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CONSTRUCTION OF BROOKS–LINDENSTRAUSS KERNELS ON AFFINE BUILDINGS OF ARBITRARY REDUCED TYPE, WITH APPLICATIONS

JEAN-PHILIPPE ANKER, BERTRAND RÉMY, AND BARTOSZ TROJAN

ABSTRACT. This article deals with harmonic analysis on affine buildings. Its main goal is to construct suitable kernels associated to a discrete multitemporal wave equations on the latter spaces, the long-standing motivation being to contribute to progress in arithmetic quantum unique ergodicity (AQUE) on certain Riemannian manifolds.

INTRODUCTION

Quantum unique ergodicity (QUE) deals with certain limit measures on compact (or finite-volume) negatively curved manifolds or locally symmetric manifolds of non-compact type. More precisely, on a negatively curved compact manifold M we consider a sequence $(\varphi_j : j \in \mathbb{N})$ of L^2 -normalized eigenfunctions of the positive Laplacian Δ corresponding to eigenvalues $\lambda_j \rightarrow \infty$. In this context, Z. Rudnick and P. Sarnak [26] conjectured that the only weak- \star cluster value of the sequence of probability measures $|(\varphi_j)_j|^2 d\text{vol}_M$ is $d\text{vol}_M$ itself. When M is a locally symmetric space whose universal cover $\tilde{M} = G/K$ is an irreducible Riemannian symmetric space of non-compact type, the analogous conjecture has a more group-theoretic formulation since then the identity component $G = \text{Isom}(\tilde{M})^\circ$ is a simple Lie group and K is a maximal compact subgroup (they are all conjugated). The conjectured statement on limit measures is the same but one makes the assumption that the functions $(\varphi_j : j \in \mathbb{N})$ are joint eigenfunctions of the algebra \mathbb{D} of all G -invariant differential operators on \tilde{M} descended to $M = \Gamma \backslash \tilde{M}$.

Though it is possible to attack these conjectures from a purely analytic perspective (see *e.g.* [2]), in the locally symmetric case some specific choices of eigenfunctions $(\varphi_j : j \in \mathbb{N})$ can be made, relying on group- and representation-theoretic considerations. Indeed, if one assumes that Γ the fundamental group of M is an arithmetic lattice in G , then an additional commutative algebra of difference operators becomes available. More precisely, since G can be seen as the Archimedean points of an algebraic group \mathbf{G} over a number field F , there is a Hecke algebra \mathcal{H} associated with an adelic realization of the homogeneous space $\Gamma \backslash G$ whose elements commute with \mathbb{D} . Thus it makes sense to work with joint eigenfunctions φ_j of both \mathbb{D} and \mathcal{H} which is the context of arithmetic quantum unique ergodicity (AQUE). These additional arithmetic invariance conditions may be considered as a strong restriction, but they are automatically fulfilled if the \mathbb{D} -spectrum is multiplicity free, which is actually conjectured under suitable assumptions. The most striking and complete results in AQUE are those obtained by E. Lindenstrauss for compact hyperbolic surfaces [18] together with the finite volume case proved by K. Soundararajan [32], but more and more cases became available in recent years.

Still in the arithmetic situation, there is an intermediate context in which the additional arithmetic invariance assumption only involves a single place v of F . In this case, the corresponding subalgebra \mathcal{H}_v of \mathcal{H} is a convolution algebra of functions on the non-Archimedean Lie group $\mathbf{G}(F_v)$ where F_v denotes the completion of F with respect to v . Equivalently, \mathcal{H}_v can be seen as some commutative algebra \mathcal{A}_0 of averaging operators acting on functions defined on the vertices of the Bruhat–Tits building attached to $\mathbf{G}(F_v)$. This roughly explains why the main technique of the present paper is the purely geometric (building-theoretic) counterpart to harmonic analysis of bi- K_v -invariant functions on $\mathbf{G}(F_v)$ where K_v is a suitably chosen maximal compact subgroup of $\mathbf{G}(F_v)$, as developed by I. Satake and I.G. Macdonald [19].

We need to introduce additional concepts in order to state our results whose main outcome is the construction of a wave kernel on an affine building. Let us recall that an affine building [34] is a simplicial

complex covered by subcomplexes called *apartments* which are all isomorphic to a given Euclidean tiling. The (discrete) symmetry group of the model tiling is called the *Weyl group* of the building, and apartments in the building are requested to satisfy the following incidence conditions:

- any two simplices must be contained in an apartment;
- given any two apartments, there must be an isomorphism between them fixing their intersection.

These axioms allow one to define a real-valued distance between arbitrary points making an affine building a non-positively curved contractible space. They also lead to a vectorial distance between vertices (*i.e.* 0-simplices) of particular type and called special vertices: A vertex is called *special* if its stabilizer in the Weyl group (of any apartment containing it) is the full linear part of the Weyl group; we denote by V_s the set of special vertices.

Given an affine building \mathcal{X} a (purely geometric) harmonic analysis can be developed and leads to a Gelfand–Fourier transform with no group involved (see Subsection 1.4 and [24] for more details). First of all, the geometry of an apartment embedded in \mathcal{X} defines a root system Φ with Weyl group W . Let us denote by \mathfrak{a} the Euclidean space containing it; this space contains the co-weight lattice P in which the vectorial distance $\sigma : V_s \times V_s \rightarrow P^+$ takes its values, see Section 1.3 for details. For the purpose of applications, we consider a commutative Banach algebra \mathcal{A}_1 densely containing \mathcal{A}_0 whose spectrum Σ is contained in the complexification $\mathfrak{a}_{\mathbb{C}}$ of \mathfrak{a} , so that the Gelfand–Fourier transform \mathcal{F} maps \mathcal{A}_1 into the algebra $C(\Sigma)$ of continuous functions on Σ , which is defined as (6).

Theorem A. *Assume that the root system Φ of the affine building \mathcal{X} is reduced. Set $\rho = \sum_{i=1}^r \lambda_i$ where $\{\lambda_1, \dots, \lambda_r\}$ is the canonical co-weight basis in \mathfrak{a} . Let $z_0 = \zeta_0 + i\theta_0$ be an element of Σ . Then there exist $M(\zeta_0), N(\zeta_0) \in \mathbb{N}$ such that for all integers $M \geq M(\zeta_0)$ and $N \geq N(\zeta_0)$, there is a function $k : V_s \times V_s \rightarrow \mathbb{C}$ with the following properties:*

- (i) *for all $x, y \in V_s$ the value of $k(x, y)$ depends only on the vectorial distance $\sigma(x, y)$; moreover, $k(x, y)$ vanishes when*

$$\sigma(x, y) \not\leq cM^{r+1}N^r \rho$$

where $c > 0$ is a constant depending only on the root system Φ ;

- (ii) *there are $C > 0$ and $c > 0$ such that*

$$\sup_{x, y \in V_s} |k(x, y)| \leq Ce^{-cN};$$

- (iii) *for all $x \in V_s$ and $z \in \Sigma$, we have*

$$\mathcal{F}(k(x, \cdot))(z) \geq -1 \quad \text{while} \quad \mathcal{F}(k(x, \cdot))(z_0) \geq M.$$

Such kernels were first constructed by Sh. Brooks and E. Lindenstrauss in [9–11] for homogeneous trees and next by Z. Shem-Tov in [28] for affine buildings of type \tilde{A}_r with $r \geq 2$. Our construction works for any affine building of reduced type and our kernel differs from the one in [28].

Let us briefly comment on the proof of Theorem A. To construct the kernel we seek for building blocks satisfying (i) and (ii). For this purpose in the case of homogeneous trees, Sh. Brooks and E. Lindenstrauss used a fundamental solution to the discrete wave equation. Notice that, on real hyperbolic spaces, fundamental solutions to the shifted wave equation are known to possess similar properties (see for instance [33]). Therefore we introduce a discrete multitemporal wave equation on affine buildings which is the analog of the system studied by M.A. Semenov-Tian-Shansky [27] on non-compact symmetric spaces. We consider a particular non-trivial solution to this equation and show that it satisfies finite propagation speed and uniform exponential space-time decay (see Theorem 2.3). Inspired by Sh. Brooks and E. Lindenstrauss [9–11], we define a kernel satisfying (i), (ii), and (iii). The last property is proved by a careful study of its Gelfand–Fourier transform on the spectrum Σ . Lastly let us note again that the statement of Theorem A and its proof make no use of any group action.

Let us explain now how the above result fits in a general strategy elaborated to prove some cases of AQUE. The general scheme of proof introduced by E. Lindenstrauss [18] consists of three steps which are quite independent from one another (see [31, Introduction] for more details).

- **Microlocal lift.** This step consists in lifting the problem from the locally symmetric space $M = \Gamma \backslash G/K$ to a geometric bundle over M (the unit tangent bundle in rank one) and eventually to the homogeneous space $\Gamma \backslash G$. However, even if initially it was proved geometrically, there is now a representation-theoretic approach which works in fairly general setup [30]. The outcome is a reformulation of the problem where the main objects of study are limits $\mu = \lim_{j \rightarrow \infty} |\varphi_j|^2 dx$ where dx is the normalized Haar measure on $\Gamma \backslash G$, and where φ_j are L^2 -normalized eigenfunctions of both $\mathcal{Z}(\mathfrak{g})$ and \mathcal{H} (or part of it). Here \mathfrak{g} denotes the Lie algebra of G , and $\mathcal{Z}(\mathfrak{g})$ the center of its universal enveloping algebra.
- **Positive entropy.** A consequence of the previous step is that any cluster value μ of $(|\varphi_j|^2 dx : j \in \mathbb{N})$ is invariant under a maximal split torus of G , denoted by A in the sequel. In this step, the goal is to prove that for all regular elements $a \in A$ (or at least for some specific ones), almost every ergodic component of a limit measure μ has positive entropy. This is the place where our construction may play a rôle.
- **Classification of homogeneous measures.** The next step consists in exploiting results from a wide program which is interesting in itself, namely the classification of invariant and ergodic probability measures on homogeneous spaces such as $\Gamma \backslash G$. This program is far from being completed and it is usually the main limiting step in proving instances of AQUE. Some intermediate results often state that measures satisfying suitable positivity entropy conditions are measures associated to homogeneous spaces of algebraic subgroups of G : the latter measures are usually called *algebraic*.

In this general scheme, the use of a kernel as in Theorem A occurs in the second step, following ideas initially due to Sh. Brooks and E. Lindenstrauss [9–11] and enables one to deal with suitable "explicit" partitions of $\Gamma \backslash G$. This leads to the following positive entropy result whose proof consists of checking that the lemmas used by Z. Shem-Tov in [28, Theorem 1.1] to deal with the case $G = \mathrm{SL}_n(\mathbb{R})$ can be used verbatim once his specific kernel for SL_n is replaced by ours.

