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# NEGATIVE BLOC RULE: AN AXIOMATIC AND PROBABILISTIC ANALYSIS

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## Abstract

In single winner voting elections, the plurality rule is one of the most studied rules. Plurality has been extended to the field of multiwinner voting elections where instead of electing one candidate,  $k$  candidates have to be elected. This extension is called the Bloc rule and consists in voting for the  $k$  top preferred candidates. Antiplurality is another common voting rule, where voters vote for all the candidates except their last ranked candidate. In this paper, we introduce an extension of antiplurality in the field of multiwinner elections and call it Negative bloc rule. Axiomatic properties of this new rule are checked and compared to other multiwinner voting rules ( $k$ -plurality,  $k$ -antiplurality,  $k$ -Borda and Bloc). This axiomatic analysis is completed with a probabilistic approach on the similarity of results between Negative bloc rule and the four other rules. Finally, the behavior of Negative bloc rule according to some Condorcet properties is investigated.

KEYWORDS: Multiwinner voting rules; Bloc; Negative Bloc; Condorcet.

*JEL Classification:* D71, D72

## 1 Introduction

Traditional social choice theory is mainly focused on single winner voting election. Nevertheless, some papers deal with elections with multiple winners (Droop (1881), Young (1975), Fishburn (1981), Debord (1992)). This multiwinner elections setting

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has even attracted more attention in the recent years (Elkind *et al.* (2017), Skowron *et al.* (2016), Diss & Doghmi (2016) among others). In this setting, the objective is not to elect a single winner but a fixed number  $k$  of winners, which is called in the literature a *committee*. Among the multiwinner voting rules, there exist different ways to take voters opinions into account and aggregate them. One of these ways is to use approval based-rules, in which case each voter has a set of approved candidates, and this set possibly consists of all candidates. This method has been studied in numerous of papers (Aziz *et al.* (2017), Lackner & Skowron (2021)). Another method is to use preference-based rules, in which case each voter reports a ranking over the set of candidates. We study *committee scoring rules* in this work, which are preference based rules. They are also studied in Skowron *et al.* (2016), Faliszewski *et al.* (2018) and Diss *et al.* (2020).

Committee scoring rules give points to the candidates according to their position in the rankings of the voters. For each candidate we then sum up the number of points obtained over the voters, and the  $k$  candidates with the highest scores are declared to be the winners. Under the well known plurality rule, each voter votes for his top choice and the candidate with the highest score wins. Plurality has been extended in the multiwinner field in two ways, a straightforward way with  $k$ -plurality and an adapted way with Bloc rule. In the first one, each voter votes for his top choice and the  $k$  candidates with the highest scores win. In the second way, instead of voting for their top choice, each voter votes for their  $k$  top choices. These approaches favor candidates that are top ranked in the voters preferences, no matter if these candidates are also bottom ranked by a large number of voters. Antiplural-ity allows to elect candidates that are not so hated (Boehm (1976), Brams (1977)).  $k$ -antiplurality selects the  $k$  candidates with the  $k$  smallest number of last places in the voters' preferences. We propose in this paper an extension of antiplurality in the multiwinner setting with the Negative bloc rule. In the same spirit as bloc with plurality, Negative bloc gives points to all candidates except the  $k$  last ranked ones. This rule amounts to giving support to every candidate except the members of the committee the voters do not want to be elected. These approaches favor candidates that are barely often ranked among the  $k$  last candidates. The main contribution of this paper is the introduction of this new rule and its analysis in different ways.

Two approaches will be used to analyze this new rule, an axiomatic method and a probabilistic one. The axiomatic method consists in checking the behavior of the voting rule toward some properties. These properties (axioms) have been developed in the literature and are considered as suitable for a voting rule. Some papers even

propose a characterization of scoring rules (Merlin (2003), Skowron *et al.* (2016), Faliszewski *et al.* (2018)). In this paper we will use the axioms developed in Elkind *et al.* (2017), and extend their analysis to check whether the axioms are fulfilled by Negative bloc and  $k$ -antiplurality. The probabilistic method studies how often a voting event occurs in a certain situation or under a certain voting rule. One of the first examples is the probability of a cycle to occur under the method of majority decision with 3 candidates (DeMeyer & Plott (1970)). Since then, the method has been used to investigate the probability of many voting events in various settings. In our work, we focus on four voting events corresponding to the four following questions. The first question is how often Negative bloc rule gives results close to those obtained for four other scoring rules i.e. how often Negative bloc elects the same committee as another scoring rule. The four other scoring rules have been compared with each other according to the proximity of their results in Diss *et al.* (2020). The second question is how often Negative bloc rule elects the *Condorcet Committee à la Gehrlein* when there is one (Gehrlein (1985)). This probability is called the Condorcet Committee efficiency (CCE) and has been calculated for other scoring rules in Diss & Doghmi (2016). The third question is how often Negative bloc rule elects the *Condorcet winner* when there is one. Formally, this amounts to evaluate how often the Condorcet winner belongs to the winning committee under the Negative bloc rule. And the last question is how often Negative bloc rule elects the Condorcet loser when it exists. Formally, this is equivalent to evaluating how often the Condorcet loser belongs to the winning committee under the Negative bloc rule.

This paper is structured as follows. Section 2 introduces basic notations and definitions of the scoring rules studied in this work. Section 3 presents the behavior of the rules toward some axioms. Section 4 presents our results on the probabilities of agreement between Negative bloc rule and the four other scoring rules. Section 5 presents our probabilities concerning the two Condorcet properties. Section 6 concludes the paper.

## 2 Preliminaries

### Notations

Let  $E = (A, V)$  be an election with  $A = \{a, b, c, \dots\}$  the finite set of  $m$  candidates and  $V = \{1, \dots, n\}$  the set of  $n$  voters. In this paper we consider the preference of each voter to be a linear order on the set of candidates from the more preferred candidate

to the less preferred one. A linear order is a binary relation that is transitive, complete and antisymmetric. We denote by  $\pi$  the preference profile of voters and  $\pi(i)$  the preference of a specific voter  $i$ . We have  $m!$  possible linear orders for  $m$  candidates.

**Example 1.** Election with  $n = 34$  voters and  $m = 3$  candidates

4	9	5	6	3	7
$a$	$a$	$b$	$b$	$c$	$c$
$b$	$c$	$a$	$c$	$a$	$b$
$c$	$b$	$c$	$a$	$b$	$a$

In this paper, we display elections as in Example 1. For instance, the first column means that four voters prefer  $a$  to  $b$  and  $b$  to  $c$  and they prefer  $a$  to  $c$ .

