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ALEA, Lat. Am. J. Probab. Math. Stat. **22**, 93–129 (2025)
DOI: 10.30757/ALEA.v22-03

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Abstract. This article is devoted to the study of products of random operators of the form $M_{0,n} = M_0 \cdots M_{n-1}$, where $(M_n)_{n\geqslant 0}$ is an ergodic sequence of positive operators acting on the space of signed measures on some set $\mathbb{X}$. Under suitable conditions, in particular, a Doeblin-type minoration suited for non conservative operators, we obtain asymptotic results of the form

$$\mu M_{0,n} \simeq \mu(h) r_n \pi_n,$$

for any positive measure μ, where $\tilde{h}$ is a random bounded function, $(r_n)_{n\geqslant 0}$ is a random non negative sequence and (π_n) is a random sequence of probability measures on $\mathbb{X}$. Moreover, $\tilde{h}$, (r_n) and (π_n) do not depend on the choice of the measure μ. We prove additionally that $n^{-1}\log(r_n)$ converges almost surely to the Lyapunov exponent λ of the process $(M_{0,n})_{n\geqslant 0}$ and that the sequence of random probability measures (π_n) converges weakly towards a random probability measure. These results are analogous to previous estimates from Hennion (1997) in the case of $d \times d$ matrices, that were obtained with different techniques, based on a projective contraction in Hilbert distance. In the case where the sequence (M_n) is i.i.d, we additionally exhibit an expression of the Lyapunov exponent λ as an integral with respect to the weak limit of the sequence of random probability measures (π_n) and exhibit an oscillation behavior of r_n and $\|\mu M_{0,n}\|$ when $\lambda = 0$. We provide a detailed comparison of our assumptions with the ones of Hennion (1997) and present an example of application of our results to the modelling of an age structured population.

Contents

Received by the editors December 23rd, 2023; accepted November 28th, 2024.
2010 *Mathematics Subject Classification.* 60F99,60K37,92D25.
Key words and phrases. Ergodicity, Projective contraction, Non conservative operators, Discrete time semi groups, Matrix population models, Random environment.
The author received the support of the Chair "Modélisation Mathématique et Biodiversité" of VEOLIA-Ecole Polytechnique-MnHn-FX, and of the ANR projects NOLO (ANR 20-CE40-0015) and RAWABRANCH (ANR-23-CE40-0008), funded by the French ministry of research.

1. Introduction

1.1. *General introduction.* The study of products of random linear operators can be traced back to the seminal article of Furstenberg and Kesten (1960), studying products of the form

$$M_{0,n} = M_0 \dots M_{n-1},$$

where $(M_n)_{n\geqslant 0}$ is a stationary and metrically transitive sequence of $p \times p$ real or complex random matrices. Under a mild irreducibility assumption, the authors exhibit a strong law of large numbers on the norm of $M_{0,n}$, in the form

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mu M_{0,n}\| = \lambda,$$

where λ is a deterministic number called Lyapunov exponent of the sequence $(M_n)_{n\geqslant 0}$, defined as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\log \|M_{0,n}\|\right] = \inf_{n\geqslant 1} \frac{1}{n} \mathbb{E}\left[\log \|M_{0,n}\|\right],$$

where the norm $\|\cdot\|$ can be chosen to be any submultiplicative norm. Under additional positivity and boundedness assumptions on the entries of the matrices (M_n), Furstenberg and Kesten (1960) also prove a strong law of large numbers for the entries $M_{0,n}(i,j)$ of the products : for any $i, j \in \{1, \dots, d\}$, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \log M_{0,n}(i,j) = \lambda.$$

These estimates on the behavior of the entries of $M_{0,n}$ were then extended to the case of products of invertible matrices, see e.g. Guivarc'h and Liu (2001) and Bougerol and Lacroix (1985). These works rely on a careful study of the action of invertible matrices on the projective space $\mathcal{P}(\mathbb{R}^d)$.

To strengthen the results from Furstenberg and Kesten (1960) on products of matrices with non negative entries and relax their assumptions, Hennion (1997) studied the action of $\mathcal{M}_d(\mathbb{R}_+)$ on the projective space $\mathcal{P}(\mathbb{R}_+^d)$, endowed with the so called (pseudo)-Hilbert distance d_H previously defined in Busemann and Kelly (1953) and Birkhoff (1957). This distance is particularly well adapted to this problem, since the contraction coefficient of the projective action of a matrix with respect to d_H is explicit in terms of its entries, in particular, any matrix with nonnegative entries in 1-contracting and any matrix with positive entries is strictly contracting. Under the assumption that almost surely, for n large enough, $M_{0,n}$ has all positive entries, Hennion obtains the asymptotic decomposition

$$M_{0,n}(i,j) = \lambda_n R_n(i) L_n(j) + \underset{n\to\infty}{o}(\lambda_n),$$

where λ_n is the dominant eigenvalue of $M_{0,n}$ and L_n, R_n are the associated left and right eigenvectors, with the normalizations $\|R_n\| = 1$ and $\langle L_n, R_n\rangle = 1$. Moreover $(R_n)_{n\geqslant 0}$ almost surely converges to a random vector R, $(L_n/\|L_n\|)_n$ converges in distribution, and $\left(n^{-1}\log\lambda_n\right)_{n\geqslant 1}$ almost surely converges to the Lyapunov exponent λ.

Such results have important implications, in particular in the field of populations dynamics. Indeed, a population composed of d types of individuals, evolving in a fluctuating environment, without interacting which each other, can be modelled by a linear model of the form

$$x_n = x_{n-1} M_{n-1}, \tag{1.1}$$

where x_n is a row vector of $\mathbb{R}_+^d$ encoding the mass of individuals of each type at time n and $M_{n-1} = (M_{n-1}(i,j))_{1\leqslant i,j\leqslant d}$ is a random matrix encoding the rates at which individuals of each type i create individuals of each type j between times $n-1$ and n. In such a time-inhomogeneous population model, the understanding of the asymptotics of x_n amounts to the understanding of the

matrix product $M_{0,n}$.

Moreover, such products also appear in the study of multitype Galton-Watson processes in random environment (MGWRE), which were introduced in Athreya and Karlin (1971). They are a generalization of Galton-Watson processes to the case where the distribution of the (random) offspring of an individual depends on a notion of type and on a random environment that changes through time. When conditioning a MGWRE on the environment sequence, one obtains a so-called quenched population model, which satisifies (1.1), where x_n is the expectation of the population conditionally on the environmental sequence. The value of the Lyapunov exponent λ of the underlying matrix product separates three regimes of the MGWRE : subcritical ($\lambda < 0$), critical ($\lambda = 0$), supercritical ($\lambda > 0$). These three regimes have different properties. In particular, when $\lambda \leqslant 0$, the MGWRE goes extinct with probability 1, when $\lambda > 0$, the MGWRE survives with positive probability. This separation between regimes was established in Athreya and Karlin (1971) and Kaplan (1974), using results from Furstenberg and Kesten (1960). More recent advances in the study of random matrix products - in particular Hennion's article- were key to the last developments of the theory of MGWRE in random environments, see e.g. Pham (2018); Le Page et al. (2018); Grama et al. (2023).

Products of random infinite dimensional operators have also been the subject of some investigation. In the case where the involved operators are the transition matrices of some Markov chain, in other words if they are conservative (in the sense that $M\mathbb{1} = \mathbb{1}$), ergodicity results are obtained by Cogburn (1984), completed with some more precise results in Orey (1991). Various limit theorems (law of large numbers, existence of Lyapunov exponents, central limit theorem, local central limit theorem, Large Deviation principle) have also been obtained on products of ergodic sequences of infinite dimensional operators using spectral techniques by Dragičević et al. (2018). The stability of the Lyapunov exponents under perturbation is studied in Atnip et al. (2022); Froyland et al. (2019). Ergodicity of products of random operators was also obtained in Kifer (1996) in the case where they act on some compact space $\mathbb{X}$.

In this paper, we would like to obtain similar ergodicity results for products of infinite dimensional positive operators, thus extending the results of Kifer (1996) without any topological assumption on $\mathbb{X}$, in particular without compacity. We have in particular in mind applications to population models with an infinite number of types. We first consider such a set $\mathbb{X}$, typically infinite, endowed with a σ-algebra $\mathcal{X}$, and build a set $\mathcal{K}^+$ of positive linear operators acting both on the space of signed measures $\mathcal{M}(\mathbb{X})$ on the left and the space of measurable bounded functions $\mathcal{B}(\mathbb{X})$ on the right. Then, we let (M_n) be a stationary, ergodic sequence of elements of $\mathcal{K}^+$ and define the products $M_{0,n} = M_0 \cdots M_{n-1}$. The approach of Hennion (1997) can be extended to this infinite dimensional setup. Indeed, it is possible to define the Hilbert distance d_H on the projective positive cone of an infinite dimensional vector space and to obtain a nice characterisation of the operators that are (strictly) contracting with respect to d_H. We refer the reader to Ligonnière (2023) for a proof of these facts. However, as we explain in Section 4, this characterisation leads to stronger positivity assumptions in an infinite dimensional context than it did in the finite dimensional one. For example, such an extension of Hennion's approach would not be able to deal with products of infinite Leslie matrices that we present in Section 5.

For this reason, we use a different contraction method to obtain a projective contraction. This method aims at extending the Doeblin contraction techniques for Markov operators to a product of non conservative operators (that is operators M such that $M\mathbb{1} \neq \mathbb{1}$ in general). To do so, we consider the auxiliary family of Markov operators $P^N_{k,n}$, defined for each $k \leqslant n \leqslant N$ as

$$\delta_x P^N_{k,n} f = \frac{\delta_x M_{k,n}(f m_{n,N})}{m_{k,N}(x)},$$

where, for $x \in \mathbb{X}$,

$$m_{k,n}(x) = \delta_x M_{k,n} \mathbb{1}.$$

These Markov operators are related to the projective action $\mu \cdot M_{k,n} = \frac{\mu M_{k,n}}{\|\mu M_{k,n}\|}$ of $M_{k,n}$ on measures by :

$$\delta_x \cdot M_{k,n} = \delta_x P^n_{k,n} = \delta_x P^n_{k,k+1} \cdots P^n_{n-1,n},$$

for any $x \in \mathbb{X}$. We provide sufficient conditions for the Markov operators $(P^n_{k,k+1})_{k<n}$ to satisfy Doeblin minorations of the form $\delta_x P(f) \geqslant c\nu(f)$, which guarantees that they are contracting in total variation. This allows to obtain a notion of projective contraction of the $M_{k,n}$ on the set of signed measures. Such auxiliary operators were already introduced and already applied to study both homogeneous semi-groups of operators, e.g. in Del Moral and Miclo (2002); Champagnat and Villemonais (2016) and inhomogeneous ones, e.g. in Bansaye et al. (2020). We consider here a random sequence of operators (M_n), i.e a discrete time, random, time-inhomogeneous semi group, and assume this sequence is stationary and ergodic. The stationary and ergodic framework allows us to provide more explicit assumptions, and we obtain a finer asymptotic analysis of the sequence $(M_{0,n})_{n\geqslant 0}$. Namely, we prove that the following almost sure approximation in total variation holds for n large enough

$$\|\mu M_{0,n} - \mu(h) r_n \pi_n\|_{TV} \leqslant \delta^n \|\mu M_{0,n}\|_{TV}, \tag{1.2}$$

where $\delta < 1$, h is a random bounded function, r_n is a positive random number and (π_n) is a sequence of random probability measures on $\mathbb{X}$, which are all independent of the measure μ. We prove additionally that $(n^{-1}\log(r_n))$ converges almost surely to the Lyapunov exponent of the process $M_{0,n}$, and that the sequence (π_n) of random probabilities converges in distribution with respect to the total variation topology towards a random probability measure Λ on $\mathcal{M}_1(\mathbb{X})$. Additionally, we show in Theorem 2.2 that when the sequence (M_n) is i.i.d, the probability distribution Λ and the Lyapunov exponent λ are related as follows :

$$\lambda = \int \log \|\mu M\| d\Lambda(\mu) d\mathcal{P}(M),$$

where $\mathcal{P}$ refers to the law of the operators (M_n), thus extending a result stated in Bougerol and Lacroix (1985) in finite dimension. Finally, still under the assumption that (M_n) is i.i.d, we show in Theorem 2.3 that, when $\lambda = 0$, it holds almost surely, for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,

$$\limsup_{n\to\infty} \log \|\mu M_{0,n}\| = -\liminf_{n\to\infty} \|\mu M_{0,n}\| = +\infty, \tag{OSC}$$

except in a situation knowed as Null-Homology.
These results should allow to extend many known results on MGWRE with a finite type set to a class of MGWRE with an infinite type set $\mathbb{X}$. In particular, our results imply that when the Lyapunov exponent λ is nonpositive, outside of Null-Homology, the quenched population size $\mu M_{0,n}\mathbb{1}$ satisfies $\liminf_{n\to\infty} \mu M_{0,n}\mathbb{1} = 0$ almost surely. By a classical first moment argument, this is a sufficient condition for the almost sure extinction of the population. The survival of the population when $\lambda > 0$ is a more delicate problem and will be the object of a forecoming article.

1.2. *Framework and notations.* Let $(\mathbb{X}, \mathcal{X})$ be a measurable set of arbitrary cardinality, such that for all $x \in \mathbb{X}$, the singleton $\{x\} \in \mathcal{X}$. We denote by $\mathcal{B}(\mathbb{X})$ the Banach space of bounded measurable functions on $\mathbb{X}$, endowed with the supremum norm, and $\mathcal{B}_+(\mathbb{X})$ the cone of nonnegative functions of $\mathcal{B}(\mathbb{X})$. The vector space of signed measures, noted $\mathcal{M}(\mathbb{X})$, and the cone of nonnegative elements of $\mathcal{M}(\mathbb{X})$, noted $\mathcal{M}_+(\mathbb{X})$, are endowed with the total variation norm $\|\cdot\|_{TV}$. Note that $\|\mu\|_{TV} = \mu(\mathbb{X})$ for any $\mu \in \mathcal{M}_+(\mathbb{X})$. Let $\mathcal{M}_1(\mathbb{X})$ be the set of probability measures on $(\mathbb{X}, \mathcal{X})$. For any measurable set $A \in \mathcal{X}$, let $\mathbb{1}_A \in \mathcal{B}(\mathbb{X})$ be the indicator function of A. For short, we note $\mathbb{1} = \mathbb{1}_{\mathbb{X}} \in \mathcal{B}(\mathbb{X})$ the unit function on $\mathbb{X}$. We also note $\delta_x \in \mathcal{M}_1(\mathbb{X})$ the Dirac measure at x.

Let $\mathcal{K}^+$ be the set of maps Q of the form:

$$Q : \mathbb{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+,$$

such that, for any $x \in \mathbb{X}$, for any $A \in \mathcal{X}$, the map $x \mapsto Q(x, A)$ is measurable, the map $A \mapsto Q(x, A)$ is a positive and finite measure on $(\mathbb{X}, \mathcal{X})$ and $\|\!|Q|\!\| := \sup_{x\in\mathbb{X}} Q(x, \mathbb{X}) < \infty$.

Such a map $Q \in \mathcal{K}^+$ naturally operates on $\mathcal{B}(\mathbb{X})$ by setting, for any $f \in \mathcal{B}(\mathbb{X})$, and any $x \in \mathbb{X}$,

$$Qf(x) = \int_{\mathbb{X}} f(y)Q(x, dy).$$

Note that $|Qf(x)| \leqslant \|f\|_\infty \|\!|Q|\!\|$, thus $Qf \in \mathcal{B}_+(\mathbb{X})$ as soon as $f \in \mathcal{B}_+(\mathbb{X})$ and Q acts as a bounded positive operator with norm $\|\!|Q|\!\|$ on $\mathcal{B}(\mathbb{X})$. Moreover, for any positive measure $\mu \in \mathcal{M}_+(\mathbb{X})$, and any $Q \in \mathcal{K}^+$, the positive measure μQ on $\mathbb{X}$ is well defined by setting, for any nonnegative function f,

$$\mu Q(f) = \mu(Qf) = \int_{\mathbb{X}} Qf(x)\mu(dx).$$

Note that μQ has indeed finite mass $\mu Q(\mathbb{1}) = \mu(Q\mathbb{1}) \leqslant \|\!|Q|\!\| \mu(\mathbb{1}) < \infty$ since μ is assumed to be a finite measure. This action can therefore naturally be extended to the set of signed measures $\mathcal{M}(\mathbb{X})$, where Q acts as a bounded linear operator with norm $\|\!|Q|\!\|$.

Thus, the elements of $\mathcal{K}^+$ operate as positive linear operators both on the sets of bounded measurable functions and on the set of signed measures on $\mathbb{X}$, with a duality relation between these two actions. Moreover, it is also possible to define a projective action $\cdot$ of $\mathcal{K}^+$ onto the projective space associated with $\mathcal{M}_+(\mathbb{X})$, ie the set of probability measures $\mathcal{M}_1(\mathbb{X})$, by setting, for any $\mu \in \mathcal{M}_+(\mathbb{X})$ and any $M \in \mathcal{K}^+$ such that $\mu M \neq 0$,

$$\mu \cdot M = \frac{\mu M}{\|\mu M\|_{TV}} \in \mathcal{M}_1(\mathbb{X}).$$

Finally, the set $\mathcal{K}^+$ is naturally endowed with an associative, non commutative product, defined by : for any $Q_1, Q_2 \in \mathcal{K}^+$, any $x \in \mathbb{X}$ and any $A \in \mathcal{X}$,

$$Q_1Q_2(x, A) = \int_y Q_1(x, dy)Q_2(y, A).$$

This product is compatible with the left and right actions defined above, in other words, for any $Q_1, Q_2 \in \mathcal{K}^+$, any $\mu \in \mathcal{M}_+(\mathbb{X})$, and $f \in \mathcal{B}_+(\mathbb{X})$,

$$\mu(Q_1Q_2) = (\mu Q_1)Q_2 \text{ and } Q_1Q_2(f) = Q_1(Q_2f),$$

and whenever $\mu \in \mathcal{M}_+(\mathbb{X})$ and $\mu Q_1Q_2 \neq 0$,

$$\mu \cdot (Q_1Q_2) = (\mu \cdot Q_1) \cdot Q_2.$$

The operator norm $\|\!| \cdot |\!\|$ satisfies the submultiplicativity relation

$$\|\!|Q_1Q_2|\!\| \leqslant \|\!|Q_1|\!\| \, \|\!|Q_2|\!\|.$$

Remark 1.1. In the case of a finite or countable set $\mathbb{X}$, any measure on $\mathbb{X}$ is atomic thus an operator $Q \in \mathcal{K}^+$ on $\mathbb{X}$ corresponds to a matrix indexed by $\mathbb{X}$ with nonnegative entries. The product of operators of $\mathcal{K}^+$ corresponds to the matrix product and the respective left and right actions of $\mathcal{K}^+$ on signed measures and bounded functions correspond to the product of matrices respectively with the vectors of $\ell^1(\mathbb{X})$ (seen as row vectors) and of $\ell^\infty(\mathbb{X})$ (seen as column vectors).