Theorem B. *Let G be a simple real Lie group which is \mathbb{R} -split, and let Γ be an arithmetic subgroup of G . Let μ be an A -invariant probability measure on the homogeneous space $X = \Gamma \backslash G$. Assume that $\mu = c \lim_{j \rightarrow \infty} |\varphi_j|^2 dx$ is a weak- \star limit where $c > 0$ and φ_j are some L^2 -normalized joint eigenfunctions of both $\mathcal{Z}(\mathfrak{g})$ and of the Hecke algebra at some fixed prime p . Then any regular element $a \in A$ acts with positive entropy on each ergodic component of μ .*

As already mentioned, modulo some progresses on the classification of invariant measures on homogeneous spaces, Theorem B may contribute to provide some additional cases of AQUE involving a single place. In particular, it may be useful in the following situation (see Subsection 4.2 for more details): Let \mathbf{G} be an absolutely almost simple group over \mathbb{Q} and let p be a prime number. We set $G_\infty = \mathbf{G}(\mathbb{R})$ and $G_p = \mathbf{G}(\mathbb{Q}_p)$, we choose a maximal compact subgroup K_p in G_p . Let us denote by $\Gamma[\frac{1}{p}]$ (resp. by Γ) the $\mathbb{Z}[\frac{1}{p}]$ -points (resp. the \mathbb{Z} -points) of \mathbf{G} , for instance defined by means of some embedding $\mathbf{G} < \mathrm{GL}_m$. In good cases (e.g. when \mathbf{G} is simply connected), we have $G_p = \Gamma[\frac{1}{p}]K_p$ and $\Gamma[\frac{1}{p}] \cap K_p = \Gamma$, hence a bijection $\Gamma g_\infty \mapsto \Gamma[\frac{1}{p}](g_\infty, 1)K_p$ between $\Gamma \backslash G_\infty$ and $\Gamma[\frac{1}{p}] \backslash (G_\infty \times G_p)/K_p$. As recalled in Subsection 4.1, since we are eventually interested in G_∞ only, we can adjust the choice of \mathbf{G} so that G_p is split over \mathbb{Q}_p , hence the Bruhat–Tits building of G_p has a reduced root system and thus admits kernels as in our Theorem A. This will lead to a non-compact but finite volume quotient space $\Gamma \backslash G_\infty$. To deal with compact spaces $\Gamma \backslash G_\infty$ one has to assume that \mathbf{G} is anisotropic over \mathbb{Q} . This can be achieved either by choosing \mathbf{G} so that $G_{p'}$ is compact at some other non-Archimedean place p' (implying strong restrictions on the absolute type of \mathbf{G}) or by assuming that the rational points of \mathbf{G} at some Archimedean place are compact (with no restriction on the absolute root system but requiring to pass from \mathbb{Q} to a number field F for the ground field of \mathbf{G}).

The structure of the paper goes as follows: In Section 1, we introduce the basics about affine buildings and harmonic analysis thereon (again, we do it without calling on Bruhat–Tits theory, in fact without using any group action). In Section 2, we define a discrete multitemporal wave equation and use Fourier analysis to study a particular solution for which we prove finite propagation speed and uniform space-time estimates.

In Section 3, we give a detailed proof of Theorem A. In Section 4, we describe the general context in which constructed kernels could be used and prove Theorem B.

We conclude the introduction with some notation. In the whole paper, \mathbb{N} (resp. \mathbb{N}^*) is the set of integers ≥ 0 (resp. > 0).

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1. AFFINE BUILDINGS

1.1. Buildings. A family \mathcal{X} of non-empty finite subsets of some set V is an *abstract simplicial complex* if for all $\sigma \in \mathcal{X}$, each subset $\gamma \subseteq \sigma$ also belongs to \mathcal{X} . The elements of \mathcal{X} are called *simplices*. The dimension of a simplex σ is $\#\sigma - 1$. Zero dimensional simplices are called *vertices*. The set $V(\mathcal{X}) = \bigcup_{\sigma \in \mathcal{X}} \sigma$ is the *vertex set* of \mathcal{X} . The dimension of the complex \mathcal{X} is the maximal dimension of its simplices. A *face* of a simplex σ is a non-empty subset $\gamma \subseteq \sigma$. For a simplex σ we denote by $\text{St}(\sigma)$ the collection of simplices containing σ ; in particular, $\text{St}(\sigma)$ is a simplicial complex. Two abstract simplicial complexes \mathcal{X} and \mathcal{X}' are *isomorphic* if there is a bijection $\psi : V(\mathcal{X}) \rightarrow V(\mathcal{X}')$ such that for all $\sigma = \{x_1, \dots, x_k\} \in \mathcal{X}$ we have $\psi(\sigma) = \{\psi(x_1), \dots, \psi(x_k)\} \in \mathcal{X}'$. With every abstract simplicial complex \mathcal{X} one can associate its *geometric realization* $|\mathcal{X}|$ in the vector space of functions $V \rightarrow \mathbb{R}$ with finite support, see e.g. [22, §2].

A set \mathcal{C} equipped with a collection of equivalence relations $\{\sim_i : i \in I\}$ where $I = \{0, \dots, r\}$, is called a *chamber system* and the elements of \mathcal{C} are called *chambers*. A *gallery* of type $f = i_1 \dots i_k$ in \mathcal{C} is a sequence of chambers (c_1, \dots, c_k) such that for all $j \in \{1, 2, \dots, k\}$, we have $c_{j-1} \sim_{i_j} c_j$, and $c_{j-1} \neq c_j$. If $J \subseteq I$, a *residue* of type J is a subset of \mathcal{C} such that any two chambers can be joined by a gallery of type $f = i_1 \dots i_k$ with $i_1, \dots, i_k \in J$. From a chamber system \mathcal{C} we can construct an abstract simplicial complex where each residue of type J corresponds to a simplex of dimension $r - \#J$. Then, for a given vertex x , we denote by $C(x)$ the set of chambers containing x .

A *Coxeter group* is a group W given by a presentation

$$\langle r_i : (r_i r_j)^{m_{i,j}} = 1, \text{ for all } i, j \in I \rangle$$

where $M = (m_{i,j})_{I \times I}$ is a symmetric matrix with entries in $\mathbb{Z} \cup \{\infty\}$ such that for all $i, j \in I$,

$$m_{i,j} = \begin{cases} \geq 2 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

For a word $f = i_1 \dots i_k$ in the free monoid I we denote by r_f an element of W of the form $r_f = r_{i_1} \dots r_{i_k}$. The length of $w \in W$, denoted $\ell(w)$, is the smallest integer k such that there is a word $f = i_1 \dots i_k$ and $w = r_f$. We say that f is *reduced* if $\ell(r_f) = k$. A Coxeter group W may be turned into a chamber system by introducing in W the following collection of equivalence relations: $w \sim_i w'$ if and only if $w = w'$ or $w = w' r_i$. The corresponding simplicial complex Σ is called *Coxeter complex*.

A simplicial complex \mathcal{X} is called a *building of type Σ* if it contains a family of subcomplexes called *apartments* such that

- (Bi) each apartment is isomorphic to Σ ,
- (Bii) any two simplices of \mathcal{X} lie in a common apartment,
- (Biii) for any two apartments \mathcal{A} and \mathcal{A}' having a chamber in common there is an isomorphism $\psi : \mathcal{A} \rightarrow \mathcal{A}'$ fixing $\mathcal{A} \cap \mathcal{A}'$ pointwise.

The rank of the building is the cardinality of the set I . We always assume that \mathcal{X} is irreducible. A simplex c is a chamber in \mathcal{X} if it is a chamber in any of its apartments. By $C(\mathcal{X})$ we denote the set of chambers in \mathcal{X} . Using the building axioms we see that $C(\mathcal{X})$ has a chamber system structure. However, it is not unique. A geometric realization of the building \mathcal{X} is its geometric realization as an abstract simplicial complex. In

this article we assume that the system of apartments in \mathcal{X} is *complete* meaning that any subcomplex of \mathcal{X} isomorphic to Σ is an apartment. We denote by $\text{Aut}(\mathcal{X})$ the group of automorphisms of the building \mathcal{X} .

1.2. Affine Coxeter complexes. In this section we recall basic facts about root systems and Coxeter groups. A general reference is [8], which deals with Coxeter systems attached to reduced root systems. Since at the beginning the root system may be non-reduced we will also refer to [20, 23].

Let Φ be an irreducible, but not necessarily reduced, finite root system in a Euclidean space \mathfrak{a} . The \mathbb{Z} -span of Φ is the *root lattice* L . Let $\{\alpha_1, \dots, \alpha_r\}$ be a base of Φ , and let Φ^+ denote the corresponding subset of *positive roots*. We denote

$$\mathfrak{a}_+ = \{x \in \mathfrak{a} : \langle \alpha, x \rangle > 0 \text{ for every } \alpha \in \Phi^+\}$$

the *positive Weyl chamber*.

For a positive root $\alpha \in \Phi^+$, its *height* is defined as

$$\text{ht}(\alpha) = \sum_{i=1}^r n_i$$

wherever $\alpha = \sum_{i=1}^r n_i \alpha_i$ with $n_i \in \mathbb{N}$. Since Φ is irreducible, there is a unique *highest root* $\alpha_0 = \sum_{i=1}^r m_i \alpha_i$ with $m_i \in \mathbb{N}$. We set

$$I_0 = \{1, \dots, r\} \quad \text{and} \quad I_g = \{0\} \cup \{i \in I_0 : m_i = 1\}.$$

Let $\Phi^\vee = \{\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha : \alpha \in \Phi\}$ be the dual root system. The \mathbb{Z} -span of Φ^\vee is the *co-root lattice* \mathcal{Q} . Let $\mathcal{Q}^+ = \sum_{\alpha \in \Phi^+} \mathbb{N} \alpha^\vee$. The dual basis $\{\lambda_1, \dots, \lambda_r\}$ to $\{\alpha_1, \dots, \alpha_r\}$ consists of *fundamental co-weights* and its \mathbb{Z} -span is the *co-weight lattice* P . Let $P^+ = \sum_{i=1}^r \mathbb{N} \lambda_i$ be the cone of *dominant co-weights*, and let $\sum_{i=1}^r \mathbb{N}^* \lambda_i$ be the subcone of *strongly dominant co-weights*. For instance

$$\rho = \sum_{i=1}^r \lambda_i$$

is a strongly dominant co-weight. Notice that

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^{++}} \alpha^\vee$$

where Φ^{++} denotes the set of indivisible positive roots in Φ , see [8, Proposition 10.29, §1, VI]

Let \mathcal{H} be the family of affine hyperplanes, called *walls*, being of the form

$$H_{\alpha;k} = \{x \in \mathfrak{a} : \langle x, \alpha \rangle = k\}$$

where $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$. Each wall determines two half-apartments

$$H_{\alpha;k}^- = \{x \in \mathfrak{a} : \langle x, \alpha \rangle \leq k\} \quad \text{and} \quad H_{\alpha;k}^+ = \{x \in \mathfrak{a} : \langle x, \alpha \rangle \geq k\}.$$

Note that for a given α , the family $H_{\alpha;k}^-$ is increasing in k while the family $H_{\alpha;k}^+$ is decreasing. To each wall we associate $r_{\alpha;k}$ the orthogonal reflection in \mathfrak{a} defined by

$$r_{\alpha;k}(x) = x - (\langle x, \alpha \rangle - k) \alpha^\vee.$$

Set $r_0 = r_{\alpha_0;1}$, and $r_i = r_{\alpha_i;0}$ for each $i \in I_0$.

The *finite Weyl group* W is the subgroup of $\text{GL}(\mathfrak{a})$ generated by $\{r_i : i \in I_0\}$. Let us denote by w_0 the longest element in W . The *fundamental sector* in \mathfrak{a} defined as

$$\begin{aligned} S_0 &= \overline{\mathfrak{a}_+} = \bigoplus_{i \in I_0} \mathbb{R}_+ \lambda_i \\ &= \bigcap_{i \in I_0} H_{\alpha_i;0}^+ \end{aligned}$$

is the fundamental domain for the action of W on \mathfrak{a} .

The *affine Weyl group* W^a is the subgroup of $\text{Aff}(\mathfrak{a})$ generated by $\{r_i : i \in I\}$. Observe that W^a is a Coxeter group. The hyperplanes \mathcal{H} give the geometric realization of its Coxeter complex Σ_Φ . To see this,

let $C(\Sigma_\Phi)$ be the family of closures of the connected components of $\mathfrak{a} \setminus \bigcup_{H \in \mathcal{H}} H$. By C_0 we denote the *fundamental chamber* (or *fundamental alcove*), i.e.

$$\begin{aligned} C_0 &= \{x \in \mathfrak{a} : \langle x, \alpha_0 \rangle \leq 1 \text{ and } \langle x, \alpha_i \rangle \geq 0 \text{ for all } i \in I_0\} \\ &= \left(\bigcap_{i \in I_0} H_{\alpha_i; 0}^+ \right) \cap H_{\alpha_0; 1}^- \end{aligned}$$

which is the fundamental domain for the action of W^a on \mathfrak{a} . Moreover, the group W^a acts simply transitively on $C(\Sigma_\Phi)$. This allows us to introduce a chamber system in $C(\Sigma_\Phi)$: For two chambers C and C' and $i \in I$, we set $C \sim_i C'$ if and only if $C = C'$ or there is $w \in W^a$ such that $C = w.C_0$ and $C' = wr_i.C_0$.