For each pair  $a, b \in A$ , if the number of voters who rank candidate  $a$  higher than candidate  $b$  is greater than those who rank candidate  $b$  higher than candidate  $a$ , we say that the candidate  $a$  beats  $b$  in terms of a pairwise majority comparison, which we denote  $aMb$ .

We focus on multiwinner elections, so we will use other specific notations. We denote by  $k$  the size of the committee (i.e the number of candidates to be elected) with  $k \in [1, \dots, m]$ . A  $k$ -element subset of  $A$  is a committee denoted by  $C$ .

**Definition 1.** A multiwinner voting rule is a correspondence  $f$  that associates with each preference profile  $\pi$  one or several  $k$ -element subsets  $C$  of  $A$ .

We denote  $\mathcal{W} = (E, k)$  or simply  $\mathcal{W}$  the set of winning committees possibly a singleton. We will write  $W, W', \dots$  to designate elements of  $\mathcal{W}$ .

## Committee scoring rules

In this work, we focus on some specific multiwinner voting rules. These rules are part of a class of rules called *committee scoring rules* introduced by Elkind *et al.* (2017) and characterized by Faliszewski *et al.* (2018). The rules give a score to each candidate and the  $k$  candidates with the highest scores are elected. We define the score of a candidate  $a$  by:

$$S(a) = \sum_{i=1}^n s_{\text{pos}_i(a)}$$

where  $pos_i(a)$  is the position of the candidate  $a$  in the ranking of the voter  $i$ , and  $s_{pos_i(a)}$  is the score corresponding to the position of the candidate  $a$  in the ranking of the voter  $i$ .

**Definition 2.** A score vector  $s = (s_1, \dots, s_m)$  associates a number of points with each position, with  $s_j \geq s_k$  for  $j \leq k$  and  $s_1 > s_m$ .

Among the different voting rules of this paper only the score vector  $s$  will change from one rule to another. We then define the different score vectors corresponding to the five studied rules:

- Plurality score: Only the candidate ranked first in the preference gets one point. Formally we have  $s = (1, 0, \dots, 0)$ . The  $k$ -plurality rule (also called SNTV (single nontransferable Vote)) elects the  $k$  candidates with the highest plurality scores.
- Antiplurality score: Each candidate gets one point except the candidate ranked last in the preference. Formally we write  $s = (1, \dots, 1, 0)$ . The  $k$ -antiplurality rule (also called  $k$ -negative plurality) elects the  $k$  candidates with the highest antiplurality scores.
- $k$ -approval score: One point is giving to each of the  $k$  first candidates in the preference. Formally we have  $s = (\underbrace{1, \dots, 1}_{k \text{ times}}, 0, \dots, 0)$ . The Bloc rule (also called Limited voting or Constant scoring rule) elects the  $k$  candidates with the highest  $k$ -approval scores.
- $k$ -negative approval score: Each candidate gets one point except the  $k$  last candidates in the preference. Formally we write  $s = (1, \dots, 1, \underbrace{0, \dots, 0}_{k \text{ times}})$ . The Negative bloc rule elects the  $k$  candidates with the highest  $k$ -negative approval scores.
- Borda score: Each candidate gets a number of points equal to the number of candidates ranked behind him. Formally we have  $s = (m - 1, m - 2, \dots, 0)$ . The  $k$ -Borda rule elects the  $k$  candidates with the highest Borda scores.

Table 1 summarizes the score obtained by each candidate and the winning committee associated with each voting rule in Example 1. We can note that  $k$ -plurality and Negative bloc elect the same committee and furthermore, they give the same score to each candidate. The same happens for  $k$ -antiplurality and Bloc.

Table 1: Results of Example 1

	$a$	$b$	$c$	winning committee
$k$ -plurality	13	11	10	$\{a, b\}$
$k$ -antiplurality	21	22	25	$\{b, c\}$
Bloc	21	22	25	$\{b, c\}$
Negative bloc	13	11	10	$\{a, b\}$
$k$ -Borda	34	33	35	$\{a, c\}$

**Remark 1.** We can identify special cases with specific values of  $k$ .

If  $k = 1$ , Bloc is equivalent to  $k$ -plurality and Negative bloc is equivalent to  $k$ -antiplurality.

If  $k = m - 1$ , (which is the case in Example 1) Bloc is equivalent to  $k$ -antiplurality and Negative bloc is equivalent to  $k$ -plurality.

If  $k = \frac{m}{2}$ , Bloc and Negative bloc are equivalent.

### 3 Axioms

We focus on the axioms developed by Elkind *et al.* (2017)<sup>1</sup>. We extend their analysis with two more rules,  $k$ -antiplurality and Negative bloc. We start this section with the definition of the different axioms.

**Committee Monotonicity (CM).** For each election  $E = (A, V)$  the following conditions hold: (1) For each  $k \in [1, \dots, m - 1]$ , if  $C \in \mathcal{W}$  then there exists a  $C' \in \mathcal{W}' = (E, k + 1)$  such that  $C \subseteq C'$ ; (2) For each  $k \in [2, \dots, m]$ , if  $C \in \mathcal{W}$ , then there exists a  $C' \in \mathcal{W}' = (E, k - 1)$  such that  $C' \subseteq C$ .

This axiom states that (1) if  $C$  is a winning committee and we want to elect one more candidate, we just have to add one candidate to  $C$  and (2) if  $C$  is a winning committee and we want to elect less candidates, we just have to remove one candidate from  $C$ .

**Solid Coalition (SC).** For each election  $E = (A, V)$  and each  $k$ , if at least  $\frac{n}{k}$  voters rank some candidate  $c$  first then for all  $W \in \mathcal{W}$ ,  $c \in W$ .

This axiom states that if a candidate is ranked first by at least  $\frac{n}{k}$  voters, then this candidate should belong to every winning committee.

<sup>1</sup>The five rules studied are all committee scoring rules that are characterized in the paper of Faliszewski *et al.* (2018), thus they respect the same axioms: Symmetry, Committee Consistency, Committee Neutrality, Committee Dominance and Continuity.

