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ERGODIC BEHAVIOR OF PRODUCTS OF RANDOM POSITIVE OPERATORS

MAXIME LIGONNIÈRE

ABSTRACT. This article is devoted to the study of products of random operators of the form $M_{0,n} = M_0 \cdots M_{n-1}$, where $(M_n)_{n \in \mathbb{N}}$ is an ergodic sequence of positive operators on the space of signed measures on a space \mathbb{X} . Under suitable conditions, in particular, a Doeblin-type minoration suited for non conservative operators, we obtain asymptotic results of the form

$$\mu M_{0,n} \simeq \mu(\tilde{h}) r_n \pi_n,$$

where \tilde{h} is a random bounded function, $(r_n)_{n \geq 0}$ is a random non negative sequence and π_n is a random probability measure on \mathbb{X} . Moreover, \tilde{h} , (r_n) and π_n do not depend on the choice of the measure μ . We prove additionally that $n^{-1} \log(r_n)$ converges almost surely to the Lyapunov exponent λ of the process $(M_{0,n})_{n \geq 0}$ and that the sequence of random probability measures (π_n) converges weakly towards a random probability measure. These results are analogous to previous estimates from [Hen97] in the case of $d \times d$ matrices, that were obtained with different techniques, based on a projective contraction in Hilbert distance. In the case where the sequence (M_n) is i.i.d, we additionally exhibit an expression of the Lyapunov exponent λ as an integral with respect to the weak limit of the sequence of random probability measures (π_n) and exhibit an oscillation behavior of r_n when $\lambda = 0$. We provide a detailed comparison of our assumptions with the ones of [Hen97] and present some example of applications of our results, in particular in the field of population dynamics.

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1. INTRODUCTION

1.1. General introduction. The study of products of random linear operators can be traced back to the seminal article of Furstenberg and Kesten [FK60], studying products of the form

$$M_{0,n} = M_0 \dots M_{n-1},$$

where $(M_n)_{n \geq 0}$ is a stationary sequence of $d \times d$ real or complex random matrices. Under a mild irreducibility assumption, the authors exhibit a law of large numbers on the norm of $M_{0,n}$, involving a deterministic number called Lyapunov exponent, defined as

$$\lambda = \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{n} \log \|M_{0,n}\| \right] = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} [\log \|M_{0,n}\|],$$

where the norm $\|\cdot\|$ can be chosen to be any submultiplicative norm. Under additional positivity and boundedness assumptions on the entries of the matrices (M_n) , [FK60] also proves a law of large numbers for the entries $M_{0,n}(i, j)$ of the products : almost surely,

$$\lim_{n \rightarrow \infty} n^{-1} \log M_{0,n}(i, j) = \lambda.$$

These estimates on the behavior of the entries of $M_{0,n}$ were then extended to the case of products of invertible matrices, see e.g. [GL01] and [BL85]. These works rely on a careful study of the action of invertible matrices on the projective space $\mathcal{P}(\mathbb{R}_d)$.

To strengthen the results from [FK60] on products of matrices with non negative entries and relax their assumptions, Hennion [Hen97] studied the action of $\mathcal{M}_d(\mathbb{R}_+)$ on the projective space $\mathcal{P}(\mathbb{R}_+^d)$, endowed with the so called (pseudo)-Hilbert distance d_H previously defined in [BK53] and [Bir57]. This distance is particularly well adapted to this problem, since the contraction coefficient of the projective action of a matrix with respect to d_H is explicit in terms of its entries, in particular, any matrix with positive entries is strictly contracting. Under the assumption that almost surely, for n large enough, $M_{0,n}$ has all positive entries, Hennion obtains the asymptotic decomposition

$$M_{0,n}(i, j) = \lambda_n R_n(i) L_n(j) + o_{n \rightarrow \infty}(\lambda_n),$$

where λ_n is the dominant eigenvalue of $M_{0,n}$ and L_n, R_n are the associated left and right eigenvectors, with the normalizations $\|R_n\| = 1$ and $\langle L_n, R_n \rangle = 1$. Moreover $(R_n)_{n \geq 0}$ almost surely converges to a random vector R , $(L_n / \|L_n\|)_n$ converges in distribution, and $(n^{-1} \log \lambda_n)_{n \geq 1}$ almost surely converges to the Lyapunov exponent λ .

Such results have important implications, in particular in the field of populations dynamics. Indeed, a population composed of d types of individuals, evolving in a fluctuating environment, without interacting with each other, can be modelled by a linear model of the form

$$(1.1) \quad x_n = x_{n-1} M_{n-1},$$

where x_n is a row vector of \mathbb{R}_+^d encoding the mass of individuals of each type at time n and $M_{n-1} = (M_{n-1}(i, j))_{1 \leq i, j \leq d}$ is a random matrix encoding the rates at which individuals of

each type i create individuals of each type j between times $n - 1$ and n . In such a time-inhomogeneous population model, the understanding of the asymptotics of x_n amounts to the understanding of the matrix product $M_{0,n}$.

Moreover, such products also appear in the study of multitype Galton-Watson processes in random environment (MGWRE), which were introduced in [AK71]. They are a generalization of Galton-Watson processes to the case where the distribution of the (random) offspring of an individual depends on a notion of type and on a random environment that changes through time. When conditioning a MGWRE on the environment sequence, one obtains a so-called quenched population model, which satisfies (1.1) where x_n is the expectation of the population conditionally on the environmental sequence. The value of the Lyapunov exponent λ of the underlying matrix product separates three regimes of the MGWRE : subcritical ($\lambda < 0$), critical ($\lambda = 0$), supercritical ($\lambda > 0$). These three regimes have different properties, in particular, when $\lambda \leq 0$, the MGWRE goes extinct with probability 1, when $\lambda > 0$, the MGWRE survives with positive probability. This separation between regimes was established in [AK71] and [Nor74], using results from [FK60]. More recent advances in the study of random matrix products - in particular Hennion's article- were key to the last developments of the theory of MGWRE in random environments, see e.g. [Cam18; LPD18; GLP23].

In this paper, we study the infinite dimensional counterpart of products of random matrices, having in particular in mind applications to population models with an infinite number of types. We first consider a set \mathbb{X} , typically infinite, and build a set \mathcal{K}^+ of positive linear operators acting both on the space of signed measures $\mathcal{M}(\mathbb{X})$ on the left and the space of measurable bounded functions $\mathcal{B}(\mathbb{X})$ on the right. Then, we let (M_n) be a stationary, ergodic sequence of elements of \mathcal{K}^+ and define the products $M_{0,n} = M_0 \cdots M_{n-1}$. The approach of [Hen97] can be extended to this infinite dimensional setup. Indeed, it is possible to define the Hilbert distance d_H on the projective positive cone of an infinite dimensional vector space and to obtain a nice characterisation of the operators that are (strictly) contracting with respect to d_H . We refer the reader to [Lig] for a proof of these facts. However, as we explain in Section 4, this characterisation leads to stronger positivity assumptions in an infinite dimensional context than it did in the finite dimensional one. For example, such an extension of Hennion's approach would not be able to deal with products of infinite Leslie matrices that we present in Section 5.