The vertices of C_0 are $\{0, \lambda_1/m_1, \dots, \lambda_r/m_r\}$. Let us denote the set of vertices of all $C \in C(\Sigma_\Phi)$ by $V(\Sigma_\Phi)$. Under the action of W^a , the set $V(\Sigma_\Phi)$ is made up of $r+1$ orbits $W^a.0$ and $W^a.(\lambda_i/m_i)$ for all $i \in I_0$. Thus setting $\tau_{\Sigma_\Phi}(0) = 0$, and $\tau_{\Sigma_\Phi}(\lambda_i/m_i) = i$ for $i \in I_0$, we obtain the unique labeling $\tau_{\Sigma_\Phi} : V(\Sigma_\Phi) \rightarrow I$ such that any chamber $C \in C(\Sigma_\Phi)$ has one vertex with each label.

For each simplicial automorphism $\varphi : \Sigma_\Phi \rightarrow \Sigma_\Phi$ there is a permutation π of the set I such that for all chambers C and C' , we have $C \sim_i C'$ if and only if $\varphi(C) \sim_{\pi(i)} \varphi(C')$, and

$$\tau_{\Sigma_\Phi}(\varphi(v)) = \pi(\tau_{\Sigma_\Phi}(v)) \quad \text{for all } v \in V(\Sigma_\Phi).$$

A vertex v is called *special* if for each $\alpha \in \Phi^+$ there is k such that v belongs to $H_{\alpha; k}$. The set of all special vertices is denoted by $V_s(\Sigma_\Phi)$.

Given $\lambda \in P$ and $w \in W^a$, the set $S = \lambda + w.S_0$ is called a *sector* in Σ_Φ with a *base vertex* λ .

Moreover, by [1, Corollary 3.20], an affine Coxeter complex Σ_Φ uniquely determines the affine Weyl group W^a but not a finite root system Φ . In fact, the root systems C_r and BC_r have the same affine Weyl group.

1.3. Affine buildings. A building \mathcal{X} of type Σ is called an *affine building* if Σ is a Coxeter complex corresponding to an affine Weyl group. Select a chamber c_0 in $C(\mathcal{X})$ and an apartment \mathcal{A}_0 containing c_0 . Using an isomorphism $\psi_0 : \mathcal{A}_0 \rightarrow \Sigma$ such that $\psi_0(c_0) = C_0$, we define the labeling in \mathcal{A}_0 by

$$\tau_{\mathcal{A}_0}(v) = \tau_\Sigma(\psi_0(v)) \quad \text{for all } v \in V(\mathcal{A}_0).$$

Now, thanks to the building axioms the labeling can be uniquely extended to $\tau : V(\mathcal{X}) \rightarrow I$. To turn $C(\mathcal{X})$ into a chamber system over I we declare that two chambers c and c' are i -adjacent if they share all vertices except the one of type i (equivalently, they intersect along an i -panel). For each $c \in C(\mathcal{X})$ and $i \in I$, we define

$$q_i(c) = \#\{c' \in C(\mathcal{X}) : c' \sim_i c\} - 1.$$

In all the paper, we *assume* that $q_i(c)$ only depends on i , i.e. that $q_i(c)$ is independent of c , and therefore the building \mathcal{X} is *regular*; we henceforth write q_i instead of $q_i(c)$. We also assume that $1 < q_i(c) < \infty$ and therefore the building \mathcal{X} is *thick* and *locally finite*. A vertex of \mathcal{X} is special if it is special in any of its apartments. The set of special vertices is denoted by V_s . We choose the finite root system Φ in such a way that Σ is its Coxeter complex. In all cases except when the affine group has type C_r or BC_r , the choice is unique. In the remaining cases we select C_r if $q_0 = q_r$, otherwise we take BC_r . This guarantees that $q_{\tau(\lambda)} = q_{\tau(\lambda+\lambda')}$ for all $\lambda, \lambda' \in P$, see the discussion in [20, Section 2.13]. In this article all buildings have reduced type.

Given two special vertices $x, y \in V_s$, let \mathcal{A} be an apartment containing x and y and let $\psi : \mathcal{A} \rightarrow \Sigma$ be a type-rotating isomorphism such that $\psi(x) = 0$ and $\psi(y) \in S_0$, see [23, Definition 4.1.1]. We set $\sigma(x, y) = \psi(y) \in P^+$. For $\lambda \in P^+$ and $x \in V_s$, we denote by $V_\lambda(x)$ the set of all special vertices $y \in V_s$ such that $\sigma(x, y) = \lambda$. The building axioms entail that the cardinality of $V_\lambda(x)$ depends only on λ , see [24, Proposition 1.5]. Let N_λ be the common value.

We fix once and for all an origin o which is a special vertex of the chamber c_0 . Let us define a multiplicative function on \mathfrak{a} by

$$\chi_0(\lambda) = \prod_{\alpha \in \Phi^+} q_\alpha^{\langle \lambda, \alpha \rangle} \quad \text{with } \lambda \in \mathfrak{a}$$

where $q_\alpha = q_i$ whenever $\alpha \in W.\alpha_i$ for $i \in I_0$. For $w \in W^a$ having the reduced expression $w = r_{i_1} \cdots r_{i_k}$, we set $q_w = q_{i_1} \cdots q_{i_k}$. Then

$$(1) \quad N_\lambda = \frac{W(q^{-1})}{W_\lambda(q^{-1})} \chi_0(\lambda)$$

where $W_\lambda = \{w \in W : w.\lambda = \lambda\}$, and for any subset $U \subseteq W$ we have set

$$U(q^{-1}) = \sum_{w \in U} q_w^{-1}.$$

1.4. Spherical harmonic analysis. In this subsection we summarize spherical harmonic analysis on affine buildings (see [19, 24]).

We consider the averaging operators

$$A_\lambda f(x) = \frac{1}{N_\lambda} \sum_{y \in V_\lambda(x)} f(y) \quad \text{for all } \lambda \in P^+$$

acting on functions $f : V_s \rightarrow \mathbb{C}$. Let $\mathcal{A}_0 = \mathbb{C}\text{-span}\{A_\lambda : \lambda \in P^+\}$ be a commutative unital involutive algebra⁽¹⁾ whose characters can be expressed in terms of Macdonald functions (see [24])

$$(2) \quad P_\lambda(z) = \frac{\chi_0^{-\frac{1}{2}}(\lambda)}{W(q^{-1})} \sum_{w \in W} \mathbf{c}(w.z) e^{\langle w.z, \lambda \rangle} \quad \text{with } \lambda \in P^+ \text{ and } z \in \mathfrak{a}_\mathbb{C}$$

where

$$\mathbf{c}(z) = \prod_{\alpha \in \Phi^+} \frac{1 - q_\alpha^{-1} e^{-\langle z, \alpha^\vee \rangle}}{1 - e^{-\langle z, \alpha^\vee \rangle}}$$

is the so-called \mathbf{c} -function; the values of P_λ where the denominator of the \mathbf{c} -function vanishes can be obtained by taking appropriate limits. Specifically, every multiplicative functional on \mathcal{A}_0 is given by the evaluation

$$h_z(A_\lambda) = P_\lambda(z) \quad \text{for all } \lambda \in P^+,$$

at certain $z \in \mathfrak{a}_\mathbb{C}$. Moreover, $h_z = h_{z'}$ if and only if $W.z + i2\pi L = W.z' + i2\pi L$ (see [19, Theorem 3.3.12(ii)]).

Recall that Macdonald functions as given by the formula (2) are linear combinations

$$(3) \quad P_\lambda = \sum_{\mu \leq \lambda} c_{\lambda, \mu} m_\mu$$

(with nonnegative coefficients) of monomial symmetric functions

$$m_\mu = \sum_{\nu \in W.\mu} e^\nu \quad \text{for all } \mu \in P^+.$$

Conversely, every m_λ is a linear combination of P_μ 's with $\mu \leq \lambda$.

Let \mathcal{A}_2 be the closure of \mathcal{A}_0 in the operator norm on $\ell^2(V_s)$. Then \mathcal{A}_2 is a commutative unital C^* -algebra whose spectrum $\widehat{\mathcal{A}}_2$ equals $i\mathfrak{a}$ modulo $W \times i2\pi L$ ⁽²⁾, and the Gelfand–Fourier transform

$$\mathcal{F}A(\theta) = h_{i\theta}(A) \quad \text{with } \theta \in \mathfrak{a}$$

identifies \mathcal{A}_2 with $C(\widehat{\mathcal{A}}_2)$ the C^* -algebra of (bounded) continuous functions on $\widehat{\mathcal{A}}_2$. Finally, the following inversion formula holds: for every $A \in \mathcal{A}_0$ and $x, y \in V_s$;

$$(4) \quad (A\delta_x)(y) = \left(\frac{1}{2\pi}\right)^r \frac{W(q^{-1})}{|W|} \int_{\mathcal{D}} \mathcal{F}A(\theta) \overline{P_\lambda(i\theta)} \frac{d\theta}{|\mathbf{c}(i\theta)|^2}$$

where δ_x denotes the Dirac measure at x , $\lambda = \sigma(x, y)$ and

$$(5) \quad \mathcal{D} = \{\theta \in \mathfrak{a} : \langle \theta, \alpha^\vee \rangle \leq 2\pi \text{ for all } \alpha \in \Phi\},$$

¹The involution is induced by $A_\lambda^* = A_{\lambda^*}$ where $\lambda^* = -w_0.\lambda$.

²Specifically, h_z extends to a bounded character on \mathcal{A}_2 if and only if $z \in i\mathfrak{a}$.

is a fundamental domain for the lattice $2\pi L$ in \mathfrak{a} , see [24].

Actually we need another completion of \mathcal{A}_0 : As in [12, Section 6], we consider the following commutative unital involutive Banach algebra:

$$\mathcal{A}_1 = \left\{ A = \sum_{\lambda \in P^+} c_\lambda A_\lambda : \|A\|_1 = \sum_{\lambda \in P^+} |c_\lambda| < \infty \right\}.$$

Notice that on the one hand the character h_z extends continuously to \mathcal{A}_1 if and only if $\sup_{\lambda \in P^+} |P_\lambda(z)| < \infty$; according to [19, Theorem 4.7.1] this happens if and only if $\operatorname{Re} z$ belongs to the convex hull of $W \cdot \log \chi_0^{\frac{1}{2}}$. On the other hand h_z is Hermitian if and only if $P_{\lambda^*}(z) = \overline{P_\lambda(z)}$ for every $\lambda \in P^+$; as $P_{\lambda^*}(z) = P_\lambda(-z)$ and $\overline{P_\lambda(z)} = P_\lambda(\bar{z})$, we deduce that $-\bar{z} \in W \cdot z + i2\pi L$. In summary, the continuous Hermitian characters of \mathcal{A}_1 are parametrized by the set

$$(6) \quad \Sigma = \left\{ z \in \operatorname{co}(W \cdot \log \chi_0^{\frac{1}{2}}) + i\mathfrak{a} : -\bar{z} \in W \cdot z + i2\pi L \right\}$$

modulo $W \times i2\pi L$ ⁽³⁾, which we call the *spectrum* of \mathcal{X} . Finally, the Gelfand–Fourier transform extends to a continuous homomorphism from \mathcal{A}_1 into $C(\Sigma)$.