**Consensus Committee (CC).** For each election  $E = (A, V)$  and each  $k$ , if there is a  $C$  such that each voter ranks some member of  $C$  first and each member of  $C$  is ranked first by either  $\lfloor \frac{n}{k} \rfloor$  or  $\lceil \frac{n}{k} \rceil$  voters then  $\mathcal{W} = \{C\}$ .

This axiom states that if all voters rank a candidate of  $C$  first and all candidates of  $C$  are ranked first by  $\lfloor \frac{n}{k} \rfloor$  or  $\lceil \frac{n}{k} \rceil$  voters, then  $C$  is the only winning committee.

**Unanimity (U).** For each election  $E = (A, V)$  and each  $k$ , if each voter ranks the same  $k$  candidates  $C$  on top (possibly in different order), then  $\mathcal{W} = \{C\}$  (strong Unanimity) or  $C \in \mathcal{W}$  (weak Unanimity).

This axiom states that if all candidates of  $C$  occupy the  $k$  first spots in the preference of all voters, then  $C$  is a winning committee for weak Unanimity and  $C$  is the only winning committee for strong Unanimity.

**Fixed Majority (FM).** For each election  $E = (A, V)$  and each  $k$ , if a majority of voters ranks the same  $k$  candidates  $C$  on top (possibly in different order), then  $\mathcal{W} = \{C\}$ .

This axiom states that if all candidates of  $C$  occupy the  $k$  first spots in the preference of a strict majority of voters, then  $C$  is the only winning committee.

**Monotonicity (M).** For each election  $E = (A, V)$ , each  $c \in A$  and each  $k$ , if  $c \in W$ , then for each  $E'$  obtained from  $E$  by shifting  $c$  one position forward in some voter  $i$  preference it holds that: (1) for *candidate monotonicity*:  $c \in W'$  and (2) for *non-crossing monotonicity*: if  $c$  was ranked immediately below some  $b \notin W$ , then  $W \in \mathcal{W}'$ .

This axiom states that if we improve the ranking of a candidate  $c$  belonging to a winning committee  $W$  then (1)  $c$  still belong to a winning committee; and (2) if this improvement is obtained without switching  $c$  with another candidate from  $W$ ,  $W$  is still a winning committee.

**Homogeneity (H).** For each election  $E = (A, V)$  and each  $k$ , and  $E' = tE$  with  $t \in \mathbb{N}$ , then  $\mathcal{W} = \mathcal{W}'$ .

This axiom states that the winning committees in an election made of multiple copies of  $E$  are the same than in  $E$ .

**Consistency (C).** For every pair of election  $E_1 = (A, V_1)$ ,  $E_2 = (A, V_2)$  and each  $k$ , if  $\mathcal{W}_1 \cap \mathcal{W}_2 \neq \emptyset$  then  $\mathcal{W}_{1+2} = \mathcal{W}_1 \cap \mathcal{W}_2$ .

This axiom states that winning committees in an election that merges two elections are those that are winning committees in both elections.



## Negative bloc

We start our analysis with the Negative bloc rule.

**Proposition 1.** *Negative bloc rule violates CM, SC, CC, FM and the strong version of U.*

*Proof.* • CM: in Table 2, when  $k=2$ , the winning committee is  $\mathcal{W} = \{\{a, c\}\}$ . When  $k=3$ , the winning committee is  $\mathcal{W}' = \{\{a, b, d\}\}$ , and  $\mathcal{W} \not\subseteq \mathcal{W}'$ ; then the rule does not fulfill the Committee monotonicity axiom.

Table 2: Election with  $n = 4$  and  $m = 4$

1	1	1	1
$a$	$a$	$b$	$d$
$c$	$c$	$c$	$c$
$b$	$d$	$d$	$a$
$d$	$b$	$a$	$b$

Table 3: Election with  $n = 4$  and  $m = 4$  for  $k = 2$

1	1	1	1
$a$	$b$	$c$	$c$
$b$	$a$	$a$	$b$
$d$	$d$	$b$	$a$
$c$	$c$	$d$	$d$

Table 4: Election with  $n = 4$  and  $m = 5$  for  $k = 2$

1	1	1	1
$a$	$a$	$b$	$b$
$c$	$e$	$c$	$e$
$e$	$c$	$d$	$d$
$b$	$d$	$e$	$c$
$d$	$b$	$a$	$a$

Table 5: Election with  $n = 3$  and  $m = 5$  for  $k = 2$

1	1	1
$a$	$a$	$b$
$b$	$b$	$a$
$c$	$c$	$c$
$d$	$e$	$d$
$e$	$d$	$e$

- SC: in Table 3,  $c$  is ranked first by  $\frac{n}{k}$  voters, whereas the winning committee is  $\mathcal{W} = \{\{a, b\}\}$ ; then the rule does not fulfill the Solid Coalition axiom.
- CC: in Table 4, the committee  $C = \{a, b\}$  fulfills the conditions of the Consensus Committee axiom, whereas with Negative bloc, the winning committee is  $\mathcal{W} = \{\{c, e\}\}$ ; then the rule violates the Consensus Committee axiom .
- FM and U: in Table 5, the committee  $C = \{a, b\}$  fulfills the conditions of the Fixed majority and Unanimity axioms, whereas with Negative bloc, the winning committees are  $\mathcal{W} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ .  $\mathcal{W} \neq \{C\}$ ; then the rule does not fulfill the Fixed majority axiom, and the strong version of the Unanimity axiom.

□

**Proposition 2.** *Negative bloc rule satisfies the two versions of M, H, C and the weak version of U.*

*Proof.* • M: let us consider  $c \in W$ . If  $c$  is permuted forward with another candidate in the ranking of at least one voter *ceteris paribus*, then  $c$  increases his score or keeps the same score; and the other candidates have the same score or a worse one than previously. Thus,  $c$  belongs to at least one winning committee.

If  $c$  has been permuted with  $b \notin W$ ,  $c$  increases his score and the committee  $C$  remains a winning committee:  $C \in \mathcal{W}'$ .