We consider a dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability set and $\theta : \Omega \longrightarrow \Omega$ is a measurable transformation, which preserves the probability $\mathbb{P}$, i.e $\mathbb{P} \circ \theta^{-1} = \mathbb{P}$.
Let $M : \Omega \longrightarrow \mathcal{K}^+$ be a measurable map. We denote as $\mathbb{N}_0$ the set of nonnegative integers and note, for each $n \in \mathbb{N}_0$,

$$M_n = M \circ \theta^n.$$

Note that the sequence $(M_n)_{n\in\mathbb{N}_0}$ is stationary. For each $k < n$, $\omega \in \Omega$, let us define the random product

$$M_{k,n}(\omega) = M_k(\omega) \cdots M_{n-1}(\omega) = (M \circ \theta^k(\omega)) \cdots \left(M \circ \theta^{n-1}(\omega)\right) \in \mathcal{K}^+.$$

with the convention $M_{k,k}(\omega) = \mathrm{Id} \in \mathcal{K}^+$. Notice that $M_{k,k+n}(\omega) = M_{0,n} \circ \theta^k(\omega)$. The operators satisfy the following semi group property : for any $k \leqslant n \leqslant N$, any $\mu \in \mathcal{M}(\mathbb{X})$, any $f \in \mathcal{B}(\mathbb{X})$

$$\mu M_{k,N}(\omega) f = \mu M_{k,n}(\omega) M_{n,N}(\omega) f. \tag{1.3}$$

Moreover, for any $x \in \mathbb{X}$, $k \leqslant n$, $\omega \in \Omega$, we set

$$m_{k,n}(x,\omega) = \delta_x M_{k,n}(\omega)\mathbb{1} = \|\delta_x M_{k,n}(\omega)\|_{TV}.$$

Notice in particular that for any positive measure μ,

$$\|\mu M_{k,N}\| = \mu(m_{k,N}) = \mu M_{k,N}\mathbb{1} = \mu M_{k,n} m_{n,N}.$$

Let us point out additionally that $|||M_{k,n}(\omega)||| = \sup_{x\in\mathbb{X}} m_{k,n}(x,\omega) = \|m_{k,n}(\cdot,\omega)\|_\infty$, and that for any $k \leqslant n \leqslant N$, $|||M_{k,N}||| \leqslant |||M_{k,n}|||\,|||M_{n,N}|||$. Finally, to shorten the notations, we often omit the dependence in ω, writing for example $m_{k,n}(x) = m_{k,n}(x,\omega)$, and $|||M_{k,n}||| = \|m_{k,n}\|_\infty = \sup_{x\in\mathbb{X}} m_{k,n}(x,\omega)$.

1.3. *Assumptions.* We list here several hypotheses that will be used in the rest of the article.

*A*1. The dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ is ergodic.

We recall that a dynamical system is ergodic when any measurable set $A \in \mathcal{A}$ such that $\theta^{-1}(A) = A$ satisfies $\mathbb{P}(A) \in \{0,1\}$.

*A*2. For almost all $\omega \in \Omega$, the function $x \mapsto m_{0,1}(x,\omega)$ is a positive function.

By stationarity, **A2** implies that $\mathbb{P}(d\omega)$-almost surely, the product $M_{k,n}(\omega)$ is a continuous, non zero, positive linear operator. We introduce the integrability property

*A*3. $\mathbb{E}\left[\log^+ \|m_{0,1}\|_\infty\right] < \infty$.

In particular, recalling that $\|m_{k,n}\|_\infty = |||M_{k,n}|||$, by submultiplicativity of the norm $|||\cdot|||$, **A1** and **A3** imply that $\mathbb{E}\log^+(|||M_{k,n}|||) < \infty$ for all $k \leqslant n$.

We call admissible coupling constants a measurable map $(\nu, c, d) : \omega \in \Omega \mapsto (\nu_\omega, c(\omega), d(\omega)) \in \mathcal{M}_1(\mathbb{X}) \times [0,1]^2$ such that for $\mathbb{P}$-almost any $\omega \in \Omega$,

i) for all $x \in \mathbb{X}$ and all $f \in \mathcal{B}_+(\mathbb{X})$, the couple $(\nu_\omega, c(\omega))$ satisfies

$$\delta_x M(\omega)(f) \geqslant c(\omega)\|\delta_x M(\omega)\|\nu_\omega(f) \tag{1.4}$$

ii) for all $n \geqslant 0$, the couple $(\nu_\omega, d(\omega))$ satisfies

$$\nu_\omega(m_{1,n}) \geqslant d(\omega)|||M_{1,n}||| \tag{1.5}$$

When an admissible triplet is defined, we define the random variable

$$\gamma(\omega) = c(\omega)d(\omega).$$

Note that taking $c = d = 0$ and any measurable map $\omega \mapsto \nu_\omega$ defines an admissible triplet, however in this case $\gamma = 0$. Our main assumption is therefore

*A*4. There exists an admissible triplet $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ such that $\mathbb{P}[\gamma > 0] > 0$.

2. Statement of the results and structure of the paper

2.1. *Main results.* Set

$$\bar\gamma := \exp\left(\mathbb{E}\left[\log(1-\gamma)\right]\right) \in [0,1],$$

and notice that **A4** yields $\bar\gamma < 1$. Under the previous assumptions, we prove the following Theorem:

Theorem 2.1. *Let $M : \Omega \longrightarrow \mathcal{K}^+$ be a measurable map and assume that Assumptions **A1**,**A2** and **A4** hold. Then,*

i) $\mathbb{P}(d\omega)$*-almost surely, there exists a random function* $h \in \mathcal{B}(\mathbb{X})$ *such that, for any* $\delta \in (\bar{\gamma}^{\mathbb{P}[\gamma>0]}, 1)$*, for* n *large enough, for any finite measures* $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$*,*

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n} \right\|_{TV} \leqslant \delta^n \|\mu_1 M_{0,n}\|. \tag{2.1}$$

Such a function h *is unique up to a multiplicative constant.*

ii) There exists a probability measure Λ *on the space* $\mathcal{M}_1(\mathbb{X})$*, such that for any probability measure* μ*, the sequence of random probability measures* $(\mu \cdot M_{0,n})$ *converges in distribution towards* Λ*, in the space* $\mathcal{M}_1(\mathbb{X})$*, endowed with the total variation norm.*

iii) Assuming additionally **A3***, for almost any* $\omega \in \Omega$ *and any finite, positive, non-zero measure* μ*,*

$$\frac{1}{n} \log \|\mu M_{0,n}\| \underset{n\to\infty}{\longrightarrow} \inf_{N\in\mathbb{N}} \frac{1}{N} \mathbb{E}\left[\log \|M_{0,N}\|\right] = \lambda \in [-\infty, \infty). \tag{2.2}$$

Note that the estimate (1.2) can be derived from Theorem 2.1 by a choice of an arbitrary measure μ_2, and by setting

$$\pi_n = \mu_2 \cdot M_{0,n}, \ r_n = \frac{\|\mu_2 M_{0,n}\|}{\mu_2(h)},$$

The rest of this paper focuses on the independent case, that is, the case where the sequence of operators (M_n) is i.i.d, with a law called $\mathcal{P}$. This can be obtained by setting $(\Omega, \mathcal{A}, \mathbb{P})$ to be the product space $\Omega = (\mathcal{K}^+)^{\mathbb{N}}$, $\mathbb{P} = \mathcal{P}^{\otimes\mathbb{N}}$ and $M : (\mathcal{K}^+)^{\mathbb{N}} \to \mathcal{K}^+, (N_k)_{k\geqslant 0} \mapsto N_0$. In this independent case, we are able to characterize the measure Λ on the projective space $\mathcal{M}_1(\mathbb{X})$ as the only invariant measure left invariant by the projective action of the matrices (M_n). In other words, it is the only invariant probability measure of the Markov chain $(\mu_n)_{n\geqslant 0}$ defined by

$$\mu_n = \frac{\mu_0 M_{0,n}}{\|\mu_0 M_{0,n}\|} \in \mathcal{M}_1(\mathbb{X}).$$

A classical cocyle property yields the decomposition

$$\log \|\mu_0 M_{0,n}\|_{TV} = \sum_{k=0}^{n-1} \log(\|\mu_k M_k\|) = \sum_{k=0}^{n-1} \rho(\mu_k, M_k), \tag{2.3}$$

where (μ_k, M_k) is a Markov chain on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$, with the unique invariant probability measure $\Lambda \otimes \mathcal{P}$. These property allows us to obtain additional insight over the asymptotic behavior of the mass of the measure $\mu M_{0,n}$. To do so, we introduce the following strengthened versions of assumptions **A3** and **A4**:

A3+. $\mathbb{E}\left[\left|\log \|m_{0,1}\|_\infty\right|\right] < \infty$.

A4+. There exists an admissible triplet (ν, c, d) such that $\mathbb{E}\left|\log(\gamma)\right| < \infty$.

Note that **A4+** implies in particular that $\mathbb{P}[\gamma > 0] = 1$. Under these assumptions we are able to link the Lyapunov exponent λ, which governs the exponential growth of the mass of the measure $\|\mu M_{k,n}\|$, with the asymptotic projective distribution Λ on $\mathcal{M}_1(\mathbb{X})$.

Theorem 2.2. *Consider an i.i.d sequence* (M_n) *of elements of* $\mathcal{K}^+$ *with law* $\mathcal{P}$*, suppose assumptions* **A1***,* **A2***,* **A3+** *and* **A4+** *hold. Then, the almost sure convergence* (2.2) *also holds in* $\mathcal{L}^1(\Lambda \otimes \mathbb{P})$*, that is*

$$\int \left| \frac{1}{n} \log \|\mu M_{0,n}\| - \lambda \right| d\Lambda(\mu) d\mathcal{P}^{\otimes n}(M_0, \cdots, M_{n-1}) \underset{n\to\infty}{\longrightarrow} 0.$$

As a consequence,

$$\lambda = \int \log \|\mu M\| d\Lambda(\mu) d\mathcal{P}(M). \tag{2.4}$$

Moreover, in the critical case where $\lambda = 0$, the law of large numbers (2.2) is not enough to know whether $\liminf \|\mu M_{0,n}\|$ and $\limsup \|\mu M_{0,n}\|$ are 0, a positive real number, or $+\infty$. Answering these questions is the objective of our last theorem. When $\lambda = 0$, the sequence (M_n) is i.i.d and $\mu_0 \sim \Lambda$, the increments $\rho(\mu_k, M_k)$ of the sum (2.3) are centered and form an ergodic sequence. By analogy with a centered random walk with i.i.d increments, we expect their sum $\log \|\mu_0 M_{0,n}\|$ to oscillate between $-\infty$ and $+\infty$. More formally, we use the theory of Markov random walks, and in particular Alsmeyer (2001), which establishes this oscillation property exists, except in some case called Null-Homology. In our context, we say that there is Null Homology when there exists some function $\eta : \mathcal{M}_1(\mathbb{X}) \mapsto \mathbb{R}$ such that $d(\Lambda \otimes \mathcal{P})(\mu, M)$-a.s.

$$\log \|\mu M\| = \eta(\mu \cdot M) - \eta(\mu). \tag{NH}$$

When (NH) does not hold, using results from Alsmeyer (2001), we indeed obtain the oscillation of $\log \|\mu M_{0,n}\|$ for any initial measure μ. On the contrary, when (NH) holds, it is clear that $\log \|\mu_0 M_{0,n}\| = \eta(\mu_n) - \eta(\mu_0)$, thus $\log \|\mu_0 M_{0,n}\|$ may or may not oscillate, depending on η. More formally, our result is

Theorem 2.3. *Consider an i.i.d sequence (M_n) of elements of $\mathcal{K}^+$ with law $\mathcal{P}$, suppose assumptions **A 1**, **A 2**, **A 3** and **A 4+** hold. Assume additionally that $\lambda = 0$. Then, if (NH) does not hold, we have*

$$\liminf_{n\to\infty} \log \|\mu M_{0,n}\| = -\infty \text{ and } \limsup_{n\to\infty} \log \|\mu M_{0,n}\| = +\infty \tag{OSC}$$

for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$. If (NH) holds, let $\mu_0 \sim \Lambda$, and let us note $a < b \in [-\infty, +\infty]$ the respective infinimum and supremum of the support of the random variable $\eta(\mu_0)$. It holds

$$a = -\infty \Leftrightarrow \mathbb{P} - a.s., \text{ for any } \mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}, \liminf_{n\to\infty} \log \|\mu M_{0,n}\| = -\infty$$

and

$$b = +\infty \Leftrightarrow \mathbb{P} - a.s., \text{ for any } \mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}, \limsup_{n\to\infty} \log \|\mu M_{0,n}\| = +\infty.$$

The notion of Null Homology already appears in the framework of products of $p \times p$ non-negative matrices, and Hennion (1997) provides some geometric condition which prevents Null-Homology (see in particular Theorem 5 of Hennion (1997)). We are unfortunately not able to generalize this condition in the infinite dimensional case.

2.2. *Structure of the paper.* Section 3 contains the proofs of Theorems 2.1, 2.2 and 2.3. More precisely, in Subsection 3.1, we recall how the coefficient γ allow to control some contraction rates of the operators $M_{0,n}$. These results are adapted from Bansaye et al. (2020). In Subsection 3.2, we use the ergodic structure, in particular Assumptions **A1** and **A4** to obtain a geometric decay of the error terms that appeared in our previous estimations. In Subsection 3.3 we derive the three claims of Theorem 2.1. Finally, in Subsection 3.4, we focus on the case where the sequence (M_n) is i.i.d. In this case, a study of the invariant measures and the ergodicity properties of the Markov chains $(\mu_0 \cdot M_{0,n})_{n\geqslant 0}$ and $(\mu_0 \cdot M_{0,n}, M_n)_{n\geqslant 0}$, allows to prove Theorems 2.2 and 2.3.

Section 4 is dedicated to a comparison of our results with those obtained based on Hilbert contractions. More precisely, we show how natural conditions coming from Hilbert contractions techniques provide more tractable sufficient conditions for our Assumptions (in particular **A 4**), both in finite and infinite dimension.

In Section 5, we apply our results to study products of infinite Leslie Matrices. This constitutes an example of an interesting class of systems that cannot be studied using the Hilbert metric. More precisely, we provide in Subsection 5.2 reasonable sufficient conditions under which a product of Leslie matrices modelling the behavior of an age structured population satisfies assumptions **A1** to **A4**. However, when these conditions are not satisfied, it can be quite difficult to exhibit an admissible triplet such that $\gamma > 0$, even on a deterministic and constant sequence of Leslie Matrices.

To illustrate this fact, we present in Subsection 5.3 an example of system where $\gamma = 0$ even if all the other assumptions are satisfied.

3. Proofs

3.1. *Contraction results based on inhomogeneous Doeblin minoration.* Given an admissible triplet (ν, c, d), we define for every $k \geqslant 0$ the $[0,1]$-valued random variables $c_k = c_k(\omega) = c(\theta^k(\omega))$, $d_{k+1} = d_{k+1}(\omega) = d(\theta^k(\omega))$ and $\gamma_k = \gamma_k(\omega) = \gamma(\theta^k(\omega))$, as well as the $\mathcal{M}_1(\mathbb{X})$ valued random variable $\nu_k = \nu_{\theta^k(\omega)}$. Notice that $c = c_0$ and $d = d_1$, that the sequences $(c_k)_{k\geqslant 0}$, $(d_k)_{k\geqslant 1}$, $(\gamma_k)_{k\geqslant 0}$ are all stationary sequences and that for all $k \geqslant 0$,

$$\gamma_k = c_k d_{k+1}.$$

Moreover the random variables ν_k, c_k, d_{k+1} satisfy some time-shifted versions of (1.4) and (1.5). Indeed, for any $f \in \mathcal{B}_+(\mathbb{X})$ and any $x \in \mathbb{X}$, it holds

$$\delta_x M_{k,k+1}(f) \geqslant c_k m_{k,k+1}(x)\nu_k(f),\ \mathbb{P}-\text{a.s.} \tag{3.1}$$

and for any $n \geqslant k$,

$$\nu_k(m_{k+1,n}) \geqslant d_{k+1}\|M_{k+1,n}\| = d_{k+1}\sup_{x\in\mathbb{X}} m_{k+1,n}(x),\ \mathbb{P}-\text{a.s.} \tag{3.2}$$

The first step towards proving Theorem 2.1 is establishing

Proposition 3.1. *Suppose A2 and let $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ be an admissible triplet. Then, $\mathbb{P}(d\omega)$-almost surely, for any $k \leqslant n \leqslant N$ and any finite measures $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, it holds*

$$\|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i), \tag{3.3}$$

and, if $n \geqslant 1$, it also holds $\mathbb{P}(d\omega)$-a.s.

$$\gamma_{n-1}\left|\frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\right| \leqslant 2\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\prod_{i=k}^{n-1}(1-\gamma_i). \tag{3.4}$$

This result was already introduced in Bansaye et al. (2020) in a somewhat different setup. We have chosen to state and prove it here for the sake of completeness. Its proof is based on performing a Doeblin minoration on a well-chosen sequence of auxiliary Markov operators $(P^N_{k,n})$. This Doeblin property yields (3.3), a contraction property for the projective action of $M_{k,n}$ on the space of measures $\mathcal{M}_+(\mathbb{X})$. We derive then Equation (3.4), which describes how the growth of the mass $\|\mu M_{k,t}\|$ between times $t = n$ and $t = N$ depends on the initial measure μ.

