the shooting function lead to the same optimal cost, and therefore only one of them need to be found. On the opposite, without this hypothesis, it would be necessary to find all the zeros and to compare them in terms of cost to get the optimal one.

Equation (6) is solved using a classical Newton-like solver, and it is well known that finding a good initial guess for the costate is a critical issue. However, given a solution  $X^*$  of  $(\text{BOCP})$ , if  $V_i$  is differentiable at  $(X_i^*, X_{i+1}^*)$ , it is established [4] that the vector  $-\nabla_a V_i(X_i^*, X_{i+1}^*)$  is a zero of  $S_{i, X_i^*, X_{i+1}^*}$ . Unfortunately,  $V_i$  is not known, but since  $C_i$  is an approximation of  $V_i$ , the vector  $-\nabla_a C_i(\hat{X}_i, \hat{X}_{i+1})$  could be a good initial guess for the associated  $(\text{Micro})$  problem shooting function [7].

Now we have described how to solve the  $(\text{Micro})$  problem, we will focus on the creation of a database for the definition of  $C_i$  approximation functions.

### 3 Data generation

In this section, we shall describe the creation of a database  $\mathbb{D}_i$  containing optimal values of Problems  $(\text{OCP}_{i,x_i,x_{i+1}})$  for various pairs  $(x_i, x_{i+1})$ :

$$\mathbb{D}_i \subset \left\{ (x_i, x_{i+1}, c) \mid (x_i, x_{i+1}) \text{ admissible and } V_i(x_i, x_{i+1}) = c \right\}.$$

Two methods are presented to achieve this goal. They are compared in terms of prior information, computation cost and data distribution.

### 3.1 Method 1

The first method consists in computing the value function  $V_i$ . Let  $\mathbb{X}_i$  be a discretization of admissible initial and final states. For all  $(x_i, x_{i+1}) \in \mathbb{X}_i$ , we solve the optimal control problem ( $\text{OCP}_{i,x_i,x_{i+1}}$ ) by an indirect simple shooting method and store the corresponding optimal value  $V_i(x_i, x_{i+1})$  in  $\mathbb{D}_i$ . Algorithm 1 provides an overview of these steps.

---

#### Algorithm 1 Method 1

---

**Require:**  $\mathbb{X}_i$

**Ensure:**  $\mathbb{D}_i$

**for all**  $(x_i, x_{i+1}) \in \mathbb{X}_i$  **do**

$p_i \leftarrow \text{solve}(S_{i,x_i,x_{i+1}}(p_i) = 0)$  ▷ Newton solver

$c \leftarrow \bar{c}_i(x_i, p_i)$  ▷ Optimal cost

$\mathbb{D}_i.\text{append}(x_i, x_{i+1}, c)$  ▷ Storage

**end for**

---

**Prior information** First of all, the set  $\mathbb{X}_i$  is created as follows: the initial state is first discretized, then for each of these values, the set of associated reachable final states is determined and discretized. This ensures that the space of initial and final state is fully explored and that all pairs  $(x_i, x_{i+1}) \in \mathbb{X}_i$  are admissible, which prevents the Newton solver to fail. The difficulty in this method is the determination of the reachable final states.

**Computation cost** For each iteration of the Newton solver used to find a zero of the shooting function, at least one integration of the Hamiltonian vector field is required.

**Data distribution** By construction, the distribution of the pairs  $(x_i, x_{i+1})$  is fully controlled inside the admissibility set. For example a uniform distribution may be chosen for an easy interpolation.

### 3.2 Method 2

The second method, described in Algorithm 2, is based on a discretization  $\mathbb{Z}_i$  of the initial state and costate space. For all  $z_i = (x_i, p_i) \in \mathbb{Z}_i$ , the exponential map  $z_{i+1} = (x_{i+1}, p_{i+1}) = \exp_{\vec{H}}(t_{i+1}, t_i, z_i)$  is computed, and Hypothesis 1 ensures that  $V_i(x_i, x_{i+1}) = \bar{c}_i(x_i, p_i)$ .