- H: let us consider  $W$  and  $t \in \mathbb{N}$ . We denote  $E' = (A, tN)$ , in this case the score of all the candidates is multiplied by  $t$ , that does not change the order of the scores. So  $W \in \mathcal{W}'$ , for all  $W \in \mathcal{W}$ , so  $\mathcal{W} = \mathcal{W}'$ .
- C: let us consider the same winning committee  $W$  for elections  $E_1$  and  $E_2$ . When we merge these two elections, the score of each candidate is the sum of his scores in the two elections. The  $k$  candidates with the highest scores are the same in election  $E_1$  and in election  $E_2$ . These  $k$  candidates remain the  $k$  candidates with the highest scores in the unified election. So  $W$  is a winning committee in the unified election  $E_{1+2}$ .
- U: let us take a committee  $C$  in which voters rank the members of  $C$  first in their preferences i.e. for each voter, the  $k$  first ranks are occupied by the members of  $C$ . We distinguish two cases:
  - If  $k < \frac{m}{2}$ , all members of  $C$  get the maximum of possible points, then  $C \in \mathcal{W}$ .
  - If  $k \geq \frac{m}{2}$ , only the members of  $C$  get points, then  $C \in \mathcal{W}$

□

## ***k*-antiplurality**

We continue our analysis with *k*-antiplurality.

**Proposition 3.** *k*-antiplurality rule violates SC, CC, FM and the strong version of U.

*Proof.* • SC: let us consider Table 3;  $c$  is ranked first by  $\frac{n}{k}$  voters. The winning committee is  $\mathcal{W} = \{a, b\}$ . The rule does not fulfill the Solid Coalition axiom.

- CC: now let us consider Table 4; the committee  $C = \{a, b\}$  fulfills the conditions of the Consensus Committee axiom, whereas with *k*-antiplurality, the winning committee is  $\mathcal{W} = \{c, e\}$ . The rule violates the Consensus Committee axiom.

- FM and U: in Table 5, the committee  $C = \{a, b\}$  fulfills the conditions of the Fixed majority and Unanimity axioms. Whereas with  $k$ -antiplurality, the winning committees are  $\mathcal{W} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ .  $\mathcal{W} \neq \{C\}$ ; then the rule does not fulfill the Fixed majority axiom, and the strong version of the Unanimity axiom.

□

**Proposition 4.**  $k$ -antiplurality rule satisfies CM, M, H and C.

*Proof.* • CM: suppose we have an election for a committee of size  $k - 1$ ,  $k$  or  $k + 1$ , the scores of the candidates remain the same in all cases. If we want to increase the size of the committee by 1, we add to the winning committee the candidate with the highest score which is not yet in that committee. If we want to decrease the size of the committee by 1, we remove from the winning committee the candidate with the lowest score.

- M, H and C: for the same reasons as for Negative bloc rule,  $k$ -antiplurality rule satisfies the Monotonicity, Homogeneity and Consistency axioms.
- U: consider a committee  $C$  in which voters rank the members of  $C$  first in their preferences i.e. for each voter, the  $k$  first ranks are occupied by the members of  $C$ . In this case, all members of  $C$  get the maximum possible points and  $C \in \mathcal{W}$ . Thus  $k$ -antiplurality rule satisfies the weak version of the Unanimity axiom.

□

Table 6 extends the analysis of Elkind *et al.* (2017) with two more rules: Negative bloc and  $k$ -antiplurality.

Rule	Committee Monotonicity	Solid Coalition	Consensus Committee	Unanimity	Monotonicity	Homogeneity	Consistency
<b>Negative bloc</b>	×	×	×	weak	C\NC	√	√
Bloc	×	×	×	fix maj.	C\NC	√	√
$k$ -Borda	√	×	×	strong	C\NC	√	√
$k$ -plurality	√	√	√	weak	C\NC	√	√
<b><math>k</math>-antiplurality</b>	√	×	×	weak	C\NC	√	√

Table 6: √ (resp. ×) indicates that the rule has (resp. does not have) the corresponding property. C means Candidate Monotonicity and NC Non-Crossing Monotonicity (C\NC means satisfying both conditions). "fix maj." in the Unanimity column means that the rules satisfy Fixed Majority but also that they satisfy Unanimity in the strong sense.

Concerning axioms introduced by Elkind *et al.* (2017), we are going to compare Negative Bloc rule with the four other rules, Bloc,  $k$ -Borda,  $k$ -plurality and  $k$ -antiplurality:

**Bloc:** Negative bloc rule is close to Bloc; however we notice two differences concerning Fixed Majority and Unanimity axioms.

**$k$ -Borda:** Similarly Negative bloc rule is close to  $k$ -Borda; here also there are only two differences and they concern Committee Monotonicity and Unanimity.

**$k$ -plurality:** Negative bloc rule is relatively close to  $k$ -plurality with three differences in axioms in Table 6: Committee Monotonicity, Solid Coalition and Consensus Committee.

**$k$ -antiplurality:** Negative bloc rule is very close to  $k$ -antiplurality with only one difference in the axiom: Committee Monotonicity.

In light of the results in Table 6, we can expect closer results between Negative bloc and  $k$ -antiplurality than between Negative bloc and the other rules, in terms of frequencies at which they select the same committee, which is the topic of the next section.

## 4 Agreement between Negative bloc rule and some multiwinner scoring rules

As stated in Remark 1, Negative bloc is equivalent to  $k$ -antiplurality when  $k = 1$ , to  $k$ -plurality when  $k = m - 1$  and to Bloc when  $k = \frac{m}{2}$ . Except for these values of  $k$ , Negative bloc rule does not correspond exactly with any other scoring rule. Our objective in this section is to find for any pair  $(k, m)$ , how often Negative bloc rule gives similar results with  $k$ -plurality,  $k$ -antiplurality, Bloc and  $k$ -Borda. We look for a probability (occurrence frequency) for  $m = 3, 4, 5, 6$  and different values of  $k$ . In order to find these probabilities we assume that each voting situation (see subsection 4.1 below) appears with the same probability, so we are going to use the Impartial and Anonymous Culture (IAC) assumption. This model was introduced by Gehrlein & Fishburn (1976) and remains one of the most used in social choice theory for computing the likelihood of voting events.

## 4.1 Methodology

In this section, we will use a methodology very close to Diss *et al.* (2020). We define by  $\tilde{n} = (n_1, \dots, n_{m!})$  a voting situation with  $\sum_{t=1}^{m!} n_t = n$  with  $n_t$  referring to the number of voters endowed with the preference order numbered  $t$ , among the  $m!$  linear orders. Obtaining the probability of a voting event under IAC requires two elements. The first one is the total number of voting situations and the second one is the number of cases in which the voting event appears. The total number of voting situations is calculated with the binomial coefficient  $\binom{n+m!-1}{m!-1}$ . For instance, with  $m = 4$  and  $n = 7$  we have 2 035 800 voting situations and with the same value of  $m$  and  $n = 11$  we have 286 097 760 voting situations. The number of voting situations grows even faster with a greater number  $m$  of candidates. Enumerating all voting situations is very demanding in terms of memory for our computer, so we need to use computer simulations to address this challenge. We will not obtain exact results but with a number of iterations high enough (1 000 000 in our case) our approximations will be of good quality. The process consists of two steps that will be repeated 1 000 000 times to evaluate the probability of agreement between Negative bloc rule and the other scoring rules.