For this reason, we use a different contraction method to obtain a projective contraction. This method aims at extending the Doeblin contraction techniques for Markov operators to a product of non conservative operators (that is operators M such that $M\mathbf{1} \neq \mathbf{1}$ in general). To do so, we consider the auxiliary family of Markov operators $P_{k,n}^N$, defined for each $k \leq n \leq N$ as

$$\delta_x P_{k,n}^N f = \frac{\delta_x M_{k,n}(f m_{n,N})}{\delta_x M_{k,N}(\mathbf{1})},$$

where, for $x \in \mathbb{X}$, $m_{k,n}(x) = \delta_x M_{k,n} \mathbf{1}$. These Markov operators are related to the projective action $\mu \cdot M_{k,n} = \frac{\mu M_{k,n}}{\|\mu M_{k,n}\|}$ of $M_{k,n}$ on measures by :

$$\mu \cdot M_{k,n} = \mu P_{k,n}^n = \mu P_{k,k+1}^n \cdots P_{n-1,n}^n.$$

We provide sufficient conditions for these Markov operators to satisfy Doeblin minorations of the form $\delta_x P(f) \geq c\nu(f)$, which guarantees that they are contracting in total variation. This allows to obtain a notion of projective contraction of the $M_{k,n}$ on the set of signed measures. Such auxiliary operators were already introduced and already applied to study both homogeneous semi-groups of operators [DM02; CV16] and inhomogeneous ones [BCG20]. We consider here a random sequence of operators (M_n) , i.e a discrete time, random, time-inhomogeneous semi group, and assume this sequence is stationary and ergodic. The stationary and ergodic framework allows us to provide more explicit assumptions, and we obtain more developed asymptotic results on (M_n) . Namely, we prove that the following almost sure approximation in total variation holds for n large enough

$$(1.2) \quad \left\| \mu M_{0,n} - \mu(\tilde{h})r_n\pi_n \right\|_{TV} \leq \eta^n \|\mu M_{0,n}\|_{TV},$$

where $\eta < 1$, \tilde{h} is a random bounded function, r_n is a positive random number and (π_n) is a sequence of random probability measures on \mathbb{X} , which are all independent of the measure μ . We prove additionally that $(n^{-1} \log(r_n))$ converges almost surely to the Lyapunov exponent of the process $M_{0,n}$, and that the sequence (π_n) of random probabilities converges in distribution with respect to the total variation topology towards a random probability measure Λ .

Additionally, we show in Theorem 2 that when the sequence (M_n) is i.i.d, the probability distribution Λ and the Lyapunov exponent λ are related as follows :

$$\lambda = \int \log \|\mu M\| d\Lambda(\mu) d\mathcal{P}(M),$$

where \mathcal{P} refers to the law of the operators (M_n) , thus extending a result stated in [BL85] in finite dimension. Finally, still under the assumption that (M_n) is i.i.d, we show that, when $\lambda = 0$ it holds almost surely, for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,

$$\limsup_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = -\liminf_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = +\infty,$$

except in a situation known as Null-Homology.

These results should allow to extend many known results on MGWRE with a finite type set to a class of MGWRE with an infinite type set \mathbb{X} . In particular, our results imply that when the Lyapunov exponent λ is nonpositive, the quenched population size $\mu M_{0,n} \mathbf{1}$ satisfies $\liminf_{n \rightarrow \infty} \mu M_{0,n} \mathbf{1} = 0$ almost surely. By a classical first moment argument, this is a sufficient condition for the almost sure extinction of the population. The survival of the population when $\lambda > 0$ is a more delicate problem and will be the object of a forthcoming article.

1.2. Framework and notations. Let $(\mathbb{X}, \mathcal{X})$ be a measurable set of arbitrary cardinality, such that for all $x \in \mathbb{X}$, the singleton $\{x\} \in \mathcal{X}$. We denote by $\mathcal{B}(\mathbb{X})$ the Banach space of bounded measurable functions on \mathbb{X} , endowed with the supremum norm, and $\mathcal{B}_+(\mathbb{X})$ the cone of nonnegative functions of $\mathcal{B}(\mathbb{X})$. The vector space of signed measures, noted $\mathcal{M}(\mathbb{X})$ and the cone of nonnegative elements of $\mathcal{M}(\mathbb{X})$, noted $\mathcal{M}_+(\mathbb{X})$ are endowed with the total variation norm $\|\cdot\|_{TV}$. Note that $\|\mu\|_{TV} = \mu(\mathbb{X})$ for any $\mu \in \mathcal{M}_+(\mathbb{X})$. Let $\mathcal{M}_1(\mathbb{X})$ be the

set of probability measures on $(\mathbb{X}, \mathcal{X})$. For any measurable set $A \in \mathcal{X}$, let $\mathbf{1}_A \in \mathcal{B}(\mathbb{X})$ be the indicator function of A . For short, we note $\mathbf{1} = \mathbf{1}_{\mathbb{X}} \in \mathcal{B}(\mathbb{X})$ the unit function on \mathbb{X} . We also note $\delta_x \in \mathcal{M}(\mathbb{X})$ the Dirac measure at x .

Let \mathcal{K}^+ be the set of maps Q of the form:

$$Q : \mathbb{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+,$$

such that, for any $x \in \mathbb{X}$, for any $A \in \mathcal{X}$, the map $x \mapsto Q(x, A)$ is measurable, the map $A \mapsto Q(x, A)$ is a positive and finite measure on $(\mathbb{X}, \mathcal{X})$ and $\|Q\| := \sup_{x \in \mathbb{X}} Q(x, \mathbb{X}) < \infty$.

Such a map $Q \in \mathcal{K}^+$ naturally operates on $\mathcal{B}(\mathbb{X})$ by setting, for any $f \in \mathcal{B}(\mathbb{X})$, and any $x \in \mathbb{X}$,

$$Qf(x) = \int_{\mathbb{X}} f(y)Q(x, dy).$$

Note that $|Qf(x)| \leq \|f\|_{\infty} \|Q\|$, thus $Qf \in \mathcal{B}_+(\mathbb{X})$ as soon as $f \in \mathcal{B}_+(\mathbb{X})$ and Q acts as a bounded positive operator with norm $\|Q\|$ on $\mathcal{B}(\mathbb{X})$. Moreover, for any positive measure $\mu \in \mathcal{M}_+(\mathbb{X})$, and any $Q \in \mathcal{K}^+$, the positive measure μQ on \mathbb{X} is well defined by setting, for any nonnegative function f ,

$$\mu Q(f) = \mu(Qf) = \int_{\mathbb{X}} Qf(x)\mu(dx).$$

Note that μQ has indeed finite mass $\mu Q(\mathbf{1}) = \mu(Q\mathbf{1}) \leq \|Q\| \mu(\mathbf{1}) < \infty$ since μ is assumed to be a finite measure. This action can therefore naturally be extended to the set of signed measures $\mathcal{M}(\mathbb{X})$, where Q acts as a bounded linear operator with norm $\|Q\|$.