Let us introduce now the operators $P^N_{k,n}$ upon which we perform the desired Doeblin minoration. Under assumption **A2**, $\mathbb{P}(d\omega)$-almost surely, for any $k \leqslant n \leqslant N$, $x \mapsto m_{k,n}(x)$ and $x \mapsto m_{n,N}(x)$ are positive functions on $\mathbb{X}$. For each $k \leqslant n \leqslant N$, an operator $P^N_{k,n}(\omega)$ can be defined $\mathbb{P}(d\omega)$-almost surely, as follows : for each $x \in \mathbb{X}$, for each positive measurable $f : \mathbb{X} \longrightarrow \mathbb{R}$,

$$\delta_x P^N_{k,n} f = \frac{\delta_x M_{k,n}(f m_{n,N})}{m_{k,N}(x)},$$

$P^N_{k,n}$ is a positive and conservative operator (i.e. $P^N_{k,n}\mathbb{1} = \mathbb{1}$). Indeed, by Equation (1.3), for any $x \in \mathbb{X}$,

$$\delta_x P^N_{k,n}\mathbb{1} = \frac{\delta_x M_{k,n} m_{n,N}}{\delta_x M_{k,N}\mathbb{1}} = 1.$$

Moreover, $P^N_{k,n}$ satisfies the relation :

$$P^N_{k,n} = P^N_{k,k+1}\cdots P^N_{n-1,n}.$$

Note that $P^N_{k,n}$ is a matrix when $\mathbb{X}$ is countable and then, for any $x, y \in \mathbb{X}$,

$$P^N_{k,n}(x,y) = \frac{m_{n,N}(y)}{m_{k,N}(x)} M_{k,n}(x,y).$$

These operators satisfy a Doeblin contraction property summed up in

Lemma 3.2. *Assume Assumption A 2 holds and let $\omega \mapsto (\nu_\omega, c(\omega), d(\omega))$ be an admissible triplet. Then $\mathbb{P}(d\omega)$-almost surely, all the $P^N_{k,n}(\omega)$ are well defined, and it holds*

i) For any $n \leqslant N-1$, there exists a random probability measure $\nu_{n,N}$ on $\mathbb{X}$ such that, for any $x \in \mathbb{X}$,

$$\delta_x P^N_{n,n+1} \geqslant c_n d_{n+1} \nu_{n,N} = \gamma_n \nu_{n,N}.$$

ii) For any signed measures ρ_1, ρ_2, of same mass and any $n \leqslant N-1$,

$$\left\|\rho_1 P^N_{n,n+1} - \rho_2 P^N_{n,n+1}\right\|_{TV} \leqslant (1-\gamma_n)\|\rho_1 - \rho_2\|_{TV}.$$

iii) For any $k \leqslant n \leqslant N$ and any signed measures ρ_1, ρ_2 of same mass,

$$\left\|\rho_1 P^N_{k,n} - \rho_2 P^N_{k,n}\right\|_{TV} \leqslant \prod_{i=k}^{n-1} (1-\gamma_i) \left\|\rho_1 - \rho_2\right\|_{TV}.$$

Notice that in this lemma, our single assumption is **A2**. It allows the $(P^N_{k,n})$ to be defined $\mathbb{P}(d\omega)$-almost surely. In particular, **A4** is not assumed, we allow $\gamma_n(\omega) = 0$, in which case we just obtain that $P^N_{n,n+1}$ is 1-contracting.

Proof of Lemma 3.2: Let $\omega \in \Omega$ such that all the $P^N_{k,n}$ are well defined. For any $x \in \mathbb{X}$ and any $f \in \mathcal{B}_+(\mathbb{X})$, it holds,

$$\delta_x M_{n,n+1}(f m_{n+1,N}) \geqslant c_n m_{n,n+1}(x) \nu_n(f m_{n+1,N}),$$

thus

$$\delta_x P^N_{n,n+1} f = \frac{\delta_x M_{n,n+1}(f m_{n+1,N})}{m_{n,N}(x)} \geqslant c_n \frac{\nu_n(f m_{n+1,N}) m_{n,n+1}(x)}{m_{n,N}(x)},$$

with, by definition of d_{n+1} :

$$d_{n+1} m_{n,N}(x) = d_{n+1} \delta_x M_{n,n+1}(m_{n+1,N}) \leqslant \nu_n(m_{n+1,N}) m_{n,n+1}(x).$$

Therefore,

$$\delta_x P^N_{n,n+1} f \geqslant c_n d_{n+1} \frac{\nu(f m_{n+1,N})}{\nu(m_{n+1,N})} = c_n d_{n+1} \nu_{n,N}(f) = \gamma_n \nu_{n,N}(f),$$

setting

$$\nu_{n,N}(\cdot) = \frac{\nu_n(\cdot\, m_{n+1,N})}{\nu_n(m_{n+1,N})},$$

which is a probability measure. This concludes the proof of *i)*.
Let us prove now *ii)*. This result is a classical consequence of the previous point using the theory of Markov operators. A Markov operator P is said to be δ-Doeblin (with $\delta > 0$) when there exists a probability measure μ such that $\delta_x P f \geqslant \delta \mu(f)$ for any x in the state space and any $f \in \mathcal{B}_+(\mathbb{X})$. Furthermore, such an operator is $1-\delta$ contracting in total variation : for any signed measure ρ of mass 0,

$$\|\rho P\|_{TV} \leqslant (1-\delta)\|\rho\|_{TV}.$$

This property trivially holds for $\delta = 0$: any positive operator satisfies $\delta_x P f \geqslant 0$ when f is a non negative function, and any Markov operator is $1 = (1-0)$-contracting in total variation. In our context, the previous point of the lemma yields that $\mathbb{P}(d\omega)$-almost surely, for any $n \leqslant N-1$, and

any $\nu \in \mathcal{M}_1(\mathbb{X})$, the Markov operator $P^N_{n,n+1}$ is γ_n-Doeblin. Therefore, for any $\rho_1, \rho_2 \in \mathcal{M}(\mathbb{X})$, such that $\rho_1(\mathbb{1}) = \rho_2(\mathbb{1})$, noting $\rho = \rho_1 - \rho_2$, it holds

$$\left\|\rho P^N_{n,n+1}\right\|_{TV} \leqslant (1-\gamma_n)\|\rho\|_{TV}.$$

This proves *ii)*, let us move now to *iii)*. Since all the $P^N_{n,n+1}$ are conservative operators, the image of a measure of mass 0 by $P^N_{n,n+1}$ is a measure of mass 0. The equality $P^N_{k,n} = P^N_{k,k+1}\cdots P^N_{n-1,n}$, yields $\left\|\rho P^N_{k,n}\right\|_{TV} \leqslant (1-\gamma_{n-1})\|\rho P^N_{k,n-1}\|_{TV}$. By induction, we deduce

$$\left\|\rho P^N_{k,n}\right\|_{TV} \leqslant \prod_{i=k}^{n-1}(1-\gamma_i)\left\|\rho\right\|_{TV}.$$

This concludes the proof. □

Proof of Proposition 3.1: Let us prove first Inequality (3.3). Applying Lemma 3.2, *iii)* with $n = N$ and $\rho = \delta_x - \delta_y$, we get, $\mathbb{P}(d\omega)$-almost surely

$$\left\|\delta_x P^n_{k,n} - \delta_y P^n_{k,n}\right\|_{TV} \leqslant \prod_{i=k}^{n-1}(1-\gamma_i)\left\|\delta_x - \delta_y\right\|_{TV} \leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i).$$

Hence, for any $f \in \mathcal{B}(\mathbb{X})$ and $x, y \in \mathbb{X}$,

$$\left|\frac{\delta_x M_{k,n}(f)}{m_{k,n}(x)} - \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i).$$

Let μ_1 and μ_2 be two positive measures. The inequality

$$\left|\delta_x M_{k,n}(f) - m_{k,n}(x)\frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2m_{k,n}(x)\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i),$$

yields, after integrating with respect to $\mu_1(dx)$:

$$\left|\mu_1 M_{k,n}(f) - \mu_1(m_{k,n})\frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2\mu_1(m_{k,n})\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i),$$

so that

$$\left|\mu_1 \cdot M_{k,n}(f) - \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)}\right| \leqslant 2\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i).$$

Integrating now with respect to $\mu_2(dy)$, we obtain

$$\left|\mu_1 \cdot M_{k,n}(f) - \mu_2 \cdot M_{k,n}(f)\right| \leqslant 2\left\|f\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i),$$

and finally

$$\left\|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\right\|_{TV} \leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i).$$

Let us move now to the proof of Inequality (3.4). Applying Inequality (3.3) to the function $x \mapsto m_{n,N}(x)$, one gets

$$\left|\frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}\right| \leqslant 2\left\|m_{n,N}\right\|_\infty \prod_{i=k}^{n-1}(1-\gamma_i) = 2\left\|\left\|M_{n,N}\right\|\right\| \prod_{i=k}^{n-1}(1-\gamma_i).$$

By (3.1) and (3.2) it holds,

$$d_n \left\|\left\|M_{n,N}\right\|\right\| \leqslant \nu_{n-1}(m_{n,N})$$

and

$$\mu_2(m_{k,N}) = \mu_2 M_{k,n-1} M_{n-1,n} m_{n,N} \geqslant c_{n-1}\mu_2(m_{k,n})\nu_{n-1}(m_{n,N}).$$

Combining these identities we obtain, $\mathbb{P}(d\omega)$-almost surely,

$$c_{n-1}d_n \left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leqslant 2\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \prod_{i=k}^{n-1}(1-\gamma_i).$$

Thus taking an infimum in ν, this yields, $\mathbb{P}(d\omega)$-almost surely,

$$\gamma_{n-1} \left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leqslant 2\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \prod_{i=k}^{n-1}(1-\gamma_i).$$

This ends the proof. □

3.2. *Asymptotic estimates under ergodicity assumptions.* For any n such that $\gamma_{n-1} \neq 0$ and any $k \leqslant n$, we define

$$\Gamma_{k,n} = \frac{1}{\gamma_{n-1}} \prod_{i=k}^{n-1}(1-\gamma_i), \tag{3.5}$$

and we set $\Gamma_{k,n} = +\infty$ when $\gamma_{n-1} = 0$. With these notations, Equation (3.4) can be rewritten

$$\left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leqslant 2\Gamma_{k,n}\frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})}. \tag{3.6}$$

In this subsection, we use the ergodicity Assumption **A1**, as well as Assumption **A4**, which provide a control on the sequence of Doeblin coefficients (γ_n). In the following lemma, we use these assumptions to establish a geometric decay of both $\prod_{i=k}^{n-1}(1-\gamma_i)$ and $\Gamma_{k,n}$ as $n \to \infty$. More precisely, (3.8) provides the geometric decay of $\prod_{k=1}^{n-1}(1-\gamma_i)$ using Birkhoff's ergodic theorem. To derive the decay of $\Gamma_{k,n}$ as $n \to \infty$ we need additionally to avoid that γ_{n-1} is either 0 or too close to 0. Under Assumption **A4+** the coefficients $(\gamma_n)_{n\geqslant 0}$ are almost surely all nonzero, and (3.10) provides a sufficient control for the geometric decay of $\Gamma_{k,n}$. Under the weaker assumption **A4**, it is possible however that $\gamma_{n-1} = 0$ for some values of n. We can however focus on some random times at which γ_{n-1} is greater that some predetermined level $\varepsilon > 0$. We define those random times by

$$\{n\}_\varepsilon = \max\{i \leqslant n | \gamma_{i-1} \geqslant \varepsilon\} \cup \{-\infty\} \tag{3.7}$$

for all $n \geqslant 0$ and all $\varepsilon > 0$. (3.9) establishes that the subsequence of $(\Gamma_{k,n})_{n\geqslant 0}$ associated with the random times $(\{n\}_\varepsilon)_{n\geqslant 0}$ still enjoys some slower geometric decay.

Lemma 3.3. *Assume that assumptions **A1**, **A2** and **A4** hold. Then for any $k \geqslant 0$,*

$$\left(\prod_{i=k}^{k+n-1}(1-\gamma_i)\right)^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} \bar{\gamma} < 1, \mathbb{P}(d\omega) - \textit{almost surely} \tag{3.8}$$

Moreover, for all $\varepsilon > 0$ such that $\mathbb{P}[\gamma \geqslant \varepsilon] > 0$, it holds

$$\limsup_{n\to+\infty} \left(\Gamma_{k,\{n\}_\varepsilon}\right)^{\frac{1}{n}} \leqslant \bar{\gamma}^{\mathbb{P}[\gamma\geqslant\varepsilon]} < 1, \mathbb{P}(d\omega) - \textit{almost surely.} \tag{3.9}$$

*Finally, assuming additionally **A4+**, it holds*

$$\lim_{n\to\infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} = 1, \mathbb{P}(d\omega) - \textit{almost surely.} \tag{3.10}$$

Proof of Lemma 3.3: We recall first that by definition, for any $\omega \in \Omega$

$$(1-\gamma_i)(\omega) = (1-\gamma)\circ\theta^i(\omega).$$

Notice then that, by Assumption **A4**, for almost every $\omega \in \Omega$, $\gamma(\omega) \in (0,1]$. Thus $\log(1-\gamma(\omega)) \in [-\infty, 0)$ and $\bar{\gamma} = \exp\left(\mathbb{E}\left[\log(1-\gamma(\omega))\right]\right) \in [0,1)$.

Thus for any k,

$$\log\left[\left(\prod_{i=k}^{k+n-1}(1-\gamma_i)\right)^{\frac{1}{n}}\right] = \frac{1}{n}\sum_{i=0}^{n-1}\log(1-\gamma_k)\circ\theta^i.$$

Since θ is an ergodic map, by Birkhoff's ergodic theorem, for any $k \geqslant 0$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\log(1-\gamma_{k+i}) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\log(1-\gamma)\right] = \log\bar{\gamma},\ \mathbb{P}(d\omega) - \text{ almost surely.}$$

This yields (3.8). Moreover, under **A4**, for ε small enough, by Birkhoff's ergodic theorem, it holds almost surely

$$\lim_{n\to\infty}\frac{1}{n}\#\{1 \leqslant k \leqslant n | \gamma_{k-1} \geqslant \varepsilon\} = \mathbb{P}[\gamma \geqslant \varepsilon] > 0.$$

However

$$\{n\}_\varepsilon \geqslant \#\{1 \leqslant k \leqslant n | \gamma_{k-1} \geqslant \varepsilon\}$$

by definition of $\{n\}_\varepsilon$. Thus, almost surely

$$\liminf_{n\to\infty}\frac{\{n\}_\varepsilon}{n} \geqslant \mathbb{P}[\gamma \geqslant \varepsilon] > 0,$$

in particular, $\lim_{n\to\infty}\{n\}_\varepsilon = +\infty$ almost surely.

As a consequence, for any $k \geqslant 0$, it holds $\mathbb{P}$-a.s., for n large enough

$$\left(\Gamma_{k,\{n\}_\varepsilon}\right)^{\frac{1}{n}} = \left(\frac{1}{\gamma_{\{n\}_\varepsilon-1}}\prod_{i=k}^{\{n\}_\varepsilon-1}(1-\gamma_i)\right)^{\frac{1}{n}} \leqslant \left(\frac{1}{\varepsilon}\right)^{\frac{1}{n}}\exp\left(\frac{\{n\}_\varepsilon-k}{n}\frac{1}{\{n\}_\varepsilon-k}\sum_{i=k}^{\{n\}_\varepsilon-1}\log(1-\gamma_i)\right)$$

where almost surely

$$\lim_{n\to\infty}\frac{1}{\{n\}_\varepsilon-k}\sum_{i=k}^{\{n\}_\varepsilon-1}\log(1-\gamma_i) = \log(\bar{\gamma}).$$

Hence

$$\limsup_{n\to\infty}\left(\Gamma_{k,\{n\}_\varepsilon}\right)^{\frac{1}{n}} \leqslant \exp\left(\liminf_{n\to\infty}\left(\frac{\{n\}_\varepsilon}{n}\right)\log(\bar{\gamma})\right) \leqslant \bar{\gamma}^{\mathbb{P}[\gamma\geqslant\varepsilon]}.$$

This concludes the proof of (3.9). Let us move to the proof of Inequality (3.10).
Notice first that since $\gamma_n \leqslant 1$ for all n,

$$\liminf_{n\to\infty}\left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \geqslant 1.$$

Let us prove now the converse inequality. Let us define for each $b > 1$,

$$Y_n(b) = \frac{-\log(\gamma_n)}{\log(b)} \geqslant 0,$$

and

$$N_b = \sum_{n\in\mathbb{N}_0}\mathbb{1}_{(1/\gamma_n)^{\frac{1}{n}}>b} = \sum_{n\in\mathbb{N}_0}\mathbb{1}_{Y_n(b)>n}.$$

For a given value of b, the sequence $(Y_n(b))_{n\in\mathbb{N}_0}$ is stationary, thus

$$\mathbb{E}(N_b) = \sum_{n\in\mathbb{N}_0} \mathbb{P}\left[Y_n(b) > n\right] = \sum_{n\in\mathbb{N}_0} \mathbb{P}\left[Y_0(b) > n\right].$$

It is a well known fact that for a nonnegative random variable Y,

$$\mathbb{E}[Y] < \infty \Leftrightarrow \sum_{n\geqslant 0} \mathbb{P}(Y > n) < \infty.$$

By Assumption **A4**, it holds $\mathbb{E}\left[Y_0(b)\right] < \infty$ and $\mathbb{E}\left[N_b\right] < \infty$ for all $b > 1$. Therefore, $\mathbb{P}(d\omega)$-almost surely, $N_b < \infty$, thus $\left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \leqslant b$ for n large enough and $\limsup_{n\to\infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \leqslant 1$. Finally,

$$\lim_{n\to\infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} = 1,\ \mathbb{P}(d\omega) - \text{almost surely.}$$

□

Putting the estimates from Lemma 3.3 together with Proposition 3.1, we obtain

Proposition 3.4. *Assume **A1**, **A2**, **A4** hold. Let $\varepsilon > 0$ such that $\mathbb{P}[\gamma \geqslant \varepsilon] > 0$. Then, $\mathbb{P}(d\omega)$-almost surely, for any $k \in \mathbb{N}_0$, there exists a bounded, non-negative measurable function h_k such that, for n large enough, for any $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,*

$$\left\| \mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2 M_{k,n} \right\|_{TV} \leqslant \Delta^{\varepsilon}_{k,n} \|\mu_1 M_{k,n}\|, \tag{3.11}$$

where almost surely, for n large enough

$$\Delta^{\varepsilon}_{k,n} := \frac{8\Gamma_{k,\{n\}_\varepsilon}}{1 - 2\Gamma_{k,\{n\}_\varepsilon}}.$$

is well-defined and positive, and

$$\limsup_{n\to\infty} \left(\Delta^{\varepsilon}_{k,n}\right)^{\frac{1}{n}} \leqslant \bar{\gamma}^{\mathbb{P}[\gamma\geqslant\varepsilon]} < 1. \tag{3.12}$$

Furthermore, $\mathbb{P}(d\omega)$-almost surely, a function h_k satisfying (3.11) is unique up to a multiplicative constant.