Let us detail the process for a specific situation with  $m, n$  and  $k$  fixed. The first step is to generate randomly a voting situation of length  $m!$  with  $n$  voters. The second step is to check whether Negative bloc rule agrees or not with the other rules at that voting situation. The probability of agreement is then calculated as the quotient of the number of voting situations where Negative bloc agrees with each other rules over the total number of iterations. We have to repeat this process for all values of  $m, k$  and  $n$ . In this work we will go up to  $n=100\,000$ .

## 4.2 Main results

Before commenting our results we need to give a finer definition of agreement. We will define two types of agreements, that can be viewed as degrees of agreement:

- Agreement of type 1: This is the more natural agreement we can think of, the strongest one as well. When Negative bloc rule and the other rule select the same committee(s), their sets of winning committees  $\mathcal{W}$  are the same. The probability of such an agreement of type 1 will be denoted  $Pr1$ .
- Agreement of type 2: This agreement is softer than the one before. When a scoring rule selects one or several committees, then those committees are in-

cluded in the set of winning committees of the Negative bloc rule. The probability of such an agreement of type 2 will be denoted  $Pr2$ .

It is clear that if agreement of type 1 holds, agreement of type 2 holds as well. But the converse is not true. It clearly follows that  $Pr2 \geq Pr1$  for any value of  $m, k$  and  $n$ . Our results for probability of agreement of type 1 (resp. type 2) are available in Figure 1 (resp. Figure 2).

Before presenting our results, some preliminary and general remarks will be useful. With  $(k, m) = (2, 3)$ , it is clear that  $k = m - 1$ , which means that  $k$ -antiplurality and Bloc are equivalent and  $k$ -plurality and Negative bloc are equivalent too. This case is very particular, for this reason we choose not to display it in Figure 1 and Figure 2. For the pairs  $(2, 4)$  and  $(3, 6)$ ,  $k = \frac{m}{2}$ ; then Bloc and Negative bloc are equivalent and thus the line corresponding to Bloc is always equal to one, and for this reason we have not found it useful to show it on the figure. A more surprising fact is that in these two cases above,  $k$ -plurality and  $k$ -antiplurality have the same probability of agreement with the Negative bloc rule, then the two solid bold lines coincide.

We can read these figures as follows: for the pair  $(2, 4)$  for instance, when  $n=100\ 000$ , the probability of agreement of type 1 ( $Pr1$ ) of  $k$ -Borda rule with Negative bloc rule is 69.35%.

$Pr1$  increases with the number of voters, for any pair  $(k, m)$ . For example with the pair  $(2, 5)$ , for  $k$ -plurality the probability rises from 6.37% when  $n = 4$  to 23.06% when  $n=100\ 000$ ; for  $k$ -Borda it rises from 37.23% to 56.74%; for Bloc it rises from 17.57% to 37.34% and for  $k$ -antiplurality it rises from 17.32% to 33.43%. For a fixed size of committee and a fixed number of voters ( $n$  large), we can see that  $Pr1$  decreases as the number of candidates increases. For  $k$ -Borda, the probability for  $k = 2$  and  $n=100\ 000$  decreases from 69.44% when  $m=4$ , to 56.74% when  $m=5$ , and to 46.62% when  $m=6$ .

In terms of agreement with the Negative bloc rule, except for the special cases where  $k = m - 1$  and  $k = \frac{m}{2}$ , the  $k$ -Borda rule is clearly the one that gives the closest results. Obviously when  $k = \frac{m}{2}$ , Bloc is the rule that gives the closest results since it is completely equivalent to Negative bloc. For the second closest rule in terms of agreement with Negative bloc rule, we need to distinguish when  $k < \frac{m}{2}$  and when  $k > \frac{m}{2}$ :

- $k < \frac{m}{2}$ :  $k$ -antiplurality tends to become the second closest rule as the number of candidates increases.

