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GROWTH OF FREQUENTLY OR LOG-FREQUENTLY HYPERCYCLIC FUNCTIONS

A. MOUZE, V. MUNNIER

ABSTRACT. We obtain the optimal boundary behavior of the log-frequently hypercyclic functions with respect to the Taylor shift acting on $H(\mathbb{D})$ in terms of average L^p -norms. In passing we establish some new results on the growth of frequently or log-frequently hypercyclic functions for the differentiation operator on $H(\mathbb{C})$. All these results highlight the similarities and the differences between the lower and upper bounds on the growth of frequently and log-frequently hypercyclic functions, on the one hand in the case of the Taylor shift operator on $H(\mathbb{D})$ and on the other hand in the case of the differentiation operator on $H(\mathbb{C})$.

1. INTRODUCTION

Let \mathbb{D} be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane and $H(\mathbb{D})$ be the set of analytic function in \mathbb{D} endowed with the topology of uniform convergence on compact subsets. For $0 < r < 1$ and $f \in H(\mathbb{D})$, we consider the classical integral means

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \quad (1 \leq p < \infty) \text{ and } M_\infty(r, f) = \sup_{0 \leq t \leq 2\pi} |f(re^{it})|.$$

In the same spirit, for any holomorphic polynomial P let us define, for all $p \geq 1$,

$$\|P\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p} \text{ and } \|P\|_\infty = \sup_{0 \leq t \leq 2\pi} |P(e^{it})|.$$

Moreover for all $p > 1$, q will always stand for the exponent conjugate to p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For

any f in $H(\mathbb{D})$ and any z in \mathbb{D} , we will write $f(z) = \sum_{k \geq 0} a_k z^k$. In the present paper, we consider the

Taylor shift operator T acting on $H(\mathbb{D})$ defined as follows:

$$T : H(\mathbb{D}) \rightarrow H(\mathbb{D}), \quad f(z) = \sum_{k \geq 0} a_k z^k \mapsto T(f)(z) = \sum_{k \geq 0} a_{k+1} z^k.$$

We know that T is frequently hypercyclic, that means there exists a function $f \in H(\mathbb{D})$ such that for every non-empty open set $U \subset H(\mathbb{D})$ the set $\{n \in \mathbb{N} : T^n f \in U\}$ has a positive (natural) lower density, i.e.

$$\underline{d}(\{n \in \mathbb{N} : T^n f \in U\}) > 0,$$

where, for a subset $A \subset \mathbb{N}$,

$$\underline{d}(A) = \liminf_{N \rightarrow +\infty} \left(\frac{\#\{k \in \{0, \dots, N\} \cap A\}}{N+1} \right).$$

Such a function f is called a frequently hypercyclic function for T . Generally speaking, an operator S on a Fréchet space X is called *frequently hypercyclic* if there is some $x \in X$ such that, for any nonempty open subset U of X , $\underline{d}(\{n \in \mathbb{N} : S^n x \in U\}) > 0$. The notion of frequent hypercyclicity was introduced by Bayart and Grivaux in [1] and since then became a very active branch of mathematical analysis. We refer the reader to [2] or [15] for some background in linear dynamics and we refer the reader to [3, 4, 14, 19, 22] for the dynamical properties of the Taylor shift operator

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T acting on $H(\mathbb{D})$. In particular, in [19] we gave the optimal rate of growth of the frequently hypercyclic functions for the Taylor shift operator in terms of average L^p -norms for $p \in [1, \infty]$. The result is stated as follows. The notation \log_l stands for the iterated function $\log \circ \dots \circ \log$ where \log appears l times.

Theorem 1.1. ([19]) *The following assertions hold.*

- (1) *Given $1 < p \leq +\infty$, there is a frequently hypercyclic function $f \in H(\mathbb{D})$ for T satisfying the following estimate: there exists $C > 0$ such that for every $0 < r < 1$*

$$M_p(r, f) \leq \frac{C}{(1-r)^{1/\max(2,p)}}.$$

This estimate is optimal: every frequently hypercyclic function $f \in H(\mathbb{D})$ for T satisfies $\liminf_{r \rightarrow 1} \left((1-r)^{1/\max(2,p)} M_p(r, f) \right) > 0$.

- (2) *For any $l \geq 1$, there is a frequently hypercyclic function $f \in H(\mathbb{D})$ for T satisfying the following estimate: there exists $C > 0$ such that for every $0 < r < 1$ sufficiently large*

$$M_1(r, f) \leq C |\log(1-r)| \log_l(-\log(1-r)).$$

This estimate is optimal in the following sense: every frequently hypercyclic function $f \in H(\mathbb{D})$ for T satisfies $\liminf_{r \rightarrow 1} \left(\frac{M_1(r, f)}{-\log(1-r)} \right) > 0$.

For results on the growth of frequently hypercyclic functions in the same vein as for the previous theorem, we refer the reader to [5],[6, 8], [12] or [20], where the authors deal respectively with the Dunkl operator on $H(\mathbb{C})$, the differentiation operator on $H(\mathbb{C})$, the partial differentiation operator on the space of harmonic functions on \mathbb{R}^N or weighted Taylor shifts on $H(\mathbb{D})$. For the Taylor shift operator acting on $H(\mathbb{D})$, it is important to observe that for $p \in (1, \infty]$ the lower growth of frequently hypercyclic functions coincides with the smallest possible maximum growth. There is a kind of regularity in their boundary behavior that contrasts with the irregularity of their orbit under the action of T . In the present paper, we are interested in the log-frequently hypercyclic functions for T . The natural lower density quantifies how often the iterated $T^n f$ visit a non-empty open set. If we replace the natural lower density by the lower logarithmic density given by, for any subset $A \subset \mathbb{N}$,

$$\underline{d}_{\log}(A) = \liminf_{N \rightarrow +\infty} \left(\frac{1}{\log(N)} \sum_{\substack{1 \leq n \leq N \\ n \in A}} \frac{1}{n} \right),$$

we obtain another way to measure the frequency of visits. Thus a function $f \in H(\mathbb{D})$ is called log-frequently hypercyclic for T if for every non-empty open set $U \subset H(\mathbb{D})$

$$\underline{d}_{\log}(\{n \in \mathbb{N} : T^n f \in U\}) > 0.$$

The inequality $\underline{d} \leq \underline{d}_{\log}$ ensures that a frequently hypercyclic function for T is necessarily log-frequently hypercyclic. The converse implication is not necessarily true since there exist sets $A \subset \mathbb{N}$ with $\underline{d}(A) = 0$ and $\underline{d}_{\log}(A) > 0$. We refer the reader to the survey [13] for the notion of lower logarithmic density and to [7, 10, 11, 17, 18] for the recent use of concepts of weighted densities as the logarithmic density in linear dynamics. Here we are going to obtain the boundary behavior of the log-frequently hypercyclic functions for T in terms of average L^p -norms. As the orbit of a log-frequently hypercyclic function should visit in some sense less often the non-empty open sets, one can expect a weaker growth of its norms. This intuition will only be partially verified. In particular, we will prove the following general result with the most interesting case $p = \infty$ that we highlight.

Theorem 1.2. *Let f be a log-frequently hypercyclic function in $H(\mathbb{D})$ for the Taylor shift. Then we have the following estimates*

- (1) *for $p = \infty$,*
 (a) $\limsup_{r \rightarrow 1} (\sqrt{1-r} M_{\infty}(r, f)) > 0,$

(b) there exists $C \geq 1$ such that $\liminf_{r \rightarrow 1} \left((1-r)^{1/(2C)} M_\infty(r, f) \right) > 0$.

These estimates are optimal: for all positive integer a large enough, there exists a log-frequently hypercyclic function $f_a \in H(\mathbb{D})$ for T and $C > 0$ such that

$$M_\infty(r, f_a) \leq \frac{C}{\sqrt{1-r}} \text{ and } \liminf_{r \rightarrow 1} \left((1-r)^{1/(2a)} M_\infty(r, f_a) \right) < +\infty.$$

(2) more generally, for $p > 1$,

(a) $\limsup_{r \rightarrow 1} \left((1-r)^{\frac{1}{\max(2,q)}} M_p(r, f) \right) > 0$,

(b) there exists $C \geq 1$ such that $\liminf_{r \rightarrow 1} \left((1-r)^{1/\max(2C, qC)} M_p(r, f) \right) > 0$.

These estimates are optimal: for all positive integer a large enough, there exists a log-frequently hypercyclic function $f_a \in H(\mathbb{D})$ for T and $C > 0$ such that

$$M_p(r, f_a) \leq \frac{C}{(1-r)^{1/\max(2,q)}} \text{ and } \liminf_{r \rightarrow 1} \left((1-r)^{1/\max(2a, qa)} M_p(r, f_a) \right) < +\infty.$$

In contrast with the case of the frequently hypercyclic functions, the lower growth of log-frequently hypercyclic functions is different from their smallest possible maximum growth. Nevertheless one can observe that this latter is the same as in the case of frequent hypercyclicity. To obtain the estimates of Theorem 1.2 we use in particular classical inequalities with L^p -norms. To show the assertions of optimality in Theorem 1.2, our proof is constructive and inspired by the ideas of [8] where the authors studied the minimal growth of frequently hypercyclic functions for the differentiation operator on the space of entire functions denoted by $H(\mathbb{C})$. A crucial tool is the Rudin-Shapiro polynomials. Theorem 1.2 will be obtained by combining Theorem 2.1 with Theorem 3.5 below. For $p = 1$, we will obtain the same rate of growth as in the case of frequent hypercyclicity: we refer the reader to Theorem 2.2. Finally we are interested in the case of differentiation operator D on $H(\mathbb{C})$, called MacLane's operator. Notice that the operator D can also be regarded as a particular weighted Taylor shift operator. It is known that the frequently hypercyclic functions f for D satisfy the following optimal estimate [6, 8]: for $p \in [1, \infty]$,

$$(1) \quad \limsup_{r \rightarrow \infty} \left(r^{\frac{1}{2\min(2,p)}} e^{-r} M_p(r, f) \right) > 0.$$