2. MULTITEMPORAL WAVE EQUATION ON AFFINE BUILDINGS

In [27] Semenov-Tian-Shansky introduced on Riemannian symmetric space $X = G/K$ of non-compact the following multitemporal wave equation on Riemannian symmetric spaces $X = G/K$ of non-compact type:

$$(7) \quad \Gamma(D) \left(\frac{\partial}{\partial t} \right) u(t, x) = D_x u(t, x) \quad \text{for all } D \in \mathbb{D}, t \in \mathfrak{a} \text{ and } x \in X.$$

Here Γ denotes the Harish–Chandra isomorphism between the algebra \mathbb{D} of G -invariant differential operators on X , and the algebra $\mathcal{P}(\mathfrak{a})^W$ of W -invariant polynomials on the Cartan subspace \mathfrak{a} . This equation was further studied in [25], [16], [17, Ch. V, §§ 5.7–5.11]; see the latter reference for more details.

In this section we introduce a discrete analog of (7) on affine buildings \mathcal{X} of reduced type, and produce a particular non-trivial solution enjoying two basic properties: finite propagation speed and uniform space-time exponential decay. Specifically, the algebra \mathbb{D} of invariant differential operators on X is replaced by the algebra \mathcal{A}_0 of averaging operators on \mathcal{X} and the Harish–Chandra isomorphism Γ by the isomorphism induced by $A_\lambda \mapsto P_\lambda$ ($\lambda \in P^+$) between \mathcal{A}_0 and the algebra of W -invariant trigonometric polynomials on $i\mathfrak{a}$. Thus (7) becomes

$$(8) \quad \sum_{\substack{\mu \in P^+ \\ \mu \leq \lambda}} c_{\lambda, \mu} \sum_{\omega \in W \cdot \mu} u(\nu + \omega, x) = A_\lambda u(\nu, x) \quad \text{for all } \lambda, \nu \in P^+ \text{ and } x \in V_\mathfrak{s}$$

where the coefficients $c_{\lambda, \mu}$ are defined by (3).

Next, let us consider the function $u : P^+ \times V_\mathfrak{s} \rightarrow \mathbb{C}$ whose partial Gelfand–Fourier transform with respect to the second variable is given by

$$u(\nu, \hat{\theta}) = \sum_{w \in W} e^{i\langle w \cdot \theta, \nu \rangle} = |W_\nu| m_\nu(i\theta) \quad \text{for all } \nu \in P^+ \text{ and } \theta \in \mathfrak{a}.$$

In particular, $u(\nu, \cdot)$ belongs to \mathcal{A}_0 . We claim that u solves (8) which writes

$$(9) \quad \sum_{\mu \leq \lambda} c_{\lambda, \mu} \sum_{\omega \in W \cdot \mu} u(\nu + \omega, \hat{\theta}) = P_\lambda(i\theta) u(\nu, \hat{\theta}) \quad \text{for all } \nu \in P^+ \text{ and } \theta \in \mathfrak{a},$$

³Notice that we may therefore restrict to $z \in \operatorname{co}(W \cdot \log \chi_0^{\frac{1}{2}}) \cap \overline{\mathfrak{a}_+} + i\mathcal{D}$.

after taking the partial Gelfand–Fourier transform. Indeed,

$$\begin{aligned}
 \sum_{\mu \leq \lambda} c_{\lambda, \mu} \sum_{\omega \in W \cdot \mu} u(\nu + \omega, \hat{\theta}) &= \sum_{\mu \leq \lambda} c_{\lambda, \mu} \sum_{\omega \in W \cdot \mu} \sum_{w \in W} e^{i\langle w \cdot \theta, \nu + \omega \rangle} \\
 &= \sum_{\mu \leq \lambda} c_{\lambda, \mu} \sum_{w \in W} e^{i\langle w \cdot \theta, \nu \rangle} \sum_{\omega \in W \cdot \mu} e^{i\langle w \cdot \theta, \omega \rangle} \\
 &= \sum_{\mu \leq \lambda} c_{\lambda, \mu} m_{\mu}(i\theta) \sum_{w \in W} e^{i\langle w \cdot \theta, \nu \rangle} \\
 &= P_{\lambda}(i\theta) u(\nu, \hat{\theta}).
 \end{aligned}$$

If $\sigma(o, x) = \omega$, then by the inversion formula (4) we obtain

$$\begin{aligned}
 u(\mu, x) &= \left(\frac{1}{2\pi}\right)^r \frac{W(q^{-1})}{|W|} \int_{\mathcal{D}} u(\mu, \hat{\theta}) \overline{P_{\omega}(i\theta)} \frac{d\theta}{|\mathbf{c}(i\theta)|^2} \\
 &= \left(\frac{1}{2\pi}\right)^r \frac{W(q^{-1})}{|W \cdot \mu|} \int_{\mathcal{D}} m_{\mu}(i\theta) \overline{P_{\omega}(i\theta)} \frac{d\theta}{|\mathbf{c}(i\theta)|^2}.
 \end{aligned}$$

Now, using the explicit formula for Macdonald functions (2), we get

$$\begin{aligned}
 u(\mu, x) &= \chi_0(\omega)^{-\frac{1}{2}} \frac{1}{|W \cdot \mu|} \sum_{w \in W} \left(\frac{1}{2\pi}\right)^r \int_{\mathcal{D}} m_{\mu}(i\theta) e^{-i\langle w \cdot \theta, \omega \rangle} \frac{d\theta}{\mathbf{c}(iw \cdot \theta)} \\
 &= \chi_0(\omega)^{-\frac{1}{2}} \underbrace{\frac{|W|}{|W \cdot \mu|}}_{|W_{\mu}|} \left(\frac{1}{2\pi}\right)^r \int_{\mathcal{D}} m_{\mu}(i\theta) e^{-i\langle \omega, \theta \rangle} \frac{d\theta}{\mathbf{c}(i\theta)}.
 \end{aligned}$$

The last integral vanishes if $\omega \not\leq \mu$; in other words, $u(\mu, \cdot)$ is supported in

$$\{x \in V_s : \sigma(o, x) \leq \mu\}.$$

This property is a higher rank analog of finite propagation speed for the wave equation on homogeneous trees (see [3, Section 4] or [9, Lemma 2]).

Next, we write

$$\frac{1 - e^{-\alpha^{\vee}}}{1 - q_{\alpha}^{-1} e^{-\alpha^{\vee}}} = 1 + (1 - q_{\alpha}) \sum_{j_{\alpha}=1}^{\infty} q_{\alpha}^{-j_{\alpha}} e^{-j_{\alpha} \alpha^{\vee}},$$

thus

$$\frac{1}{\mathbf{c}} = \sum_{\nu \in Q^+} C(\nu) e^{-\nu}$$

where for $\nu \in Q^+$ we have set

$$C(\nu) = \sum_{(j_{\alpha} : \alpha \in \Phi^+) \in \mathcal{P}(\nu)} \prod_{\substack{\alpha \in \Phi^+ \\ j_{\alpha} > 0}} (1 - q_{\alpha}) q_{\alpha}^{-j_{\alpha}}$$

with

$$\mathcal{P}(\nu) = \left\{ (j_{\alpha} : \alpha \in \Phi^+) \in \mathbb{N}^{|\Phi^+|} : \sum_{\alpha \in \Phi^+} j_{\alpha} \alpha^{\vee} = \nu \right\}.$$

Therefore,

$$\begin{aligned}
u(\mu, x) &= \chi_0(\omega)^{-\frac{1}{2}} \frac{|W|}{|W \cdot \mu|} \left(\frac{1}{2\pi} \right)^r \sum_{\nu \in Q^+} C(\nu) \int_{\mathcal{D}} m_\mu(i\theta) e^{-i\langle \omega + \nu, \theta \rangle} d\theta \\
&= \chi_0(\omega)^{-\frac{1}{2}} \frac{|W|}{|W \cdot \mu|} \sum_{\nu \in Q^+} \frac{1}{|W \cdot (\omega + \nu)|} C(\nu) \langle m_\mu, m_{\omega + \nu} \rangle \\
(10) \quad &= \chi_0(\omega)^{-\frac{1}{2}} \frac{|W|}{|W \cdot \mu|} \sum_{\substack{w \in W \\ \omega \leq w \cdot \mu}} C(w \cdot \mu - \omega).
\end{aligned}$$

Our next aim is to estimate

$$\chi_0(\omega)^{-\frac{1}{2}} \sum_{\substack{w \in W \\ \omega \leq w \cdot \mu}} |C(w \cdot \mu - \omega)|.$$

Let us recall that the condition $w \cdot \mu \geq \omega$ implies that $w \cdot \mu - \omega$ belongs to the cone Q^+ generated by the positive co-roots.

Lemma 2.1. *There are $C > 0$ and $A > 1$ such that for each $\nu \in Q^+$,*

$$|C(\nu)| \leq CA^{-\langle \nu, \tilde{\rho} \rangle}$$

where

$$\tilde{\rho} = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Proof. Let

$$K = \max \{ \langle \alpha^\vee, \tilde{\rho} \rangle : \alpha \in \Phi^+ \}.$$

Since $K \geq 1$, we have

$$\sum_{\alpha \in \Phi^+} j_\alpha \geq \frac{1}{K} \sum_{\alpha \in \Phi^+} j_\alpha \langle \alpha^\vee, \tilde{\rho} \rangle = \frac{1}{K} \langle \nu, \tilde{\rho} \rangle.$$

Hence, ⁽⁴⁾

$$\begin{aligned}
|C(\nu)| &\leq \sum_{(j_\alpha) \in \mathcal{P}(\nu)} \prod_{\alpha: j_\alpha > 0} (q_\alpha - 1) q_\alpha^{-j_\alpha} \\
&\lesssim \sum_{k \geq \frac{1}{K} \langle \nu, \tilde{\rho} \rangle} \# \left\{ (j_\alpha) \in \mathcal{P}(\nu) : \sum_{\alpha \in \Phi^+} j_\alpha = k \right\} A^{-2Kk}
\end{aligned}$$

where $A^{2K} = \min \{ q_\alpha : \alpha \in \Phi^+ \}$. Consequently,

$$\begin{aligned}
|C(\nu)| &\lesssim \sum_{k \geq \frac{1}{K} \langle \nu, \tilde{\rho} \rangle} k^{\#\Phi^+} A^{-2Kk} \\
&\lesssim A^{-\langle \nu, \tilde{\rho} \rangle}
\end{aligned}$$

and the lemma follows. □

Lemma 2.2. *There are $C > 0$ and $c > 0$ such that for all $\mu, \omega \in P^+$,*

$$\chi_0(\omega)^{-\frac{1}{2}} \sum_{\substack{w \in W \\ w \cdot \mu \geq \omega}} |C(w \cdot \mu - \omega)| \leq Ce^{-c|\mu|}.$$

⁴We write $f \lesssim g$ if there is $C > 0$ such that $f \leq Cg$.

Proof. Since

$$\chi_0(\omega)^{-\frac{1}{2}} \leq \prod_{\alpha \in \Phi^+} q_\alpha^{-\frac{1}{2}\langle \omega, \alpha \rangle} \leq A_1^{-\langle \omega, \tilde{\rho} \rangle},$$

provided that $1 < A_1 \leq \min\{q_\alpha : \alpha \in \Phi^+\}$, by Lemma 2.1, we can write

$$\sum_{\substack{w \in W \\ w \cdot \mu \geq \omega}} \chi_0(\omega)^{-\frac{1}{2}} |C(w \cdot \mu - \omega)| \lesssim \sum_{\substack{w \in W \\ w \cdot \mu \geq \omega}} A^{-\langle w \cdot \mu - \omega, \tilde{\rho} \rangle} A_1^{-\langle \omega, \tilde{\rho} \rangle}.$$

Since $w \cdot \mu \geq \omega$ and $\omega \in P^+$, we have

$$\langle w \cdot \mu, \tilde{\rho} \rangle \geq |w \cdot \mu| = |\mu|,$$

hence there is $c > 0$ such that

$$\sum_{\substack{w \in W \\ w \cdot \mu \geq \omega}} \chi_0(\omega)^{-\frac{1}{2}} |C(w \cdot \mu - \omega)| \lesssim e^{-c|\mu|}$$

which completes the proof. \square

Using (10) and Lemma 2.2, we can easily deduce the following consequence.