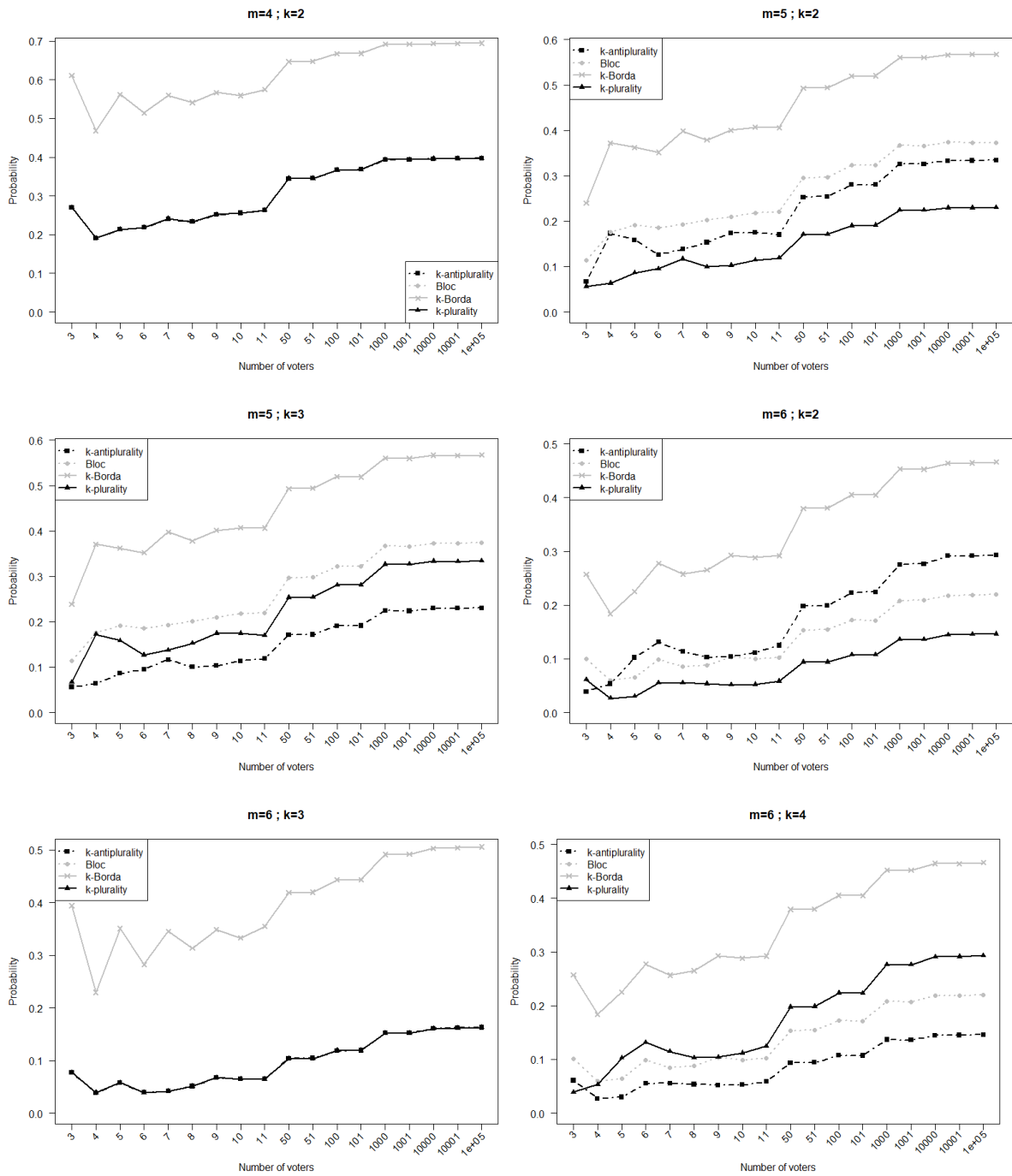


Figure 1: Probability of agreement of type 1 between Negative bloc and other multiwinner scoring rules

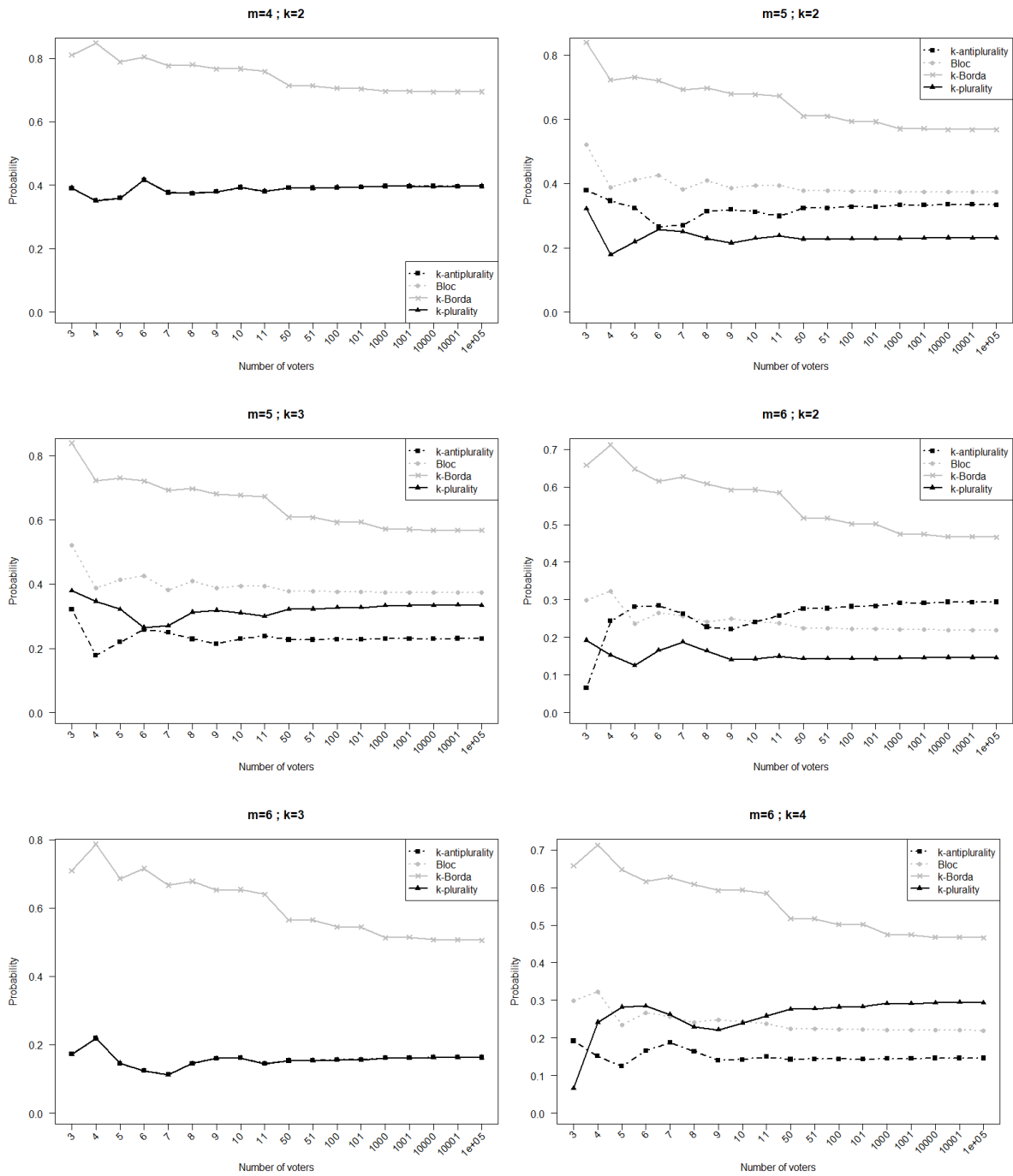


Figure 2: Probability of agreement of type 2 between Negative bloc and other multiwinner scoring rules



- $k > \frac{m}{2}$ :  $k$ -plurality tends to become the second closest rule as the number of candidates increases.

For  $Pr2$ , depending on the pair  $(k, m)$  it is not clear whether the probability increases or decreases with the number of voters. What is clear is that  $Pr2$  converges to  $Pr1$  as the number of voters increases.

In terms of agreement with the Negative bloc rule, we have the same conclusions as with  $Pr1$ .