Thus, the elements of \mathcal{K}^+ operate as positive linear operators both on the sets of bounded measurable functions and on the set of signed measures on \mathbb{X} , with a duality relation between these two actions. Moreover, it is also possible to define a projective action \cdot of \mathcal{K}^+ onto the projective space associated with $\mathcal{M}_+(\mathbb{X})$, ie the set of probability measures $\mathcal{M}_1(\mathbb{X})$, by setting, for any $\mu \in \mathcal{M}_+(\mathbb{X})$ and any $M \in \mathcal{K}^+$ such that $\mu M \neq 0$,

$$\mu \cdot M = \frac{\mu M}{\|\mu M\|_{TV}} \in \mathcal{M}_1(\mathbb{X}).$$

Finally, the set \mathcal{K}^+ is naturally endowed with an associative, non commutative product, defined by $:$ for any $Q_1, Q_2 \in \mathcal{K}^+$, any $x \in \mathbb{X}$ and any $A \in \mathcal{X}$,

$$Q_1 Q_2(x, A) = \int_y Q_1(x, dy) Q_2(y, A).$$

This product is compatible with the left and right actions defined above, in other words, for any $Q_1, Q_2 \in \mathcal{K}^+$, any $\mu \in \mathcal{M}_+(\mathbb{X})$, and $f \in \mathcal{B}_+(\mathbb{X})$,

$$\mu(Q_1 Q_2) = (\mu Q_1) Q_2 \text{ and } Q_1 Q_2(f) = Q_1(Q_2 f),$$

and whenever $\mu \in \mathcal{M}_+(\mathbb{X})$ and $\mu Q_1 Q_2 \neq 0$,

$$\mu \cdot (Q_1 Q_2) = (\mu \cdot Q_1) \cdot Q_2.$$

The operator norm $\|\cdot\|$ satisfies the submultiplicativity relation

$$\|Q_1 Q_2\| \leq \|Q_1\| \|Q_2\|.$$

Remark 1. *In the case of a finite or countable set \mathbb{X} , any measure on \mathbb{X} is atomic, an operator $Q \in \mathcal{K}^+$ on \mathbb{X} corresponds to a matrix indexed by \mathbb{X} with nonnegative entries. The product of operators of \mathcal{K}^+ corresponds to the matrix product and the respective left and right actions of \mathcal{K}^+ on signed measures and bounded functions correspond to the product of matrices respectively with the vectors of $\ell^1(\mathbb{X})$ (seen as row vectors) and of $\ell^\infty(\mathbb{X})$ (seen as column vectors).*

We consider a dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability set and $\theta : \Omega \rightarrow \Omega$ is a measurable transformation, which preserves the probability \mathbb{P} , i.e $\mathbb{P} \circ \theta^{-1} = \mathbb{P}$. Let $M : \Omega \rightarrow \mathcal{K}^+$ be a measurable map. We denote as \mathbb{N}_0 the set of nonnegative integers and note, for each $n \in \mathbb{N}_0$,

$$M_n = M \circ \theta^n.$$

Note that the sequence $(M_n)_{n \in \mathbb{N}_0}$ is stationary. For each $k < n$, $\omega \in \Omega$, let us define the random product

$$M_{k,n}(\omega) = M_k(\omega) \cdots M_{n-1}(\omega) = (M \circ \theta^k(\omega)) \cdots (M \circ \theta^{n-1}(\omega)) \in \mathcal{K}^+.$$

with the convention $M_{k,k}(\omega) = \text{Id} \in \mathcal{K}^+$. Notice that $M_{k,k+n}(\omega) = M_{0,n} \circ \theta^k(\omega)$. The operators satisfy the following semi group property : for any $k \leq n \leq N$, any $\mu \in \mathcal{M}(\mathbb{X})$, any $f \in \mathcal{B}(\mathbb{X})$

$$(1.3) \quad \mu M_{k,N}(\omega) f = \mu M_{k,n}(\omega) M_{n,N}(\omega) f.$$

Moreover, for any $x \in \mathbb{X}$, $k \leq n$, $\omega \in \Omega$, we set

$$m_{k,n}(x, \omega) = \delta_x M_{k,n}(\omega) \mathbf{1} = \|\delta_x M_{k,n}(\omega)\|_{TV}.$$

Notice in particular that for any positive measure μ ,

$$\|\mu M_{k,n}\| = \mu(m_{k,n}) = \mu M_{k,N} \mathbf{1} = \mu M_{k,n} m_{n,N},$$

Let us point out additionally that $\|M_{k,n}(\omega)\| = \sup_{x \in \mathbb{X}} m_{k,n}(x, \omega) = \|m_{k,n}(\cdot, \omega)\|_\infty$, and that for any $k \leq n \leq N$, $\|M_{k,N}\| \leq \|M_{k,n}\| \|M_{n,N}\|$. Finally, to shorten the notations, we often omit the dependence in ω , writing for example $m_{k,n}(x) = m_{k,n}(x, \omega)$, and $\|M_{k,n}\| = \|m_{k,n}\|_\infty = \sup_{x \in \mathbb{X}} m_{k,n}(x, \omega)$.

1.3. Assumptions. We list here several hypotheses that will be used in the rest of the article.

A1. *The dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ is ergodic.*

We recall that a dynamical system is ergodic when any measurable set $A \in \mathcal{A}$ such that $\theta^{-1}(A) = A$ satisfies $\mathbb{P}(A) \in \{0, 1\}$.

A2. *For almost all $\omega \in \Omega$, the function $x \mapsto m_{0,1}(x, \omega)$ is a positive function.*

By stationarity, **A2** implies that $\mathbb{P}(d\omega)$ -almost surely, the product $M_{k,n}(\omega)$ is a continuous, non zero, positive linear operator. We introduce the integrability properties

A3. $\mathbb{E} [\log^+ \|m_{0,1}\|_\infty] = \mathbb{E} \log^+ (\|M_{0,1}\|) < \infty.$

and

A3+. $\mathbb{E} [\lceil \log \|m_{0,1}\|_\infty \rceil] = \mathbb{E} \log (\|M_{0,1}\|) < \infty$.

In particular, by submultiplicativity of the norm $\|\cdot\|$, **A1** and **A3** imply that $\mathbb{E} \log^+ (\|M_{k,n}\|) < \infty$ for all $k \leq n$.

In this work, we control the products $(M_{0,n})_{n \in \mathbb{N}}$ using the random sequence $(\gamma_k)_{k \in \mathbb{N}}$ defined as

$$\gamma_k(\omega) = \sup_{\nu \in \mathcal{M}_1(\mathbb{X})} c_k^\nu(\omega) d_{k+1}^\nu(\omega),$$

where for any $k \in \mathbb{N}_0$, for any probability measure $\nu \in \mathcal{M}_1(\mathbb{X})$, we set

$$c_k^\nu(\omega) = \inf_{x \in \mathbb{X}, f \in \mathcal{B}_+(\mathbb{X}), \nu(f) > 0} \frac{\delta_x M_{k,k+1}(\omega)(f)}{\nu(f) m_{k,k+1}(x)},$$

$$d_{k,n}^\nu(\omega) = \frac{\nu M_{k,n} \mathbb{1}}{\sup_{x \in \mathbb{X}} \delta_x M_{k,n} \mathbb{1}} = \frac{\nu(m_{k,n}(\cdot, \omega))}{\|M_{k,n}(\omega)\|} \in [0, 1],$$

and

$$d_k^\nu(\omega) = \inf_{n \geq k} d_{k,n}^\nu(\omega).$$

Note that $c_k^\nu(\omega)$ is the largest element of $[0, 1]$ satisfying

$$\delta_x M_{k,k+1}(\omega)(f) \geq c_k^\nu(\omega) m_{k,k+1}(x, \omega) \nu(f)$$

for all $x \in \mathbb{X}$ and $f \in \mathcal{B}_+(\mathbb{X})$. Similarly, $d_k^\nu(\omega)$ is the largest element of $[0, 1]$ satisfying

$$\nu(m_{k,n}) \geq d_k^\nu(\omega) \|M_{k,n}\| \geq d_k^\nu(\omega) m_{k,n}(x)$$

for all $x \in \mathbb{X}$ and $n \in \mathbb{N}_0$. For each $k \in \mathbb{N}_0$, each $\nu \in \mathcal{M}_1(\mathbb{X})$, the quantities $c_k^\nu, d_{k,n}^\nu, d_k^\nu$ are measurable functions of $\omega \in \Omega$ with values in $[0, 1]$. Moreover, $c_k^\nu = c_0^\nu \circ \theta^k$, $d_{k,k+n}^\nu = d_{0,n}^\nu \circ \theta^k$, $d_k^\nu = d_0^\nu \circ \theta^k$, and $\gamma_k = \gamma_0 \circ \theta^k$. We assume

A4. For \mathbb{P} -almost any ω , $\gamma_0(\omega) > 0$, and moreover $\mathbb{E} |\log(\gamma_0)| < \infty$.