---

#### Algorithm 2 Method 2

---

**Require:**  $\mathbb{Z}_i$

**Ensure:**  $\mathbb{D}_i$

**for all**  $z_i = (x_i, p_i) \in \mathbb{Z}_i$  **do**

$x_{i+1} \leftarrow \bar{x}_{i+1}(z_i)$  ▷ Final state

$c \leftarrow \bar{c}_i(z_i)$  ▷ Optimal cost

$\mathbb{D}_i.\text{append}(x_i, x_{i+1}, c)$  ▷ Storage

**end for**

---

**Prior information** Since there is no admissibility constraints on  $(x_i, p_i)$ , the set  $\mathbb{Z}_i$  can be the cartesian product between discretizations of initial state and costate. Nevertheless, the range of the costate discretization must be well chosen in order to ensure a full exploration of the reachable states  $x_{i+1}$  while limiting the number of useless integrations of the Hamiltonian vector field.

**Computation cost** In contrast to Method 1, no solver is involved here. Hence, only one integration of the Hamiltonian vector field is needed for each pair  $(x_i, x_{i+1})$ .

**Data distribution** In this method, the distribution of the reached final states  $x_{i+1}$  is no more controlled. The main consequence is the risk of empty zones. Fortunately, we shall see in Section 4 that a sufficient density in the set  $\mathbb{Z}_i$  ensures a satisfactory distribution of reached states  $x_{i+1}$ .

## 4 Results

The considered application is the following industrial problem (detailed in [6]): the optimal control of the torque split and the gear shift of a Hybrid Electric Vehicle (HEV) on the *Worldwide harmonized Light vehicles Test Cycle* (WLTC). The goal is to minimize the fuel consumption of the Internal Combustion Engine (ICE) for fixed initial condition and fixed or free final state of charge of the battery by acting on the gear and on the torque split between the ICE and the Electric Motor (EM) (see [9, 10, 12] for more informations).

This problem can be formulated as (OCP) and we consider that we stand in the frame of the assumptions stated in Section 2.3. The bilevel method can be applied in this context, and the time interval (1800 seconds of the WLTC) is arbitrary decomposed into 17 time sub-intervals of 100s.

The goal of this section is to compare the two methods presented in Sections 3.1 and 3.2 on our HEV application. For this purpose and to clarify the presentation of the results, we only focus on the first sub-interval  $[0, 100]$  and we fix the initial state, denoted  $x_0$ . So next, we apply the two methods to create the database  $\mathbb{D}_0$ .

The evolution of the function  $p_0 \mapsto \bar{x}_1(x_0, p_0)$  is shown in Figure 1. It can be noticed that for all  $p_0 \in [p_0^-, p_0^+]$ , the function (5) is injective. Moreover, for all  $p_0 \leq p_0^-$  (respectively  $p_0 \geq p_0^+$ ), the extremal resulting from the integration of the Hamiltonian flow is associated to a unique  $(x, u)$ . Hence, the Hypothesis 1 can be considered numerically valid.

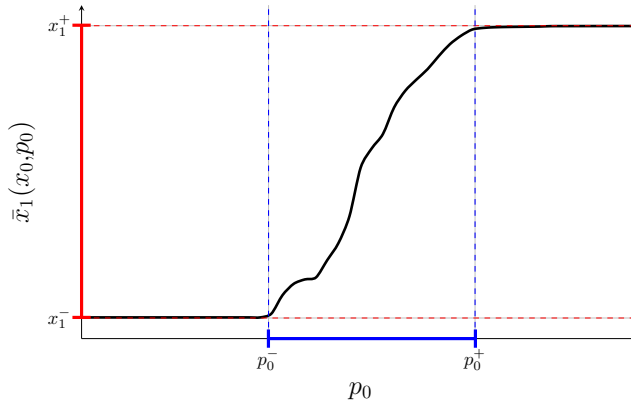


Figure 1: Evolution of the function (5), with a fixed initial state  $x_0$ . The red interval on the y-axis corresponds to the admissible initial state set at time  $t_1$ , and the blue interval on the x-axis corresponds to the useful costate initialization set, mentioned in the Prior Information paragraph of Method 2.

**Remark 3.** As discussed in Section 3, the final states must be chosen in  $[x_1^-, x_1^+]$  for Method 1. In contrast, for Method 2 they naturally belong to this interval, but although the initial costate can be taken in  $\mathbb{R}^n$ , its useful range is limited to the interval  $[p_0^-, p_0^+]$ , cf. Figure 1.

In order to compare both methods in terms of number of Hamiltonian flow integrations, 475 points  $(x_i, x_{i+1}, c)$  have been generated for various initial and final admissible states with Method 1.

	Method 1	Method 2
Pros	Controlled distribution	Computation cost
Cons	Prior knowledge of admissible domain Computation cost	Prior knowledge of initial costate range Non controlled distribution

Table 1: Pros and Cons of Method 1 and Method 2.