Theorem 2.3. *Let u be the real function on $P^+ \times V_s$ with Fourier–Gelfand transform satisfying*

$$\mathcal{F}u(\mu, \theta) = |W_\mu| m_\mu(i\theta) \quad \text{for all } \mu \in P^+ \text{ and } \theta \in \mathfrak{a}.$$

Then for each $\mu \in P^+$, the function $u(\mu, \cdot)$ is supported in $\{x \in V_s : \sigma(o, x) \leq \mu\}$. Moreover, there are $C > 0$ and $c > 0$ such that

$$(11) \quad |u(\mu, x)| \leq C e^{-c|\mu|} \quad \text{for all } \mu \in P^+ \text{ and } x \in V_s.$$

Remark 2.4. *In the case of \tilde{A}_2 buildings we obtained sharper estimates than (11): Namely, there is $C > 0$ such that for all $\mu \in P^+$ and $x \in V_s$,*

$$|u(\mu, x)| \leq C q^{-\max\{\langle \alpha_1, \mu \rangle, \langle \alpha_2, \mu \rangle\}}$$

where $q + 1$ denotes the thickness of the building.

Remark 2.5. *It would be interesting to extend Theorem 2.3 to all fundamental solutions of (8). We intend to return to this question in the future.*

3. KERNEL CONSTRUCTION

In this section we prove our main result stated in the introduction as Theorem A which holds for any (regular, thick, locally finite) affine building of reduced type. The proof, inspired by the kernel construction of Brooks and Lindenstrauss in [9, Section 2], [10, Section 3] and [11, Section 3.1], uses the solution of the multitemporal wave equation described in Theorem 2.3.

Theorem 3.1. *Let $z_0 = \zeta_0 + i\theta_0 \in \Sigma$. There are $M(\zeta_0), N(\zeta_0) \in \mathbb{N}$ such that, for all integers $M \geq M(\zeta_0)$ and $N \geq N(\zeta_0)$, there is a function $k : V_s \times V_s \rightarrow \mathbb{C}$ such that*

- (i) *for all $x, y \in V_s$ the value of $k(x, y)$ depends only on the vectorial distance $\sigma(x, y)$; moreover, $k(x, y)$ vanishes when*

$$\sigma(x, y) \not\leq 16M(8\pi hMN)^r \rho$$

where

$$h = \sum_{\alpha \in \Phi^+} \text{ht}(\alpha^\vee)$$

and $\text{ht}(\alpha^\vee)$ denotes the height of the coroot α^\vee with respect to the coroot basis $\{\alpha_1^\vee, \dots, \alpha_r^\vee\}$;

- (ii) *there are $C > 0$ and $c > 0$ such that*

$$\sup_{x, y \in V_s} |k(x, y)| \leq C e^{-cN};$$

(iii) for all $x \in V_s$ and $z \in \Sigma$, we have

$$\mathcal{F}(k(x, \cdot))(z) \geq -1 \quad \text{while} \quad \mathcal{F}(k(x, \cdot))(z_0) \geq M.$$

Proof. We are going to define a kernel with desired properties by describing it on the Gelfand–Fourier side. In view of Theorem 2.3, the convenient building blocks are the elementary symmetric functions which essentially reduces the proof to showing (iii).

Let b and M be two positive integers whose values will be specified later. Let $K_{M,b}$ be the complex function on $\mathfrak{a}_{\mathbb{C}}$ defined by

$$(12) \quad K_{M,b}(\theta) = \frac{1}{1 + 4|W|M} \left(\sum_{k=0}^{4M} m_{k\rho}(ib\theta) \right)^2 - 1.$$

It is clear that $K_{M,b}$ is invariant under the action of W as well as it is invariant under translations by vectors from $2\pi iL$. Given $z \in \Sigma$, there are $\tilde{w} \in W$ and $\ell \in L$ such that $\tilde{w}.z + i2\pi\ell = -\bar{z}$. We claim that for all $k \in \mathbb{N}$,

$$(13) \quad m_{k\rho}(z) \in \mathbb{R}.$$

Indeed, we have

$$\begin{aligned} \overline{m_{k\rho}(z)} &= \sum_{w \in W} e^{k\langle \bar{z}, w.\rho \rangle} \\ &= \sum_{w \in W} e^{k\langle -\tilde{w}.z - i2\pi\ell, w.\rho \rangle} \\ &= \sum_{w \in W} e^{k\langle z, \tilde{w}^{-1}ww_0.\rho \rangle} = m_{k\rho}(z) \end{aligned}$$

where w_0 is the longest element of W and $w_0.\rho = -\rho$. Using (13) we easily deduce that $K_{M,b} \geq -1$ on Σ .

Our next step is to show that there are $M(\zeta_0), N(\zeta_0) \in \mathbb{N}$ such that for all integers $M \geq M(\zeta_0)$ and $N \geq N(\zeta_0)$ there is an integer $b \geq N$ such that

$$(14) \quad K_{M,b}(z_0) \geq M.$$

First, let us see how (14) allows us to complete the proof. Using the inversion formula (4), we define the kernel $k : V_s \times V_s \rightarrow \mathbb{C}$ by

$$k(x, y) = \left(\frac{1}{2\pi} \right)^r \frac{W(q^{-1})}{|W|} \int_{\mathcal{D}} K_{M,b}(\theta) \overline{P_\omega(i\theta)} \frac{d\theta}{|\mathbf{c}(i\theta)|^2}$$

where $\omega = \sigma(x, y)$. It follows from the above that $k(x, y)$ satisfies (iii). Next, we deduce from (12) that

$$\int_{\mathcal{D}} K_{M,b}(\theta) \overline{P_\omega(i\theta)} \frac{d\theta}{|\mathbf{c}(i\theta)|^2}$$

vanishes whenever $\omega \not\leq 8Mb\rho$. Hence, $k(x, y) \neq 0$ implies that $\sigma(x, y) \leq 8Mb\rho$, which yields (i) because $b \leq 2B^r$. Thus it remains to prove (ii). Observe that, according to Corollary 2.3, there are $C > 0$ and $c > 0$ such that for all $\mu \in P^+$,

$$\|\mathcal{F}^{-1}m_\mu\|_\infty \leq Ce^{-c|\mu|}.$$

We claim that

$$(15) \quad K_{M,b} = \sum_{\substack{\nu \in P^+ \\ 0 < \nu \leq 8M\rho}} c_\nu m_{b\nu}$$

with $|c_\nu| \leq 1$. For the proof, notice that

$$\sum_{k=0}^{4M} m_{k\rho}(ib\theta)$$

is the Euclidean Fourier transform of the characteristic function of the set

$$X = \{kbw.\rho : w \in W_0, 0 \leq k \leq 4M\}.$$

Consequently, $K_{M,b}$ is Euclidean Fourier transform of

$$\frac{1}{1+4|W|M} \mathbb{1}_X * \mathbb{1}_X - \delta_0$$

where δ_0 denotes the Dirac measure at 0. Therefore,

$$0 \leq c_\nu \leq \|\mathbb{1}_X\|_{\ell^1} \|\mathbb{1}_X\|_{\ell^\infty} = \frac{1}{1+4|W|M}$$

as stated. Lastly, by (15), we get

$$\begin{aligned} \sup_{x,y \in V_g} |k(x,y)| &\leq \sum_{\substack{\nu \in P^+ \\ 0 < \nu \leq 8M\rho}} \|\mathcal{F}^{-1} m_{b\nu}\|_\infty \\ &\leq C' \sum_{j=1}^{8M|\rho|} j^{r-1} e^{-cbj} = O(e^{-cb}) \end{aligned}$$

which is $O(e^{-cN})$. This proves (ii).

It remains to show (14). The proof is divided into three cases.

Case 1: If $\theta_0 = 0$, then for all $k \in \mathbb{N}$,

$$\begin{aligned} m_{k\rho}(b\zeta_0) &= \sum_{w \in W} e^{kb\langle \zeta_0, w \cdot \rho \rangle} \\ &= \frac{1}{2} \sum_{w \in W} e^{kb\langle \zeta_0, w \cdot \rho \rangle} + \frac{1}{2} \sum_{w \in W} e^{kb\langle \zeta_0, ww_0 \cdot \rho \rangle} \\ &= \sum_{w \in W} \cosh(kb\langle \zeta_0, w \cdot \rho \rangle) \\ &\geq |W|. \end{aligned}$$

Thus

$$(16) \quad K_{M,b}(\zeta_0) \geq \frac{1}{1+4|W|M} \left(\sum_{k=0}^{4M} m_{k\rho}(b\zeta_0) \right)^2 - 1 \geq 4M|W|$$

for every $b, M \in \mathbb{N}$. In this case we set $b = N$.

Case 2: Assume that $\zeta_0 = 0$. We start by the following observation: if

$$0 < |x| \leq \frac{\pi}{8M+1},$$

then

$$\begin{aligned} D_{4M}(ix) &= \sum_{k=-4M}^{4M} e^{ikx} = \frac{\sin(4M + \frac{1}{2})x}{\sin \frac{1}{2}x} \\ &= (8M+1) \frac{\sin(4M + \frac{1}{2})x}{(4M + \frac{1}{2})x} \frac{\frac{1}{2}x}{\sin \frac{1}{2}x} \\ &\geq \frac{2}{\pi} (8M+1). \end{aligned}$$

If $x = 0$ then

$$D_{4M}(ix) = \sum_{k=-4M}^{4M} e^{ikx} = 8M+1.$$

Therefore

$$D_{4M}(ix) \geq \frac{2}{\pi} (8M+1) \quad \text{for } x \in \left[-\frac{\pi}{8M+1}, \frac{\pi}{8M+1} \right].$$

Using the last inequality, we obtain

$$\begin{aligned}
\sum_{k=0}^{4M} m_{k\rho}(i\theta) &= 1 - \frac{1}{2}|W| + \frac{1}{2} \sum_{w \in W} D_{4M}(i\langle \theta, w \cdot \rho \rangle) \\
&\geq 1 - \frac{1}{2}|W| + \frac{8}{\pi}|W|M + \frac{1}{\pi}|W| \\
(17) \qquad \qquad \qquad &\geq 1 + 2|W|M
\end{aligned}$$

provided that

$$\langle \theta, w \cdot \rho \rangle \in \left[-\frac{\pi}{8M+1}, \frac{\pi}{8M+1} \right] \quad \text{for all } w \in W.$$

Next, we are going to find b such that

$$(18) \qquad b\langle \theta_0, w \cdot \rho \rangle \in \left[-\frac{\pi}{8M+1}, \frac{\pi}{8M+1} \right] + 2\pi\mathbb{Z} \quad \text{for all } w \in W.$$

Let

$$(19) \qquad B = 2h(8M+1)N.$$

By Dirichlet's approximation theorem, there is $b' \in \{1, 2, \dots, B^r\}$ such that

$$b'\langle \theta_0, \alpha_j^\vee \rangle \in \left[0, \frac{2\pi}{B} \right) + 2\pi\mathbb{Z} \quad \text{for all } j \in \{1, 2, \dots, r\}.$$

Hence,

$$b'\langle \theta_0, \alpha^\vee \rangle \in \left[0, \text{ht}(\alpha^\vee) \frac{2\pi}{B} \right) + 2\pi\mathbb{Z} \quad \text{for all } \alpha \in \Phi^+,$$

and so

$$2b'\langle \theta_0, w \cdot \rho \rangle \in \left(-\frac{2\pi h}{B}, \frac{2\pi h}{B} \right) + 2\pi\mathbb{Z} \quad \text{for all } w \in W.$$

Let $b = 2b' \lceil N/(2b') \rceil$. Then $2N \leq b \leq 2b'N$. Moreover, if $N \leq 2b'$ then

$$b = 2b' \leq 2B^r.$$

Otherwise $2b' < N$ and

$$b \leq N + 2b' < 2N \leq B \leq 2B^r.$$

Hence, we get (18). Now, in view of (17),

$$\begin{aligned}
K_{M,b}(\theta_0) &= \frac{1}{1+4|W|M} \left(\sum_{k=0}^{4M} m_{k\rho}(ib\theta_0) \right)^2 - 1 \\
&\geq \frac{1}{1+4|W|M} \left(1 + 2|W|M \right)^2 - 1 \\
&\geq \frac{1}{2}|W|M \geq M
\end{aligned}$$

for every $M \in \mathbb{N}$.