Once again, by stationarity, under **A4**, it holds $\mathbb{P}(d\omega)$ -almost surely, for any $k \geq 0$, $\gamma_k(\omega) > 0$.

2. STATEMENT OF THE RESULTS AND STRUCTURE OF THE PAPER

2.1. Main results. Set

$$\tilde{\eta} := \exp(\mathbb{E} [\log(1 - \gamma_0)]) \in [0, 1],$$

and notice that **A4** yields $\tilde{\eta} < 1$. Under the previous assumptions, we prove the following Theorem:

Theorem 1. *Let $M : \Omega \rightarrow \mathcal{K}^+$ be a measurable map and assume that Assumptions **A1**, **A2** and **A4** hold. Then,*

- i) $\mathbb{P}(d\omega)$ -almost surely, there exists a random function $h \in \mathcal{B}(\mathbb{X})$ such that, for any $\eta \in (\tilde{\eta}, 1)$, for n large enough, for any finite measures $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$,

$$(2.1) \quad \left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)} \mu_2 M_{0,n} \right\|_{TV} \leq \eta^n \|\mu_1 M_{0,n}\|.$$

Such a function h is unique up to a multiplicative constant.

- ii) There exists a probability measure Λ on the space $\mathcal{M}_1(\mathbb{X})$, such that for any probability measure μ , the sequence of random probability measures $(\mu \cdot M_{0,n})$ converges in distribution towards Λ , in the space $\mathcal{M}_1(\mathbb{X})$, endowed with the total variation norm.
- iii) Assuming additionally **A3**, for almost any $\omega \in \Omega$ and any finite, positive, non-zero measure μ ,

$$(2.2) \quad \frac{1}{n} \log \|\mu M_{0,n}\| \xrightarrow{n \rightarrow \infty} \inf_{N \in \mathbb{N}} \frac{1}{N} \mathbb{E} [\log \|M_{0,N}\|] = \lambda \in [-\infty, \infty).$$

Note that the estimate (1.2) can be derived from Theorem 1 by a choice of an arbitrary measure μ_2 , and by setting

$$\pi_n = \mu_2 \cdot M_{0,n}, \tilde{h} = \frac{h}{\mu_2(h)}, r_n = \|\mu_2 M_{0,n}\|,$$

Part of our study focuses on the independent case, that is, when the sequence of operators (M_n) is i.i.d, with a law called \mathcal{P} . This can be obtained by setting $(\Omega, \mathcal{A}, \mathbb{P})$ to be the product space $\Omega = (\mathcal{K}^+)^{\mathbb{N}}$, $\mathbb{P} = \mathcal{P}^{\otimes \mathbb{N}}$ and $M : (\mathcal{K}^+)^{\mathbb{N}} \rightarrow \mathcal{K}^+$, $(N_k)_{k \geq 0} \mapsto N_0$.

In this independent case, we can obtain additional insight over the asymptotical behavior of the mass of the measure $\mu M_{0,n}$.

Theorem 2. Consider an i.i.d sequence (M_n) of elements of \mathcal{K}^+ with law \mathcal{P} , suppose assumptions **A1**, **A2**, and **A4** hold.

- i) Under the additional assumption **A3+**, the almost sure convergence (2.2) also holds in $\mathcal{L}^1(\Lambda \otimes \mathcal{P})$, that is $\int \left\| \frac{1}{n} \log \|\mu M_{0,n}\| - \lambda \right\| d\Lambda(\mu) d\mathcal{P}^{\otimes n}(M_0, \dots, M_{n-1}) \xrightarrow{n \rightarrow \infty} 0$. As a consequence,

$$(2.3) \quad \lambda = \int \log \|\mu M\| d\Lambda(\mu) d\mathcal{P}(M) = \int \rho(\mu, M) d\Lambda \otimes \mathcal{P}(\mu, M).$$

- ii) If $\lambda = 0$, under Assumption **A3**, the following dichotomy holds.

Either for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, almost surely

$$(OSC) \quad \liminf_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = -\infty \text{ and } \limsup_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = +\infty$$

or there exists a function $\eta : \mathcal{M}_1(\mathbb{X}) \mapsto \mathbb{R}$ such that $d(\Lambda \otimes \mathcal{P})(\mu, M)$ -almost surely

$$(NH) \quad \log \|\mu M\| = \eta(\mu \cdot M) - \eta(\mu).$$

When (NH) holds, let $\mu_0 \sim \Lambda$. If the infimum of the support of $\eta(\mu_0)$ is finite (resp is infinite), then almost surely, for any $\mu \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, $\liminf_{n \rightarrow \infty} \log \|\mu M_{0,n}\| \in \mathbb{R}$ (resp. $\liminf_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = -\infty$). The same holds for $\limsup_{n \rightarrow \infty} \log \|\mu M_{0,n}\|$.

2.2. Structure of the paper. Section 3 contains the proofs of Theorems 1 and 2. More precisely, in Subsection 3.1, we recall how the coefficients γ_n allow to control some contraction rates of the operators $M_{0,n}$. These results are adapted from [BCG20]. In Subsection 3.2, we use the ergodic structure, in particular Assumptions A1 and A4 to obtain a geometric decay of the error terms that appeared in our previous estimations. Finally, in Subsection 3.3 we derive the three claims of Theorem 1. In Subsection 3.4, we focus on the case where the sequence (M_n) is i.i.d. In this case, a study of the invariant measures and the ergodicity properties of the Markov chains $(\mu_0 \cdot M_{0,n})_{n \geq 0}$ and $(\mu_0 \cdot M_{0,n}, M_n)_{n \geq 0}$, allows to prove Theorem 2.

Section 4 is dedicated to a comparison of our results with those obtained based on Hilbert contractions. More precisely, we show how natural conditions coming from Hilbert contractions techniques provide more tractable sufficient conditions for our Assumptions (in particular A4), both in finite and infinite dimension.

In Section 5, we apply our results to study products of Leslie Matrices. This constitutes an example of an interesting class of systems that cannot be studied using the Hilbert metric. More precisely, we provide in Subsection 5.2 reasonable sufficient conditions under which a product of Leslie matrices modelling the behavior of an age structured population satisfies assumptions A1 to A4. However, when these conditions are not satisfied, it can be quite difficult to check that $\gamma_0 > 0$, even on a deterministic and constant sequence of Leslie Matrices (M_n) . To illustrate this fact, we present in Subsection 5.3 an example of system where $\gamma_0 = 0$ even if all the other assumptions are satisfied.

3. PROOFS

3.1. Contraction results based on inhomogeneous Doeblin minoration. The first step towards proving Theorem 1 is establishing

Proposition 3. *Suppose A2. Then, $\mathbb{P}(d\omega)$ -almost surely, for any $k \leq n \leq N$ and any finite measures $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$, it holds*

$$(3.1) \quad \|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \leq 2 \prod_{i=k}^{n-1} (1 - \gamma_i),$$

and

$$(3.2) \quad \gamma_{n-1} \left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leq \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} 2 \prod_{i=k}^{n-1} (1 - \gamma_i).$$

This result was already introduced in [BCG20] in a somewhat different setup. We have chosen to state and prove it here for the sake of completeness. Its proof is based on performing a Doeblin minoration on a well-chosen sequence of auxiliary Markov operators $(P_{k,n}^N)$. This Doeblin property yields (3.1), a contraction property for the projective action of $M_{k,n}$ on the space of measures $\mathcal{M}_+(\mathbb{X})$. We derive then Equation (3.2), which describes how the growth of the mass $\|\mu M_{k,t}\|$ between times $t = n$ and $t = N$ depends on the initial measure μ .