Nevertheless Remark 4.3 of [8] asserts that in general a frequently hypercyclic function f for D need not satisfy $\liminf_{r \rightarrow \infty} \left(r^{\frac{1}{2\min(2,p)}} e^{-r} M_p(r, f) \right) > 0$. In the last section, we complete this remark by giving the lower bound of such functions and we prove the optimality. In particular, we obtain the following statement

Theorem 1.3. *Let $1 \leq p \leq \infty$. Let f be a frequently hypercyclic function for the differentiation operator on $H(\mathbb{C})$. Then we have the following estimate: for all $1 \leq p \leq +\infty$, there exists $C > 1$ such that*

$$\liminf_{r \rightarrow \infty} \left(e^{-\frac{r}{C}} M_p(r, f) \right) > 0.$$

Moreover this estimate is optimal in the following sense: for all positive integer $a \in \mathbb{N}$ large enough, there exists a frequently hypercyclic function f_a for the differentiation operator and a real number $C_a > 1$, with $C_a \rightarrow +\infty$ as $a \rightarrow +\infty$, such that

$$\liminf_{r \rightarrow \infty} \left(e^{-\frac{r}{C_a}} M_p(r, f_a) \right) < +\infty.$$

To conclude we show that the log-frequently hypercyclic functions for D necessarily satisfy the estimate (1). In addition we obtain the optimal lower bounds of such functions in the same spirit as the statement of Theorem 1.3. We refer the reader to Theorem 4.6.

The paper is organized as follows: Section 2 is devoted to the study of the boundary behavior of log-frequently hypercyclic functions for the Taylor shift operator. In Section 3, we prove the optimality of the results obtained in Section 2. Finally in Section 4 we give the new results regarding the frequently or log-frequently hypercyclic entire functions for the differentiation operator.

Throughout the paper, we will use the following well-known property: a strictly increasing sequence (n_k) of positive integers is of positive lower natural density if and only if $\sup_{k \geq 1} \left(\frac{n_k}{k} \right) < \infty$.

Moreover, whenever A and B depend on some parameters, we will use the notation $A \lesssim B$ (resp. $A \gtrsim B$) to mean $A \leq CB$ (resp. $A \geq CB$) for some constant $C > 0$ that does not depend on the involved parameters.

2. BOUNDARY BEHAVIOR OF LOG-FREQUENTLY HYPERCYCLIC FUNCTIONS FOR THE TAYLOR SHIFT OPERATOR

In this section we deal with the growth of log-frequently hypercyclic functions for the Taylor shift operator in terms of L^p -average. We begin with the case $p > 1$ and we establish the following result.

Theorem 2.1. *Let f be a log-frequently hypercyclic function in $H(\mathbb{D})$ for the Taylor shift. Then we have the following estimates,*

- (1) for all $2 \leq p \leq \infty$,
 - (a) $\limsup_{r \rightarrow 1} (\sqrt{1-r} M_p(r, f)) > 0$,
 - (b) there exists $C \geq 1$ such that $\liminf_{r \rightarrow 1} \left((1-r)^{1/(2C)} M_p(r, f) \right) > 0$;
- (2) and, for $1 < p < 2$,
 - (a) $\limsup_{r \rightarrow 1} \left((1-r)^{1/q} M_p(r, f) \right) > 0$,
 - (b) there exists $C \geq 1$ such that $\liminf_{r \rightarrow 1} \left((1-r)^{1/(qC)} M_p(r, f) \right) > 0$.

Proof. Let us write $f(z) = \sum_{k \geq 0} a_k z^k$. Since f is log-frequently hypercyclic, there exists a subsequence

$(n_k) \subset \mathbb{N}$ with positive lower logarithmic density satisfying $|T^{n_k} f(0) - \frac{3}{2}| = |a_{n_k} - \frac{3}{2}| < 1/2$ for all $k \geq 1$. Hence for all $k \geq 1$ we have $|a_{n_k}| \geq 1$. Moreover since the sequence (n_k) has positive lower logarithmic density, there exists $0 < C_1 < 1$ such that for all n large enough

$$(2) \quad \sum_{1 \leq n_j \leq 2^n} \frac{1}{n_j} \geq C_1 n \log(2).$$

Let us consider the set $I = \{n_k, k \geq 1\}$ and, for all $n \in \mathbb{N}$, the set I_n given by $I_n = \{2^n, \dots, 2^{n+1} - 1\} \cap I$. Observe that, for all positive integer n ,

$$(3) \quad \log(2) \leq \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \leq \log(2) + \frac{1}{2^{n+1}}.$$

For all $N \in \mathbb{N}$, we define the subsets $A(N)$ and A of \mathbb{N} as follows

$$A(N) = \left\{ n \in \{1, \dots, N\}; \sum_{k \in I_n} \frac{1}{k} \geq \frac{C_1}{2} \log(2) \right\} \text{ and } A = \bigcup_{N \geq 1} A(N).$$

First observe that

$$(4) \quad \sum_{1 \leq n_j \leq 2^N} \frac{1}{n_j} = \sum_{\substack{1 \leq k \leq 2^N \\ k \in I}} \frac{1}{k} \leq \sum_{n=1}^N \sum_{k \in I_n} \frac{1}{k} = \sum_{n \in A(N)} \sum_{k \in I_n} \frac{1}{k} + \sum_{n \in A(N)^c} \sum_{k \in I_n} \frac{1}{k}.$$

Combining (2), (3) and (4) with the definition of the set $A(N)$ we get

$$(5) \quad \begin{aligned} C_1 N \log(2) &\leq \sum_{n \in A(N)} \sum_{k \in I_n} \frac{1}{k} + \sum_{n \in A(N)^c} \sum_{k \in I_n} \frac{1}{k} \\ &\leq \#A(N) (\log(2) + 2^{-(N+1)}) + \frac{C_1}{2} (N - \#A(N)) \log(2) \\ &\leq \#A(N) \left(\left(1 - \frac{C_1}{2}\right) \log(2) + 2^{-(N+1)} \right) + \frac{C_1}{2} N \log(2). \end{aligned}$$

The inequality (5) ensures that the set A has a positive lower natural density. Therefore the set A is infinite. Let (N_l) be an increasing sequence of positive integers of A such that (N_l) has a positive lower natural density. Hence there exists $C_2 \geq 1$ such that, for all $l \geq 1$,

$$(6) \quad N_l \leq C_2 l$$

(1) On the one hand let us consider a sequence (r_l) with $1 - \frac{1}{2^{N_l}} \leq r_l < 1 - \frac{1}{2^{N_l+1}}$, we have by applying Parseval Theorem

$$(7) \quad (M_2(r_l, f))^2 = \sum_{k \geq 0} |a_k|^2 r_l^{2k} \geq \sum_{n \geq 1} \sum_{k \in I_n} k \frac{|a_k|^2}{k} r_l^{2k}.$$

Since we have both for all $k \in I_n$, $|a_k| \geq 1$ and for all $l \geq 1$ $\sum_{k \in I_{N_l}} \frac{1}{k} \geq \frac{C_1}{2} \log(2)$, the inequality (7) becomes

$$(8) \quad \begin{aligned} (M_2(r_l, f))^2 &\geq \sum_{n \geq 1} \sum_{k \in I_n} k \frac{|a_k|^2}{k} r_l^{2k} \\ &\gtrsim \sum_{n=1}^{N_l} \left(2^n \sum_{k \in I_n} \frac{1}{k} \right) \\ &\gtrsim 2^{N_l}. \end{aligned}$$

Since we have the following inequality $2^{N_l} \leq \frac{1}{1-r_l} \leq 2^{N_l+1}$, we deduce, for all $l \geq 1$,

$$(9) \quad (M_2(r_l, f))^2 \gtrsim \frac{1}{1-r_l}.$$

Hence, we obtain

$$\limsup_{r \rightarrow 1} (\sqrt{1-r} M_2(r, f)) > 0.$$

(2) On the other hand, let us consider $0 < r < 1$. Again we apply Parseval Theorem to write

$$(10) \quad (M_2(r, f))^2 = \sum_{k \geq 0} |a_k|^2 r^{2k} \geq \sum_{l \geq 1} \sum_{k \in I_{N_l}} |a_k|^2 r^{2k}.$$

Since for all $k \in I_{N_l}$, we have $|a_k| \geq 1$, we deduce, taking into account (6),

$$(11) \quad \begin{aligned} (M_2(r, f))^2 &\geq \sum_{l \geq 1} \sum_{k \in I_{N_l}} r^{2k} \\ &\gtrsim \sum_{l=1}^{+\infty} r^{2^{C_2 l+1}} \#I_{N_l}. \end{aligned}$$

By construction we have

$$(12) \quad \frac{C_1}{2} \log(2) \leq \sum_{k \in I_{N_l}} \frac{1}{k} \leq \frac{\#I_{N_l}}{2^{N_l}} \leq \frac{\#I_{N_l}}{2^l}.$$

From $(N_l) \subset A$ and (12) we get

$$(M_2(r, f))^2 \gtrsim \sum_{l=1}^{+\infty} 2^l r^{2^{C_2 l+1}} \gtrsim \frac{1}{(1-r)^{1/C_2}}.$$

Therefore we have the following conclusion: if f is a log-frequently hypercyclic function in $H(\mathbb{D})$ for the Taylor shift, there exists $C \geq 2$ such that

$$\liminf_{r \rightarrow 1} \left((1-r)^{1/C} M_2(r, f) \right) > 0.$$

Moreover, since

$$M_2(f, r) \leq M_p(f, r) \text{ for } 2 < p \leq \infty,$$

we obtain the result for $p \geq 2$. Finally, from the Hausdorff-Young inequality

$$M_p(f, r) \gtrsim \left(\sum_{j \geq 1} |a_j|^q r^{jq} \right)^{\frac{1}{q}} \text{ for } 1 < p < 2,$$

and arguing as in the proof for the case $p = 2$ (with the same notations), we get, for $1 < p < 2$,

$$(M_p(f, r_l))^q \gtrsim \sum_{j \geq 1} |a_j|^q r_l^{jq} \gtrsim \frac{1}{1 - r_l} \text{ for all } 1 - \frac{1}{2^{N_l}} \leq r_l < 1 - \frac{1}{2^{N_l+1}}$$

and, for any $0 < r < 1$,

$$(M_p(f, r))^q \gtrsim \sum_{j \geq 1} |a_j|^q r^{jq} \gtrsim \frac{1}{(1 - r)^{1/C_2}}.$$

This gives the desired conclusion. \square

Now we deal with the case $p = 1$.