Case 3: It remains to consider $z_0 = \zeta_0 + i\theta_0 \in \Sigma$ with $\zeta_0 \in \overline{\mathfrak{a}_+} \setminus \{0\}$ and $\theta_0 \in \mathfrak{a}$. In fact this case requires more work. First, let us observe that $\langle \zeta_0, \rho \rangle > 0$. As $z_0 \in \Sigma$, there are $w_1 \in W$ and $\ell \in L$ such that $w_1 \cdot \zeta_0 = -\zeta_0$ and $w_1 \cdot \theta_0 + 2\pi\ell = \theta_0$. Let us consider

$$W(\zeta_0) = \{w \in W : w \cdot \zeta_0 = \zeta_0\}.$$

We claim the following

Claim 3.2.

$$W(\zeta_0) = w_0 W(\zeta_0) w_1 = \{w \in W : \langle w \cdot \zeta_0, \rho \rangle = \langle \zeta_0, \rho \rangle\}.$$

To prove the first equality, let $w \in W(\zeta_0)$. Then

$$\langle w_0 w w_1 \cdot \zeta_0, \rho \rangle = \langle w \cdot \zeta_0, \rho \rangle = \langle \zeta_0, \rho \rangle$$

where we have used $w_1 \cdot \zeta_0 = -\zeta_0$ and $w_0 \cdot \rho = -\rho$. To justify the second equality, it is enough to prove the inclusion \supseteq . Since $\zeta_0 \in \overline{\mathfrak{a}_+} \setminus \{0\}$, for each $w \in W$, there are $\gamma_j \geq 0$ such that

$$(20) \quad \zeta_0 - w \cdot \zeta_0 = \sum_{j=1}^r \gamma_j \alpha_j.$$

Thus if $\langle w \cdot \zeta_0, \rho \rangle = \langle \zeta_0, \rho \rangle$, then

$$0 = \langle \zeta_0 - w \cdot \zeta_0, \rho \rangle = \sum_{j=1}^r \gamma_j,$$

and so $\gamma_1 = \dots = \gamma_r = 0$, that is $w \in W(\zeta_0)$, proving the claim

For future use, let us observe that

$$(21) \quad \kappa = \min \{ \langle \zeta_0 - w \cdot \zeta_0, \rho \rangle : w \in W \setminus W(\zeta_0) \} > 0.$$

Since for each $b \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned} \sum_{w \in W(\zeta_0)} e^{ikb \langle \theta_0, w \cdot \rho \rangle} &= \sum_{w \in W(\zeta_0)} e^{ikb \langle \theta_0, w_1^{-1} w w_0 \cdot \rho \rangle} \\ &= \sum_{w \in W(\zeta_0)} e^{-ikb \langle \theta_0, w \cdot \rho \rangle}, \end{aligned}$$

we obtain

$$\begin{aligned} \sum_{w \in W(\zeta_0)} e^{kb \langle z_0, w \cdot \rho \rangle} &= e^{kb \langle \zeta_0, \rho \rangle} \sum_{w \in W(\zeta_0)} e^{ikb \langle \theta_0, w \cdot \rho \rangle} \\ &= e^{kb \langle \zeta_0, \rho \rangle} \frac{1}{2} \left\{ \sum_{w \in W(\zeta_0)} e^{ikb \langle \theta_0, w \cdot \rho \rangle} + \sum_{w \in W(\zeta_0)} e^{-ikb \langle \theta_0, w \cdot \rho \rangle} \right\} \\ &= e^{kb \langle \zeta_0, \rho \rangle} \sum_{w \in W(\zeta_0)} \cos \langle \theta_0, w \cdot \rho \rangle. \end{aligned}$$

Now, using $\cos t = 1 - 2(\sin \frac{t}{2})^2$, we arrive at

$$\sum_{w \in W(\zeta_0)} e^{kb \langle z_0, w \cdot \rho \rangle} = |W(\zeta_0)| e^{kb \langle \zeta_0, \rho \rangle} - 2e^{kb \langle \zeta_0, \rho \rangle} \sum_{w \in W(\zeta_0)} \left(\sin \frac{kb}{2} \langle \theta_0, w \cdot \rho \rangle \right)^2.$$

In order to estimate $K_{M,b}(z_0)$, we write

$$(22) \quad \sum_{k=0}^{4M} m_{k\rho}(bz_0) = 1 + S_1 - S_2 + S_3 + S_4 + S_5$$

where we have set

$$(23) \quad S_1 = |W(\zeta_0)| e^{4Mb \langle \zeta_0, \rho \rangle},$$

and

$$\begin{aligned}
S_2 &= 2e^{4Mb\langle \zeta_0, \rho \rangle} \sum_{w \in W(\zeta_0)} (\sin 2Mb\langle \theta_0, w \cdot \rho \rangle)^2 \\
S_3 &= \sum_{k=1}^{4M-1} e^{kb\langle \zeta_0, \rho \rangle} \sum_{w \in W(\zeta_0)} \cos kb\langle \theta_0, w \cdot \rho \rangle \\
S_4 &= \sum_{k=1}^{4M} \sum_{\substack{w \in W \setminus W(\zeta_0) \\ \langle \zeta_0, w \cdot \rho \rangle > 0}} e^{kb\langle z_0, w \cdot \rho \rangle} \\
S_5 &= \sum_{k=1}^{4M} \sum_{\substack{w \in W \setminus W(\zeta_0) \\ \langle \zeta_0, w \cdot \rho \rangle \leq 0}} e^{kb\langle z_0, w \cdot \rho \rangle}.
\end{aligned} \tag{24}$$

We claim that S_1 is the leading term. To see this, let $M(\zeta_0)$ and $N(\zeta_0)$ be two positive integers such that the following inequalities holds for all $M \geq M(\zeta_0)$ and $N \geq N(\zeta_0)$,

$$\begin{aligned}
\frac{e^{4Mb\langle \zeta_0, \rho \rangle}}{M} &\geq \frac{32|W|}{|W(\zeta_0)|}, \\
e^{4Mb\kappa} &\geq \frac{\kappa'}{|W(\zeta_0)|}, \\
e^{N\langle \zeta_0, \rho \rangle} &\geq 9
\end{aligned} \tag{25}$$

where κ is defined in (21), and

$$\kappa' = \sum_{\substack{w \in W \setminus W(\zeta_0) \\ \langle \zeta_0, w \cdot \rho \rangle > 0}} \frac{1}{1 - e^{-\langle \zeta_0, w \cdot \rho \rangle}} > 0.$$

The estimate for S_5 is straightforward:

$$|S_5| \leq \sum_{k=1}^{4M} \sum_{\substack{w \in W \setminus W(\zeta_0) \\ \langle \zeta_0, w \cdot \rho \rangle \leq 0}} e^{kb\langle \zeta_0, w \cdot \rho \rangle} \leq 4M|W| \leq \frac{S_1}{8}.$$

To bound S_4 we proceed as follows:

$$\begin{aligned}
|S_4| &\leq \sum_{\substack{w \in W \setminus W(\zeta_0) \\ \langle \zeta_0, w \cdot \rho \rangle > 0}} \sum_{k=1}^{4M} e^{kb\langle \zeta_0, w \cdot \rho \rangle} = \sum_{\substack{w \in W \setminus W(\zeta_0) \\ \langle \zeta_0, w \cdot \rho \rangle > 0}} \frac{e^{4Mb\langle \zeta_0, w \cdot \rho \rangle} - 1}{1 - e^{-b\langle \zeta_0, w \cdot \rho \rangle}} \\
&\leq e^{4Mb(\langle \zeta_0, \rho \rangle - \kappa)} \kappa' \leq \frac{S_1}{8}.
\end{aligned}$$

For S_3 we have

$$\begin{aligned}
|S_3| &\leq |W(\zeta_0)| \sum_{k=1}^{4M-1} e^{kb\langle \zeta_0, \rho \rangle} = |W(\zeta_0)| \frac{e^{4Mb\langle \zeta_0, \rho \rangle} - e^{b\langle \zeta_0, \rho \rangle}}{e^{b\langle \zeta_0, \rho \rangle} - 1} \\
&\leq \frac{1}{e^{b\langle \zeta_0, \rho \rangle} - 1} S_1,
\end{aligned}$$

which is smaller than $\frac{S_1}{8}$ if $b \geq N$.

To control S_2 , we show that there is a positive integer $b \geq N$, such that

$$\sum_{w \in W(\zeta_0)} (\sin 2Mb\langle \theta_0, w \cdot \rho \rangle)^2 \leq \frac{|W(\zeta_0)|}{16}. \tag{26}$$

For this purpose we resume the arguments in Case 2 based on the Dirichlet approximation theorem. Let

$$B = 8\pi hMN.$$

Then there is $1 \leq b' \leq B^r$, such that

$$b' \langle \theta_0, \vee \alpha_j \rangle \in \left[0, \frac{2\pi}{B}\right) + 2\pi\mathbb{Z} \quad \text{for all } j \in \{1, 2, \dots, r\}.$$

Hence,

$$2b' \langle \theta_0, w \cdot \rho \rangle \in \left(-\frac{2\pi h}{B}, \frac{2\pi h}{B}\right) + 2\pi\mathbb{Z} \quad \text{for all } w \in W.$$

We again set $b = 2b' \lceil N/(2b') \rceil$. Then

$$N \leq b \leq \min \{2b'N, 2B^r\},$$

and

$$2Mb \langle \theta_0, w \cdot \rho \rangle \in \left(-\frac{1}{4}, \frac{1}{4}\right) + 2\pi\mathbb{Z} \quad \text{for all } w \in W,$$

which implies (26).

Combining all the estimates on S_j , we deduce that

$$\sum_{k=1}^{4M} m_{k\rho}(bz_0) \geq 1 + \frac{1}{2}S_1 \geq 1 + 4M|W|.$$

Hence, $K_{M,b}(z_0) \geq 4|W|M \geq M$. This completes the proof of (14), and the theorem follows. \square

4. APPLICATIONS

In what follows, we describe some potentially new situations where arithmetic quantum unique ergodicity (AQUE) could be proved, using the kernel constructed in Section 3 and taking advantage of the fact that we are no more limited by the type of the underlying (reduced) root system. This would require additional progress in various steps of the known strategies for AQUE, and more particularly in the strategy elaborated by Brooks and Lindenstrauss (for more details, see the introduction). Nevertheless, an already available result is a positive entropy theorem as proved by Shem-Tov in [28].

Theorem 4.1. *Let G be a simple real Lie group which is \mathbb{R} -split, let Γ be an arithmetic subgroup of G and let μ be an A -invariant probability measure on the homogeneous space $X = \Gamma \backslash G$. Assume that $\mu = c \lim_{j \rightarrow \infty} |\varphi_j|^2 dx$ is a weak- \star limit where $c > 0$ and φ_j are some L^2 -normalized joint eigenfunctions of both $\mathcal{E}(\mathfrak{g})$ and the Hecke algebra at some fixed prime p . Then any regular element $a \in A$ acts with positive entropy on each ergodic component of μ .*

This statement is merely a generalization of [28, Theorem 1.1], passing from $\mathrm{SL}_n(\mathbb{R})$ to any split simple \mathbb{R} -group. Its proof consists in checking that Shem-Tov's arguments in [28, Section 4] can be used, once we replace his kernels constructed in [28, Section 3.1] by our kernels from Theorem 3.1.