Let us introduce now the operators $P_{k,n}^N$ upon which we perform the desired Doeblin minoration. Under assumption **A2**, $\mathbb{P}(d\omega)$ -almost surely, for any $k \leq n \leq N$, $x \mapsto m_{k,n}(x)$ and $x \mapsto m_{n,N}(x)$ are positive functions on \mathbb{X} . For each $k \leq n \leq N$, an operator $P_{k,n}^N(\omega)$ can be defined $\mathbb{P}(d\omega)$ -almost surely, as follows : for each $x \in \mathbb{X}$, for each positive measurable $f : \mathbb{X} \rightarrow \mathbb{R}$,

$$\delta_x P_{k,n}^N f = \frac{\delta_x M_{k,n}(f m_{n,N})}{m_{k,n}(x)},$$

$P_{k,n}^N$ is a positive and conservative operator (i.e. $P_{k,n}^N \mathbf{1} = \mathbf{1}$). Indeed, by Equation (1.3), for any $x \in \mathbb{X}$,

$$\delta_x P_{k,n}^N \mathbf{1} = \frac{\delta_x M_{k,n} m_{n,N}}{\delta_x M_{k,n} \mathbf{1}} = 1.$$

Moreover, $P_{k,n}^N$ satisfies the relation :

$$P_{k,n}^N = P_{k,k+1}^N \cdots P_{n-1,n}^N.$$

Note that $P_{k,n}^N$ is a matrix when \mathbb{X} is countable and then, for any $x, y \in \mathbb{X}$,

$$P_{k,n}^N(x, y) = \frac{m_{n,N}(y)}{m_{k,n}(x)} M_{k,n}(x, y).$$

These operators satisfy a Doeblin contraction property summed up in

Lemma 4. *Assume Assumption **A2** holds, then $\mathbb{P}(d\omega)$ -almost surely, all the $P_{k,n}^N(\omega)$ are well defined, and it holds*

- i) *For any $\nu \in \mathcal{M}_1(\mathbb{X})$, any $n \leq N - 1$, there exists a random probability measure $\nu_{n,N}$ on \mathbb{X} such that, for any $x \in \mathbb{X}$,*

$$\delta_x P_{n,n+1}^N \geq c_n^\nu d_{n+1,N}^\nu \nu_{n,N} \geq c_n^\nu d_{n+1}^\nu \nu_{n,N}.$$

- ii) *For any signed measures ρ_1, ρ_2 , of same mass and any $n \leq N - 1$,*

$$\|\rho_1 P_{n,n+1}^N - \rho_2 P_{n,n+1}^N\|_{TV} \leq (1 - \gamma_n) \|\rho_1 - \rho_2\|_{TV}.$$

- iii) *For any $k \leq n \leq N$ and any signed measures ρ_1, ρ_2 of same mass,*

$$\|\rho_1 P_{k,n}^N - \rho_2 P_{k,n}^N\|_{TV} \leq \prod_{i=k}^{n-1} (1 - \gamma_i) \|\rho_1 - \rho_2\|_{TV}.$$

Notice that in this lemma, our single assumption is **A2**. It allows the $(P_{k,n}^N)$ to be defined $\mathbb{P}(d\omega)$ -almost surely. In particular, **A4** is not assumed, we allow $\gamma_n(\omega) = 0$, in which case we just obtain that $P_{n,n+1}^N$ is 1-contracting.

Proof of Lemma 4. Let $\omega \in \Omega$ such that all the $P_{k,n}^N$ are well defined. Let $\nu \in \mathcal{M}_1(\mathbb{X})$. By definition of c_n^ν , for any $x \in \mathbb{X}$ and any $f \in \mathcal{B}_+(\mathbb{X})$, it holds,

$$\delta_x M_{n,n+1}(f m_{n+1,N}) \geq c_n^\nu m_{n,n+1}(x) \nu(f m_{n+1,N}),$$

thus

$$\delta_x P_{n,n+1}^N f = \frac{\delta_x M_{n,n+1}(f m_{n+1,N})}{m_{n,N}(x)} \geq c_n^\nu \frac{\nu(f m_{n+1,N}) m_{n,n+1}(x)}{m_{n,N}(x)},$$

with, by definition of $d_{n+1,N}^\nu$:

$$d_{n+1,N}^\nu m_{n,N}(x) = d_{n+1,N}^\nu \delta_x M_{n,n+1}(m_{n+1,N}) \leq \nu(m_{n+1,N}) m_{n,n+1}(x).$$

Therefore,

$$\delta_x P_{n,n+1}^N f \geq c_n^\nu d_{n+1,N}^\nu \frac{\nu(f m_{n+1,N})}{\nu(m_{n+1,N})} = c_n^\nu d_{n+1,N}^\nu \nu_{n,N}(f),$$

setting

$$\nu_{n,N}(\cdot) = \frac{\nu(\cdot m_{n+1,N})}{\nu(m_{n+1,N})},$$

which is a probability measure. This concludes the proof of *i*).

Let us prove now *ii*). This result is a classical consequence of the previous point using the theory of Markov operators. A Markov operator P is said to be δ -Doebelin (with $\delta > 0$) when there exists a probability measure μ such that $\delta_x P f \geq \delta \mu(f)$ for any x in the state space and any $f \in \mathcal{B}_+(\mathbb{X})$. Furthermore, such an operator is $1 - \delta$ contracting in total variation : for any signed measure ρ of mass 0,

$$\|\rho P\|_{TV} \leq (1 - \delta) \|\rho\|_{TV}.$$

This property trivially holds for $\delta = 0$: any positive operator satisfies $\delta_x P f \geq 0$ when f is a non negative function, and any Markov operator is $1 = (1 - 0)$ -contracting in total variation. In our context, the previous point of the lemma yields that $\mathbb{P}(d\omega)$ -almost surely, for any $n \leq N - 1$, and any $\nu \in \mathcal{M}_1(\mathbb{X})$, the Markov operator $P_{n,n+1}^N$ is $c_n^\nu d_{n+1,N}^\nu$ -Doebelin. Therefore, for any $\rho_1, \rho_2 \in \mathcal{M}(\mathbb{X})$, such that $\rho_1(\mathbb{1}) = \rho_2(\mathbb{1})$, noting $\rho = \rho_1 - \rho_2$, it holds

$$\|\rho P_{n,n+1}^N\|_{TV} \leq (1 - c_n^\nu(\omega) d_{n+1,N}^\nu(\omega)) \|\rho\|_{TV}.$$