Theorem 2.2. *Let f be a log-frequently hypercyclic function in $H(\mathbb{D})$ for the Taylor shift. Then we have*

$$\liminf_{r \rightarrow 1} \left(\frac{M_1(r, f)}{-\log(1 - r)} \right) > 0.$$

Proof. Let us write $f(z) = \sum_{k \geq 0} a_k z^k$. We construct the sets $A(N)$, A and the sequences (n_k) and (N_l) as in the proof of Theorem 2.1. Hardy's inequality (see for instance [9]) ensures that, for all $0 < r < 1$,

$$M_1(r, f) \gtrsim \sum_{k \geq 1} \frac{|a_k|}{k} r^k.$$

Thus we can write, for all $0 < r < 1$,

$$M_1(r, f) \gtrsim \sum_{l=1}^{+\infty} \sum_{k \in I_{N_l}} \frac{|a_k|}{k} r^k.$$

Since for all $k \in I_{N_l}$, we have $|a_k| \geq 1$, we deduce

$$(13) \quad M_1(r, f) \gtrsim \sum_{l=1}^{+\infty} r^{2^{N_l+1}} \sum_{k \in I_{N_l}} \frac{1}{k} \gtrsim \sum_{l=1}^{+\infty} r^{2^{N_l+1}}.$$

Taking into account (6), we get, using integral comparison,

$$M_1(r, f) \gtrsim \sum_{l=1}^{+\infty} r^{2^{C_2 l+1}} \gtrsim -\log(1 - r).$$

This finishes the proof. \square

Remark 2.3. Notice that we obtain for $p = 1$ the same lower bound as in the case of frequent hypercyclicity (see Theorem 1.1). Thus this bound is necessarily optimal since a frequently hypercyclic function for T is log-frequently hypercyclic and Theorem 1.1 ensures the optimality.

3. OPTIMALITY OF THE GROWTH OF LOG-FREQUENTLY HYPERCYCLIC FUNCTIONS FOR THE TAYLOR SHIFT OPERATOR

3.1. Definitions and notations. Following the ideas of a construction originally presented in [8], we are going to obtain a family of log-frequently hypercyclic functions for the Taylor shift operator on $H(\mathbb{D})$, with the most moderate possible rate of explosion near the boundary. As in [8], we will need the so-called *flat polynomials* or Rudin-Shapiro polynomials. First, let us recall some important facts about these polynomials [21, 16].

Lemma 3.1. (1) For each $N \geq 1$, there is a trigonometric polynomial $p_N = \sum_{k=0}^{N-1} \varepsilon_{N,k} e^{ik\theta}$ where $\varepsilon_{N,k} = \pm 1$ for all $0 \leq k \leq N-1$ with at least half of the coefficients being $+1$ and with

$$\|p_N\|_p \leq 5\sqrt{N} \text{ for } p \in [2, +\infty].$$

(2) For each $N \geq 1$, there is a trigonometric polynomial $p_N^* = \sum_{k=0}^{N-1} a_{N,k} e^{ik\theta}$ where $|a_{N,k}| \leq 1$ for all $0 \leq k \leq N-1$ with at least $\lfloor \frac{N}{4} \rfloor$ coefficients being $+1$ and with

$$\|p_N^*\|_p \leq 3N^{1/q} \text{ for } p \in [1, 2].$$

In the sequel, we are going to build a log-frequently hypercyclic function for the Taylor shift operator. We will use elongation of sequences in order to repeat relatively frequently any given function that belongs to a dense set in a suitable way.

We write $2\mathbb{N}^* := \bigcup_{k \geq 1} \mathcal{A}_k$ where for any $k \geq 1$, $\mathcal{A}_k := \{2^k(2j-1); j \geq 1\}$. Let us consider

pairs (Q, l) with $Q := \sum_{k=0}^d b_k z^k$ a polynomial with rational coefficients and $l \in \mathbb{N}$ satisfying $\|Q\|_{\ell_1} := \sum_{k=0}^d |b_k| \leq l$ that we display as a single sequence $(q_k, l_k)_{k \geq 1}$ where (l_k) tends to infinity. Clearly (q_k) is a dense set in $H(\mathbb{D})$. For any $k \geq 1$, we denote by d_k the degree of q_k .

Let $a \in \mathbb{N}$. Set, for any $n \in \mathbb{N}$, $\lambda_n = a^{a^n}$. Let also $0 < \varepsilon < 1$ and $\delta = 1 - \varepsilon$. Let us choose $a \geq 4$ to have the inequality $\lambda_n^{1+\delta} < \lambda_{n+1}$. We set now $f_a := \sum_{j \geq 0} a_j z^j = \sum_{n \geq 1} P_n(z)$ where (P_n) is a sequence of polynomials defined as follows, using the polynomials of Lemma 3.1,

(1) for any $2 \leq p \leq +\infty$

$$P_n(z) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is in } \mathcal{A}_k \text{ and } \lambda_n^\delta < \alpha_k. \\ z^{\lambda_n} p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}(z^{\alpha_k}) q_k(z) & \text{otherwise,} \end{cases}$$

where $\alpha_k = 1 + \lfloor \max(l_k^2, d_k + 3l_k \log(1 + l_k)) \rfloor$;

(2) for any $1 < p \leq 2$

$$P_n(z) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is in } \mathcal{A}_k \text{ and } \lambda_n^\delta < \alpha_k. \\ z^{\lambda_n} p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}^*(z^{\alpha_k}) q_k(z) & \text{otherwise,} \end{cases}$$

where $\alpha_k = 1 + \lfloor \max(l_k^q, d_k + 3l_k \log(1 + l_k)) \rfloor$.

3.2. Rate of growth of the constructed function. It is easy to check that the function f_a belongs to $H(\mathbb{D})$. The following lemma gives an estimate of the supremum of the growth of f_a .

Lemma 3.2. Under the previous hypotheses, we have, for all $0 < r < 1$,

$$(1) \text{ for } 2 \leq p \leq +\infty, M_p(r, f_a) \lesssim \frac{1}{\sqrt{1-r}};$$

$$(2) \text{ for } 1 < p \leq 2, M_p(r, f_a) \lesssim \frac{1}{(1-r)^{1/q}}.$$

Proof. For any l large enough, let us consider $1 - \frac{1}{\lambda_l^\delta} \leq r \leq 1 - \frac{1}{\lambda_{l+1}^\delta}$. Observe that $\lambda_l^\delta \leq (1-r)^{-1}$.

- (1) Case $2 \leq p \leq +\infty$: From Minkowski's inequality, Lemma 3.1, the definition of polynomials P_n and the fact that $r \mapsto M_p(r, f_a)$ is a non decreasing function (see [9, Theorem 1.5]), we get,

$$\begin{aligned}
M_p(r, f_a) &\leq \sum_{n \geq 1} M_p(r, P_n) \\
&\leq \sum_{n \geq 1} \left(1 - \frac{1}{\lambda_{l+1}^\delta}\right)^{\lambda_n} \|p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}\|_p \|q_k\|_{\ell^1} \\
&\lesssim \sum_{n \geq 1} \left(1 - \frac{1}{\lambda_{l+1}^\delta}\right)^{\lambda_n} l_k \sqrt{\frac{\lambda_n^\delta}{\alpha_k}} \\
&\lesssim \sum_{n \leq l} \lambda_n^{\frac{\delta}{2}} + \sum_{n \geq l+1} \left(1 - \frac{1}{\lambda_{l+1}^\delta}\right)^{\lambda_n} \lambda_n^{\frac{\delta}{2}} \\
&\lesssim \lambda_l^{\frac{\delta}{2}} \left(1 + \sum_{n \leq l-1} \left(\frac{\lambda_n}{\lambda_l}\right)^{\frac{\delta}{2}}\right) + \sum_{n \geq l+1} e^{-\lambda_n^{1-\delta}} \lambda_n^{\frac{\delta}{2}} \\
&\lesssim \lambda_l^{\frac{\delta}{2}} \lesssim \frac{1}{\sqrt{1-r}}.
\end{aligned}$$

We obtain the desired estimate.

- (2) Case $1 < p \leq 2$: we use a similar analysis, now replacing the polynomial p_m by p_m^* again with $m = \lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor$. The computation takes a similar form to the previous calculation

$$\begin{aligned}
M_p(r, f_a) &\leq \sum_{n \geq 1} M_p(r, P_n) \\
&\leq \sum_{n \geq 1} r^{\lambda_n} \|p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}^*\|_p \|q_k\|_{\ell^1} \\
&\lesssim \sum_{n \geq 1} r^{\lambda_n} l_k \left(\frac{\lambda_n^\delta}{\alpha_k}\right)^{\frac{1}{q}} \\
&\lesssim \lambda_l^{\frac{\delta}{q}} \lesssim \left(\frac{1}{1-r}\right)^{\frac{1}{q}}.
\end{aligned}$$

This finishes the proof. □

In the same spirit we establish an estimate of the infimum of the growth of f_a .