Proof of Proposition 3.4: We recall first that by (3.9), it holds almost surely

$$\lim_{n\to\infty} \Gamma_{k,\{n\}_\varepsilon} = 0. \tag{3.13}$$

As a consequence, almost surely, for n large enough, $2\Gamma_{k,\{n\}_\varepsilon} < 1$, thus $\Delta^{\varepsilon}_{k,n}$ is indeed well-defined and positive. We also derive (3.12) from (3.9) and (3.13). As a consequence, $\lim_{n\to\infty} \Delta^{\varepsilon}_{k,n} = 0$ almost surely.

Let us assume now that there exists a positive function h_k satisfying Inequality (3.11). Then, if $x, y \in \mathbb{X}$, setting $\mu_1 = \delta_x, \mu_2 = \delta_y$ and applying this inequality to the constant function $\mathbb{1}$, we almost surely get, for n large enough

$$\left| m_{k,n}(x) - \frac{h_k(x)}{h_k(y)} m_{k,n}(y) \right| \leqslant \Delta^{\varepsilon}_{k,n} m_{k,n}(x) = \underset{n\to\infty}{o}(m_{k,n}(x)).$$

Thus

$$\lim_{n\to\infty} \frac{h_k(x)}{h_k(y)} \frac{m_{k,n}(y)}{m_{k,n}(x)} = 1,$$

which readily implies that

$$\frac{h_k(x)}{h_k(y)} = \lim_{n\to\infty} \frac{m_{k,n}(x)}{m_{k,n}(y)}.$$

This yields the unicity of h_k up to a multiplicative constant, when it exists. Let us now prove the existence of a function h_k satisfying (3.11).

By Inequality (3.4), with $\mu_1 = \delta_x, \mu_2 = \delta_y$, one gets, $\mathbb{P}(d\omega)$-almost surely, for any $k \leqslant n \leqslant N$:

$$\left|\frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| \leqslant 2\Gamma_{k,n}\frac{m_{k,n}(x)}{m_{k,n}(y)}, \tag{3.14}$$

where the right-hand-side term is infinite on the event $\{\gamma_{n-1} = 0\}$. Setting

$$\mathbf{diam}_{k,n}(x,y) = \sup_{N_1,N_2 \geqslant n}\left|\frac{m_{k,N_1}(x)}{m_{k,N_1}(y)} - \frac{m_{k,N_2}(x)}{m_{k,N_2}(y)}\right|,$$

this yields, for any $x, y \in \mathbb{X}$,

$$\mathbf{diam}_{k,n}(x,y) \leqslant 4\Gamma_{k,n}\frac{m_{k,n}(x)}{m_{k,n}(y)}.$$

Exchanging the roles of x, y, one gets :

$$\min\left[\mathbf{diam}_{k,n}(x,y), \mathbf{diam}_{k,n}(y,x)\right] \leqslant 4\Gamma_{k,n}. \tag{3.15}$$

For all $k \geqslant 0$, $x, y \in \mathbb{X}$, both the sequences $(\mathbf{diam}_{k,n}(x,y))_{n\geqslant k}, (\mathbf{diam}_{k,n}(y,x))_{n\geqslant k}$ are non-increasing, and as a consequence from the definition of $\{n\}_\varepsilon$, $\{n\}_\varepsilon \leqslant n$, thus it holds

$$\begin{aligned}\min\left[\mathbf{diam}_{k,n}(x,y), \mathbf{diam}_{k,n}(y,x)\right] &\leqslant \min\left[\mathbf{diam}_{k,\{n\}_\varepsilon}(x,y), \mathbf{diam}_{k,\{n\}_\varepsilon}(y,x)\right] \\ &\leqslant \left(4\Gamma_{k,\{n\}_\varepsilon}\right) \underset{n\to\infty}{\longrightarrow} 0,\end{aligned} \tag{3.16}$$

$\mathbb{P}(d\omega)$-almost surely, for n large enough, by (3.13). Thus one of the sequences $(\mathbf{diam}_{k,n}(x,y))_{n\geqslant k}$, $(\mathbf{diam}_{k,n}(y,x))_{n\geqslant k}$ has 0 as an adherence value. Since these sequences are non decreasing and positive, they converge, thus, $\mathbb{P}$-a.s., one of them tends to 0. Without loss of generality, suppose that

$$\mathbf{diam}_{k,n}(x,y) \underset{n\to\infty}{\longrightarrow} 0.$$

Then, the sequence of positive real numbers $\left(\frac{m_{k,n}(x)}{m_{k,n}(y)}\right)_{n\geqslant k}$ is a Cauchy sequence, it converges to a nonnegative limit $l_k(x,y)$. Moreover, as a consequence from the definition of $\{n\}_\varepsilon$, for all $k \leqslant n \leqslant N$, it holds $\{n\}_\varepsilon \leqslant n \leqslant N$ and thus $k \leqslant k \wedge \{n\}_\varepsilon \leqslant N$. Applying (3.14) yields therefore

$$\left|\frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)}\right| \leqslant 2\Gamma_{k,k\wedge\{n\}_\varepsilon}\frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)}. \tag{3.17}$$

In particular, letting $N \to \infty$ in (3.17) proves that the limit $l_k(x,y) = \lim_{N\to\infty}\frac{m_{k,N}(x)}{m_{k,N}(y)}$ satisfies

$$\left|l_k(x,y) - \frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)}\right| \leqslant 2\Gamma_{k,k\wedge\{n\}_\varepsilon}\frac{m_{k,k\wedge\{n\}_\varepsilon}(x)}{m_{k,k\wedge\{n\}_\varepsilon}(y)} \tag{3.18}$$

Almost surely, it holds moreover for n large enough

$$k \wedge \{n\}_\varepsilon = \{n\}_\varepsilon \text{ and } 2\Gamma_{k,k\wedge\{n\}_\varepsilon} \leqslant \frac{1}{4}.$$

Plugging this into Equation (3.18) yields, for n large enough

$$\left|\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} - l_k(x,y)\right| \leqslant \frac{1}{4}\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}.$$

Since $\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} > 0$, this implies that $l_k(x,y) = \lim_{n\to\infty} \frac{m_{k,n}(x)}{m_{k,n}(y)} > 0$ and consequently,

$$\frac{m_{k,n}(y)}{m_{k,n}(x)} \underset{n\to\infty}{\longrightarrow} \frac{1}{l_k(x,y)} < \infty.$$

Note that Proposition 3.1 allows to prove that $\mathbb{P}(d\omega)$-almost surely, (3.14) holds jointly for any $k \leqslant n \leqslant N$ and any $x, y \in \mathbb{X}$, thus so does (3.16). Thus $\mathbb{P}(d\omega)$-almost surely, all the sequences of the form $\left(\frac{m_{k,n}(x)}{m_{k,n}(y)}\right)_{n\geqslant k}$ for all $k \geqslant 0, x, y \in \mathbb{X}$ converge to a positive limit as $n \longrightarrow \infty$.

Now, let us fix an arbitrary element $x_0 \in \mathbb{X}$, and set $h_k(x) = \lim_{n\to\infty} \frac{m_{k,n}(x)}{m_{k,n}(x_0)}$ for all x. The function h_k is positive and satisfies, for any $x, y \in \mathbb{X}$,

$$\frac{m_{k,N}(x)}{m_{k,N}(y)} \underset{N\to\infty}{\longrightarrow} \frac{h_k(x)}{h_k(y)}.$$

Plugging this into Equation (3.17), we obtain almost surely, for n large enough

$$\left|\frac{h_k(x)}{h_k(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}\right| \leqslant 2\Gamma_{k,\{n\}_\varepsilon} \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}. \tag{3.19}$$

Consequently,

$$\begin{aligned} h_k(x) &\leqslant h_k(y)\left(1 + 2\Gamma_{k,\{n\}_\varepsilon}\right) \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \\ &\leqslant h_k(y)\left(1 + 2\Gamma_{k,\{n\}_\varepsilon}\right) \frac{\|m_{k,\{n\}_\varepsilon}\|_\infty}{m_{k,\{n\}_\varepsilon}(y)} < \infty, \end{aligned}$$

by **A2**, which implies that h_k is bounded.
Notice now that for n large enough, taking $N = n$ in (3.17) yields

$$\left|\frac{m_{k,n}(x)}{m_{k,n}(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}\right| \leqslant 2\Gamma_{k,\{n\}_\varepsilon} \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \tag{3.20}$$

Therefore, combining (3.20) with (3.19), we obtain

$$\begin{aligned} \left|\frac{h_k(x)}{h_k(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| &\leqslant \left|\frac{m_{k,n}(x)}{m_{k,n}(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}\right| + \left|\frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)}\right| \\ &\leqslant 4\Gamma_{k,\{n\}_\varepsilon} \frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \end{aligned} \tag{3.21}$$

Once again, we recall that almost surely, for n large enough

$$2\Gamma_{k,\{n\}_\varepsilon} < 1.$$

Therefore (3.17) yields $\mathbb{P}$-a.s., for n large enough

$$\frac{m_{k,\{n\}_\varepsilon}(x)}{m_{k,\{n\}_\varepsilon}(y)} \leqslant \frac{1}{1 - 2\Gamma_{k,\{n\}_\varepsilon}} \frac{m_{k,n}(x)}{m_{k,n}(y)}. \tag{3.22}$$

Plugging this into (3.21) finally yields

$$\left|\frac{h_k(x)}{h_k(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| \leqslant \frac{4\Gamma_{k,\{n\}_\varepsilon}}{1 - 2\Gamma_{k,\{n\}_\varepsilon}} \frac{m_{k,n}(x)}{m_{k,n}(y)} = \frac{\Delta^\varepsilon_{k,n}}{2} \frac{m_{k,n}(x)}{m_{k,n}(y)} \tag{3.23}$$

Moreover, for any positive and finite measure $\mu_1 \in \mathcal{M}_+(\mathbb{X})$, any $y \in \mathbb{X}$, integrating (3.23) with respect to $\mu_1(dx)$, one gets

$$\left|\frac{\mu_1(h_k)}{h_k(y)} - \frac{\mu_1(m_{k,n})}{m_{k,n}(y)}\right| \leqslant \frac{\Delta^{\varepsilon}_{k,n}}{2}\frac{\mu_1(m_{k,n})}{m_{k,n}(y)},$$

Thus

$$|m_{k,n}(y)\mu_1(h_k) - \mu_1(m_{k,n})h_k(y)| \leqslant \frac{\Delta^{\varepsilon}_{k,n}}{2}\mu_1(m_{k,n})h_k(y).$$

Integrating with respect to any positive and finite measure $\mu_2(dy)$, this yields

$$|\mu_2(m_{k,n})\mu_1(h_k) - \mu_1(m_{k,n})\mu_2(h_k)| \leqslant \frac{\Delta^{\varepsilon}_{k,n}}{2}\mu_1(m_{k,n})\mu_2(h_k),$$

and finally,

$$\left|\frac{\mu_1(h_k)}{\mu_2(h_k)} - \frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})}\right| \leqslant \frac{\Delta^{\varepsilon}_{k,n}}{2}\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})}. \tag{3.24}$$

Let us prove now that h_k satisfies Inequality (3.11). It holds

$$\begin{aligned}\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} \\ &\quad + \left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV}.\end{aligned}$$

On the one hand, applying Inequality (3.3), one has, almost surely, for n large enough

$$\begin{aligned}\|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} &\leqslant \mu_1(m_{k,n})\,\|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \\ &\leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i)\mu_1(m_{k,n}) \\ &\leqslant 2\prod_{i=k}^{\{n\}_{\varepsilon}-1}(1-\gamma_i)\mu_1(m_{k,n}) \\ &\leqslant \frac{\Delta^{\varepsilon}_{k,n}}{2}\mu_1(m_{k,n}).\end{aligned}$$

On the other hand, by Equation (3.24), it holds $\mathbb{P}$-a.s., for n large enough

$$\begin{aligned}\left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \mu_2(m_{k,n})\left|\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\right| \\ &\leqslant \frac{\Delta^{\varepsilon}_{k,n}}{2}\mu_1(m_{k,n}).\end{aligned}$$

Finally, for n large enough, it indeed holds almost surely for n large enough

$$\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \leqslant \Delta^{\varepsilon}_{k,n}\|\mu_1 M_{k,n}\|_{TV}.$$

This ends the proof. □

The control provided in (3.11) only holds for large enough values of n which are random and depend on k. Under the stronger assumption **A 4+**, it is possible to obtain a control which is uniform in $k \leqslant n$.

Proposition 3.5. *Assume A1, A2, A4+ hold. Then almost surely, for any $k \leqslant n$ and any measure $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, it holds*

$$\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \leqslant 4\Gamma_{k,n}\|\mu_1 M_{k,n}\| = \underset{n\to\infty}{o}(\delta^n\|\mu_1 M_{k,n}\|), \tag{3.25}$$

for any $\delta \in (\bar{\gamma}, 1)$.

Proof of Proposition 3.5: We start by letting N go to infinity in (3.14) and obtain, almost surely, for all $k, n \in \mathbb{N}_0$ and all $x, y \in \mathbb{X}$

$$\left|\frac{h_k(x)(x)}{h_k(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)}\right| 2\Gamma_{k,n}\frac{m_{k,n}(x)}{m_{k,n}(y)}, \tag{3.26}$$

where by **A4+** almost surely, $\gamma_{n-1} > 0$ for all $n \geqslant 1$. Replacing (3.23) by (3.26) in the proof of Proposition 3.4, we obtain first

$$\left|\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\right| \leqslant 2\Gamma_{k,n}\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} \tag{3.27}$$

for any non-zero positives measures μ_1, μ_2. From this we derive both

$$\begin{aligned}\|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} &\leqslant \mu_1(m_{k,n})\,\|\mu_2 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \\ &\leqslant 2\prod_{i=k}^{n-1}(1-\gamma_i)\mu_1(m_{k,n}) \\ &\leqslant 2\Gamma_{k,n}\mu_1(m_{k,n})\end{aligned}$$

and

$$\begin{aligned}\left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \mu_2(m_{k,n})\left|\frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\right| \\ &\leqslant 2\Gamma_{k,n}\mu_1(m_{k,n}),\end{aligned}$$

which we combine to obtain

$$\begin{aligned}\left\|\mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} &\leqslant \|\mu_1 M_{k,n} - \mu_1(m_{k,n})\mu_2 \cdot M_{k,n}\|_{TV} \\ &\quad + \left\|\mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n}\right\|_{TV} \\ &\leqslant 4\Gamma_{k,n}\|\mu_1 M_{k,n}\|.\end{aligned}$$

□

3.3. *Proof of Theorem 2.1.*

Proof of assertion i) : Uniform geometric ergodicity: Let us take $k = 0$ in Proposition 3.4. Then, for any $\varepsilon > 0$, $\mathbb{P}(d\omega)$-almost surely, noting $h = h_0$, it holds for any finite and positive measures μ_1, μ_2, on $\mathbb{X}$,

$$\left\|\mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)}\mu_2 M_{0,n}\right\|_{TV} \leqslant \Delta^{\varepsilon}_{0,n}\|\mu_1 M_{0,n}\|,$$

where, $\mathbb{P}(d\omega)$-almost surely,

$$\limsup_{n\to\infty}\left(\Delta^{\varepsilon}_{0,n}\right)^{\frac{1}{n}} \leqslant \bar{\gamma}^{\mathbb{P}[\gamma\geqslant\varepsilon]} \in [0,1).$$

Let now $\delta \in (\bar{\gamma}^{\mathbb{P}[\gamma>0]}, 1)$. For $\varepsilon > 0$ small enough, $\delta > \bar{\gamma}^{\mathbb{P}[\gamma\geqslant\varepsilon]}$. As a consequence, $\mathbb{P}(d\omega)$-almost surely, for n large enough, (depending on ω),

$$\Delta^{\varepsilon}_{0,n} \leqslant \delta^n.$$

Thus, $\mathbb{P}(d\omega)$-almost surely, for any $\delta \in (\bar{\gamma}, 1)$, for n large enough and any positive and finite measures μ_1, μ_2,

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n} \right\|_{TV} \leqslant \delta^n \|\mu_1 M_{0,n}\|_{TV}.$$

This proves Equation (2.1). □

Proof of assertion ii): The proof relies on a classical time-reversal technique, see e.g. Cogburn (1984); Orey (1991), or Hennion (1997) for a version that is closer to our context. As stated in Cornfeld et al. (1982, II.10.4, pp.239-241), the ergodic system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ can be extended as an invertible ergodic system $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}}, \overline{\theta})$, such that $\Omega \subset \overline{\Omega}$, $\overline{\theta}|_{\Omega} = \theta$, and $\overline{\theta}$ is a bijective, bimeasurable, measure preserving and ergodic mapping. The definitions of $M_{k,n}$, $c_n, d_n, \nu_n, \gamma_n$ can be naturally extended to all $k \leqslant n$ in $\mathbb{Z}$, and one still has $c_n = c \circ \overline{\theta}^n$, $d_n = d \circ \overline{\theta}^n$, $\gamma_n = \gamma \circ \overline{\theta}^n$ for $n \in \mathbb{Z}$. Assumption **A4** implies that all the $(\gamma_n)_{n\in\mathbb{Z}}$ are almost surely positive and have log-moments.

Therefore, Lemma 3.2 and Proposition 3.1 extend to indexes $k \leqslant n \leqslant N \in \mathbb{Z}$.