Finally, recalling that

$$\gamma_n = \sup_{\nu \in \mathcal{M}_1(\mathbb{X})} c_n^\nu d_{n+1,N}^\nu,$$

and taking an infimum, we get :

$$\|\rho P_{n,n+1}^N\|_{TV} \leq (1 - \gamma_n) \|\rho\|_{TV}.$$

This proves *ii*), let us move now to *iii*). Since all the $P_{n,n+1}^N$ are conservative operators, the image of a measure of mass 0 by $P_{n,n+1}^N$ is a measure of mass 0. The equality $P_{k,n}^N = P_{k,k+1}^N \cdots P_{n-1,n}^N$, yields $\|\rho P_{k,n}^N\|_{TV} \leq (1 - \gamma_{n-1}) \|\rho P_{k,n-1}^N\|_{TV}$. By induction, we deduce

$$\|\rho P_{k,n}^N\|_{TV} \leq \prod_{i=k}^{n-1} (1 - \gamma_i) \|\rho\|_{TV}.$$

This concludes the proof. \square

Proof of Proposition 3. Let us prove first Inequality (3.1). Applying Lemma 4, *iii*) with $n = N$ and $\rho = \delta_x - \delta_y$, we get, $\mathbb{P}(d\omega)$ -almost surely

$$\|\delta_x P_{k,n}^n - \delta_y P_{k,n}^n\|_{TV} \leq \prod_{i=k}^{n-1} (1 - \gamma_i) \|\delta_x - \delta_y\|_{TV} \leq 2 \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Hence, for any $f \in \mathcal{B}(\mathbb{X})$ and $x, y \in \mathbb{X}$,

$$\left| \frac{\delta_x M_{k,n}(f)}{m_{k,n}(x)} - \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)} \right| \leq 2 \|f\|_\infty \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Let μ_1 and μ_2 be two positive measures. The inequality

$$\left| \delta_x M_{k,n}(f) - m_{k,n}(x) \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)} \right| \leq 2 m_{k,n}(x) \|f\|_\infty \prod_{i=k}^{n-1} (1 - \gamma_i),$$

yields, after integrating with respect to $\mu_1(dx)$:

$$\left| \mu_1 M_{k,n}(f) - \mu_1(m_{k,n}) \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)} \right| \leq 2 \mu_1(m_{k,n}) \|f\|_\infty \prod_{i=k}^{n-1} (1 - \gamma_i),$$

so that

$$\left| \mu_1 \cdot M_{k,n}(f) - \frac{\delta_y M_{k,n}(f)}{m_{k,n}(y)} \right| \leq 2 \|f\|_\infty \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Integrating now with respect to $\mu_2(dy)$, we obtain

$$|\mu_1 \cdot M_{k,n}(f) - \mu_2 \cdot M_{k,n}(f)| \leq 2 \|f\|_\infty \prod_{i=k}^{n-1} (1 - \gamma_i),$$

and finally

$$\|\mu_1 \cdot M_{k,n} - \mu_2 \cdot M_{k,n}\|_{TV} \leq 2 \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Let us move now to the proof of Inequality (3.2). Applying Inequality (3.1) to the function $x \mapsto m_{n,N}(x)$, one gets

$$\left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leq 2 \|m_{n,N}\|_\infty \prod_{i=k}^{n-1} (1 - \gamma_i) = 2 \|M_{n,N}\| \prod_{i=k}^{n-1} (1 - \gamma_i).$$

By definition of c_{n-1}^ν and d_n^ν , it holds, for any $\nu \in \mathcal{M}_1(\mathbb{X})$,

$$d_n^\nu \|M_{n,N}\| \leq \nu(m_{n,N})$$

and

$$\mu_2(m_{k,N}) = \mu_2 M_{k,n-1} M_{n-1,n} m_{n,N} \geq c_{n-1}^\nu \mu_2(m_{k,n}) \nu(m_{n,N}).$$

Combining these identities we obtain, $\mathbb{P}(d\omega)$ -almost surely, for any $\nu \in \mathcal{M}_1(\mathbb{X})$:

$$c_{n-1}^\nu d_n^\nu \left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leq 2 \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Thus taking an infimum in ν , this yields, $\mathbb{P}(d\omega)$ -almost surely,

$$\gamma_{n-1} \left| \frac{\mu_1(m_{k,N})}{\mu_1(m_{k,n})} - \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \right| \leq 2 \frac{\mu_2(m_{k,N})}{\mu_2(m_{k,n})} \prod_{i=k}^{n-1} (1 - \gamma_i).$$

This ends the proof. \square

3.2. Asymptotic estimates under ergodicity assumptions. In this subsection, we use the ergodicity Assumption **A1**, as well as Assumption **A4**, which provide a control on the sequence of Doeblin coefficients (γ_n) . Together, these assumptions allow to prove that the right-hand side terms of (3.1) and (3.2) decay at least geometrically fastly.

Proposition 5. *Assume that assumptions **A1**, **A2** and **A4** hold. Then for any $k \geq 1$,*

$$(3.3) \quad \left(\prod_{i=k}^{k+n-1} (1 - \gamma_i) \right)^{\frac{1}{n}} \xrightarrow[n \rightarrow \infty]{} \tilde{\eta} < 1, \text{ for almost all } \omega \in \Omega$$

and

$$(3.4) \quad \lim_{n \rightarrow \infty} \left(\frac{1}{\gamma_n} \right)^{\frac{1}{n}} = 1, \text{ for almost all } \omega \in \Omega.$$

Proof of Proposition 5. We recall first that by definition, for any $\omega \in \Omega$

$$(1 - \gamma_i)(\omega) = (1 - \gamma_0) \circ \theta^i(\omega).$$

Notice then that, by Assumption **A4**, for almost every $\omega \in \Omega$, $\gamma_0(\omega) \in (0, 1]$. Thus $\log(1 - \gamma_0(\omega)) \in [-\infty, 0)$ and $\tilde{\eta} = \exp(\mathbb{E}[\log(1 - \gamma_0(\omega))]) \in [0, 1)$.

Thus for any k ,

$$\log \left[\left(\prod_{i=k}^{k+n-1} (1 - \gamma_i) \right)^{\frac{1}{n}} \right] = \frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_i) \circ \theta^{k+i}.$$

Since θ is an ergodic map, by Birkhoff's ergodic theorem, for any $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_{k+i}) \xrightarrow[n \rightarrow \infty]{} \mathbb{E}[\log(1 - \gamma_0)] = \log \tilde{\eta}, \mathbb{P}(d\omega) - \text{almost surely.}$$

This yields (3.3).

Let us move to the proof of Inequality (3.4). Notice first that since $\gamma_n \leq 1$ for all n ,

$$\liminf_{n \rightarrow \infty} \left(\frac{1}{\gamma_n} \right)^{\frac{1}{n}} \geq 1.$$

Let us prove now the converse inequality. Let us define for each $b > 1$,

$$Y_n(b) = \frac{-\log(\gamma_n)}{\log(b)} \geq 0,$$

and

$$N_b = \sum_{n \in \mathbb{N}_0} \mathbb{1}_{(1/\gamma_n)^{\frac{1}{n}} > b} = \sum_{n \in \mathbb{N}_0} \mathbb{1}_{Y_n(b) > n}.$$

For a given value of b , the sequence $(Y_n(b))_{n \in \mathbb{N}_0}$ is stationary, thus

$$\mathbb{E}(N_b) = \sum_{n \in \mathbb{N}_0} \mathbb{P}[Y_n(b) > n] = \sum_{n \in \mathbb{N}_0} \mathbb{P}[Y_0(b) > n].$$

It is a well known fact that for a nonnegative random variable Y ,

$$\mathbb{E}[Y] < \infty \Leftrightarrow \sum_{n \geq 0} \mathbb{P}(Y > n) < \infty.$$

By Assumption **A4**, it holds $\mathbb{E}[Y_0(b)] < \infty$ and $\mathbb{E}[N_b] < \infty$ for all $b > 1$. Therefore, $\mathbb{P}(d\omega)$ -almost surely, $N_b < \infty$, thus $\left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \leq b$ for n large enough and $\limsup_{n \rightarrow \infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} \leq 1$. Finally,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\gamma_n}\right)^{\frac{1}{n}} = 1, \mathbb{P}(d\omega) - \text{almost surely.}$$

□

Putting the estimates from Proposition **5** together with Proposition **3**, we obtain

Proposition 6. *Assume **A1**, **A2**, **A4** hold. Then, $\mathbb{P}(d\omega)$ -almost surely, for any $k \in \mathbb{N}_0$, there exists a measurable function h_k such that, for any n for any $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{X})$,*

$$(3.5) \quad \left\| \mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2 M_{k,n} \right\|_{TV} \leq \frac{4}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \|\mu_1 M_{k,n}\| = \underset{n \rightarrow \infty}{o}(\|\mu_1 M_{k,n}\|).$$

Furthermore, $\mathbb{P}(d\omega)$ -almost surely, such a function is unique up to a multiplicative constant.