Lemma 3.3. *Under the previous hypotheses, we have*

- (1) for $2 \leq p \leq +\infty$, $\liminf_{r \rightarrow 1} \left((1-r)^{1/(2a)} M_p(r, f_a)\right) < +\infty$;
(2) for $1 < p \leq 2$, $\liminf_{r \rightarrow 1} \left((1-r)^{1/(qa)} M_p(r, f_a)\right) < +\infty$.

Proof. For any l large enough, we set $r_l = 1 - \frac{1}{\lambda_l^\delta}$. We use the same strategy as in the proof of Lemma 3.2 to obtain the following estimates.

- (1) Case $2 \leq p \leq +\infty$:

$$\begin{aligned}
M_p(r_{l+1}, f_a) &\leq \sum_{n \geq 1} M_p(r_{l+1}, P_n) \\
&\leq \sum_{n \geq 1} (r_{l+1})^{\lambda_n} \|p_{\lfloor \frac{\lambda_n}{\alpha_k} \rfloor}\|_p \|q_k\|_{\ell^1} \\
&\lesssim \sum_{n \geq 1} (r_{l+1})^{\lambda_n} l_k \sqrt{\frac{\lambda_n^\delta}{\alpha_k}} \\
&\lesssim \sum_{n \leq l} \lambda_n^{\frac{\delta}{2}} + \sum_{n \geq l+1} \left(1 - \frac{1}{\lambda_{l+1}^\delta}\right)^{\lambda_n} \lambda_n^{\frac{\delta}{2}} \\
&\lesssim \lambda_l^{\frac{\delta}{2}} \left(1 + \sum_{n \leq l-1} \left(\frac{\lambda_n}{\lambda_l}\right)^{\frac{\delta}{2}}\right) + \sum_{n \geq l+1} e^{-\lambda_n^{1-\delta}} \lambda_n^{\frac{\delta}{2}} \\
&\lesssim \lambda_l^{\frac{\delta}{2}}.
\end{aligned}$$

Observe that $r_{l+1} = 1 - \lambda_l^{-a\delta}$. Hence, we obtain

$$\liminf_{r \rightarrow 1} \left((1-r)^{\frac{1}{2a}} M_p(r, f_a) \right) < +\infty.$$

(2) Case $1 < p \leq 2$:

$$\begin{aligned}
M_p(r_{l+1}, f_a) &\leq \sum_{n \geq 1} M_p(r_{l+1}, P_n) \\
&\leq \sum_{n \geq 1} (r_{l+1})^{\lambda_n} \|p_{\lfloor \frac{\lambda_n}{\alpha_k} \rfloor}^*\|_p \|q_k\|_{\ell^1} \\
&\lesssim \sum_{n \geq 1} (r_{l+1})^{\lambda_n} l_k \left(\frac{\lambda_n^\delta}{\alpha_k}\right)^{\frac{1}{q}} \\
&\lesssim \lambda_l^{\frac{\delta}{q}} \\
&\lesssim \left(\frac{1}{1-r_{l+1}}\right)^{\frac{1}{qa}}.
\end{aligned}$$

Hence, we obtain

$$\liminf_{r \rightarrow 1} \left((1-r)^{\frac{1}{qa}} M_p(r, f_a) \right) < +\infty.$$

The proof is complete. \square

3.3. Log-frequent hypercyclicity of the constructed function and conclusion. We are going to show that the functions f_a defined in the previous subsection are log-frequently hypercyclic functions for the Taylor shift operator.

Proposition 3.4. *Under the previous hypotheses, for all $a \in \mathbb{N}$ large enough, the function f_a is log-frequently hypercyclic for the Taylor shift operator on $H(\mathbb{D})$.*

Proof. The proof is divided into two parts.

(1) *Construction of a set with positive lower log-density:*

For k large enough, let us consider n in \mathcal{A}_k such that $\lambda_n^\delta \geq \alpha_k$. We denote by \mathcal{B}_n the set of

s in $[\lambda_n, \lambda_n^{1+\delta}]$ such that the coefficient of z^s in $z^{\lambda_n} p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}(z^{\alpha_k})$ (or in $z^{\lambda_n} p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}^*(z^{\alpha_k})$) is equal to 1. Then we define the set \mathcal{T}_k as follows

$$\mathcal{T}_k = \left\{ s : s \in \mathcal{B}_n \text{ for } n \in \mathcal{A}_k \text{ and } \lambda_n^\delta \geq \alpha_k \right\}.$$

By the choice of α_k and by the fact that the flat polynomials possess at least half (or $\lfloor \frac{\lambda_n^\delta}{4\alpha_k} \rfloor$) coefficients being 1, we have

$$\sum_{s \in \mathcal{B}_n} \frac{1}{s} \geq C_{k,\delta} \log(\lambda_n) \gtrsim \sum_{s \in [\lambda_n, \lambda_{n+1}]} \frac{1}{s}$$

where $C_{k,\delta}$ only depends on k and δ . Notice that if N is large enough so that $\lambda_n \leq N < \lambda_{n+1}$ for some positive integer n then we get, combining the previous estimate with the fact that the elements of \mathcal{A}_k are in arithmetic progression,

$$\sum_{s \in \mathcal{T}_k \cap \{1, \dots, N\}} \frac{1}{s} \gtrsim \sum_{l=1}^{n-1} \sum_{s \in \mathcal{B}_l} \frac{1}{s} \geq C_{k,\delta} \sum_{l=1}^{n-1} \log(\lambda_l) \gtrsim \log(N).$$

We finally obtain

$$\liminf_{N \rightarrow +\infty} \left(\frac{1}{\log(N)} \sum_{s \in \mathcal{T}_k \cap \{1, \dots, N\}} \frac{1}{s} \right) > 0.$$

(2) *The log-frequent hypercyclicity of f_a along a well-chosen sequence:*

Let k be a positive integer large enough to be fixed later. Let s be in \mathcal{B}_n for n in \mathcal{A}_k satisfying $\lambda_n^\delta \geq \alpha_k$. We are going to prove that the following inequality holds

$$\sup_{|z|=1-\frac{1}{l_k}} |T^s(f_a)(z) - q_k(z)| \lesssim \frac{1}{l_k},$$

provided that k is chosen large enough, the involved constant being uniform in s and k . It will allow us to conclude to the log-frequent hypercyclicity of f_a for T since the sequence (q_k) enumerates a dense set of $H(\mathbb{D})$ and since the sequence (l_k) tends to $+\infty$.

We write $s = \lambda_n + m\alpha_k$ for some m in $\left\{ 0, \dots, \left\lfloor \frac{\lambda_n^\delta}{\alpha_k} \right\rfloor \right\}$. With this notation, we have

$$z^{\lambda_n} p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}(z^{\alpha_k}) = \dots + \mathbf{1}z^s + \dots$$

and the same property remains valid for $z^{\lambda_n} p_{\lfloor \frac{\lambda_n^\delta}{\alpha_k} \rfloor}^*(z^{\alpha_k})$. By the choice of α_k , the next block of coefficients is dissociated from the present one because the condition $\alpha_k + d_k < 2\alpha_k$. So, we have that the first d_k Taylor coefficients of $T^s(f_a)$ are precisely those of q_k . Notice again by the choice of α_k that the Taylor coefficients a_j of $T^s(f_a)$ of index j satisfying $s + d_k + 1 \leq j \leq s + 2l_k \log(l_k)$ are null. Again, by the construction of each block P_n (even in the case p in $[1, 2]$) and since n is even, all the coefficients a_j in the range $[\lambda_{n+1}, \lambda_{n+2} - 1]$ are null. Indeed, the last index which is in \mathcal{B}_n is bounded from above by $\lambda_n^{1+\delta} + d_k < \lambda_{n+1}$ (by the choice of n such that $\lambda_n^\delta \geq \alpha_k \geq d_k$). Therefore, we can write

$$T^s(f_a) - q_k = \sum_{s+3l_k \log(l_k) \leq j \leq \lambda_{n+1}-1} a_j z^{j-s} + \sum_{j=\lambda_{n+2}}^{+\infty} a_j z^{j-s} := S_1(z) + S_2(z).$$

By construction of f_a , we observe in the range of index j lying in $[s+3l_k \log(l_k), \lambda_{n+1}-1] \cap \mathbb{N}$ that all the coefficients a_j coincide with the coefficients of q_k eventually signed (or multiplied by an unimodular coefficient in the case p in $[1, 2]$). Thus we obtain

$$\forall j \in [s+3l_k \log(l_k), \lambda_{n+1}-1] \cap \mathbb{N}, \quad |a_j| \leq \|q_k\|_{\ell^1} \leq l_k.$$

Then the triangle inequality and a classical inequality of convexity imply

$$\sup_{|z|=1-\frac{1}{l_k}} |S_1(z)| \leq l_k \sum_{j \geq s+3l_k \log(l_k)} \left(1 - \frac{1}{l_k}\right)^{j-s} \leq l_k \sum_{j \geq 3l_k \log(l_k)} e^{-\frac{j}{l_k}} \leq l_k^2 e^{-3 \log(l_k)} \leq \frac{1}{l_k}.$$

Using Lemma 3.2 and Cauchy estimates, we get

$$\forall j \geq 1, |a_j| \lesssim \sqrt{j}.$$

For a large enough, we have

$$\lambda_{n+1} \geq 2a\alpha_k \geq 2al_k \log(l_k) \text{ and } \{j-s; j \geq \lambda_{n+2}\} \subset \{j \geq \lambda_{n+1}\},$$

and we finally obtain again by the triangle inequality and an inequality of convexity

$$\begin{aligned} \sup_{|z|=1-\frac{1}{l_k}} |S_2(z)| &\lesssim \sum_{j \geq \lambda_{n+1}} \sqrt{j+s} \left(1 - \frac{1}{l_k}\right)^j \\ &\lesssim \sqrt{2} \sum_{j \geq \lambda_{n+1}} \sqrt{j} e^{-\frac{j}{l_k}} \\ &\lesssim \sqrt{2} \int_{2al_k \log(l_k)}^{+\infty} \sqrt{t} e^{-\frac{t}{l_k}} dt \\ &\lesssim l_k^{\frac{3}{2}} \sqrt{\log(l_k)} e^{-2a \log(l_k)} \\ &\lesssim \frac{1}{l_k}. \end{aligned}$$

Finally we obtain the desired result applying the triangle inequality again. Hence, since \mathcal{T}_k has a positive lower log-density, the vector f_a is log-frequently hypercyclic. \square

Combining Lemma 3.2 with Lemma 3.3 and Proposition 3.4, we have established the following result.