For nonnegative $n \leqslant N$, for any positive measures μ_1, μ_2 on $\mathbb{X}$, one has in particular

$$\|\mu_1 \cdot M_{-n,0} - \mu_2 \cdot M_{-n,0}\|_{TV} \leqslant 2 \prod_{i=0}^{n-1} (1 - \gamma_{-i-1}). \tag{3.28}$$

With $\mu_2 = \mu_1 M_{-N,-n}$, this yields :

$$\|\mu_1 \cdot M_{-n,0} - \mu_1 \cdot M_{-N,0}\|_{TV} \leqslant 2 \prod_{i=0}^{n-1} (1 - \gamma_{-i-1}).$$

Noticing that $\overline{\theta}$ is now an ergodic automorphism of the measured space $\overline{\Omega}$, and applying Birkhoff-Khinchin Ergodic Theorem as stated in Cornfeld et al. (1982, Theorem 1, p.11), one gets, for almost any $\omega \in \overline{\Omega}$

$$\frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_{-i-1}) \circ \overline{\theta}^{-i} \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[\log(1 - \gamma_{-1})\right] = \mathbb{E}\left[\log(1 - \gamma)\right].$$

Thus

$$\left(\prod_{i=0}^{n-1} (1 - \gamma_{-i-1}) \right)^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} \exp\left(\mathbb{E}\left[\log(1 - \gamma)\right]\right) = \bar{\gamma} < 1.$$

Therefore, almost surely, the sequence $(\mu_1 \cdot M_{-n,0})_{n\in\mathbb{N}_0}$ is a Cauchy sequence in the space $\mathcal{M}_1(\mathbb{X})$ of probabilities on $\mathbb{X}$, endowed with the total variation norm. It thus converges almost surely to a random probability π_{μ_1} on $\mathbb{X}$. For any finite, positive non-zero measures μ_1, μ_2, plugging π_{μ_1}, π_{μ_2} into (3.28), one proves that for almost any ω,

$$\pi_{\mu_1} = \pi_{\mu_2}.$$

Thus, there exists a random probability π, such that, almost surely, for any positive measure μ

$$\mu \cdot M_{-n,0} \underset{n\to\infty}{\longrightarrow} \pi \text{ in total variation distance.}$$

By stationarity of $\overline{\theta}$,

$$\mu \cdot M_{-n,0} \overset{d}{=} \mu \cdot M_{0,n},$$

which proves that, noting Λ the distribution of π,

$$\mu \cdot M_{0,n} \underset{n\to\infty}{\overset{d}{\longrightarrow}} \Lambda.$$

□

Before proving the last assertion of Theorem 2.1, we need to present the following lemma which establishes a sharp lower bound on some triangular inequality.

Lemma 3.6. *Assume A2 holds. Then, for any $0 \leqslant k < n$ and any $\mu \in \mathcal{M}_+(\mathbb{X})$*

$$\gamma_k \|\mu M_{0,k+1}\|_{TV} \|\!| M_{k+1,n} |\!\| \leqslant \|\mu M_{0,n}\|_{TV} \leqslant \|\mu M_{0,k+1}\|_{TV} \|\!| M_{k+1,n} |\!\|. \tag{3.29}$$

Proof of Lemma 3.6: Let $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$ and $0 \leqslant k < n$.

$$\mu M_{0,n} = (\mu M_{0,k}) M_k M_{k+1,n} \geqslant c_k \|\mu M_{0,k+1}\|_{TV} \nu_k M_{k+1,n},$$

thus

$$\|\mu M_{0,n}\|_{TV} = \mu M_{0,n} \mathbb{1} \geqslant c_k \|\mu M_{0,k+1}\| (\nu_k M_{k+1,n})(\mathbb{1}) = c_k \|\mu M_{0,k+1}\|_{TV} \|\nu_k M_{k+1,n}\|_{TV}.$$

By definition of d_{k+1}, it holds additionally

$$\|\nu_k M_{k+1,n}\| \geqslant d_{k+1} |\!\| M_{k+1,n} |\!\|.$$

Combining the two previous inequalities, we get

$$\|\mu M_{0,n}\|_{TV} \geqslant c_k d_{k+1} \|\mu M_{0,k+1}\|_{TV} |\!\| M_{k+1,n} |\!\| = \gamma_k \|\mu M_{0,k+1}\|_{TV} |\!\| M_{k+1,n} |\!\|$$

To obtain the second inequality of the lemma we simply write

$$\begin{aligned} \|\mu M_{0,n}\| &= \mu M_{0,n} \mathbb{1} \\ &= \mu M_{0,k+1} M_{k+1,n} \mathbb{1} \\ &\leqslant \sup_{x \in \mathbb{X}} \delta_x M_{k+1,n} \mathbb{1} \times \mu M_{0,k+1} \mathbb{1} \\ &\leqslant |\!\| M_{k+1,n} |\!\| \|\mu M_{0,k+1}\|_{TV}. \end{aligned}$$

□

Proof of assertion iii): We notice first that for any fixed integer, the $\mathbb{P}-$almost sure convergence

$$n^{-1} \log |\!\| M_{0,n} |\!\| \underset{n \to \infty}{\longrightarrow} \lambda := \inf_{N \geqslant 1} \frac{1}{N} \mathbb{E} \left[\log |\!\| M_{0,N} |\!\| \right]$$

is a classical consequence of Kingsman's subbaditive ergodic theorem and the subadditivity property:

$$\log |\!\| M_{0,n+p} |\!\| \leqslant \log |\!\| M_{0,n} |\!\| + \log |\!\| M_{n,n+p} |\!\|.$$

Note that applying this theorem requires Assumption **A3** to ensure the integrability of $\log^+ |\!\| M_{0,n} |\!\|$. It is also classical that the same convergence holds for shifted sequences : for any $k \geqslant 0$,

$$n^{-1} \log |\!\| M_{k,n} |\!\| \underset{n \to \infty}{\longrightarrow} \lambda \tag{3.30}$$

almost surely.
Let now μ be a positive, finite measure on $\mathbb{X}$. Let $k \geqslant 0$. From Lemma 3.6, it holds for $n \geqslant k$

$$\gamma_k \|\mu M_{0,k}\|_{TV} |\!\| M_{k,n} |\!\| \leqslant \|\mu M_{0,n}\|_{TV} \leqslant \|\mu M_{0,k}\|_{TV} |\!\| M_{k,n} |\!\|.$$

Thus

$$\left| \frac{1}{n} \log \|\mu M_{0,n}\|_{TV} - \frac{1}{n} \log |\!\| M_{k,n} |\!\| \right| \leqslant \frac{1}{n} \left(|\log \|\mu M_{0,k}\|| + |\log \gamma_k| \right) \underset{n \to +\infty}{\longrightarrow} 0$$

almost surely on the event $\{\gamma_k > 0\}$. Combining this last estimate with (3.30) proves that the convergence

$$\frac{1}{n} \log \|\mu M_{0,n}\| \underset{n \to \infty}{\longrightarrow} \lambda$$

holds almost surely on the event $\{\gamma_k > 0\}$, thus also on the event $\bigcup_{k \geqslant 0} \{\gamma_k > 0\}$. Under Assumptions **A1** and **A4**, $\mathbb{P}[\bigcup_{k \geqslant 0} \{\gamma_k > 0\}] = 1$, by Birkhoff's ergodic theorem, thus the convergence (2.2) holds $\mathbb{P}(d\omega)$-almost surely. □

3.4. *The independent case : proof of Theorems 2.2 and 2.3.* Let us introduce the Markov chain $(\mu_n)_{n\geqslant 0}$ with state space $\mathcal{M}_1(\mathbb{X})$, defined by $\mu_{n+1} = \mu_n \cdot M_n = \mu_0 \cdot M_{0,n+1}$. The process $(\mu_n, M_n)_{n\geqslant 0}$ is then clearly also a Markov chain with state space $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$ and transition kernel :

$$Qf(\mu, M) = \int f(\mu \cdot M, N) d\mathcal{P}(N).$$

We denote $\mathbb{P}_{\bar{\chi}}$ the law of the Markov chain $((\mu_n, M_n))_{n\geqslant 0}$ when (μ_0, M_0) is distributed according to a measure $\bar{\chi}$ on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$. Theorems 2.2 and 2.3 rely on the study of the invariant measures and the ergodicity properties of the Markov chains $(\mu_n)_{n\geqslant 0}$ and $(\mu_n, M_n)_{n\geqslant 0}$. In particular, we show that the limit distribution Λ of the Markov chain (μ_n) is its only invariant distribution. This is stated in the following proposition.

Proposition 3.7. *Suppose $(M_n)_{n\geqslant 0}$ is an i.i.d sequence of elements of $\mathcal{K}^+$ distributed according to $\mathcal{P}$, and satisfying the assumptions of Theorem 2.1. Then*

i) For any initial distribution χ on $\mathcal{M}_1(\mathbb{X})$, the Markov chain $(\mu_n)_{n\geqslant 0}$ converges weakly to Λ.
ii) Λ is the only invariant measure of the Markov chain $(\mu_n)_{n\geqslant 0}$.
iii) $\Lambda \otimes \mathcal{P}$ is the only invariant measure of the Markov chain $(\mu_n, M_n)_{n\geqslant 0}$.

As a consequence, the dynamical systems associated with the Markov chains $(\mu_n)_{n\geqslant 0}$ and $(\mu_n, M_n)_{n\geqslant 0}$ are ergodic.

To prove Proposition 3.7, we need to define the convolution operation $\star$ between probability measures on $\mathcal{M}_1(\mathcal{K}^+)$ as follows : For any $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{M}_1(\mathcal{K}^+)$, $\mathcal{Q}_1 \star \mathcal{Q}_2$ is the law of $N_1 N_2$, where $(N_1, N_2) \sim \mathcal{Q}_1 \otimes \mathcal{Q}_2$. We note, for any $\mathcal{Q} \in \mathcal{M}_1(\mathcal{K}^+)$, $\mathcal{Q}^{\star n}$ the n-th convolution power of $\mathcal{Q}$. As an example, if $N_0, \ldots, N_{n-1}$ are i.i.d with law $\mathcal{Q}$, $\mathcal{Q}^{\star n}$ is simply the distribution of $N_{0,n} = N_0 \cdots N_{n-1}$. Given a probability distribution χ on $\mathcal{M}_1(\mathbb{X})$ and $\mathcal{Q}$ on $\mathcal{K}^+$, we also note $\chi \dot{\star} \mathcal{Q}$ the law of $\mu \cdot N$, where $(\mu, N) \sim \chi \otimes \mathcal{Q}$. These operations, previously defined in Bougerol and Lacroix (1985) in a finite dimensional context, satisfy some elementary properties, summed up in the following lemma.

Lemma 3.8. *Let $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ be probability measures on $\mathcal{K}^+$ and χ be a probability measure on $\mathcal{M}_1(\mathbb{X})$, it holds*

i) $(Q_1 \star Q_2) \star Q_3 = Q_1 \star (Q_2 \star \mathcal{Q}_3)$,
ii) $(\chi \dot{\star} \mathcal{Q}_1) \dot{\star} \mathcal{Q}_2 = \chi \dot{\star} (\mathcal{Q}_1 \star \mathcal{Q}_2)$,
iii) For each $\mathcal{Q} \in \mathcal{M}_1(\mathcal{K}^+)$, $\chi \mapsto \chi \dot{\star} \mathcal{Q}$ is continuous with respect to the topology of convergence in law on $\mathcal{M}_1(\mathcal{M}_1(\mathbb{X}))$.

Proof of Lemma 3.8: Consider $(N_1, N_2, N_3) \sim \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \mathcal{Q}_3$. It holds

$$N_1 N_2 N_3 = (N_1 N_2) N_3 = N_1 (N_2 N_3),$$

with $(N_1 N_2) N_3 \sim (Q_1 \star Q_2) \star Q_3$ and $N_1 (N_2 N_3) \sim Q_1 \star (Q_2 \star \mathcal{Q}_3)$. This yields *i)*.
Let us prove now point *ii)*. Consider $(\mu, N_1, N_2) \sim \chi \otimes \mathcal{Q}_1 \otimes \mathcal{Q}_2$. It holds

$$\mu \cdot (N_1 N_2) = (\mu \cdot N_1) \cdot N_2,$$

with $\mu \cdot (N_1 N_2) \sim \chi \dot{\star} (Q_1 \star Q_2)$ and $(\mu \cdot N_1) \cdot N_2 \sim (\chi \dot{\star} Q_1) \dot{\star} Q_2$. This yields *ii)*.
Let us move to the proof of *iii)*. Consider a sequence of probability measures (χ_n) on $\mathcal{M}_1(\mathbb{X})$, converging in distribution to χ. Let us show that $(\chi_n \dot{\star} \mathcal{Q})_{n\geqslant 0}$ converges in distribution towards $\chi \dot{\star} \mathcal{Q}$. Let f be a continuous, bounded function on $\mathcal{M}_1(\mathbb{X})$, it holds :

$$\int f(\mu) d(\chi_n \dot{\star} \mathcal{Q})(\mu) = \int \int f(\mu \cdot N) d\chi_n(\mu) d\mathcal{Q}(N) = \int \chi_n(g_N) d\mathcal{Q}(N),$$

where, for each $N \in \mathcal{K}^+$, the function $g_N : \mu \mapsto f(\mu \cdot N)$ is continuous and bounded. Thus

$$\chi_n(g_N) = \int f(\mu \cdot N) d\chi_n(\mu) \to \int f(\mu \cdot N) d\chi(\mu) = \chi(g_N).$$

This yields, by dominated convergence, as $n \to \infty$,

$$\int \chi_n(g_N) d\mathcal{Q}(N) = \int f(\mu) d(\chi_n \dot{\star} \mathcal{Q})(\mu) \longrightarrow \int \chi(g_N) d\mathcal{Q}(N) = \int \int f(\mu \cdot N) d\chi(\mu) d\mathcal{Q}(N),$$

which implies *iii)*.

□

Proof of Proposition 3.7: Let f be a continuous and bounded function on $\mathcal{M}_1(\mathbb{X})$, it holds

$$\chi \dot{\star} \mathcal{P}^{\star n}(f) = \int f(\mu) d(\chi \dot{\star} \mathcal{P}^{\star n})(\mu) = \int \int f(\mu \cdot M_{0,n}) d\chi(\mu) d\mathcal{P}^{\star n}(M_{0,n}).$$

However, for any $\mu \in \mathcal{M}_1(\mathbb{X})$, Theorem 2.1, *ii)* states that $(\delta_\mu \dot{\star} \mathcal{P}^{\star n})_{n \geqslant 0}$ converges weakly towards Λ. Thus, for any $\mu \in \mathcal{M}_1(\mathbb{X})$, as $n \to \infty$

$$\int f(\mu \cdot M_{0,n}) d\mathcal{P}^{\star n}(M_{0,n}) = (\delta_\mu \dot{\star} \mathcal{P}^{\star n})(f) \underset{n \to \infty}{\longrightarrow} \Lambda(f).$$

By dominated convergence, this yields

$$\chi \dot{\star} \mathcal{P}^{\star n}(f) = \int \int f(\mu \cdot M_{0,n}) d\chi(\mu) d\mathcal{P}^{\star n}(M_{0,n}) \underset{n \to \infty}{\longrightarrow} \Lambda(f),$$

which proves the weak convergence

$$\chi \dot{\star} \mathcal{P}^{\star n} \underset{n \to \infty}{\longrightarrow} \Lambda$$

in the metric space $\mathcal{M}_1(\mathcal{M}_1(\mathbb{X}))$, for any probability distribution χ. This proves *i)*.

Since, by Lemma 3.8, *iii)*, the map $\mu \mapsto \mu \dot{\star} \mathcal{P}$ is continuous, this proves that Λ is one of its fixed points, namely :

$$\Lambda = \Lambda \dot{\star} \mathcal{P}.$$

On the other hand, if $\chi \dot{\star} \mathcal{P} = \chi$, the sequence $(\chi \dot{\star} \mathcal{P}^{\star n})_{n \geqslant 0}$ is constant and converges to χ. By unicity of the limit, it holds

$$\chi = \Lambda.$$

This proves that Λ is the only invariant measure of the Markov chain (μ_n), i.e. *ii)*.

Let $(\mu_0, M_0) \sim \Lambda \otimes \mathcal{P}$. Then $\mu_1 = \mu_0 \cdot M_0 \sim \Lambda \dot{\star} \mathcal{P}$, $M_1 \sim \mathcal{P}$ and M_1 is independent of μ_0, M_0 and thus μ_1. Therefore $(\mu_1, M_1) \sim \Lambda \otimes \mathcal{P}$, and $\Lambda \otimes \mathcal{P}$ is thus an invariant measure of the Markov chain $(\mu_n, M_n)_{n \geqslant 0}$.

Conversely, consider now a probability measure $\bar{\chi}$ on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$, suppose it is an invariant measure of the Markov chain $((\mu_n, M_n))_{n \geqslant 0}$. The definition of the transition kernel Q implies that $\mu_1 = \mu_0 \cdot M_0$, $M_1 \sim \mathcal{P}$ and M_1 is independent of (μ_0, M_0), and therefore M_1 is independent of μ_1. However the second term (μ_1, M_1) of the Markov chain is distributed according to $\bar{\chi} Q = \bar{\chi}$ by invariance. Thus $\bar{\chi}$ is of the form $\bar{\chi} = \chi \otimes \mathcal{P}$.

Additionally, if $(\mu_0, M_0) \sim \bar{\chi} = \chi \otimes \mathcal{P}$, then $\mu_1 = \mu_0 \cdot M_0 \sim \chi \dot{\star} \mathcal{P}$. But by invariance of $\bar{\chi}$, $\mu_1 \sim \chi$, thus

$$\chi \dot{\star} \mathcal{P} = \chi.$$

By Proposition 3.7, this implies that $\chi = \Lambda$. Finally, this proves that $\Lambda \otimes \mathcal{P}$ is the only invariant measure of the Markov chain $(\mu_n, M_n)_{n \geqslant 0}$. By Corollary 5.12 of Hairer (2018), since both the processes $(\mu_n)_{n \geqslant 0}$ and $(\mu_n, M_n)_{n \geqslant 0}$ are Markov chains with a unique invariant measure, they are both ergodic. □

One additional lemma is required before proving Theorem 2.2.