Proof of Proposition 6. Let us assume that there exists a positive function h_k satisfying Inequality (3.5). Then, if $x, y \in \mathbb{X}$, setting $\mu_1 = \delta_x, \mu_2 = \delta_y$ and applying this inequality to the constant function $\mathbb{1}$, we almost surely get

$$\left| m_{k,n}(x) - \frac{h_k(x)}{h_k(y)} m_{k,n}(y) \right| \leq \frac{4}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) m_{k,n}(x) = \underset{n \rightarrow \infty}{o}(m_{k,n}(x)),$$

since by Proposition **5**, $\mathbb{P}(d\omega)$ -almost surely, $\gamma_{n-1}^{-1} \prod_{i=k}^{n-1} (1 - \gamma_i) = \underset{n \rightarrow \infty}{o}(1)$. Thus

$$\lim_{n \rightarrow \infty} \frac{h_k(x) m_{k,n}(y)}{h_k(y) m_{k,n}(x)} = 1,$$

which readily implies that

$$\frac{h_k(x)}{h_k(y)} = \lim_{n \rightarrow \infty} \frac{m_{k,n}(x)}{m_{k,n}(y)}.$$

This yields the unicity of h_k up to a multiplicative constant, when it exists. Let us now prove the existence of such a function h_k .

By Inequality (3.2), with $\mu_1 = \delta_x, \mu_2 = \delta_y$, one gets, $\mathbb{P}(d\omega)$ -almost surely, for any $k \leq n \leq N$:

$$(3.6) \quad \left| \frac{m_{k,N}(x)}{m_{k,N}(y)} - \frac{m_{k,n}(x)}{m_{k,n}(y)} \right| \leq \frac{2}{\gamma_{n-1}} \frac{m_{k,n}(x)}{m_{k,n}(y)} \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Setting

$$\mathbf{diam}_{k,n}(x, y) = \sup_{N_1, N_2 \geq n} \left| \frac{m_{k,N_1}(x)}{m_{k,N_1}(y)} - \frac{m_{k,N_2}(x)}{m_{k,N_2}(y)} \right|,$$

this yields, for any $x, y \in \mathbb{X}$,

$$\mathbf{diam}_{k,n}(x, y) \leq \frac{4}{\gamma_{n-1}} \frac{m_{k,n}(x)}{m_{k,n}(y)} \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Exchanging the roles of x, y , one gets :

$$(3.7) \quad \min [\mathbf{diam}_{k,n}(x, y), \mathbf{diam}_{k,n}(y, x)] \leq \frac{4}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i),$$

and Proposition 5 yields, $\mathbb{P}(d\omega)$ -almost surely,

$$\min [\mathbf{diam}_{k,n}(x, y), \mathbf{diam}_{k,n}(y, x)] \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence, one of the two sequences $(\mathbf{diam}_{k,n}(x, y))_{n \geq k}$ and $(\mathbf{diam}_{k,n}(y, x))_{n \geq k}$ has zero as an adherence value. Since both sequences are non increasing, it implies that one of them converges to 0. Without loss of generality, suppose that

$$\mathbf{diam}_{k,n}(x, y) \xrightarrow[n \rightarrow \infty]{} 0.$$

Then, the sequence of nonnegative real numbers $\left(\frac{m_{k,n}(x)}{m_{k,n}(y)} \right)_{n \geq k}$ is a Cauchy sequence, it converges to a nonnegative limit $l_k(x, y)$. Thus Equation (3.6) yields, for n large enough

$$\left| \frac{m_{k,n}(x)}{m_{k,n}(y)} - l_k(x, y) \right| \leq \frac{2}{\gamma_{n-1}} \frac{m_{k,n}(x)}{m_{k,n}(y)} \prod_{i=k}^{n-1} (1 - \gamma_i) \leq \frac{1}{4} \frac{m_{k,n}(x)}{m_{k,n}(y)}.$$

Since $\frac{m_{k,n}(x)}{m_{k,n}(y)} > 0$, this implies that $l_k(x, y) > 0$ and consequently,

$$\frac{m_{k,n}(y)}{m_{k,n}(x)} \xrightarrow[n \rightarrow \infty]{} \frac{1}{l_k(x, y)} < \infty.$$

Note that Proposition 3 allows to prove that $\mathbb{P}(d\omega)$ -almost surely, (3.6) holds jointly for any $k \leq n \leq N$ and any $x, y \in \mathbb{X}$, thus so does (3.7). Thus $\mathbb{P}(d\omega)$ -almost surely, all the

sequences of the form $\left(\frac{m_{k,n}(x)}{m_{k,n}(y)}\right)_{n \geq k}$ for all $x, y \in \mathbb{X}$ converge as $n \rightarrow \infty$.

Now, let us fix an arbitrary element $x_0 \in \mathbb{X}$, and set $h_k(x) = \lim_{n \rightarrow \infty} \frac{m_{k,n}(x)}{m_{k,n}(x_0)}$ for all x . The function h_k is positive and satisfies, for any $x, y \in \mathbb{X}$,

$$\frac{m_{k,n}(x)}{m_{k,n}(y)} \xrightarrow{n \rightarrow \infty} \frac{h_k(x)}{h_k(y)}.$$

Plugging this limit into Equation (3.6), we obtain

$$(3.8) \quad \left| \frac{m_{k,n}(x)}{m_{k,n}(y)} - \frac{h_k(x)}{h_k(y)} \right| \leq \frac{2}{\gamma_{n-1}} \frac{m_{k,n}(x)}{m_{k,n}(y)} \prod_{i=k}^{n-1} (1 - \gamma_i).$$

Consequently,

$$h_k(x) \leq h_k(y) \left(1 + \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \right) \frac{m_{k,n}(x)}{m_{k,n}(y)} \leq h_k(y) \left(1 + \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \right) \frac{\|m_{k,n}\|_\infty}{m_{k,n}(y)} < \infty,$$

by **A2**, which implies that h_k is bounded. Moreover, for any positive and finite measure $\mu_1 \in \mathcal{M}_+(\mathbb{X})$, any $y \in \mathbb{X}$, integrating (3.6) with respect to $\mu_1(dx)$, one gets

$$\left| \frac{\mu_1(m_{k,n})}{m_{k,n}(y)} - \frac{\mu_1(h_k)}{h_k(y)} \right| \leq \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \frac{\mu_1(m_{k,n})}{m_{k,n}(y)},$$

Thus

$$|\mu_1(m_{k,n})h_k(y) - m_{k,n}(y)\mu_1(h_k)| \leq \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \mu_1(m_{k,n})h_k(y).$$

Integrating with respect to any positive and finite measure $\mu_2(dy)$, this yields

$$|\mu_1(m_{k,n})\mu_2(h_k) - \mu_2(m_{k,n})\mu_1(h_k)| \leq \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \mu_1(m_{k,n})\mu_2(h_k),$$

and finally,

$$(3.9) \quad \left| \frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \right| \leq \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})}.$$