Before going into this specific result, we first describe the general standard context which will allow us to address some additional potential applications.

4.1. The general setting. In what follows, \mathbf{H} denotes a linear algebraic group defined over a number field F which is assumed to be absolutely almost simple and simply connected. We denote by Φ the (absolute) root system of $\mathbf{H}_{\overline{F}}$ where \overline{F} is an algebraic closure of F . The root system Φ is reduced since $\mathbf{H}_{\overline{F}}$ is split; its Dynkin diagram is connected since $\mathbf{H}_{\overline{F}}$ is almost simple.

The starting point is relevant to real homogeneous spaces. Let H be a non-compact simple linear real Lie group. Its associated Riemannian symmetric space is $\widetilde{M} = H/K$ where K is any maximal compact subgroup of H . The desired quantum unique ergodicity results are expected to hold on the homogeneous space $\Gamma \backslash H$ where Γ is a torsion-free lattice, and have geometric interpretations on the locally symmetric space $M = \Gamma \backslash H/K$.

Our goal is to explain how to find an F -group \mathbf{H} as above in order to see H as closely related to the F -rational points of \mathbf{H} for some Archimedean completion of F , and also how to construct S -arithmetic groups projecting onto lattices Γ in H . We will also discuss the issue of cocompactness of Γ and explain why, in view of the classification of non-Archimedean Lie groups, this condition either implies restrictions on the absolute root system Φ or implies to work with several Archimedean places. A convenient reference for all the material below is the first chapter in Margulis' book [21].

First, if k is a locally compact field containing F , it is well-known (see [21, Chapter I, Proposition 2.3.6]) that we have the following result, relating an algebraic property for an algebraic group and a topological one for its rational points in this case.

Theorem (Anisotropy is compactness). *The topological group of rational points $\mathbf{H}(k)$ is compact if and only if the algebraic group \mathbf{H} has k -rank 0.*

Recall that, for any field E containing the ground field F , the E -rank of \mathbf{H} , denoted by $\text{rk}_E(\mathbf{H})$, is the dimension of a maximal E -split torus in \mathbf{H} ; these tori are all conjugate by $\mathbf{H}(E)$.

We denote by $\mathcal{T}(\mathbf{H})$ the set of places v of F such that the group $\mathbf{H}(F_v)$ is compact where F_v is the completion of F with respect to v . In other words, $\mathcal{T}(\mathbf{H})$ is the set of places v for which \mathbf{H} is F_v -anisotropic: It is known to be a finite subset of the set \mathcal{R} of all places of F . Finally, we denote by \mathcal{R}_∞ (resp. \mathcal{R}_{fin}) the set of Archimedean (resp. non-Archimedean) places of F .

If we pick a finite subset S in \mathcal{R} , we denote by $F(S)$ the set of S -integral elements in F , i.e. the set of elements $x \in F$ such that $|x|_v \leq 1$ for each absolute value $|\cdot|_v$ attached to $v \in \mathcal{R}_{\text{fin}} \setminus S$.

From now on, we assume that the group \mathbf{H} is given together with an embedding in some general linear group, which we denote by $\mathbf{H} < \text{GL}_n$. This allows us to define the subgroup $\mathbf{H}(F(S))$ to be

$$\mathbf{H}(F(S)) = \mathbf{H}(F) \cap \text{GL}_n(F(S)).$$

This is well-defined up to commensurability in the sense that another embedding $\mathbf{H} < \text{GL}_m$ would lead to a commensurable subgroup $\mathbf{H}(F) \cap \text{GL}_m(F(S))$, i.e. a subgroup containing a subgroup of finite index in both $\mathbf{H}(F) \cap \text{GL}_n(F(S))$ and $\mathbf{H}(F) \cap \text{GL}_m(F(S))$. Any subgroup of the form $\mathbf{H}(F(S))$ (or, more generally, commensurable with such a subgroup) is called an S -arithmetic subgroup of $\mathbf{H}(F)$.

The key result to construct lattices in products of Lie groups is the following theorem, due to Borel–Harish-Chandra and Mostow–Tamagawa (see [21, Chapter I, Theorem 3.2.5]).

Theorem (Arithmetic groups are lattices). *For any finite set S of places of F such that $\mathcal{R}_\infty \setminus \mathcal{T}(\mathbf{H}) \subset S \subset \mathcal{R}$, the S -arithmetic group $\mathbf{H}(F(S))$ is a lattice in the topological group H_S defined to be product group $H_S = \prod_{v \in S} \mathbf{H}(F_v)$.*

We shall see in the next section, how to construct from the above statements homogeneous spaces $\Gamma \backslash G$ in the purely Archimedean Lie group setting. In addition, this result is complemented by Godement's compactness criterion, see [21, Chapter I, Theorem 3.2.4 (b)]

Theorem (Cocompactness amounts to anisotropy). *The S -arithmetic lattice $\mathbf{H}(F(S))$ in the topological group H_S is cocompact if and only if the algebraic group \mathbf{H} is F -anisotropic.*

It is clear that for \mathbf{H} to be F -anisotropic, it is sufficient that \mathbf{H} be F_v -anisotropic, hence equivalently that $\mathbf{H}(F_v)$ be compact, for some place v of F . With this remark in mind, since we are mainly targeting a suitable (cocompact) lattice inclusion $\Gamma < H$, a very standard trick consists then in finding an F -group \mathbf{H} as above such that $\mathbf{H}(\mathbb{R})$ is H for some Archimedean completion \mathbb{R} of F , and such that $\mathbf{H}(F_v)$ is compact for some other completion F_v , with v Archimedean or not. One limitation of this trick is the fact that there are very few compact non-Archimedean simple Lie groups, namely (see e.g. [34, Classification tables]):

Theorem (Very few compact non-Archimedean groups). *The only anisotropic simple groups over non-Archimedean local fields are the groups of inner type A, in other words division algebras over non-Archimedean local fields.*

This classification result explains in particular why quaternion algebras, which are fields over \mathbb{Q} , are used to construct cocompact arithmetic lattices in $\mathrm{PSL}_2(\mathbb{R})$. On the contrary, according to Weyl's unitarian trick [7, § 3 Théorème 1 p. 19], any simple real Lie group H has a (unique) anisotropic \mathbb{R} -form which we denote by $\mathbf{H}^{\mathrm{cpt}}$. Recall that an F -form of \mathbf{H} is a linear algebraic F -group \mathcal{H} such that $\mathbf{H} \otimes_F \overline{F}$ and $\mathcal{H} \otimes_F \overline{F}$ are isomorphic \overline{F} -groups.

The last important result we would like to mention in this abstract general context addresses the freedom of choice that is available when one wants to find an F -group as above with prescribed behavior at finitely many places of F . The following statement is implied by a deep result in Galois cohomology due to Borel and Harder (see [6, Theorem B]).

Theorem (Freedom in prescribing local forms). *Let Φ be an irreducible reduced root system. Let S be a finite set of places of F and, for each $v \in S$, and let \mathbf{H}_v be a simply connected F_v -group of absolute root system Φ . Then there exists a simply connected F -group \mathbf{H} such that $\mathbf{H} \otimes_F F_v$ and \mathbf{H}_v are isomorphic F_v -groups for each $v \in S$.*

The same statement holds with "adjoint" replacing "simply connected" everywhere. It is precious in the sense that, provided a natural compatibility condition is satisfied (namely, sharing the same absolute root system) we can find a suitable F -group \mathbf{H} in order to construct an S -arithmetic lattice $\mathbf{H}(F(S))$ in H_S as above; moreover, this can be done while prescribing that at least one factor for some $v \in S$ in H_S be compact, ensuring that the lattice inclusion $\mathbf{H}(F(S)) < H_S$ is cocompact. To reformulate the limitation mentioned after the previous theorem, one restriction is that if we choose a compact factor to be non-Archimedean, then the group F -group \mathbf{H} must be an F -form of SL_n ; otherwise, if we choose the compact factor to be Archimedean, then all absolute root systems are allowed but then F must be a number field $\neq \mathbb{Q}$ since several Archimedean places are needed.

4.2. Potentially new situations for AQUE. Using the theoretical framework of the previous subsection, we recall situations when AQUE, or at least positive entropy statements, are known to hold. Then we exhibit situations where the kernels from Theorem 3.1 could be used to prove new positive entropy results. We freely use here the notation used in the introduction as well as the one of the previous subsection.

We first consider the case when adelic Hecke conditions are imposed, *i.e.* when the functions φ_j are eigenfunctions of the full Hecke algebra \mathcal{H} (and form a non-degenerate sequence in the sense of [30]). Then it is known that AQUE holds for congruence cocompact lattices in $G_\infty = \mathrm{SL}_2(\mathbb{R})$, see [18], and more generally in $G_\infty = \mathrm{SL}_d(\mathbb{R})$ for d prime [31]. In the same adelic context, AQUE is also known to hold in the non-cocompact case for $G_\infty = \mathrm{SL}_2(\mathbb{R})$, since in [32] it is shown that there is no escape of mass.

We now turn to the case when local Hecke conditions are imposed, *i.e.* when $(\varphi_j : j \in \mathbb{N})$ is a non-degenerate sequence of eigenfunctions of a local Hecke algebra \mathcal{H}_p with respect to some specific prime p . Then AQUE holds for congruence cocompact lattices in $G_\infty = \mathrm{SL}_2(\mathbb{R})$, and cocompact irreducible lattices in $G_\infty = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, see [9], and more generally for congruence cocompact lattices in $G_\infty = \mathrm{SL}_d(\mathbb{R})$ for d prime, see [29]. In the same local context, AQUE is also known to hold in the non-cocompact case (up to proportionality, since then escape of mass is not disproved yet) for $\Gamma = \mathrm{SL}_d(\mathbb{Z})$ in $G_\infty = \mathrm{SL}_d(\mathbb{R})$ for d prime, see [28]. The latter reference contains a positive entropy result elaborating on ideas from [9]. We mention below that thanks to our kernels and by using Shem-Tov's ideas on $\mathrm{SL}_k(\mathbb{Z}) \setminus \mathrm{SL}_k(\mathbb{R})$ (where k is any integer ≥ 2), the positive entropy result [28, Theorem 1.1] can be generalized from the case $G_\infty = \mathrm{SL}_k(\mathbb{R})$ to the case when G_∞ is any split simple real Lie group.

As it was already mentioned in the introduction, deducing an AQUE result from a positive entropy one requires a precise understanding of measures on the homogeneous spaces $\Gamma \backslash G_\infty$. In the higher rank case, the results in this vein that are used in the references above usually come from [14] and [13].

Finally, let us explain in which general context our kernels shall be used. Let G_∞ be a non-compact simple linear real Lie group whose Lie algebra \mathfrak{g}_∞ is absolutely simple (*i.e.* $\mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$ is simple). We assume that G_∞ is the group of real points of an absolutely almost simple simply connected \mathbb{R} -group \mathbf{G} .

- As a first step, by [5, Proposition 3.8] and after choosing a totally real number field E , we can extend \mathbf{G} to an E -group, still denoted by \mathbf{G} , with $\mathbf{G}(E) = G_\infty$, and such that the twisted groups $\mathbf{G}^\sigma(E)$ are compact

for all non-trivial embeddings $\sigma : E \rightarrow \mathbb{R}$. Then the group $\Gamma = \mathbf{G}(\mathcal{O}_E)$ is a cocompact lattice in $G_\infty \times K_\infty$ where K_∞ is the product of the compact Lie groups $\mathbf{G}^\sigma(E)$.