Lemma 3.9. *Let $(M_n)_{n\geqslant 0}$ be an i.i.d sequence of elements of $\mathcal{K}^+$ with law $\mathcal{P}$, satisfying the assumptions **A1**, **A2** and **A4+**. The sets*

$$\{\forall \mu \in \mathcal{M}_1(\mathbb{X}), \limsup_{n\to\infty} \log \|\mu M_{0,n}\| = +\infty\},$$
$$\{\exists \mu \in \mathcal{M}_1(\mathbb{X}), \limsup_{n\to\infty} \log \|\mu M_{0,n}\| = +\infty\}$$

and the event

$$\{\limsup_{n\to\infty} \log |||M_{1,n}||| = +\infty\}$$

coincide up to $\mathcal{P}^{\otimes\mathbb{N}}$-negligible events. A similar statement holds replacing lim sup *by* $-$ lim inf *in the three events.*

Proof of Lemma 3.9: From Lemma 3.6, with $k = 0$, we get

$$\gamma_0 \|\mu M_0\|_{TV} |||M_{1,n}||| \leqslant \|\mu M_{0,n}\|_{TV} \leqslant \|\mu M_0\|_{TV} |||M_{1,n}|||,$$

where $\gamma_0 \|\mu M_0\|_{TV} > 0$ almost surely by **A2** and **A4+**. The lemma is a straightforward consequence of this inequality. □

Let us prove now Theorem 2.2.

Proof of Theorem 2.2, i): Let us notice first that when μ is a probability measure, $\rho : (\mu, M) \mapsto \log \|\mu M\|$ satisfies the cocycle property

$$\begin{aligned}
\rho(\mu_0, M_{0,n}) &= \log \|\mu_0 M_{0,n}\| \\
&= \sum_{k=0}^{n-1} \log \left(\frac{\|\mu_0 M_{0,k+1}\|}{\|\mu_0 M_{0,k}\|} \right) \\
&= \sum_{k=0}^{n-1} \log \|(\mu_0 \cdot M_{0,k}) M_k\| \\
&= \sum_{k=0}^{n-1} \rho(\mu_k, M_k). \qquad (3.31)
\end{aligned}$$

From Equation (3.31), we derive

$$\frac{1}{n} \log \|\mu_0 M_{0,n}\| = \frac{1}{n} \sum_{k=0}^{n-1} \rho(\mu_k, M_k).$$

By Birkhoff's Ergodic Theorem, since (μ_k, M_k) is an ergodic Markov chain, with stationary distribution $\Lambda \otimes \mathcal{P}$, this quantity converges $\mathbb{P}_{\Lambda\otimes\mathcal{P}}$-almost surely and in $\mathcal{L}^1(\mathbb{P}_{\Lambda\otimes\mathcal{P}})$ towards $\int \rho d(\Lambda \otimes \mathcal{P})$ provided ρ is an $\mathcal{L}^1$ function with respect to $\Lambda \otimes \mathcal{P}$. Let us check now this integrability property. Let $(\mu_0, (M_n)_{n\geqslant 0}) \sim \Lambda \otimes \mathcal{P}^{\mathbb{N}}$. Then, applying (3.29) with $n = 2$ yields

$$\gamma_0 \|\mu_0 M_0\| |||M_1||| \leqslant \|\mu_0 M_{0,2}\| \leqslant \|\mu_0 M_0\| |||M_1|||.$$

Noting $\mu_1 = \mu_0 \cdot M_0$, we get, since $\|\mu_0 M_0\| \neq 0$ almost surely,

$$\gamma_0 |||M_1||| \leqslant \|\mu_1 M_1\| \leqslant |||M_1|||,$$

thus

$$|\rho(\mu_1, M_1)| \leqslant |\log \|M_1\|| + |\log(\gamma_0)|.$$

Note that $(\mu_1, M_1) \sim \Lambda \otimes \mathcal{P}$, since by definition $\mu_1 = \mu_0 \cdot M_0$ and $(\mu_0, (M_n)_{n\in\mathbb{N}}) \sim \Lambda \otimes \mathcal{P}^{\otimes\mathbb{N}}$. Thus, under **A3+** and **A4+**, it holds

$$\mathbb{E}\left[|\rho(\mu_1, M_1)|\right] = \int |\rho| d(\Lambda \otimes \mathcal{P}) \leqslant \mathbb{E}|\log \|M_1\|| + \mathbb{E}|\log(\gamma_0)| < \infty.$$

This proves that ρ is integrable with respect to $\Lambda \otimes \mathcal{P}$, thus the convergence $n^{-1}\log(\|\mu_0 M_{0,n}\|) \underset{n\to\infty}{\longrightarrow} \int \rho \, d\Lambda \otimes \mathcal{P}$ holds in $\mathcal{L}^1(\Lambda \otimes \mathbb{P})$ and $\mathbb{P}_{\Lambda\otimes\mathbb{P}}$-almost surely. Since by Theorem 2.1, almost surely, for all μ, it holds $n^{-1}\log\|\mu M_{0,n}\| \underset{n\to\infty}{\longrightarrow} \lambda$, by unicity of the almost sure limit,

$$\lambda = \int \rho \, d(\Lambda \otimes \mathcal{P}).$$

□

Proof of Theorem 2.3, ii): Note now $X_n = \rho(\mu_{n-1}, M_{n-1})$, for $n \geqslant 1$. Then it holds, for $n \geqslant 0$, for any probability measure μ_0

$$(\mu_{n+1}, X_{n+1}) = (\mu_n \cdot M_n, \rho(\mu_n, M_n)).$$

Thus, $(\mu_n, X_n)_{n\geqslant 0}$ is a Markov chain on $\mathcal{M}_1(\mathbb{X}) \times \mathbb{R}$ such that

$$\mathbb{P}\left[(\mu_{n+1}, X_{n+1}) \in A \times B | (\mu_n, X_n)\right] = \int \mathbb{1}_A(\mu_n \cdot M)\mathbb{1}_B\left(\rho(\mu_n, M)\right) d\mathcal{P}(M).$$

Thus $S_n = \log\|\mu M_{0,n}\| = X_1 + \cdots + X_n$ is a Markov random walk associated with (μ_n, X_n), in the sense of Alsmeyer (2001). Suppose that $\lambda = 0$. By Theorem 2.1, it holds

$$n^{-1}S_n = n^{-1}\log\|\mu M_{0,n}\| \underset{n\to\infty}{\longrightarrow} 0,$$

$d\mathbb{P}_{\Lambda\otimes\mathcal{P}^{\otimes\mathbb{N}}}(\mu, (M_n)_{n\geqslant 0})$-almost surely, thus in probability with respect to $\mathbb{P}_{\Lambda\otimes\mathcal{P}^{\otimes\mathbb{N}}}$. Since moreover, (μ_n) is an ergodic Markov chain, the assumptions of Alsmeyer (2001) are satisfied. If there exists a function η such that $\mathbb{P}_{\Lambda\otimes\mathcal{P}}$-almost surely, for $n \geqslant 1$,

$$X_n = \eta(\mu_n) - \eta(\mu_{n-1}), \tag{3.32}$$

then taking $n = 1$ shows that we are in the case of Null Homology (NH). In this case, it holds moreover

$$\log\|\mu_0 M_{0,n}\| = X_1 + \cdots + X_n = \eta(\mu_n) - \eta(\mu_0).$$

Thus, almost surely, noting a, b the respective infimum and supremum of the support of $\eta(\mu)$, when $\mu \sim \Lambda$, since the sequence (μ_n) is a stationary and ergodic sequence with law Λ, it holds

$$\liminf_{n\to\infty} \log\|\mu_0 M_{0,n}\| = a - \eta(\mu_0), \text{ and } \limsup_{n\to\infty} \log\|\mu_0 M_{0,n}\| = b - \eta(\mu_0).$$

Thus the almost sure finiteness of these quantities are respectively equivalent to the finiteness of a and b. If Equation (3.32) does not hold, then we are in the setup of Theorem 2 or 3 of Alsmeyer (2001). These two Theorems imply that the Markov Random Walk (S_n) oscillates : $\limsup S_n = +\infty$ and $\liminf S_n = -\infty$ $\mathbb{P}_{\Lambda\otimes\mathcal{P}}-$almost surely. However, by Lemma 3.9, *ii)*, this implies that $\mathcal{P}^{\otimes\mathbb{N}}$-almost surely, for every $\mu \in \mathcal{M}_1(\mathbb{X})$,

$$\limsup_{n\to\infty} \log\|\mu M_{0,n}\| = -\liminf_{n\to\infty} \log\|\mu M_{0,n}\| = +\infty.$$

This concludes the proof. □

4. Sufficient conditions under uniform positivity assumptions

In the finite dimensional case $\mathbb{X} = \{1, \ldots, p\}$, that is when studying products of $p \times p$ matrices, similar (and actually, more complete) results are obtained in Hennion (1997). They rely on the very mild assumption

A5. $\mathbb{P}\left[\bigcup_{k\in\mathbb{N}} \left\{M_{0,k} \in \mathring{\mathcal{S}}\right\}\right] = 1,$

where $\mathring{\mathcal{S}}$ refers to the set of $p \times p$ matrices with positive entries. We expect that this approach, based on Hilbert contractions, might be extended in infinite dimensional contexts. This will require to introduce the notion of uniformly positive operators to strengthen the notion of positive matrices, and state an infinite dimensional generalization of **A5**, as we explain in Subsection 4.2.

This section aims at comparing our assumptions both with **A5**, and its natural generalization in infinite dimension.

We did not success in proving that **A5** alone is enough for our assumptions to hold. However we provide mild additional assumptions that, together with **A5**, constitute sufficient conditions for our assumptions (**A2**, **A3**, **A4**, **A4+**) to hold, and thus for Theorems 2.1, 2.2, 2.3 to apply.

4.1. *The finite dimensional case.* Let us focus in this subsection on the case where $\mathbb{X}$ is finite, let us note $p = |\mathbb{X}|$. Consider a stationary and ergodic sequence $(M_n)_{n\in\mathbb{N}_0}$ of $p \times p$ matrices with non-negative entries. Checking whether Assumptions **A1**, **A2**, **A3** are satisfied is quite straightforward, since these three assumptions only involve the law of the first matrix of the sequence. Let us see now how the additional Assumption **A5** can help exhibit an admissible triplet in order to check that Assumptions **A4** holds.

Lemma 4.1. *Consider a random, stationary sequence of $p \times p$ matrices $M_n = (M_n(x,y))_{x,y\in\mathbb{X}}$, with nonnegative entries, satisfying **A2** and **A5**. Then for any measurable map $\omega \in \Omega \mapsto \nu(\omega) \in \mathcal{M}_1(\mathbb{X})$, there exists a random variable $d : \omega \mapsto d(\omega)$ such that $\mathbb{P}[d > 0] = 1$ and (1.5) holds.*

Proof: The following decomposition holds : for any $1 \leqslant k \leqslant n$, $x \in \mathbb{X}$, $\omega \in \Omega$

$$m_{1,n}(x) = \delta_x M_{1,n}\mathbb{1} = \delta_x M_{1,k} m_{k,n} = \sum_{z\in\mathbb{X}} M_{1,k}(x,z) m_{k,n}(z).$$

Thus

$$\nu_0(m_{1,n}) = \nu_0 M_{1,n}\mathbb{1} = \sum_{y,z\in\mathbb{X}} \nu_0(y) M_{1,k}(y,z) m_{k,n}(z),$$

where we note $\nu_0 = \nu_\omega$. The fact that ν_0 is a probability measure yields, for any $1 \leqslant k \leqslant n$:

$$m_{1,n}(x) \leqslant \sup_{y,z\in\mathbb{X}} \left(\frac{M_{1,k}(x,z)}{M_{1,k}(y,z)}\right) \sum_{y\in\mathbb{X}} \nu_0(y) M_{1,k}(y,z) m_{k,n}(z) \leqslant \sup_{y,z\in\mathbb{X}} \frac{M_{1,k}(x,z)}{M_{1,k}(y,z)} \nu_0(m_{1,n}),$$

with the convention $\frac{M_{1,k}(x,z)}{M_{1,k}(y,z)} = 0$ as soon as $M_{1,k}(x,z) = 0$ and $\frac{M_{1,k}(x,z)}{M_{1,k}(y,z)} = \infty$ if $M_{1,k}(x,z) \neq 0$ and $M_{1,k}(y,z) = 0$. Thus, for any $n \geqslant k$,

$$\frac{\|\!|M_{1,n}|\!\|}{\nu_0(m_{1,n})} = \frac{\|m_{1,n}\|_\infty}{\nu_0(m_{1,n})} \leqslant \sup_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(x,z)}{M_{1,k}(y,z)}. \tag{4.1}$$

This yields

$$\inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(y,z)}{M_{1,k}(x,z)} \leqslant \inf_{n\geqslant k} \frac{\nu_0(m_{1,n})}{\|m_{1,n}\|_\infty}, \tag{4.2}$$

and therefore

$$\sup_{k\in\mathbb{N}} \inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(y,z)}{M_{1,k}(x,z)} \leqslant \liminf_n \frac{\nu_0(m_{1,n})}{\|\!|M_{1,n}|\!\|}.$$

By Assumption **A5**, $\mathbb{P}(d\omega)$- almost surely, there exists a random integer k_ω such that $M_{1,k_\omega}(\omega) \in \mathring{\mathcal{S}}$. Since $\mathbb{X}$ is finite, we get, for $\mathbb{P}$-almost any ω,

$$1 < \inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k_\omega}(y,z)}{M_{1,k_\omega}(x,z)} < \sup_{k\in\mathbb{N}} \inf_{x,y,z\in\mathbb{X}} \frac{M_{1,k}(y,z)}{M_{1,k}(x,z)} \leqslant \liminf_n \frac{\nu_0(m_{1,n})}{\|\!|M_{1,n}|\!\|}.$$

Assumption **A 2** implies moreover that that for all n, $\mathbb{P}$-almost any $\omega \in \Omega$, $\frac{\nu_0(m_{1,n})}{\|\|M_{1,n}\|\|} > 0$. Thus, setting

$$d(\omega) = \inf_{n\in\mathbb{N}} \frac{\nu_0(m_{1,n})}{\|\|M_{1,n}\|\|},$$

(ν, d) satisfies (3.1) and $d(\omega) > 0$, $\mathbb{P}(d\omega)$-a.s. □

This provides nice sufficient conditions for **A4** or **A4+** to hold.

Proposition 4.2. *Consider a random, stationary sequence of $p \times p$ matrices $M_n = (M_n(x,y))_{x,y\in\mathbb{X}}$, with nonnegative entries, satisfying **A1**, **A3**, **A5**. We assume that there exists a measurable map*

$$\omega \in \Omega \mapsto (\nu_\omega, c(\omega)) \in \mathcal{M}_1(\mathbb{X}) \times [0,1]$$

*such that $\mathbb{P}[c > 0] > 0$ and (1.4) hold. Then there exists a random variable d such that (ν, c, d) is an admissible triplet with $\mathbb{P}[\gamma > 0] > 0$. Thus assumption **A4** holds and Theorem 2.1 applies.*
If moreover

i) $\mathbb{E}[-\log c] < \infty$,
ii) there exists an deterministic integer $N \in \mathbb{N}_0$ such that

$$\mathbb{E}\left[\log \sup_{x,y,z\in\mathbb{X}} \frac{M_{0,N}(x,z)}{M_{0,N}(y,z)}\right] < \infty, \tag{4.3}$$

iii) $\int [\log \nu_\omega(m_{1,2})]^- \, d\mathbb{P}(\omega) < \infty$,

*Then **A4+** also holds.*

Note that since $\mathbb{X}$ is finite, (δ_i, c) satisfies (1.4) as soon as $c \leqslant \min_{1\leqslant j\leqslant p} \frac{M_0(i,j)}{\sum_{l=1}^p M_0(i,l)}$, thus **A 4** holds if with positive probability, M_0 has a row with only nonzero coefficients.