Based on these results, we note that the Negative bloc rule leads in most cases to a committee made of the  $k$  candidates with the highest Borda scores.

Table 7: Agreement between Negative bloc and other scoring rules when  $n \rightarrow \infty$

		$k$ -plurality						$k$ -antiplurality			
$m \rightarrow$	$k \downarrow$	3	4	5	6	$m \rightarrow$	$k \downarrow$	3	4	5	6
1	1	0.5232	0.3757	0.2908	0.2366	1	1	1	1	1	1
2	2	1	0.3970	0.2302	0.1463	2	0.5232	0.3968	0.3336	0.2934	
3	3	--	1	0.3353	0.1632	3	--	0.3754	0.2300	0.1626	
4	4	--	--	1	0.2937	4	--	--	0.2922	0.1459	
5	5	--	--	--	1	5	--	--	--	0.2378	

		Bloc						$k$ -Borda			
$m \rightarrow$	$k \downarrow$	3	4	5	6	$m \rightarrow$	$k \downarrow$	3	4	5	6
1	1	0.5232	0.3757	0.2908	0.2366	1	0.6871	0.5832	0.5125	0.4578	
2	2	0.5232	1	0.3736	0.2198	2	0.6868	0.6940	0.5673	0.4659	
3	3	--	0.3754	0.3743	1	3	--	0.5832	0.5672	0.5053	
4	4	--	--	0.2922	0.2196	4	--	--	0.5131	0.4666	
5	5	--	--	--	0.2378	5	--	--	--	0.4579	

In Table 7, we present the probability of agreement  $Pr1$  between Negative bloc and the other rules when  $n$  tends to infinity ( $n=100\ 000$ ). We can read these tables as follows: for example, when  $m = 6$  and  $k = 2$ , the  $k$ -antiplurality and Negative bloc rules select the same winning committee in 29.34% of the cases. We can make the following comments on the probabilities. First, for a fixed size of committee, the probabilities tend to decrease as the number of candidates increases, except for Bloc when  $k = \frac{m}{2}$ . Except for the special cases  $k = 1$  and  $k = m - 1$ , for a fixed number of candidates, the probabilities decrease and then increase, the minimal probability is reached at  $k = \frac{m}{2}$ ; however this is not true for Bloc and  $k$ -Borda. Second, probability of agreement between negative bloc rule and the other rules is always greater than

choosing a committee randomly, as we could suspect. To be more explicit, consider  $k$ -antiplurality for which the smallest probability is reached when  $m = 6$  and  $k = 4$  i.e. 14.56%; for these values of  $m$  and  $k$  there are 15 possible committees, then with randomness and equiprobability assumption, the probability of agreement would have been  $\frac{1}{15} \simeq 6.67\%$ . Negative bloc rule seems to give pretty close results than  $k$ -Borda. They elect the same committee in at least 45.78% of cases when  $m$  changes from 3 to 6 and  $k$  from 1 to  $m - 1$ .

## 5 On some Condorcet properties with Negative bloc rule

There exist multiple Condorcet principles in the literature. We focus in this work on three of them: the Condorcet winner, the Condorcet loser and the Condorcet committee. Based on simulations of the same type as before i.e. using IAC to generate the preferences, we evaluate how often the negative bloc rule elects the Condorcet winner, the Condorcet loser or the Condorcet committee when it exists.

### 5.1 Condorcet committee efficiency of Negative bloc rule

In this section, we aim to compute the probability that the Negative bloc rule selects the *Condorcet committee à la Gehrlein* when there is one. This gives a more complete analysis of the Negative bloc rule.

**Definition 3.** *A committee  $C$  is a Condorcet committee à la Gehrlein if each element of  $C$  beats each element not in  $C$  in a pairwise majority comparison. Formally  $C$  is a Condorcet committee à la Gehrlein if for each  $a \in C$  and each  $b \notin C$ , we have  $a \mathbf{M} b$ .*

The probability that a rule selects the Condorcet committee when it exists is called the Condorcet Committee efficiency (CCE) of that rule. The CCEs of  $k$ -plurality,  $k$ -antiplurality,  $k$ -Borda and Bloc were calculated by Diss & Doghmi (2016). To evaluate CCE, we will use a methodology similar to the one used in the previous section on agreement. We will use the IAC assumption with  $n=100\ 000$  ( $n \rightarrow \infty$ ) and we run 1 000 000 elections. The CCE is calculated by the quotient of the number of elections in which Negative bloc selects the Condorcet committee over the number of elections in which the Condorcet committee exists. We present our results in Table 8.

Table 8:  $CCE_{IAC_\infty}^{NB}(k, m)$

$m \rightarrow$ $k \downarrow$	3	4	5	6
1	0,6305	0.5515	0.5086	0.4721
2	0,6296	0.7469	0.6001	0.5157
3	--	0.5514	0.6027	0.5754
4	--	--	0.5089	0.5175
5	--	--	--	0.4719

Results in Table 8 along with those in the paper of Diss *et al.* (2020) lead us to make the following remarks. First, for a fixed size  $k$  of a committee, the probability to elect the Condorcet committee when it exists decreases with an increase in the number  $m$  of candidates. Second, for a fixed number  $m$  of candidates, the probability to elect the Condorcet Committee when it exists increases and then decreases with an increase of the size  $k$  of the winning committee; and the maximum probability is reached when  $k = \frac{m}{2}$ . Third, in the extreme cases where  $k = 1$  and  $k = m - 1$ , Negative bloc has a low CCE, the same as  $k$ -antiplurality when  $k = 1$  and the same as  $k$ -plurality when  $k = m - 1$ . Fourth, except for these two extreme cases, Negative bloc rules has a better CCE than  $k$ -plurality and  $k$ -antiplurality, and gives similar results as Bloc. For example, for the pair (3, 5), the CCEs of  $k$ -plurality and  $k$ -antiplurality are 0.4199 and 0.4521, respectively.  $k$ -Borda still stays the rule with the highest CCE (never less than 0.8172 for any pair  $(k, m)$ ).

Overall, For  $m = 3, 4, 5$  and  $6$ , Negative bloc rule always elects the Condorcet Committee with a probability higher than 47%. This probability is higher than that of common rules such as  $k$ -plurality and  $k$ -antiplurality.