- As a second step, by the Borel-Harder Theorem above and after choosing a non-Archimedean place $\mathfrak{p} \in \mathcal{R}_{\text{fin}}$, we can even assume that \mathbf{G} is split at the completion $E_{\mathfrak{p}}$ of E with respect to the place \mathfrak{p} . In particular, the Bruhat–Tits building of $G_{\mathfrak{p}} = \mathbf{G}(E_{\mathfrak{p}})$ carries a Brooks–Lindenstrauss kernel as constructed in the present paper, since its (spherical) root system is reduced.

Setting $S = \mathcal{R}_\infty \cup \{\mathfrak{p}\}$ we see that the S -integers consist of the elements in $\mathcal{O}_E[\frac{1}{\mathfrak{p}}]$ where \mathcal{O}_E is the ring of integers of the number field E . Using the notation $\Gamma[\frac{1}{\mathfrak{p}}] = \mathbf{G}(\mathcal{O}_E[\frac{1}{\mathfrak{p}}])$, and $K_{\mathfrak{p}} = \mathbf{G}(\mathcal{O}_{\mathfrak{p}})$ where $\mathcal{O}_{\mathfrak{p}}$ is the valuation ring of the local field $E_{\mathfrak{p}}$, we have $\Gamma = \Gamma[\frac{1}{\mathfrak{p}}] \cap K_{\mathfrak{p}}$. In addition, the maximal compact subgroup $K_{\mathfrak{p}}$ is actually a maximal proper subgroup since it is a maximal parahoric subgroup of an affine Tits system in the simply connected Chevalley group $\mathbf{G}(E_{\mathfrak{p}})$. This provides the equality $\mathbf{G}(E_{\mathfrak{p}}) = \Gamma[\frac{1}{\mathfrak{p}}]K_{\mathfrak{p}}$ which together with $\Gamma = \Gamma[\frac{1}{\mathfrak{p}}] \cap K_{\mathfrak{p}}$ allows us to identify (in a similar way as in the Introduction after Theorem B) the spaces $\Gamma \backslash (G_\infty \times K_\infty)$ and $\Gamma[\frac{1}{\mathfrak{p}}] \backslash (G_\infty \times K_\infty \times G_{\mathfrak{p}}) / K_{\mathfrak{p}}$. Note that not insisting on cocompactness of Γ in G_∞ allows one to skip the first step and to work with a \mathbb{Q} -group \mathbf{G} , in which case \mathfrak{p} is merely a prime number p .

We finally use the identification $\Gamma \backslash G_\infty \simeq \Gamma[\frac{1}{\mathfrak{p}}] \backslash (G_\infty \times G_{\mathfrak{p}}) / K_{\mathfrak{p}}$ to see the averaging operators A_λ defined in Section 1.4 as Hecke operators acting on $L^2(\Gamma \backslash G_\infty)$. More precisely, at the non-Archimedean place \mathfrak{p} the Cartan decomposition $G_{\mathfrak{p}} = \bigsqcup_{\lambda \in P^+} K_{\mathfrak{p}} \lambda K_{\mathfrak{p}}$ shows that the characteristic functions $\mathbb{1}_{K_{\mathfrak{p}} \lambda K_{\mathfrak{p}}}$ provide a basis for the space $\mathcal{L}(G_{\mathfrak{p}}, K_{\mathfrak{p}})$ of compactly supported bi- $K_{\mathfrak{p}}$ -invariant functions $f : G_{\mathfrak{p}} \rightarrow \mathbb{C}$; the latter space is a unital, commutative algebra for convolution (we pick the Haar measure on $G_{\mathfrak{p}}$ for which $K_{\mathfrak{p}}$ has volume 1). It follows from a classical computation that for each $\lambda \in P^+$ the map $f \mapsto f * \mathbb{1}_{K_{\mathfrak{p}} \lambda K_{\mathfrak{p}}}$ coincides with the averaging operator A_λ on continuous functions $G_{\mathfrak{p}} \rightarrow \mathbb{C}$ which are right $K_{\mathfrak{p}}$ -invariant. We will use this remark to write $A_\lambda \in \mathcal{L}(G_{\mathfrak{p}}, K_{\mathfrak{p}})$.

To go back to the quotient $\Gamma \backslash G_\infty$, we will call the operator on $L^2(\Gamma \backslash G_\infty)$ defined by the partial convolution as

$$f \mapsto \int_{K_{\mathfrak{p}} \lambda K_{\mathfrak{p}}} f(g_\infty, g_{\mathfrak{p}} h^{-1}) dh,$$

the *Hecke operator* attached to $\lambda \in P^+$. We will denote it again by A_λ . We also write $A_\lambda \in \mathcal{L}(G_{\mathfrak{p}}, K_{\mathfrak{p}})$ in this context. The connection between eigenvalues of the Hecke algebra and the spectrum Σ is explained in Appendix A.

4.3. Proof of the positive entropy theorem for all \mathbb{R} -split simple groups. We can now turn to an effective application using the kernels constructed in Theorem 3.1. We use the notation of Theorem 4.1 with G equal to G_∞ above.

We claim that Shem-Tov’s proof of [28, Theorem 1.1] works in the latter context, once one uses our kernels; in particular, the requested generalizations of [28, Proposition 3.1] (suitable kernel) and [28, Lemma 4.7] (Cauchy–Schwarz and integration on double classes mod $K_{\mathfrak{p}}$) hold. Moreover, the criterion for positive entropy [28, Proposition 4.6] is relevant to pure dynamical systems with an abstract ambient space X endowed with a probability measure μ stabilized by a measurable automorphism a (i.e. $a_* \mu = \mu$).

The remaining ingredient in the proof is [28, Lemma 4.1]. This lemma provides the partition of $X = \Gamma \backslash G_\infty$ needed in the assumptions of [28, Proposition 4.6]. The proof of the lemma itself uses four other lemmas, and we finish by arguing why these are still available. Lemma 4.2 is [31, Lemma 4.4] and holds after using a linear embedding $\mathbf{G} < \text{GL}_n$. Lemma 4.3 is a construction borrowed from [15, 7.51] and valid for general simple real Lie groups. Lemma 4.4 is also a general lemma valid for general simple real Lie groups; its proof uses [15, 7.51] which holds for the same reason. Finally, Lemma 4.5 is also a result which can be used in our context after using a linear embedding $\mathbf{G} < \text{GL}_n$.

APPENDIX A. HECKE ALGEBRA EIGENVALUES AND THE SPECTRUM Σ

In this appendix we retain the notation introduced in Section 4.

Suppose that $f \in L^2(\Gamma \backslash G_\infty)$ is a joint eigenfunction of the Hecke algebra, that is there is $\psi : P^+ \rightarrow \mathbb{C}$, such that for all $\lambda \in P^+$,

$$f * A_\lambda = \psi(\lambda)f.$$

We assume that $\|f\|_{L^2} = 1$. Then ψ is a homomorphism from $\mathcal{L}(G_p, K_p)$ into \mathbb{C} , thus in view of [19, Propositions 1.2.2 & 1.2.3], there is $z \in \mathfrak{a}_\mathbb{C}$ such that

$$\psi(\lambda) = P_\lambda(z).$$

We claim that $P_\lambda(z)$ is positive definite as a function of $\lambda \in P^+$.

Since both groups G_∞ and G_p are of type I, see e.g. [4, Theorem 6.E.19], we can use direct integral decompositions. Hence,

$$L^2\left(\Gamma\left[\frac{1}{p}\right] \backslash G_\infty \times G_p\right) = \int_{\widehat{G_\infty \times G_p}}^\oplus \mathcal{H}_\pi \mu_{\Gamma\left[\frac{1}{p}\right]}(d\pi).$$

In particular,

$$\begin{aligned} P_\lambda(z) &= \langle f * A_\lambda, f \rangle \\ &= \int_{\widehat{G \times G_p}}^\oplus \text{Tr} \left(\pi(f * A_\lambda) \pi(f)^\star \right) \mu_{\Gamma\left[\frac{1}{p}\right]}(d\pi). \end{aligned}$$

In the formulas above $\widehat{G_\infty \times G_p} = \widehat{G_\infty} \times \widehat{G_p}$ denotes the unitary dual. To compute the trace, let π be an irreducible representation of $G_\infty \times G_p$ on \mathcal{H}_π , which contains a non-trivial vector fixed by $\pi(1, k)$ for each $k \in K_p$. We have

$$\begin{aligned} \pi(f * A_\lambda) &= \int_{G_\infty \times G_p} (f * A_\lambda)(g, g_p) \pi(g, g_p) d(g, g_p) \\ &= \int_{G_\infty \times G_p} \int_{G_p} f(g, g_p h_p) A_\lambda(h_p^{-1}) \pi(g, g_p) dh_p d(g, g_p) \\ &= \int_{G_p} A_\lambda(h_p^{-1}) \int_{G_\infty \times G_p} f(g, g_p) \pi(g, g_p) d(g, g_p) \pi(1, h_p^{-1}) dh_p \\ &= \pi(f) \int_{G_p} A_\lambda(h_p^{-1}) \pi(1, h_p^{-1}) dh_p. \end{aligned}$$

The representation π is a product $\pi_\infty \times \pi_p$ acting on $\mathcal{H}_{\pi_\infty} \otimes \mathcal{H}_{\pi_p}$, where both $(\pi_\infty, \mathcal{H}_{\pi_\infty})$ and $(\pi_p, \mathcal{H}_{\pi_p})$ are irreducible unitary representations of G_∞ and G_p , respectively. Because $A_\lambda = \mathbb{1}_{K_p \lambda K_p}$, we need to understand the operator

$$\Lambda = \int_{K_p \lambda K_p} \pi_p(h) dh.$$

Let us observe that for $k \in K_p$, we have

$$\pi_p(k) \Lambda = \int_{K_p \lambda K_p} \pi_p(kh) dh = \Lambda,$$

and $\Lambda = \Lambda \pi_p(k)$. Therefore, Λ preserves $\mathcal{H}_{\pi_p}^{K_p}$, the subspace of K_p -fixed vectors of \mathcal{H}_{π_p} , which is 1-dimensional, see [19, Theorem 1.4.4(ii)]. Moreover, there is $z_{\pi_p} \in \mathfrak{a}_\mathbb{C}$ such that if $\mathcal{H}_{\pi_p}^{K_p} = \mathbb{C} \xi_{\pi_p}$ with $\langle \xi_{\pi_p}, \xi_{\pi_p} \rangle_{\pi_p} = 1$, then for each $g \in K_p \tilde{\lambda} K_p$,

$$\langle \xi_{\pi_p}, \pi_p(h) \xi_{\pi_p} \rangle = P_{\tilde{\lambda}}(z_{\pi_p}).$$

By [19, Proposition 1.4.2(i)], $P_{\tilde{\lambda}}(z_{\pi_p})$ as a function of $\tilde{\lambda}$ is positive-definite. Consequently,

$$\langle \Lambda \xi_{\pi_p}, \xi_{\pi_p} \rangle = \int_{K_p \backslash \lambda K_p} \langle \pi_p(h) \xi_{\pi_p}, \xi_{\pi_p} \rangle dh = P_{\lambda}(z_{\pi_p}).$$

Hence,

$$\mathrm{Tr} \left(\pi(f * A_{\lambda}) \pi(f)^{\star} \right) = P_{\lambda}(z_{\pi_p})$$

which leads to

$$P_{\lambda}(z) = \int_{\widehat{G_{\infty} \times G_p}}^{\oplus} P_{\lambda}(z_{\pi_p}) \mu_{\Gamma[\frac{1}{p}]}(d(\pi_{\infty}, \pi_p)),$$

and since $(P_{\lambda}(z_{\pi_p}) : \lambda \in P^+)$ is positive-definite for every $\pi_p \in \widehat{G_p}$, the same holds true for $(P_{\lambda}(z) : \lambda \in P^+)$, thus $\lambda \in \Sigma$.

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