Theorem 3.5. *Let $1 < p \leq +\infty$. For all positive integer $a \in \mathbb{N}$ large enough, there exists a log-frequently hypercyclic function (for the Taylor shift operator) f_a in $H(\mathbb{D})$ satisfying*

$$M_p(r, f_a) \lesssim \frac{1}{(1-r)^{1/\max(2,q)}} \text{ and } \liminf_{r \rightarrow 1} \left((1-r)^{1/\max(2a,q)} M_p(r, f_a) \right) < +\infty.$$

Remark 3.6. Notice that the previous result combined with Theorem 1.1 guarantee that the functions f_a cannot be frequently hypercyclic.

4. FREQUENTLY AND LOG-FREQUENTLY HYPERCYCLIC FUNCTIONS FOR THE DIFFERENTIATION OPERATOR

In this section, let us consider the particular Taylor shift operator given by the differentiation operator D on $H(\mathbb{C})$:

$$D : H(\mathbb{C}) \rightarrow H(\mathbb{C}), \quad f \mapsto f'.$$

The operator D can be regarded as a particular weighted Taylor shift operator since we have, for $f = \sum_{k \geq 0} a_k z^k \in H(\mathbb{C})$, $Df = \sum_{k \geq 0} (k+1)a_{k+1} z^k$.

4.1. Lower bound for frequently hypercyclic functions for the differentiation operator.

It is well-known that the frequently hypercyclic functions f for D satisfy the following optimal estimate [6, 8]: for $p \in [1, \infty]$,

$$(14) \quad \limsup_{r \rightarrow \infty} \left(r^{\frac{1}{2\min(2,p)}} e^{-r} M_p(r, f) \right) > 0.$$

In the Remark 4.3 of [8], the authors observe that in general a frequently hypercyclic function f for the differentiation operator need not satisfy the estimate

$$\liminf_{r \rightarrow \infty} \left(e^{-r} r^{1/(2 \min(2,p))} M_p(r, f) \right) > 0.$$

First we want to complete this result by showing that such a function still has a minimal growth. We establish the following result.

Proposition 4.1. *Let f be a frequently hypercyclic function for the differentiation operator D . Then we have the following estimate: for all $1 \leq p \leq +\infty$, there exists $C > 1$ such that*

$$\liminf_{r \rightarrow \infty} \left(e^{-\frac{r}{C}} M_p(r, f) \right) > 0.$$

Proof. Let us write $f(z) = \sum_{k \geq 0} \frac{a_k}{k!} z^k$. Since f is D -frequently hypercyclic, there exists an increasing

sequence $(n_k) \subset \mathbb{N}$ with positive lower density satisfying $|f^{(n_k)}(0) - \frac{3}{2}| = |a_{n_k} - \frac{3}{2}| < 1/2$ for all $k \geq 1$. Hence the following inequalities hold

$$(15) \quad |a_{n_k}| \geq 1 \text{ and there exists } C_1 \geq 1 \text{ such that } k \leq n_k \leq C_1 k.$$

Case $p \geq 2$: we begin by considering the case $p \geq 2$. Since we have, for all $2 < p \leq \infty$, $M_2(r, f) \leq M_p(r, f)$, it suffices to prove the result for $p = 2$. Applying Parseval's Theorem and taking into account (15) we get, for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$(M_2(r, f))^2 = \sum_{k=0}^{+\infty} \frac{|a_k|^2}{k!^2} r^{2k} \geq \sum_{k=1}^{+\infty} \frac{|a_{n_k}|^2}{(n_k!)^2} r^{2n_k} \geq \sum_{k=1}^{+\infty} \frac{r^{2n_k}}{(n_k!)^2} \geq \frac{r^{2n_j}}{(n_j!)^2} \geq \frac{n_j^{2n_j}}{(n_j!)^2}.$$

By Stirling's formula, we obtain, for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$(M_2(r, f))^2 \gtrsim \frac{e^{2n_j}}{n_j} \geq \frac{e^{2n_j}}{r}.$$

Taking into account that the sequence (n_k) has a positive lower density, we easily derive from the second part of (15) that there exists $A > 1$ such that for all $j \geq 1$ we have $n_{j+1} \leq A n_j$. Thus we obtain, for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$(M_2(r, f))^2 \gtrsim \frac{e^{2n_j}}{r} \geq \frac{e^{\frac{2}{A} n_{j+1}}}{r} \geq e^{\frac{2}{A}} r^{-1}.$$

Hence we deduce that there exists $C \geq A$ such that

$$\liminf_{r \rightarrow \infty} \left(e^{-\frac{r}{C}} M_2(r, f) \right) > 0.$$

Case $1 < p < 2$: for the case $1 < p < 2$, from the Hausdorff-Young inequality (see [9]) and (15), we can write,

$$(M_p(r, f))^q \geq \sum_{k=0}^{+\infty} \frac{|a_k|^q}{k!^q} r^{qk} \geq \sum_{k=1}^{+\infty} \frac{|a_{n_k}|^q}{(n_k!)^q} r^{qn_k} \geq \sum_{k=1}^{+\infty} \frac{r^{qn_k}}{(n_k!)^q}.$$

By Stirling's formula, we obtain, for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$(M_p(r, f))^q \gtrsim \frac{e^{qn_j}}{n_j^{q/2}}.$$

We argue as in the case $p \geq 2$ to obtain for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$M_p(r, f) \gtrsim e^{\frac{r}{A}} r^{-\frac{1}{2}},$$

which gives that there exists $C \geq A$ such that $\liminf_{r \rightarrow \infty} \left(e^{-\frac{r}{C}} M_p(r, f) \right) > 0$.

Case $p = 1$: Finally let us consider the case $p = 1$. From Hardy's inequality (see [9]) and (15), we get for all $r > 1$

$$M_1(r, f) \gtrsim \sum_{k=1}^{+\infty} \frac{|a_k|}{k(k!)} r^k \geq \sum_{k=1}^{+\infty} \frac{|a_{n_k}|}{n_k(n_k!)} r^{n_k} \geq \sum_{k=1}^{+\infty} \frac{r^{n_k}}{n_k(n_k!)}.$$

Again by Stirling's formula, we get

$$M_1(r, f) \gtrsim \sum_{k=1}^{+\infty} \frac{r^{n_k} e^{n_k}}{\sqrt{n_k} n_k^{n_k+1}}.$$

Thus, we obtain we, for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$M_1(r, f) \gtrsim \frac{n_j^{n_j} e^{n_j}}{\sqrt{n_j} n_j^{n_j+1}} = \frac{e^{n_j}}{\sqrt{n_j} n_j}.$$

We argue as in the case $p \geq 2$ to obtain for all $j \geq 1$ and for all $n_j \leq r < n_{j+1}$,

$$M_1(r, f) \gtrsim e^{\frac{r}{A}} r^{-3/2},$$

which gives there exists $C \geq A$ such that $\liminf_{r \rightarrow \infty} \left(e^{-\frac{r}{C}} M_1(r, f) \right) > 0$.

The proof is complete. \square

Now we deal with the optimality of Proposition 4.1. We keep the notations adopted in [8]. For any given polynomial q with Taylor series $q(z) = \sum_{j=0}^d \frac{q_j}{j!} z^j$ with $q_d \neq 0$, we set $\tilde{q}(z) := \sum_{j=0}^d q_j z^j$,

$d = \deg(q)$ and $\|\tilde{q}\|_{\ell^1} := \sum_{j=0}^d |q_j|$. As in section 3, we set $2\mathbb{N} = \bigcup_{k \geq 1} \mathcal{A}_k$ where for any $k \geq 1$,

$\mathcal{A}_k = \{2^k(2j-1); j \in \mathbb{N}\}$. Denote by \mathcal{P} the countable set of polynomials with rational coefficients and let us also consider pairs (q, l) with $q \in \mathcal{P}$ and $l \in \mathbb{N}$ satisfying $\|\tilde{q}\|_{\ell^1} \leq l$ displayed as a single sequence $(q_k, l_k)_{k \geq 1}$ where (l_k) tends to infinity. Clearly (q_k) is a dense set in $H(\mathbb{C})$. Hence, for any $k \geq 1$, we set $d_k = \deg(q_k)$ and we have

$$\|\tilde{q}_k\|_{\ell^1} \leq l_k \text{ for every } k \geq 1.$$

Let $p \in [1, \infty]$ be given. For $k \geq 1$, let us define the integers

$$\alpha_k := 1 + \lfloor \max(l_k^2, 2d_k + 8l_k) \rfloor$$

Let $a \geq 10$ be a positive integer. We set

$$f_a := \sum_{j=1}^{+\infty} \frac{a_j}{j!} z^j := \sum_{n \geq 0} P_{n,a}$$

where the blocks

$$P_{n,a} = \sum_{k=a^n}^{a^{n+1}-1} \frac{a_k}{k!} z^k$$

are polynomials defined thanks to $\tilde{P}_{n,a}$ as follows, using Rudin-Shapiro polynomials given by Lemma 3.1,

$$(16) \quad \tilde{P}_{n,a}(z) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \in \mathcal{A}_k \text{ and } a^n < 10\alpha_k \\ z^{a^n} p_{\lfloor a^n/\alpha_k \rfloor}(z^{\alpha_k}) \tilde{q}_k(z) & \text{if } n \in \mathcal{A}_k \text{ and } a^n \geq 10\alpha_k \end{cases}$$

In the sequel we will consider the sequence (r_n) defined as follows

$$r_n = \frac{a^{n+1}}{3}.$$

Throughout this section, we will often use the following fact:

$$(17) \quad (r_n^k/k!) \text{ is increasing for } k \leq a^n \text{ and is decreasing for } k \geq a^{n+1}.$$

For our purpose, we need the following lemma.