Proof: By Lemma 4.1, setting

$$d(\omega) = \inf_{n\in\mathbb{N}} \frac{\nu_\omega(m_{1,n})}{\|\|M_{1,n}\|\|},$$

$\omega \mapsto (\nu_\omega, d(\omega))$ satisfies (1.5) and $\mathbb{P}[d > 0] = 1$. Noting $\gamma(\omega) = c(\omega)d(\omega) = c_0(\omega)d_1(\omega)$, we notice that $\mathbb{P}[d_1 > 0] = \mathbb{P}[d > 0] = 1$ and $\mathbb{P}[c > 0] > 0$. It holds thus $\mathbb{P}[\gamma > 0] > 0$: **A4** is satisfied. This proves the first part of the proposition. Let us suppose now that *i)-iii)* hold. Since $\mathbb{E}[-\log c] < \infty$, then $\mathbb{P}[c > 0] = 1$ and $\mathbb{P}[\gamma > 0] = 1$. Let us now prove now that $\mathbb{E}|\log\gamma| < \infty$. By the inequality

$$|\log\gamma| = -\log(\gamma) \leqslant -\log(c) - \log(d)$$

and hypothesis $\mathbb{E}[|\log(c)|] < \infty$, it remains to check that $\mathbb{E}[|\log(d)|] < \infty$. Inequality (4.1) implies that $\mathbb{P}(d\omega)$-almost surely, for any $k \geqslant 1$,

$$\begin{aligned} -\log d = \log\sup_{n\in\mathbb{N}} \frac{\|\|M_{1,n}\|\|}{\nu_\omega(m_{1,n})} &\leqslant \max\left(\log \sup_{x,y,z\in\mathbb{X}} \frac{M_{1,k+1}(x,z)}{M_{1,k+1}(y,z)}, \max_{1\leqslant n\leqslant k} \log\frac{\|\|M_{1,n}\|\|}{\nu(m_{1,n})}\right) \\ &\leqslant \log \sup_{x,y,z\in\mathbb{X}} \frac{M_{1,k+1}(x,z)}{M_{1,k+1}(y,z)} + \sum_{1\leqslant n\leqslant k} \log\frac{\|\|M_{1,n}\|\|}{\nu_\omega(m_{1,n})}. \end{aligned}$$

In particular, setting $k = N$ and applying condition (4.3), it holds by stationarity

$$\mathbb{E}\left[\log \sup_{x,y,z\in\mathbb{X}} \frac{M_{1,N+1}(x,z)}{M_{1,N+1}(y,z)}\right] < \infty.$$

Consequently, it suffices to prove that $\mathbb{E}\left[\log \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})}\right] < \infty$ for any $n \geqslant 1$. Let us decompose this quantity as

$$\mathbb{E}\left[\log \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})}\right] \leqslant \underbrace{\mathbb{E}\left(\log \|M_{1,n}\|\right)^+}_{A(n)} + \underbrace{\mathbb{E}\left[(\log \nu_0(m_{1,n}))^-\right]}_{B(n)}. \tag{4.4}$$

- On the one hand, the inequality $\|M_{1,n}\| \leqslant \prod_{i=1}^{n-1} \|M_{i,i+1}\|$ readily yields

$$A(n) \leqslant \sum_{i=1}^{n-1} \mathbb{E}\left[\log(\|M_{i,i+1}\|)^+\right] = (n-1)\mathbb{E}\left[\log(\|M_{0,1}\|)^+\right] < \infty.$$

- On the other hand, for any $x \in \mathbb{X}$ and $\mathbb{P}(d\omega)$-a.s.

$$m_{1,n}(x) = \delta_x M_{1,2} m_{2,n} \geqslant c_1 m_{1,2}(x) \nu_1(m_{2,n}),$$

where $c_k = c(\theta^k(\omega))$ and $\nu_k = \nu_{\theta^k(\omega)}$. Consequently, integrating with respect to $\nu_0(dx)$, we obtain

$$\nu_0(m_{1,n}) = \nu_0 M_{1,2} m_{2,n} \geqslant c_1(\omega)\nu_0(m_{1,2})\nu_{\theta(\omega)}(m_{2,n}) \quad \mathbb{P}(d\omega)-\text{a.s},$$

which yields, by induction

$$\nu_0(m_{1,n}) \geqslant \prod_{k=1}^{n-1} c_k \nu_{k-1}(m_{k,k+1}).$$

Consequently, $\mathbb{P}(d\omega)$-a.s.,

$$(\log \nu_0(m_{1,n}))^- \leqslant \sum_{k=1}^{n-1} -\log c_k + \left[\log \nu_{k-1}(m_{k,k+1})\right]^-.$$

By stationarity, we deduce

$$\begin{aligned} B(n) &\leqslant \sum_{k=1}^{n-1} \mathbb{E}\left[-\log c_k\right] + \mathbb{E}\left[(\log \nu_{k-1}(m_{k,k+1}))^-\right] \\ &= (n-1)\mathbb{E}\left[-\log c_0\right] + \mathbb{E}\left[(\log \nu_0(m_{1,2}))^-\right] < \infty. \end{aligned}$$

Finally, combining these estimates, we get, for any $n \in \mathbb{N}_0$,

$$\mathbb{E}\left[\log \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})}\right] \leqslant (n-1)\mathbb{E}\left[-\log c_0 + \log\left[\nu_0(m_{1,2})\right]^- + (\log \|m_{0,1}\|_\infty)^+\right] < \infty.$$

□

4.2. *Extension in infinite dimension.* When $\mathbb{X}$ is infinite, we need to strengthen the notion of positive matrices as follows.

Definition 4.3. A positive linear map M on $\mathcal{B}(\mathbb{X})$ is uniformly positive if there exists $K \in \mathbb{R}_+^*$, $h \in \mathcal{B}_+(\mathbb{X})$, such that, for any $f \in \mathcal{B}_+(\mathbb{X})$ there exists $b(f) \in \mathbb{R}_+$, satisfying

$$\frac{1}{K} b(f) h \leqslant M(f) \leqslant K b(f) h.$$

Notice that when $\mathbb{X}$ is finite, a matrix of $\mathring{\mathcal{S}}$ is uniformly positive. Moreover, in Hennion's work, assumption **A5** is used as a sufficient condition to obtain projective contraction properties on the product $M_{k,n}$, with respect to a projective distance called the Hilbert distance (once again, see Busemann and Kelly (1953); Birkhoff (1957); Ligonnière (2023) for a complement on this distance). In an infinite dimensional setting, this distance can still be defined, and the projective action associated with a positive operator is contracting if and only if the operator is uniformly positive (a

proof of this claim is proposed in Ligonnière (2023)). Uniform positivity is therefore the appropriate infinite dimensional generalization of positivity in our context, and condition **A5** can thus naturally be replaced with the restrictive condition

A5'. $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}}\{M_{0,n} \gg 0\}\right) = 1$,

where we note $M \gg 0$ iff M is uniformly positive.

The present subsection aims at comparing our result with the natural extensions of Hennion's work to infinite dimensional settings. For this purpose, the following Lemma extends the idea of Lemma 4.1 to an infinite dimensional setup, assuming **A5'** instead of **A5**.

Lemma 4.4. *Consider a random stationary sequence of elements of $\mathcal{K}^+$, satisfying **A2** and **A5'**. Then for any measurable map $\omega \in \Omega \mapsto \nu(\omega) \in \mathcal{M}_1(\mathbb{X})$, there exists a random variable d such that $\mathbb{P}[d > 0] = 1$ and (1.5) holds.*

Proof: For almost any ω and any $1 \leqslant k \leqslant n$, $m_{k,n} \in \mathcal{B}(\mathbb{X})$, it holds,

$$\frac{m_{1,n}(x)}{\nu(m_{1,n})} = \frac{\delta_x M_{1,k} m_{k,n}}{\nu_0 M_{1,k} m_{k,n}} \leqslant \sup_{y\in\mathbb{X}, f\in\mathcal{B}(\mathbb{X})} \frac{\delta_y M_{1,k} f}{\nu_0 M_{1,k} f}.$$

Taking a supremum in $x \in \mathbb{X}$, we get, for any $k \leqslant n$,

$$\frac{\|M_{1,n}\|}{\nu_0(m_{1,n})} \leqslant \sup_{y\in\mathbb{X}, f\in\mathcal{B}(\mathbb{X})} \frac{\delta_y M_{1,k} f}{\nu_0 M_{1,k} f}. \tag{4.5}$$

By **A5'**, let k_ω be a random integer such that $\mathbb{P}(d\omega)$-almost surely, $M_{1,k_\omega}(\omega) \gg 0$. Then, almost surely, there is $K \in \mathbb{R}_+^*$, $h \in \mathcal{B}_+(\mathbb{X})$ such that for any $f \in \mathcal{B}_+(\mathbb{X})$, there exists $b(f) \geqslant 0$, satisfying

$$K^{-1} b(f) h \leqslant M_{1,k_\omega} f \leqslant K b(f) h. \tag{4.6}$$

From (4.6), we deduce $K^{-1} m_{1,k_\omega}(x) \leqslant h(x) b(\mathbb{1}) \leqslant K m_{1,k_\omega}(x)$. By **A2**, m_{1,k_ω} is a bounded and positive function, thus so is h. Moreover, $b(\mathbb{1}) > 0$, $\nu_0(m_{1,k}) \leqslant K\nu_0(h) b(\mathbb{1})$, thus $\nu(h) > 0$. Therefore, for any $x \in \mathbb{X}$, any $f \in \mathcal{B}_+(\mathbb{X})$, it holds $\mathbb{P}(d\omega)$ almost surely:

$$\frac{\delta_x M_{1,k_\omega} f}{\nu_0 M_{1,k_\omega} f} \leqslant K^2 \frac{h(x)}{\nu_0(h)} \leqslant K^3 \frac{\|M_{1,k_\omega}\|}{b(\mathbb{1})\nu_0(h)} \leqslant K^4 \frac{\|M_{1,k_\omega}\|}{\nu_0(m_{1,k_\omega})}. \tag{4.7}$$

Finally, combining (4.7) with (4.5), we get for almost any ω,

$$\limsup_n \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})} \leqslant \sup_{n\geqslant k_\omega} \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})} \leqslant K^4 \frac{\|M_{1,k_\omega}\|}{\nu(m_{1,k_\omega})} < \infty.$$

Since moreover almost surely, for all $n \geqslant 1$, $\frac{\|M_{1,n}\|}{\nu_0(m_{1,n})}$ is finite, then almost surely,

$$\sup_{n\in\mathbb{N}} \frac{\|M_{1,n}\|}{\nu_0(m_{1,n})} < \infty.$$

Let us set

$$d(\omega) = \inf_{n\geqslant 1} \frac{\nu_0(m_{1,n})}{\|M_{1,n}\|}.$$

Then d clearly satisfies (1.5) and $\mathbb{P}[d > 0] = 1$. □

The uniform positivity property is interesting to deal with many systems where the mass is sufficiently well mixed. We illustrate this on the following example.

Example 4.5. Take $\mathbb{X} = [0,1]^d$, choose an ergodic dymanical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ and associate with each $\omega \in \Omega$ a bounded measurable function $m_\omega : \mathbb{X} \to \mathbb{R}_+$, and a continuous function $Q_\omega : \mathbb{X}^2 \to \mathbb{R}_+^*$, such that $\int_{\mathbb{X}} Q_\omega(x,y)dy = 1$ for all x and all ω. We define M by setting, for each $\omega \in \Omega$ and $f \in \mathcal{B}(\mathbb{X})$,

$$M(\omega)(f) : x \mapsto m_\omega(x) \int_{\mathbb{X}} f(y) Q_\omega(x,y) dy.$$

Notice that $m_\omega(x) = \|\delta_x M(\omega)\|$. Our system clearly satisfies **A1** and **A2**. Assumptions **A3** and **A3+** just translate into a log-integrability assumption on $\omega \mapsto \|m_\omega\|_\infty = \|\!|M(\omega)|\!\|$.
In terms of populations, this model can represent the spatial evolution of a population in the compact domain $\mathbb{X}$. The quantity $m_\omega(x)$ represents the size of the offspring of an individual located at x, and the kernel $Q_\omega(x,y)$ represents the dispersion of its offspring in the domain $\mathbb{X}$. The dependence in ω of these quantities models an time inhomogeneity of these quantities. $M(\omega)$ is clearly uniformly positive, for each ω, since

$$K_\omega^{-1} m_\omega \mu(f) \leqslant M(\omega)(f) \leqslant m_\omega \mu(f) K_\omega,$$

where μ refers to the Lebesgue measure on $\mathbb{X}$, and

$$K_\omega = \max\left(\left(\inf_{u,v} Q_\omega(u,v)\right)^{-1}, \sup_{u,v} Q_\omega(u,v)\right) < \infty$$

since $\mathbb{X}$ is compact and Q_ω is continuous and positive. Setting $\nu_\omega = \mu$ and $c_\omega = K_\omega^{-1}$ for all ω, the left hand side inequality implies (1.4). Moreover, since $M(\omega)$ is uniformly positive, **A5'** holds, thus by Lemma 4.4 there exists a random variable d such that $\mathbb{P}[d > 0] > 0$ and $\omega \mapsto (\nu_\omega, d(\omega))$ satisfy (1.5). Therefore **A4** holds and Theorem 2.1 applies.
One can notice additionally that in this context, each matrix of the product is almost surely uniformly positive, therefore the proof of Lemma 4.4 yields the explicit control

$$d(\omega) > \frac{\int M_{1,2}(\mathbb{1})(x)dx}{K_\omega^4 \sup_{x\in\mathbb{X}} M_{1,2}(\mathbb{1})(x)} = \frac{\int m_{\theta(\omega)}(x)dx}{K_\omega^4 \|m_{\theta(\omega)}\|_\infty}.$$

Thus **A4+** reduces to a log-integrability condition both on the coefficient K_ω and on the quotient $\frac{\int m_\omega(x)dx}{\|m_\omega\|_\infty}$.

This example illustrates how Proposition 4.2 from the previous subsection can be adapted, replacing **A5** by **A5'**. To tackle the integrability of $\log\gamma$, one can replace (4.3) by

$$\mathbb{E}\left[\log \sup_{x\in\mathbb{X}, f\in\mathcal{B}(\mathbb{X})} \frac{\delta_x M_{1,N} f}{\nu_0 M_{1,N} f}\right] < \infty. \tag{4.8}$$

This yields a counterpart of Proposition 4.2 in a infinite dimensional setup.

5. Application to products of random infinite Leslie Matrices

The previous section focused on products of matrices with positive entries, and more generally, products of uniformly positive operators. This kind of products can be efficiently studied with methods based on projective contractions relatively to the Hilbert metric. The main interest of our techniques, based on Doeblin contractions, is their application to products of operators which are not uniformly positive. The goal of this section is to illustrate how such products can be studied with our theorems. We have chosen to focus here on a quite simple but natural example with no uniform positivity properties : the infinite Leslie Matrices.

5.1. *Introduction to Leslie matrices.* In this section, we set $\mathbb{X} = \mathbb{N}_0$, thus the operators of $\mathcal{K}^+$ can be represented as infinite matrices. We choose to consider infinite Leslie matrices, which have the

following form : for any $\omega \in \Omega$,

$$M(\omega) = \begin{pmatrix} f_0(\omega) & s_0(\omega) & 0 & 0 & \dots \\ f_1(\omega) & 0 & s_1(\omega) & 0 & \dots \\ f_2(\omega) & 0 & 0 & s_2(\omega) & \ddots \\ f_3(\omega) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.1}$$

where the entries $(f_k(\omega))_{k\in\mathbb{N}_0}$, $(s_k(\omega))_{k\in\mathbb{N}_0}$ are nonnegative real numbers, and $\sup_{x\in\mathbb{X}} s_x(\omega)+f_x(\omega) < \infty$. Notice that such a matrix is not uniformly positive, since there are zeros on every row and every column but the first one. Moreover, if Q is a product of k matrices of this shape, the (x,y)-entry $[Q]_{x,y} = 0$ whenever $y \geqslant x+k+1$. This prevents any product of such matrices from being uniformly positive. This example is therefore a typical situation where **A5'** does not hold.

Such matrices appear naturally when studying the dynamics of a population counting individuals according to their age. The coefficients f_x (respectively s_x) represent the mean number of individuals of age 0 (respectively of age $x+1$) created by an individual of age x, that is the mean size of the offspring of an individual of age x (respectively the survival rate of individuals of age x). Usually, only a finite number of age classes are defined, thus $\mathbb{X} = [\![0,p]\!]$, and one considers finite versions of such matrices, called Leslie matrices, see for example Caswell (2010). However, it is natural to extend their definition to an infinite number of age classes ($\mathbb{X} = \mathbb{N}_0$) obtaining infinite matrices with this shape. Indeed, several articles already study age-structured populations with an unbounded set of possible ages, see e.g. Bansaye et al. (2020); Jasińska and Kozitsky (2022); Oelschläger (1990). Therefore, products of random matrices shaped as in (5.1) model the dynamics of an age structured population evolving in a randomly changing environment which affect their reproductive behavior. This is the kind of matrices we are studying in this section. Let us note from now on

$$s_x^k(\omega) = s_x \circ \theta^k(\omega) \text{ and } f_x^k(\omega) = f_x \circ \theta^k(\omega),$$

so that $(s_x^k, f_x^k)_{x\in\mathbb{X}}$ are the nonzero entries of the random matrix $M_k(\omega) = M \circ \theta^k$. We introduce the quantities

$$d'(\omega) = \sup_{k\in\mathbb{N}_0, x\leqslant y\in\mathbb{X}} \frac{f_y^k(\omega)}{f_x^k(\omega)} \geqslant 1,$$

and

$$d''(\omega) = \sup_{x\in\mathbb{X},k\in\mathbb{N}_0} \frac{s_x^0(\omega)\cdots s_{x+k}^k(\omega)}{s_0^0(\omega)\dots s_k^k(\omega)} \geqslant 1,$$

which are useful to construct an admissible triplet.

5.2. *Ergodic behavior of products of random Leslie Matrices.* The following proposition provides sufficient conditions for assumptions **A3** and **A4** to hold in the case of products of infinite Leslie matrices.

Proposition 5.1. *Consider a random matrix product with $\mathbb{X} = \mathbb{N}_0$ and suppose that for any $\omega \in \Omega$, $M(\omega)$ is of the form of equation (5.1), with $\sup_{x\in\mathbb{X}} s_x(\omega) + f_x(\omega) < \infty$. Suppose that **A1** is satisfied, and $\mathbb{P}(d\omega)$-almost surely, it holds*

i) $f_x(\omega) + s_x(\omega) > 0$ for all $x \in \mathbb{X}$;
ii) $\mathbb{E}\left[\log^+ (\sup_{x\in\mathbb{X}} s_x + f_x)\right] < \infty$;

*then **A2** and **A3** hold. If moreover*

iii) There exists a deterministic real $A > 0$ such that $\mathbb{P}(d\omega)$-almost surely, $\sup_{x\leqslant y} \frac{f_y^0}{f_x^0} \leqslant A$,
iv) $\mathbb{P}[\sup_{x\in\mathbb{X}} \frac{s_x^0}{f_x^0} < \infty, d'' \circ \theta < \infty] > 0$

*then Assumption **A4** holds, and so do the conclusions of Theorem 2.1.*
Finally, if additionally

v) $\mathbb{P}(d\omega)$*-almost surely,* $\sup_{x\in\mathbb{X}} \frac{s_x}{f_x} < \infty$ *and* $\mathbb{E}\left[\log^+(\sup_{x\in\mathbb{X}} \frac{s_x}{f_x})\right] < \infty$,

vi) $\mathbb{P}(d\omega)$*-almost surely,* $d''(\omega) < \infty$ *and* $\mathbb{E}\,|\log d''| < \infty$,

*then M satisfies also Assumption **A4+**.*

In Proposition 5.1, we've reduced Assumptions **A2** to **A4** to a series of conditions on the law of the coefficients of the random matrix M_0, together with finiteness and integrability conditions on d''. The hardest conditions to check are the ones involving d'', since checking them requires to consider the joint law of all the $M_{0,n}$ and not only M_0. We were not able to find a general sufficient condition for the positivity and $\log-$integrability of d''. However, we provide the following quite restrictive sufficient condition.

Remark 5.2. Consider a random, stationary sequence of matrices of the form of equation (5.1), and assume that there exists an integer $x_0 \in \mathbb{X}$, such that almost surely, the sequence $(s_x(\omega))_{x\geqslant x_0}$ is non increasing. Suppose also that almost surely, for all $x \leqslant x_0$, $s_x > 0$. Then, $\mathbb{P}(d\omega)$-almost surely

$$d'' \leqslant \left(\sup_{i\leqslant x_0} \sup_{x\leqslant y\leqslant x_0} \frac{s_y^i}{s_x^i}\right)^{x_0} < \infty.$$

Moreover, if $\mathbb{E}\left|\log \frac{s_y^i}{s_x^i}\right| < \infty$ for any $x \leqslant y \leqslant x_0$, then $\mathbb{E}|\log d''| < \infty$.