We observe that the Newton solver needs in average 11.2 iterations to find a zero, which means that this method requires at least 11.2 times more integrations than Method 2.

The two methods are now compared in terms of distribution of the final states. First, 10 points of  $\mathbb{D}_0$  are generated by each method. As expected, we can observe in Figures 2a and 2b that the final states are obviously uniformly distributed with Method 1, while there are not with Method 2. Second, for the same computation cost as Method 1, 112 points of  $\mathbb{D}_0$  are generated by Method 2. The results are shown in Figure 2c. It can be observed that if the distribution remains non-uniform, the coverage of the reachable interval  $[x_1^-, x_1^+]$  is much better than that obtained by Method 1.

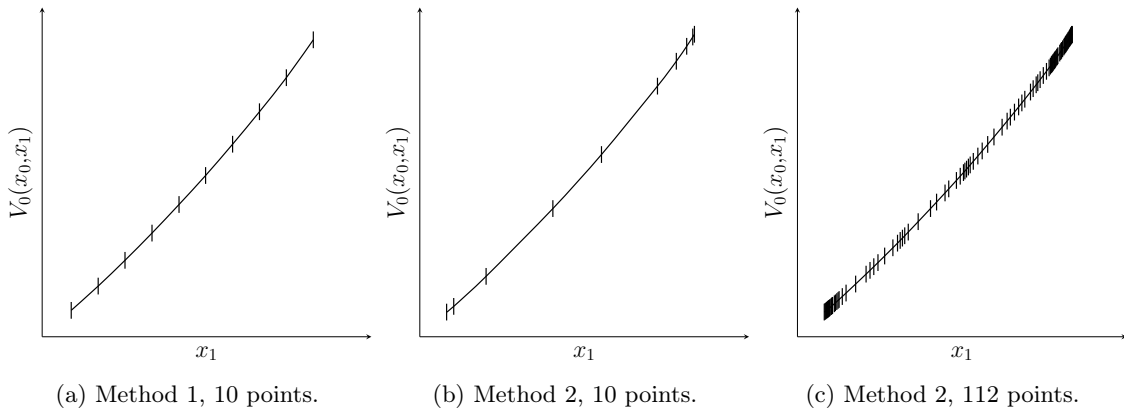


Figure 2: Value function data created by the two proposed methods.

## 5 Conclusion

The bilevel (Macro)-(Micro) method proposed in this paper requires the approximation of value functions which can be performed using a database composed of points  $(a, b, V_i(a, b))$ . We present two numerical methods to create such a database. The first one, based on the computation of the value functions  $V_i(a, b)$  for a set of given pairs  $(a, b)$ , requires to find an initial costate  $p'$  solution of the shooting equation that appears in the indirect method. The second method consists in integrating the Hamiltonian flow for a set of given pairs  $(a, p)$ , where  $p$  is the initial costate, to get a final state  $b'$  and the corresponding cost  $\bar{c}_i(a, p)$ , which under Hypothesis 1, is the optimal one:  $V_i(a, b') = \bar{c}_i(a, p)$ .

The pros and cons of both methods are gathered in Table 1. The uncontrolled distribution obtained by Method 2 can be balanced by its favorable computation cost. Indeed, for the same number of Hamiltonian flow integrations as in Method 1, it appears that the domain of reachable final states is well covered. This makes this Method 2 really competitive.

Once the database is created, the associated value function approximation  $C_i$  has to be built. A wide range of methods can be used such as neural networks models due to their generalization capacities and their ability to provide the gradient of the approximation  $C_i$  through retropropagation



technique. Indeed, the availability of this gradient should ease the resolution of the problem (**Macro**) and provide a good initialization for the resolution of the problem (**Micro**) by indirect method.

Another perspective is to use the gradient of the value function to improve the fitting of the approximations  $C_i$ . Indeed, it can be shown under some assumptions that the gradient of the value function  $V_i(x_i, x_{i+1})$  is the vector  $(-p_i, p_{i+1})$ . Therefore the computation of the Hamiltonian flow  $(x_{i+1}, p_{i+1}) = \exp_{\bar{H}}(t_{i+1}, t_i, (x_i, p_i))$  provides the optimal cost transition  $V_i(x_i, x_{i+1})$  as well as its associated gradient. An extended database containing this additional information on the gradient could then be created and used for instance to train Physics-Informed Neural Networks (PINNs).

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