Lemma 4.2. *Let $l, n \geq 1$ be positive integers. Under the previous assumptions, we consider the polynomials $g_l(\theta) = \sum_{k=a^l}^{3a^l-1} \frac{b_k}{k!} r_n^k e^{ik\theta}$. Then we have*

$$\|g_l\|_2 \leq \frac{(r_n)^{3a^l}}{(3a^l)!} \|g_l^*\|_2 \quad \text{for } 1 \leq l \leq n, \quad \text{and} \quad \|g_l\|_2 \leq \frac{(r_n)^{a^l}}{a^l!} \|g_l^*\|_2 \quad \text{for } l \geq n+1,$$

with $g_l^*(\theta) = \sum_{k=a^l}^{3a^l-1} b_k e^{ik\theta}$. More generally, we have the following inequality, for all $p \in [1, \infty]$, $p \neq 2$,

$$\|g_l\|_p \leq \sqrt{3a^l} \frac{(r_n)^{3a^l}}{(3a^l)!} \|g_l^*\|_2 \quad \text{for } 1 \leq l \leq n, \quad \text{and} \quad \|g_l\|_p \leq \sqrt{3a^l} \frac{(r_n)^{a^l}}{a^l!} \|g_l^*\|_2 \quad \text{for } l \geq n+1.$$

Proof. By Parseval's theorem, we have

$$\|g_l\|_2^2 = \sum_{k=a^l}^{3a^l-1} \frac{|b_k|^2}{k!^2} r_n^{2k}.$$

Using the fact (17) and again Parseval's theorem, we get

$$\|g_l\|_2^2 \leq \left(\frac{(r_n)^{3a^l}}{(3a^l)!} \right)^2 \sum_{k=a^l}^{3a^l-1} |b_k|^2 = \left(\frac{(r_n)^{3a^l}}{(3a^l)!} \right)^2 \|g_l^*\|_2^2 \quad \text{for } 1 \leq l \leq n,$$

and

$$\|g_l\|_2^2 \leq \left(\frac{(r_n)^{a^l}}{(a^l)!} \right)^2 \sum_{k=a^l}^{3a^l-1} |b_k|^2 = \left(\frac{(r_n)^{a^l}}{(a^l)!} \right)^2 \|g_l^*\|_2^2 \quad \text{for } l \geq n+1.$$

The result follows for $p = 2$. To conclude, observe that we have the following obvious inequalities: for all $p \geq 1$, for all polynomial P

$$\|P\|_p \leq \|P\|_\infty \leq \sqrt{\deg(P) + 1} \|P\|_2.$$

□

We are ready to estimate the entire function f_a along the radii given by the sequence (r_n) in terms of average L^p -norms with $1 \leq p \leq \infty$.

Proposition 4.3. *Under the previous assumptions, for all $1 \leq p \leq \infty$, there exists C_a , with $C_a \rightarrow +\infty$ as a tends to infinity, such that, for all n large enough,*

$$M_p(r_n, f_a) \lesssim e^{r_n/C_a}.$$

Proof. We have $f = \sum_{l \geq 1} P_{l,a} = \sum_{l=1}^{n-1} P_{l,a} + P_{n,a} + \sum_{l \geq n+1} P_{l,a}$. Thanks to the choice of α_k , we have

$$2a^l + d_k < 3a^l \text{ whenever } P_{l,a} \neq 0. \text{ Hence we can write } P_{l,a}(z) = \sum_{j=a^l}^{3a^l-1} \frac{a_j}{j!} z^j.$$

First we deal with the case $p = 2$. Using the polynomials given by (16) we apply Lemma 4.2 and Lemma 3.1 to obtain,

$$(18) \quad M_2(P_{l,a}, r_n) \lesssim \frac{(r_n)^{3a^l}}{(3a^l)!} \|p_{\lfloor a^l/\alpha_k \rfloor} \tilde{q}_k\|_2 \lesssim \frac{(r_n)^{3a^l}}{(3a^l)!} \sqrt{\frac{a^l}{\alpha_k}} l_k \quad \text{for } 1 \leq l \leq n$$

and

$$(19) \quad M_2(P_{l,a}, r_n) \lesssim \frac{(r_n)^{a^l}}{a^{l!}} \|p_{\lfloor a^l/\alpha_k \rfloor} \tilde{q}_k\|_2 \lesssim \frac{(r_n)^{a^l}}{a^{l!}} \sqrt{\frac{a^l}{\alpha_k}} l_k \quad \text{for } l \geq n+1.$$

Clearly the triangle inequality gives

$$(20) \quad M_2(f_a, r_n) \lesssim \sum_{l=1}^{n-1} M_2(P_{l,a}, r_n) + M_2(P_{n,a}, r_n) + \sum_{l \geq n+1} M_2(P_{l,a}, r_n).$$

First we have, using $\alpha_k > l_k^2$, (18), Stirling's formula and the fact that, for $1 \leq l \leq n-1$, $3a^l(n+1-l) \leq 3a^{l+1}(n+1-l-1)$,

$$(21) \quad \begin{aligned} \sum_{l=1}^{n-1} M_2(P_{l,a}, r_n) &\lesssim \sum_{l=1}^{n-1} a^{l/2} \frac{(r_n)^{3a^l}}{(3a^l)!} \lesssim \sum_{l=1}^{n-1} (a^{n+1-l})^{3a^l} \left(\frac{e}{9}\right)^{3a^l} \\ &\lesssim a^{6a^{n-1}} \sum_{l=1}^{n-1} \left(\frac{e}{9}\right)^{3a^l} \\ &\lesssim e^{r_n \frac{18 \log(a)}{a^2}}. \end{aligned}$$

Then, we deal with the second term of the sum given by (20). In the same spirit, we get

$$(22) \quad \begin{aligned} M_2(P_{n,a}, r_n) &\lesssim a^{n/2} \frac{(r_n)^{3a^n}}{(3a^n)!} \lesssim \left(\frac{a}{9}\right)^{3a^n} e^{3a^n} \\ &\lesssim e^{3a^n(1+\log(a)-\log(9))} \\ &\lesssim e^{r_n \frac{9(1+\log(a)-\log(9))}{a}}. \end{aligned}$$

Since we have $a \geq 10$, we deduce $9(1+\log(a)-\log(9)) < a$. Finally, we give the estimate of the last term of the sum (20). From (19), Stirling's formula and the inequality $\alpha_k > l_k^2$, we derive

$$(23) \quad \begin{aligned} \sum_{l \geq n+1} M_2(P_{l,a}, r_n) &\lesssim \sum_{l \geq n+1} a^{l/2} \frac{(r_n)^{a^l}}{(a^l)!} \lesssim \sum_{l \geq n+1} (a^{n+1-l})^{a^l} \left(\frac{e}{3}\right)^{a^l} \\ &\lesssim \left(\frac{e}{3}\right)^{a^{n+1}} + \sum_{l \geq n+2} \left(\frac{e}{3a}\right)^{a^l} \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned}$$

Thus the estimates (21), (22) and (23) give the announced result for $p = 2$.

For $p \in [1, \infty]$, $p \neq 2$, again thanks to Lemma 4.2 and Lemma 3.1, we can write

$$(24) \quad M_p(P_{l,a}, r_n) \lesssim \begin{cases} a^{l/2} \frac{(r_n)^{3a^l}}{(3a^l)!} \|p_{\lfloor a^l/\alpha_k \rfloor} \tilde{q}_k\|_2 \lesssim \frac{(r_n)^{3a^l}}{(3a^l)!} a^l & \text{for } 1 \leq l \leq n \\ a^{l/2} \frac{(r_n)^{a^l}}{(a^l)!} \|p_{\lfloor a^l/\alpha_k \rfloor} \tilde{q}_k\|_2 \lesssim \frac{(r_n)^{a^l}}{(a^l)!} a^l & \text{for } l \geq n+1. \end{cases}$$

Since we have $\sum_{k=1}^{+\infty} \sqrt{k} \left(\frac{e}{9}\right)^{3k} < +\infty$ and $a^{(n+1)/2} \left(\frac{e}{3}\right)^{3a^n} \rightarrow 0$ as $n \rightarrow +\infty$, we derive as in (21), (22) and (23),

$$(25) \quad \sum_{l=1}^{n-1} M_p(P_{l,a}, r_n) \lesssim a^{6a^{n-1}} \sum_{l=1}^{n-1} a^{l/2} \left(\frac{e}{9}\right)^{3a^l} \lesssim e^{r_n \frac{18 \log(a)}{a^2}},$$

$$(26) \quad M_p(P_{n,a}, r_n) \lesssim a^{n/2} \left(\frac{a}{9}\right)^{3a^n} e^{3a^n} \lesssim \sqrt{r_n} e^{3a^n(1+\log(a)-\log(9))} \lesssim e^{r_n \frac{(9+\varepsilon)(1+\log(a)-\log(9))}{a}}$$

with $0 < \varepsilon < 1$ such that, for $a \geq 10$, $(9+\varepsilon)(1+\log(a)-\log(9)) < a$, and

$$(27) \quad \sum_{l \geq n+1} M_p(P_{l,a}, r_n) \lesssim a^{(n+1)/2} \left(\frac{e}{3}\right)^{a^{n+1}} + \sum_{l \geq n+2} a^{l/2} \left(\frac{e}{3a}\right)^{a^l} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence the estimates (25), (26) and (27) give the result. \square

Now we are ready to establish the main result of this subsection that shows the optimality of Proposition 4.1.