## 5.2 Condorcet winner and Negative bloc rule

Committee scoring rules do not always elect the Condorcet winner when it exists. Election of Condorcet winner has been widely studied in the literature. This section provides the probability that Negative bloc rule elects the Condorcet winner if it exists i.e. the fact that the Condorcet winner is included in the winning committee.

**Definition 4.** A candidate  $c$  is a Condorcet winner if  $c$  beats each element  $b \in C$  in a pairwise majority comparison. Formally  $c$  is a Condorcet winner if we have  $cM_b$  for each  $b \in C, b \neq c$ .

The probability that Negative bloc rule selects the Condorcet winner when it exists as a member of the winning committee, denoted  $CW_{IAC_\infty}^{NB}(k, m)$ , is presented in Table 9. This probability was calculated for  $k$ -plurality,  $k$ -antiplurality,  $k$ -Borda and Bloc by Diss *et al.* (2020).

To evaluate  $CW_{IAC_\infty}^{NB}(k, m)$ , we will use the same methodology as in section 4 and subsection 5.1 above. Again, we use the IAC assumption with  $n=100\,000$  ( $n \rightarrow \infty$ ) and we run 1 000 000 elections. The probability is calculated by the quotient of the number of elections in which Negative bloc selects the Condorcet winner over the number of elections in which the Condorcet winner exists.

Table 9:  $CW_{IAC_\infty}^{NB}(k, m)$

$m \rightarrow$ $k \downarrow$	3	4	5	6
1	0,6305	0.5514	0.5086	0.472
2	0.9688	0.9553	0.8964	0.8531
3	--	0.9765	0.9806	0.9721
4	--	--	0.981	0.9874
5	--	--	--	0.9846

Once again, a few comments may shed some light on the results presented in Table 9. First, for a fixed number  $m$  of candidates, the probability to elect the Condorcet winner when it exists increases with the size  $k$  of the winning committee. Second, except for  $k = 1$  i.e. when Negative bloc coincides with  $k$ -antiplurality, the probability is higher than for  $k$ -plurality (see Diss *et al.* (2020)). Third, compared to  $k$ -antiplurality the probability is still higher, except when  $k = m - 1$ , where the two probabilities are very similar. For example, for the pair (3, 5), the probability of electing the Condorcet winner when it exists under  $k$ -plurality and  $k$ -antiplurality is 0.9335 and 0.9175, respectively. Bloc and Negative bloc give similar results, except for  $k = 1$  as mentioned above.  $k$ -Borda still stays the rule with the highest probability of electing the Condorcet winner when it exists (0.9998 for the pair (3, 5)).

Overall, Negative bloc rule performs well in electing the Condorcet winner when it exists, except when  $k = 1$ . Furthermore, it performs better than common and widely used rules like  $k$ -plurality and  $k$ -antiplurality.

### 5.3 Condorcet loser and Negative bloc rule

Committee scoring rules can elect the Condorcet loser when it exists. Election of a Condorcet loser is something that we want to avoid. This section provides the probability that Negative bloc rule elects the Condorcet loser when it exists i.e. the fact that the Condorcet loser is a member of the winning committee.

**Definition 5.** A candidate  $c$  is a Condorcet loser if  $c$  is beaten by each element  $b \in C$  in a pairwise majority comparison. Formally  $c$  is a Condorcet loser if we have  $bM_c$  for each  $b \in C, b \neq c$ .

The probability that Negative bloc rule selects the Condorcet loser when it exists as a member of the winning committee denoted  $CL_{IAC_\infty}^{NB}(k, m)$  is presented in Table 10. This probability has been calculated for  $k$ -plurality,  $k$ -antiplurality,  $k$ -Borda and Bloc by Diss *et al.* (2020).

Table 10:  $CL_{IAC_\infty}^{NB}(k, m)$

$m \rightarrow$ $k \downarrow$	3	4	5	6
1	0,0314	0.0241	0.0193	0.0156
2	0.3704	0.045	0.0193	0.0132
3	--	0.4491	0.1037	0.0283
4	--	--	0.4912	0.1472
5	--	--	--	0.5268

To evaluate  $CL_{IAC_\infty}^{NB}(k, m)$ , we will use a methodology similar to the one in the previous section and subsections above. We will use the IAC assumption with  $n=100\,000$  ( $n \rightarrow \infty$ ) and we run 1 000 000 elections. The probability is calculated by the quotient of the number of elections in which Negative bloc selects the Condorcet loser over the number of elections in which the Condorcet loser exists.

It appears that: first, for a fixed number  $m$  of candidates, the probability to elect the Condorcet loser when it exists increases with the size  $k$  of the winning committee; second, except for  $k = m - 1$  i.e. when Negative bloc corresponds to  $k$ -plurality, the probability is lower than for  $k$ -antiplurality; third, compared to  $k$ -plurality the probability is still lower, except when  $k = m - 1$ , where the two probabilities are very similar. For instance, for the pair (3, 5), the probability to elect the Condorcet loser

when it exists under  $k$ -plurality and  $k$ -antiplurality is 0.2249 and 0.1670, respectively. Bloc and Negative bloc give similar results, except for  $k = m - 1$ , as mentioned above.  $k$ -Borda still stays the rule with the lowest probability of electing the Condorcet loser when it exists (0.0087 for the pair (3, 5)).

Overall, Negative bloc rule performs well in not electing the Condorcet loser when it exists, except when  $k = m - 1$ . Moreover, it performs better than common and widely used rules like  $k$ -plurality and  $k$ -antiplurality.

## 6 Conclusion

In this paper we provide an axiomatic and probabilistic analysis of some Committee Scoring rules and in particular the newly introduced Negative bloc rule. We find that the five scoring rules studied are close in terms of axioms. Negative bloc rule is closer to  $k$ -antiplurality than to the other rules in terms of axioms as one may guess, because it is one of its straightforward extensions in the multiwinner setting. In terms of agreement, surprisingly, Negative bloc appears closer to  $k$ -Borda than to  $k$ -antiplurality. Regarding all the Condorcet properties,  $k$ -Borda remains the best of these five Committee Scoring rules. Negative bloc gives similar results as Bloc and better results than  $k$ -plurality and  $k$ -antiplurality in most cases. In the light of the jump in probability of election of the Condorcet loser, Negative bloc rule should be used when  $k \leq \frac{m}{2}$ . One advantage of Negative bloc as compared to  $k$ -Borda is that it requires less information on the voters preferences; thus it is easier to implement in real voting situations.

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