In the context of an age structured population, s_x represents the frequency of individuals of age x surviving to the next time step, and thus being replaced by individuals of age $x+1$. Assuming that $(s_x(\omega))_{x\geqslant x_0}$ is decreasing implies that the older individuals get, the more they tend to die, which is a reasonable assumption. However this condition is somewhat unsatisfying in a more general setting.

We split the proof of Proposition 5.1 into several lemmas that involve different groups of assumptions. Notice first that most quantities involved in Assumptions **A2** to **A4** are explicit in terms of the (f_x, s_x). Indeed :

Lemma 5.3. *Consider a product of stationary random Leslie matrices, in the form of equation (5.1). Then **A2** and **A3** are satisfied if and only if all the following conditions hold simultaneously :*

- $\mathbb{P}(d\omega)$*-almost surely, for each* $x \in \mathbb{X}$, $f_x(\omega) > 0$ *or* $s_x(\omega) > 0$
- $\mathbb{E}\left[\log^+(\sup_{x\in\mathbb{X}} f_x + s_x)\right] < \infty$.

Proof: This lemma is straightforward after noticing that for any $x \in \mathbb{X}, \omega \in \Omega$,

$$m_{0,1}(x,\omega) = f_x(\omega) + s_x(\omega).$$

□

Moreover, in this model, (1.4) is well behaved and it is quite clear how to construct a non trivial couple (ν, c).

Lemma 5.4. *Consider a product of stationary, random Leslie matrices and assume that **A2** holds. Consider a map $\omega \mapsto (\nu_\omega, c(\omega))$. Then for almost any ω such that $\nu_\omega \neq \delta_0$, (1.4) implies $c(\omega) = 0$. If $\nu_\omega = \delta_0$, then (1.4) is equivalent to*

$$c(\omega) \leqslant \inf_{x\in\mathbb{X}} \frac{f_x(\omega)}{f_x(\omega) + s_x(\omega)} = \left(1 + \sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right)^{-1}.$$

Proof: Notice that for any $x \in \mathbb{X}$, $\omega \in \Omega$,

$$\delta_x M_{0,1} = f_x(\omega)\delta_0 + s_x(\omega)\delta_{x+1}.$$

Let $\omega \in \Omega$ such that $\nu_\omega \neq \delta_0$. Then, there exists $k > 0$ such that $\nu_\omega(k) > 0$. In particular (1.4) implies

$$0 = f_k(\omega)\delta_0(\mathbb{1}_k) + s_k(\omega)\delta_{k+1}(\mathbb{1}_k) \geqslant c(\omega)m_{0,1}(k)\nu(k),$$

By **A 2**, almost surely, $m_{0,1}(k) > 0$, which implies that $c(\omega) = 0$. Conversely, if $\nu_\omega = \delta_0$, (1.4) implies

$$f_x(\omega)\delta_0 + s_x(\omega)\delta_{x+1} \geqslant c(\omega)m_{0,1}(x)\delta_0,$$

which is equivalent to

$$f_x(\omega) \geqslant c(\omega)m_{0,1}(x) = c(\omega)(f_x(\omega) + s_x(\omega))$$

for all $x \in \mathbb{X}$. This yields the desired result. □

As a consequence, we set from now on $\nu_\omega = \delta_0$ and $c_\omega = \left(1 + \sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}^{-1}\right)^{-1}$. Assumption *iv)* of Proposition 5.1 guarantees that $\mathbb{P}[c > 0] > 0$. Let us try now to exhibit a random variable d such that $(\nu = \delta_0, d)$ satisfy (1.5).

Lemma 5.5. *Consider a product of stationary random Leslie matrices, of the form of equation (5.1). Set*

$$d(\omega) = \frac{1}{d' \circ \theta(\omega) d'' \circ \theta(\omega)}.$$

Then (δ_0, d) satisfy (1.5).

Proof: Let $n \geqslant 1, x \in \mathbb{X}$, $\omega \in \Omega$, it holds

$$m_{1,n}(x) = \delta_x M_0 \cdots M_{n-1}\mathbb{1} = \sum_{i_0,i_1,\cdots i_n \in \mathbb{N}_0} \delta_x(i_0)M_1(i_0,i_1)\cdots M_{n-1}(i_{n-1},i_n).$$

Thus

$$m_{1,n}(x) = \sum_{i_1,\cdots i_n \in \mathbb{N}_0} M_1(x,i_1)\cdots M_{n-1}(i_{n-1},i_n).$$

Let us rearrange this sum according to the first index $k \leqslant n$ such that $i_k = 0$:

$$\begin{aligned} m_{1,n}(x) = &\sum_{k=1}^{n} \sum_{i_1,\cdots i_{k-1} > 0} M_0(x,i_1)\cdots M_{k-1}(i_{k-1},0) \sum_{i_{k+1},\cdots i_n \in \mathbb{N}_0} M_k(0,i_{k+1})\cdots M_{n-1}(i_{n-1},i_n) \\ &+ \sum_{i_1,\cdots i_n > 0} M_0(x,i_1)\cdots M_{n-1}(i_{n-1},i_n). \end{aligned}$$

Notice that

$$\sum_{i_{k+1},\cdots i_n \in \mathbb{N}_0} M_k(0,i_{k+1})\cdots M_{n-1}(i_{n-1},i_n) = m_{k,n}(0).$$

Moreover, the matrices M_k are shaped according to (5.1). Thus for any $i \geqslant 0, j > 0$, in order for $M_k(i,j)$ to be non zero, one must have $j = i+1$. Thus :

$$m_{0,n}(x) = \sum_{k=1}^{n} M_0(x,x+1)\cdots M_{k-1}(x+k-1,0)m_{k,n}(0) + M_0(x,x+1)\cdots M_{n-1}(x+n-1,x+n).$$

Therefore

$$m_{0,n}(x) = \sum_{i=0}^{n-1} s_x^0 s_{x+1}^1 \ldots s_{x+i-1}^{i-1} f_{x+i}^i m_{i+1,n}(0) + s_x^0 \ldots s_{x+n-1}^{n-1}.$$

This is true in particular for $x = 0$:

$$m_{0,n}(0) = \sum_{i=0}^{n-1} s_0^0 s_1^1 \ldots s_{i-1}^{i-1} f_i^i m_{i+1,n}(0) + s_0^0 \ldots s_{n-1}^{n-1}.$$

By definition of d', d'', it holds, for any $k \in \mathbb{N}_0$ and any $x \in \mathbb{N}_0$,

$$f^i_{x+i} \leqslant d' f^i_i,$$

and

$$s^0_x s^1_{x+1} \dots s^{i-1}_{x+i-1} \leqslant d'' s^0_0 s^1_1 \dots s^{i-1}_{i-1}.$$

Therefore, controlling independently each term of the sum yields

$$m_{0,n}(x) \leqslant d' d'' \sum_{i=0}^{n-1} s^0_0 s^1_1 \dots s^{i-1}_{i-1} f^i_i m_{i+1,n}(0) + d'' s^0_0 \dots s^{n-1}_{n-1} \leqslant d' d'' m_{0,n}(0),$$

thus for all $n \geqslant 0$

$$\frac{1}{d'(\omega) d''(\omega)} \leqslant \inf_{x \in \mathbb{X}} \frac{m_{0,n}(0)}{m_{0,n}(x)}.$$

By stationarity,

$$\frac{1}{d'(\theta(\omega)) d''(\theta(\omega))} \leqslant \inf_{n \geqslant 1} \frac{m_{1,n}(0)}{\|m_{1,n}\|_\infty}.$$

As a consequence, setting $d(\omega) = \left(d'(\theta(\omega)) d''(\theta(\omega))\right)^{-1}$ is enough for $\omega \mapsto (\delta_0, d(\omega))$ to satisfy (1.5). □

Let us focus on $d'(\omega)$.

Lemma 5.6. *Consider a random product of matrices of the form of equation (5.1), satisfying **A1**. Then the random variable d' is $\mathbb{P}(d\omega)$-almost surely finite if and only if there exists $A > 0$ such that*

$$\mathbb{P}\left[\sup_{x \leqslant y} \frac{f^0_y}{f^0_x} \leqslant A\right] = 1. \tag{5.2}$$

In this case $d'(\omega) \leqslant A$, $\mathbb{P}(d\omega)$-almost surely. If (5.2) fails, then $d'(\omega) = +\infty$, $\mathbb{P}(d\omega)-$ almost surely.

Proof: Notice that

$$d'(\omega) = \sup_{k \in \mathbb{N}_0} \sup_{x \leqslant y} \frac{f^k_y}{f^k_x} = \sup_{k \in \mathbb{N}_0} X \circ \theta^k(\omega).$$

where

$$X(\omega) = \sup_{x \leqslant y} \frac{f_y(\omega)}{f_x(\omega)} = \sup_{x \leqslant y} \frac{f^0_y}{f^0_x}.$$

Since θ is an ergodic mapping, $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is $\mathbb{P}(d\omega)$-almost surely equal to the supremum of the support of X. In particular $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is finite almost surely if the support of X is bounded. Conversely, if the support of X is unbounded, then $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is infinite almost surely. □

Putting these lemmas together allows to prove Proposition 5.1.

Proof of Proposition 5.1: The assumptions *i), ii)* of Proposition 5.1 are the conditions mentioned in Lemma 5.3. Hence, this lemma allows to check **A2** and **A3**. We set now for any ω, $\nu_\omega = \delta_0$

$$c(\omega) = \left(1 + \sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right)^{-1}$$

and

$$d(\omega) = \frac{1}{d'(\theta(\omega)) d''(\theta(\omega))}.$$

Lemma 5.4 and Lemma 5.5 respectively guarantee that (1.4) and (1.5) are satisfied. By Lemma 5.6, assumption *iii)* guarantees that $d'(\omega) < \infty, \mathbb{P}(d\omega)-$almost surely, and by stationarity, the same holds for $d' \circ \theta(\omega)$. As a consequence, since by *iv)* $\mathbb{P}[d'' \circ \theta > 0, c_0 > 0] > 0$, it holds with positive probability $\gamma(\omega) = c(\omega) d(\theta(\omega)) = c(\omega) \left(d'(\theta(\omega)) d''(\theta(\omega))\right)^{-1} > 0$ thus $\mathbb{P}[\gamma > 0] > 0$. This proves

that *i)-iv)* imply **A4**.
If moreover assumptions *v), vi)* hold, then $d'' \circ \theta$ and $\sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}$ are finite $\mathbb{P}(d\omega)-$a.s., thus $\gamma > 0$ almost surely. In this case it also holds

$$\mathbb{E}|\log\gamma| \leqslant \mathbb{E}\left[-\log c\right] + \mathbb{E}\left[-\log d\right].$$

On the one hand,

$$\mathbb{E}\left[-\log c\right] = \int \log\left(1 + \sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right) d\mathbb{P}(\omega).$$

Notice that for any positive real variable X, $\log(1+X)$ is integrable as soon as $\log(X)^+$ is integrable. Since we've assumed that

$$\int \log\left(\sup_{x\in\mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)}\right)^+ d\mathbb{P}(\omega) < \infty,$$

then

$$\mathbb{E}[-\log c] < \infty.$$

On the other hand,

$$\mathbb{E}\left[-\log d\right] \leqslant \mathbb{E}\log d' + \mathbb{E}\log d''.$$

Since $\sup_{x\leqslant y} \frac{f_y^0}{f_x^0} \leqslant A$ almost surely, then by stationarity, almost surely,

$$1 \leqslant d' = \sup_{k\in\mathbb{N}_0} \sup_{y\geqslant x} \frac{f_y^k}{f_x^k} \leqslant A.$$

Thus $\log d'$ is bounded and integrable. We have assumed additionally that $\log d''$ was integrable. This is enough to conclude to the integrability of $|\log\gamma|$, which proves assumption **A4+**. □

5.3. *A situation where* $\gamma = 0$. It was not clear to us how strong an assumption **A4** is, or whether it was hard to find a system breaking it while satisfying all the other assumptions. We shall present here an example of an infinite Leslie matrix, such that $\gamma = 0$ even if all other assumptions are satisfied. This example is in a deterministic environment, that is $|\mathcal{E}| = 1$, $\Omega = \mathcal{E}^{\mathbb{N}_0}$, $|\Omega| = 1$. The random matrix $M(\omega)$ is therefore constant, and $M_{0,n} = M^n$. Let us set :

$$M = \begin{pmatrix} \delta m(0) & (1-\delta)m(0) & 0 & 0 & \dots \\ \delta m(1) & 0 & (1-\delta)m(1) & 0 & \dots \\ \delta m(2) & 0 & 0 & (1-\delta)m(2) & \ddots \\ \delta m(3) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.3}$$

where $\delta \in (0,1)$, and $m(x) = \delta m(x) + (1-\delta)m(x)$ is the mean offspring size of an individual of age x. Such a model satisfies **A2** and **A3**, as soon as $x \mapsto m(x)$ is bounded and positive, since $\delta > 0$. The ergodicity and integrability properties are trivially satisfied since this model is in a constant environment. Moreover, Lemma 5.4 applies, therefore setting $\nu = \delta_0$, and $c = \left(1 + \sup_{x\in\mathbb{X}} \frac{s_x}{f_x}\right)^{-1} = \delta$, the couple (ν, c) satisfies (1.4). Let us prove that we can tune the parameters $x \mapsto m(x)$ and $c = \delta$ in such a way that the only d satisfying (1.5) is $d = 0$.

Consider a sequence of integers $(\varepsilon_x)_{x\in\mathbb{N}_0} \in \{0,1\}^{\mathbb{N}_0}$, such that :

- There are arbitrarily long subsequences of consecutive 1 in the sequence (ε_x).
- Noting $S_x = \sum_{k=0}^{x-1} \varepsilon_k$, $\frac{S_x}{x} \longrightarrow 0$ as $x \to \infty$.
- There exists $\alpha < 1$ such that for all $x \in \mathbb{N}_0$, $\frac{S_x}{x} \leqslant \alpha$.

Let a be a real number such that $a > 1$. Then, we set, for any $x \in \mathbb{X}$,

$$m(x) = 1 + (a-1)\varepsilon_x.$$

Defined as such, m is a positive and bounded function, thus Assumptions **A2** and **A3** are satisfied. This yields that for any sequence (x_i),

$$\prod_{i=0}^{n-1} m(x_i) = a^{\sum_{i=0}^{n-1} \varepsilon_{x_i}}.$$

Moreover, since this model is in constant environment, $M_{0,n} = M_{1,n+1} = M^n$, thus

$$m_{1,n+1}(x) = m_{0,n}(x) = \sum_{\substack{x_0=x,\dots x_n \in \mathbb{N} \\ x_{i+1} \in \{x_i+1,0\}}} \delta^{N(x_0,\dots x_n)}(1-\delta)^{n-N(x_0,\dots x_n)} \prod_{i=0}^{n-1} m(x_i) \geqslant (1-\delta)^n \prod_{i=0}^{n-1} m(x+i),$$

where $N(x_0, \dots x_n) = |\{1 \leqslant i \leqslant n | x_i = 0\}|$. Then

$$m_{0,n}(x) \geqslant (1-\delta)^n a^{\sum_{i=x}^{x+n-1} \varepsilon_i},$$

In particular, x can be chosen such that $\varepsilon_x = \cdots \varepsilon_{x+n-1} = 1$, which implies that

$$\|m_{0,n}\|_\infty \geqslant m_{0,n}(x) \geqslant (a(1-\delta))^n.$$

On the other hand

$$m_{0,n}(0) \leqslant 2^n \sup_{\substack{x_0=0,\dots x_n \in \mathbb{N}_0 \\ x_{i+1} \in \{x_i+1,0\}}} \prod_{i=0}^{n-1} m(x_i) \leqslant 2^n a^{\sup_{\substack{x_0=0,\dots x_n \in \mathbb{N}_0 \\ x_{i+1} \in \{x_i+1,0\}}} \sum_{i=0}^{n-1} \varepsilon_{x_i}}.$$

A sequence $(x_i)_{0 \leqslant i \leqslant n}$ of integers such that $x_0 = 0$ and for each i, $x_{i+1} \in \{x_i + 1, 0\}$ is entirely determined by the sequence $(T_k)_k$ of the lengths of its excursions away from zero. By convention, if there are only p excursions away from zero, we set T_p such that $T_0 + \cdots + T_p = n$ and $T_{p+1} = \cdots = T_n = 0$. The $(x_{T_0+\cdots+T_{i-1}})_{i \leqslant p}$ are the only zero terms in the sequence $(x_1, \dots x_n)$, and $T_0 + \cdots + T_n \leqslant n-1$. Thus

$$\sup_{\substack{x_0=0,\dots x_n \in \mathbb{N} \\ x_{i+1} \in \{x_i+1,0\}}} \sum_{i=0}^{n-1} \varepsilon_{x_i} \leqslant \sup_{T_0+\dots T_n = n} \sum_{i=0}^{n} S_{T_i} \leqslant \alpha \sum_{i=0}^{n} T_i \leqslant \alpha n,$$

and

$$m_{0,n}(0) \leqslant (2a^\alpha)^n.$$

Hence

$$\frac{\|m_{1,n+1}\|_\infty}{m_{1,n+1}(0)} = \frac{\|m_{0,n}\|_\infty}{m_{0,n}(0)} \geqslant \left(\frac{a(1-\delta)}{2a^\alpha}\right)^n = \left(\frac{a^{1-\alpha}(1-c)}{2}\right)^n \underset{n\to\infty}{\longrightarrow} \infty,$$

whenever $\frac{a^{1-\alpha}(1-c)}{2} > 1$. For (1.5) to hold, we must have $d \leqslant \inf_{n \geqslant 1} \left(\frac{m_{1,n}(0)}{\|m_{1,n}\|_\infty}\right)$. Thus for any values of $\alpha, \delta \in (0,1)$, if a is large enough, then (1.5) implies $d = 0$.

Acknowledgements

I would like to warmly thank my PhD-supervisors Vincent Bansaye and Marc Peigné for their continous guidance and support as well as their numerous feedbacks on this manuscript, which greatly helped improving it. I would also like to thank Gerold Alsmeyer for a fruitful discussion on the topic of Markov Random Walks. I am also grateful to the anonymous Referee for carefully reading this manuscript. Their useful comments and remarks enabled me to significantly improve this paper. Finally, I also thank the production editor for giving this paper its final form.

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