Theorem 4.4. *Let $1 \leq p \leq \infty$. For all positive integer $a \in \mathbb{N}$ large enough, there exists a frequently hypercyclic function f_a for the differentiation operator and a real number $C_a > 1$, with $C_a \rightarrow +\infty$ as $a \rightarrow +\infty$, such that*

$$\liminf_{r \rightarrow \infty} \left(e^{-r/C_a} M_p(r, f_a) \right) < +\infty.$$

Proof. To prove this result, it suffices to check that the function f_a built in the proof of Proposition 4.3 is frequently hypercyclic. This argument will be independent of $p \in [1, \infty]$. Thus we keep the definitions and notations of this construction. Now the proof follows that given by [8, Section 3.1] with easy modifications. For any fixed even integer $n \in \mathcal{A}_k$ with $a^n \geq \alpha_k$, denote by \mathcal{B}_n the set of indices s such that the coefficient of z^s in the polynomial $z^{a^n} p_{\lfloor a^n/\alpha_k \rfloor}(z^{\alpha_k})$ is 1. We are going to prove that

$$(28) \quad \sup_{|z|=l_k} |(D^s f_a)(z) - q_k(z)| \lesssim \frac{1}{l_k} \quad \text{for any } s \in \mathcal{B}_n, n \in \mathcal{A}_k.$$

Indeed this estimate will allow us to conclude the claim since \mathcal{P} is dense in $H(\mathbb{C})$ and for all $k \geq 1$ the set $\{s : s \in \mathcal{B}_n, n \in \mathcal{A}_k, a^n \geq \alpha_k\}$ has positive lower density. This last argument follows from the inequalities

$$\#\mathcal{B}_n \gtrsim \frac{a^n}{\alpha_k} \gtrsim \frac{1}{(a-1)\alpha_k} \#\{a^n, a^n+1, \dots, a^{n+1}-1\}$$

and the fact that \mathcal{A}_k contains an arithmetic progression of integers.

Finally we notice that the coefficients a_j of f_a satisfy for all $j \geq 1$ the estimate $|a_j| \leq j$ since we have, for $a^n \leq j \leq a^{n+1}-1$, with $n \in \mathcal{A}_k$, if $a_j \neq 0$,

$$|a_j| \leq \|\tilde{q}_k\| \leq l_k < \alpha_k \leq \frac{a^n}{10} < a^n \leq j,$$

as a consequence of the choice of l_k, α_k and the fact that the absolute values of the coefficients of Rudin-Shapiro polynomials do not exceed 1.

Now to obtain (28) it suffices to repeat with obvious modifications the proof page 3685 in [8] replacing $n^2, (n+1)^2, (n+2)^2$ by a^n, a^{n+1}, a^{n+2} respectively. We will have with the notations of [8]

$$(D^s f_a)(z) - q_k(z) = \sum_{j=s+8l_k}^{a^{n+1}} \frac{a_j}{(j-s)!} z^{j-s} + \sum_{j \geq a^{n+2}} \frac{a_j}{(j-s)!} z^{j-s} =: S_1(z) + S_2(z),$$

with $s = a^n + m\alpha_k$, where $n \in \mathcal{A}_k$, n being even, $a^n \geq \alpha_k$, $p_{\lfloor a^n/\alpha_k \rfloor}(z) = \dots + 1 \cdot z^m + \dots$ and $n+1$ being odd, $a_j = 0$ for $j = a^{n+1}, \dots, a^{n+2}-1$. For $a^n \leq j \leq a^{n+1}-1$, we have $|a_j| \leq \|\tilde{q}_k\| \leq l_k$ and we deduce

$$(29) \quad \sup_{|z|=l_k} |S_1(z)| \leq l_k \sum_{m \geq 8l_k} \frac{l_k^m}{m!} < \frac{1}{2l_k}.$$

To estimate the last term S_2 , we write $j = s + m$ with $m \geq a^{n+2} - a^{n+1} = a^{n+1}(a-1)$. We get $|a_j| = |a_{s+m}| \leq s + m \leq a^{n+1} + m \leq 2m$ and we obtain

$$(30) \quad \sup_{|z|=l_k} |S_2(z)| \leq \sum_{m \geq a^{n+1}(a-1)} \frac{2m}{m!} l_k^m \leq 2l_k \sum_{m \geq a^{n+1}(a-1)-1} \frac{l_k^m}{m!} \leq 2l_k \sum_{m \geq 8l_k} \frac{l_k^m}{m!} \leq \frac{1}{2l_k},$$

using $a^n \geq \alpha_k \geq 8l_k$.

Combining (29) with (30), we obtain (28) and the proof is complete. \square

Therefore the combination of Proposition 4.1 with Theorem 4.4 gives Theorem 1.3 stated in the introduction.

4.2. Growth of log-frequently hypercyclic functions for the differentiation operator.

Finally we conclude the paper by studying the growth of log-frequently hypercyclic functions for D . In fact, we are going to prove that these functions share the same upper estimate of the boundary behavior given by (14) as the frequently hypercyclic functions. We will also establish a bound for the lower limit of such functions. First of all we need the following lemma.

Lemma 4.5. *Let (n_k) be an increasing sequence of positive integers such that $\underline{d}_{\log}((n_k)) > 0$. Then there exists $C > 1$ such that, for all $k \geq 1$, $n_k \leq k^C$.*

Proof. According to Lemma 2.7 of [10], we have

$$(31) \quad \underline{d}_{\log}((n_k)) = \liminf_{k \rightarrow \infty} \left(\frac{1}{\log(n_k)} \sum_{i=1}^k \frac{1}{n_i} \right).$$

We can write, for all $k \geq 1$,

$$(32) \quad \frac{1}{\log(n_k)} \sum_{i=1}^k \frac{1}{n_i} \leq \frac{1}{\log(n_k)} \sum_{i=1}^k \frac{1}{i} = \frac{\log(k)}{\log(n_k)} \left(\frac{1}{\log(k)} \sum_{i=1}^k \frac{1}{i} \right).$$

Since we have $\underline{d}_{\log}((n_k)) > 0$, we get from (31) and (32) that there exists $C > 1$ such that, for all $k \geq 1$,

$$\log(n_k) \leq C \log(k).$$

Hence we deduce that there exists $C > 1$ such that, for all $k \geq 1$,

$$n_k \leq k^C.$$

This finishes the proof. □

Now we are ready to prove the following result.

Theorem 4.6. *Let f be an entire log-frequently hypercyclic function for D . Then we have the following estimates:*

(1) for all $1 \leq p \leq \infty$,

$$\limsup_{r \rightarrow \infty} \left(e^{-r} r^{\frac{1}{2 \min(2,p)}} M_p(r, f) \right) > 0;$$

(2) for all $1 \leq p \leq \infty$, there exists $C > 1$ such that

$$\liminf_{r \rightarrow \infty} \left(e^{-r^{1/C}} M_p(r, f) \right) > 0.$$

Moreover these estimates are optimal in the following sense:

- (i) for $p \in (1, \infty]$, there exists a log-frequently hypercyclic function f satisfying $M_p(r, f) \lesssim r^{-1/(2 \min(2,p))} e^r$ and for $p = 1$, given $\phi(r) \uparrow \infty$, there exists a log-frequently hypercyclic function f satisfying $M_1(r, f) \leq \phi(r) r^{-1/2} e^r$;
- (ii) for $1 \leq p \leq \infty$ and for all positive integer $a \in \mathbb{N}$ large enough, there exists a log-frequently hypercyclic function f_a and a real number C_a , with $C_a \rightarrow +\infty$ as $a \rightarrow +\infty$, such that

$$\liminf_{r \rightarrow \infty} \left(e^{-r^{1/C_a}} M_p(r, f_a) \right) < +\infty.$$

Proof. We proceed in the order given by the statement.

Proof of (1): for $p = 1$, the result is clear since there is no hypercyclic function f for D that satisfies $\limsup_{r \rightarrow \infty} (e^{-r} \sqrt{r} M_1(r, f)) < +\infty$ [6]. Moreover, since

$$M_2(f, r) \leq M_p(f, r) \text{ for } 2 < p \leq \infty,$$

it suffices to prove the result for $1 < p \leq 2$.

Let $1 < p \leq 2$. Let us write $f(z) = \sum_{k \geq 0} \frac{a_k}{k!} z^k$. Since f is log-frequently hypercyclic for D , there

exists a subsequence $(n_k) \subset \mathbb{N}$ with positive lower logarithmic density satisfying $|f^{(n_k)}(0) - \frac{3}{2}| =$

$|a_{n_k} - \frac{3}{2}| < 1/2$ for all $k \geq 1$. Hence for all $k \geq 1$ we have $|a_{n_k}| \geq 1$. Since the sequence (n_k) has positive lower logarithmic density, there exists $C_1 > 0$ such that for all n large enough

$$(33) \quad \sum_{1 \leq n_j \leq 2^n} \frac{1}{n_j} \geq C_1 n \log(2).$$

As in the beginning of the proof of Theorem 2.1 we define the sets $I = \{n_k, k \geq 1\}$, $I_n = \{2^n, \dots, 2^{n+1} - 1\} \cap I$, $A(N) = \left\{ n \in \{1, \dots, N\}; \sum_{k \in I_n} \frac{1}{k} \geq \frac{C_1}{2} \log(2) \right\}$ and $A = \bigcup_{N \geq 1} A(N)$. The

arguments of the proof of Theorem 2.1 ensure that the set A has a positive lower natural density. Hence the set A is infinite. Let (N_l) be an increasing sequence of positive integers such that $A = (N_l)$. Now, inspired by the proof of Theorem 2.4 of [6], assume that there exists a decreasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\varphi(r) \rightarrow 0$ as $r \rightarrow \infty$, such that, for $r > 0$ sufficiently large, we have

$$M_p(r, f) \leq \varphi(r) e^r r^{-\frac{1}{2p}}.$$

From the Hausdorff-Young inequality (see [9]), we get

$$\left(\sum_{j=0}^{+\infty} \frac{|a_j|^q}{j!^q} r^{jq} \right)^{\frac{1}{q}} \leq M_p(f, r) \leq \varphi(r) e^r r^{-\frac{1}{2p}},$$

for $r > 0$ large enough. Therefore we get, for $r > 0$ large enough,

$$(34) \quad \sum_{j=0}^{+\infty} |a_j|^q \frac{r^{jq+q/(2p)} e^{-qr}}{j!^q (\varphi(r))^q} \leq 1.$$

We repeat the arguments given in the proof of Theorem 2.4 of [6]: the function $r \mapsto g_j(r) = \frac{r^{jq+q/(2p)} e^{-qr}}{j!^q}$ has its maximum at $c_j := j + 1/(2p)$ with $g_j(c_j) \sim j^{-1/2}$ and an inflection point at $d_j := c_j + \sqrt{j/q + j/(2pq)}$. Thus on $K_j := [c_j, d_j]$, g_j dominates the linear function h_j defined by $h_j(c_j) = g_j(c_j)$ and $h_j(d_j) = 0$. Now for m large enough and all $2^m \leq j < 2^{m+1}$, we have $K_j \subset [2^m, 3 \cdot 2^m]$: to see this, observe that $pq \geq 4$ and we have the following inequalities:

$$c_j \leq 2^{m+1} + \frac{1}{2} \text{ and } d_j \leq 2^{m+1} + \frac{1}{2} + \sqrt{2^{m+1} + \frac{2^{m+1}}{2pq}} \leq 2^{m+1} + \frac{1}{2} + \sqrt{2^{m+1} + 2^{m-2}}.$$

Therefore we get, for all $2^m \leq j < 2^{m+1}$,

$$\int_{2^m}^{3 \cdot 2^m} \frac{g_j(r)}{(\varphi(r))^q} dr \geq \int_{c_j}^{d_j} \frac{h_j(r)}{(\varphi(r))^q} dr \gtrsim \frac{1}{(\varphi(2^m))^q}.$$

Finally integrating (34) over $[2^{N_l}, 3 \cdot 2^{N_l}]$, we get, for l large enough,

$$\sum_{j \in I_{N_l}} |a_j|^q \leq \sum_{j=2^{N_l}}^{2^{N_l+1}-1} |a_j|^q \lesssim (3 \cdot 2^{N_l} - 2^{N_l}) \varphi(2^{N_l})^q = 2^{N_l+1} \varphi(2^{N_l})^q.$$

We deduce

$$\sum_{j \in I_{N_l}} \frac{1}{j} \leq \frac{1}{2^{N_l}} \sum_{j \in I_{N_l}} |a_j|^q \lesssim \varphi(2^{N_l})^q.$$

Since $\varphi(2^{N_l}) \rightarrow 0$ as l tends to infinity, we obtain a contradiction with the definition of the sets I_{N_l} . Therefore there doesn't exist a decreasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\varphi(r) \rightarrow 0$ as r tends to infinity, satisfying $M_p(r, f) \leq \varphi(r) e^r r^{-\frac{1}{2p}}$ for all $r > 0$ sufficiently large. This implies $\limsup_{r \rightarrow \infty} \left(e^{-r} r^{\frac{1}{2p}} M_p(r, f) \right) > 0$.

Proof of (2): again using the log-frequent hypercyclicity of $\sum_{k \geq 0} \frac{a_k}{k!} z^k$, there exists an increasing sequence $(n_k) \subset \mathbb{N}$ of positive lower logarithmic density such that for all $k \geq 1$ one has $|a_{n_k}| \geq 1$. According to Lemma 4.5 there exists $C > 1$ such that, for all $k \geq 1$, $n_k \leq k^C$. Thus, one can find

$A \geq C$ satisfying, for all $k \geq 1$ $n_{k+1} \leq (k+1)^C \leq (1+n_k)^C \leq n_k^A$. Now it suffices to proceed along the same lines as in the proof of Proposition 4.1 replacing the inequality $n_{j+1} \leq An_j$ by the inequality $n_{j+1} \leq n_j^A$.

Proof of the optimality:

- (i) On one hand, since a frequently hypercyclic function for the differentiation operator is necessarily a log-frequently hypercyclic function, Theorem 1.1 of [8] ensures that the assertion (1) is optimal.
- (ii) On the other hand, to obtain the optimality for the lower bounds, it suffices to adopt the same strategy as in subsection 4.1. We keep the notations of subsection 4.1. Let a be a positive integer with $a \geq 10$. We consider α_k and r_n such that

$$\alpha_k = 1 + \lfloor \max(2d_k + 8l_k, l_k^2) \rfloor \quad \text{and} \quad r_n = a^{\frac{a^{n+1}}{3}}.$$

We replace the blocks of the function f_a by

$$(35) \quad \tilde{P}_{n,a}(z) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \in \mathcal{A}_k \text{ and } a^{a^n/2} < 10\alpha_k \\ z^{a^{a^n}} p_{\lfloor a^{a^n/2}/\alpha_k \rfloor}(z^{\alpha_k}) \tilde{q}_k(z) & \text{if } n \in \mathcal{A}_k \text{ and } a^{a^n/2} \geq 10\alpha_k \end{cases}$$

With these conditions, we have, whenever $P_{n,a} \neq 0$,

$$a^{a^n} + a^{a^n/2} + d_k \leq 2a^{a^n} + d_k \leq 3a^{a^n} < r_n < a^{a^{n+1}}.$$

By considering the polynomials $g_l(\theta) = \sum_{k=a^{a^l}}^{3a^{a^l}-1} \frac{b_k}{k!} r_n^k e^{ik\theta}$ with $l, n \geq 1$ being positive integers, then the result of Lemma 4.2 becomes,

$$\|g_l\|_2 \leq \frac{(r_n)^{3a^{a^l}}}{(3a^{a^l})!} \|g_l^*\|_2 \quad \text{for } 1 \leq l \leq n, \quad \text{and} \quad \|g_l\|_2 \leq \frac{(r_n)^{a^{a^l}}}{a^{a^l}!} \|g_l^*\|_2 \quad \text{for } l \geq n+1,$$

with $g_l^*(\theta) = \sum_{k=a^{a^l}}^{3a^{a^l}-1} b_k e^{ik\theta}$. With these estimates, we can use the same strategy as that developed in the proof of Propositions 4.3 to obtain, for all $1 \leq p \leq \infty$, the existence of real numbers C_a , with $C_a \rightarrow +\infty$ as a tends to infinity, such that, for all n large enough,

$$M_p(r_n, f_a) \lesssim e^{r_n^{1/C_a}}.$$

For $p = 2$, we sketch the proof. For the other cases the proof is similar. First we have, using $\alpha_k > d_k^2$, (18), Stirling's formula and the fact that, for $1 \leq l \leq n-1$, $a^{a^l}(a^{n+1} - 3a^{a^l}) \leq a^{a^{l+1}}(a^{n+1} - 3a^{a^{l+1}})$,

$$(36) \quad \begin{aligned} \sum_{l=1}^{n-1} M_2(P_{l,a}, r_n) &\lesssim \sum_{l=1}^{n-1} a^{a^l/4} \frac{(r_n)^{3a^{a^l}}}{(3a^{a^l})!} \lesssim \sum_{l=1}^{n-1} a^{-a^l/4} a^{a^{a^l}(a^{n+1}-3a^{a^l})} \left(\frac{e}{3}\right)^{3a^{a^l}} \\ &\lesssim a^{a^{a^{n-1}}(a^{n+1}-3a^{a^{n-1}})} \sum_{l=1}^{n-1} \left(\frac{e}{3}\right)^{3a^{a^l}} \\ &\lesssim e^{r_n^{3/a^2} 3 \log(r_n)} \lesssim e^{r_n^{4/a^2}}. \end{aligned}$$

In the same spirit, we get

$$(37) \quad \begin{aligned} M_2(P_{n,a}, r_n) &\lesssim a^{a^n/2} \frac{(r_n)^{3a^{a^n}}}{(3a^{a^n})!} \lesssim a^{-a^n/4} a^{a^{a^n}(a^{n+1}-3a^{a^n})} \left(\frac{e}{3}\right)^{3a^{a^n}} \\ &\lesssim e^{r_n^{3/a} 3 \log(r_n)} \\ &\lesssim e^{r_n^{4/a}}. \end{aligned}$$

Finally, we have

$$\begin{aligned}
 \sum_{l \geq n+1} M_2(P_{l,a}, r_n) &\lesssim \sum_{l \geq n+1} a^{a^l/4} \frac{(r_n)^{a^{a^l}}}{(a^{a^l})!} \\
 (38) \qquad \qquad \qquad &\lesssim \sum_{l \geq n+1} a^{-a^l/4} a^{a^{a^l}} (a^{n+1}/3 - a^l) e^{a^{a^l}} \\
 &\lesssim a^{-a^{n+1}/4} \left(\frac{e}{a}\right)^{a^{a^{n+1}}} + \sum_{l \geq n+2} \left(\frac{e}{a}\right)^{a^{a^l}} \longrightarrow 0 \text{ as } n \rightarrow +\infty.
 \end{aligned}$$

The inequalities (36), (37) and (38) give the announced result for $p = 2$.

To conclude inspired by the proofs of Proposition 3.4 and Theorem 4.4, it is easy to check that the constructed functions f_a are log-frequently hypercyclic for the differentiation operator. More precisely, we combine the construction of a set with positive lower logarithmic density of the proof of Proposition 3.4 with the second part of the proof of Theorem 4.4 to obtain the log-frequent hypercyclicity.

The proof is complete. \square

Remark 4.7. Notice that Theorem 1.3 ensures that the log-frequently hypercyclic functions f_a of the proof of Theorem 4.6 cannot be frequently hypercyclic.

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