Let us prove now that h_k satisfies Inequality (3.5). It holds

$$\begin{aligned} \left\| \mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2 M_{k,n} \right\|_{TV} &\leq \left\| \mu_1 M_{k,n} - \mu_1(m_{k,n}) \mu_2 \cdot M_{k,n} \right\|_{TV} \\ &\quad + \left\| \mu_1(m_{k,n}) \mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \mu_2 M_{k,n} \right\|_{TV}. \end{aligned}$$

On the one hand, applying Inequality (3.1), one has

$$\left\| \mu_1 M_{k,n} - \mu_1(m_{k,n}) \mu_2 \cdot M_{k,n} \right\|_{TV} \leq \mu_1(m_{k,n}) \left\| \mu_2 \cdot M_{k,n} - \mu_2 \cdot M_{k,n} \right\|_{TV} \leq 2 \prod_{i=k}^{n-1} (1 - \gamma_i) \mu_1(m_{k,n}).$$

On the other hand, by Equation (3.9),

$$\begin{aligned} \left\| \mu_1(m_{k,n})\mu_2 \cdot M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n} \right\|_{TV} &\leq \mu_2(m_{k,n}) \left| \frac{\mu_1(m_{k,n})}{\mu_2(m_{k,n})} - \frac{\mu_1(h_k)}{\mu_2(h_k)} \right| \\ &\leq \frac{2}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \mu_1(m_{k,n}). \end{aligned}$$

Finally,

$$\begin{aligned} \left\| \mu_1 M_{k,n} - \frac{\mu_1(h_k)}{\mu_2(h_k)}\mu_2 M_{k,n} \right\|_{TV} &\leq 2 \prod_{i=k}^{n-1} (1 - \gamma_i) \mu_1(m_{k,n}) \left(1 + \frac{1}{\gamma_{n-1}} \right) \\ &\leq \frac{4}{\gamma_{n-1}} \prod_{i=k}^{n-1} (1 - \gamma_i) \|\mu_1 M_{k,n}\|. \end{aligned}$$

This ends the proof. \square

3.3. Proof of Theorem 1.

Proof of assertion i) : Uniform geometric ergodicity. Let us take $k = 0$ in Proposition 6. Then, $\mathbb{P}(d\omega)$ -almost surely, noting $h = h_0$, it holds for any finite and positive measures μ_1, μ_2 , on \mathbb{X} ,

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)}\mu_2 M_{0,n} \right\|_{TV} \leq \frac{4}{\gamma_{n-1}} \prod_{i=0}^{n-1} (1 - \gamma_i) \|\mu_1 M_{0,n}\|.$$

By Proposition 5, $\mathbb{P}(d\omega)$ -almost surely,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\gamma_{n-1}} \right)^{\frac{1}{n}} = 1 \text{ and } \prod_{i=0}^{n-1} (1 - \gamma_i) \xrightarrow[n \rightarrow \infty]{} \tilde{\eta} \in [0, 1).$$

Thus for any $\eta \in (\tilde{\eta}, 1)$, $\mathbb{P}(d\omega)$ -almost surely, for n large enough, (depending on ω),

$$\frac{4}{\gamma_{n-1}} \prod_{i=0}^{n-1} (1 - \gamma_i) \leq \eta^n.$$

Thus, $\mathbb{P}(d\omega)$ -almost surely, for any $\eta \in (\tilde{\eta}, 1)$, for n large enough and any positive and finite measures μ_1, μ_2 ,

$$\left\| \mu_1 M_{0,n} - \frac{\mu_1(h)}{\mu_2(h)}\mu_2 M_{0,n} \right\|_{TV} \leq \eta^n \|\mu_1 M_{0,n}\| = \underset{n \rightarrow \infty}{o}(\|\mu_1 M_{0,n}\|).$$

This proves Equation (2.1). \square

Proof of assertion ii). The proof relies on a classical time-reversal technique, see e.g. [Cog84; Ore91], or [Hen97] for a version that is closer to our context. As stated in [CFS82, II.10.4, pp.239-241], the ergodic system $(\Omega, \mathcal{A}, \mathbb{P}, \theta)$ can be extended as an invertible ergodic system $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}}, \bar{\theta})$, such that $\Omega \subset \bar{\Omega}$, $\bar{\theta}|_{\Omega} = \theta$, and $\bar{\theta}$ is a bijective, bimeasurable, measure preserving and ergodic mapping. The definitions of $M_{k,n}$, c_n^ν , $d_{k,n}^\nu$, d_n^ν , γ_n can be naturally

extended to all $k \leq n$ in \mathbb{Z} , and one still has $c_n^\nu = c_0^\nu \circ \bar{\theta}^n$, $d_n^\nu = d_0^\nu \circ \bar{\theta}^n$, $\gamma_n = \gamma_0 \circ \bar{\theta}^n$ for $n \in \mathbb{Z}$. Assumption **A4** implies that all the $(\gamma_n)_{n \in \mathbb{Z}}$ are almost surely positive and have log-moments.

Therefore, Lemma 4 and Proposition 3 extend to indexes $k \leq n \leq N \in \mathbb{Z}$.

For nonnegative $n \leq N$, for any positive measures μ_1, μ_2 on \mathbb{X} , one has in particular

$$(3.10) \quad \|\mu_1 \cdot M_{-n,0} - \mu_2 \cdot M_{-n,0}\|_{TV} \leq 2 \prod_{i=0}^{n-1} (1 - \gamma_{-i-1}).$$

With $\mu_2 = \mu_1 M_{-N,-n}$, this yields :

$$\|\mu_1 \cdot M_{-n,0} - \mu_1 \cdot M_{-N,0}\|_{TV} \leq 2 \prod_{i=0}^{n-1} (1 - \gamma_{-i-1}).$$

Noticing that $\bar{\theta}$ is now an ergodic automorphism of the measured space $\bar{\Omega}$, and applying Birkhoff-Khinchin Ergodic Theorem as stated in [CFS82, Theorem 1, p.11], one gets, for almost any $\omega \in \bar{\Omega}$

$$\frac{1}{n} \sum_{i=0}^{n-1} \log(1 - \gamma_{-i-1}) \circ \bar{\theta}^{-i} \xrightarrow[n \rightarrow \infty]{} \mathbb{E}[\log(1 - \gamma_{-1})] = \mathbb{E}[\log(1 - \gamma_0)].$$

Thus

$$\left(\prod_{i=0}^{n-1} (1 - \gamma_{-i-1}) \right)^{\frac{1}{n}} \xrightarrow[n \rightarrow \infty]{} \exp(\mathbb{E}[\log(1 - \gamma_0)]) = \tilde{\eta} < 1.$$

Therefore, almost surely, the sequence $(\mu_1 \cdot M_{-n,0})_{n \in \mathbb{N}_0}$ is a Cauchy sequence in the space $\mathcal{M}_1(\mathbb{X})$ of probabilities on \mathbb{X} , endowed with the total variation norm. It thus converges almost surely to a random probability π_{μ_1} on \mathbb{X} . For any finite, positive non-zero measures μ_1, μ_2 , plugging π_{μ_1}, π_{μ_2} into (3.10), one proves that for almost any ω ,

$$\pi_{\mu_1} = \pi_{\mu_2}.$$

Thus, there exists a random probability π , such that, almost surely, for any positive measure μ

$$\mu \cdot M_{-n,0} \xrightarrow[n \rightarrow \infty]{} \pi \text{ in total variation distance.}$$

By stationarity of $\bar{\theta}$,

$$\mu \cdot M_{-n,0} \stackrel{d}{=} \mu \cdot M_{0,n},$$

which proves that, noting Λ the distribution of π ,

$$\mu \cdot M_{0,n} \xrightarrow[n \rightarrow \infty]{} \Lambda.$$

□

Proof of assertion iii). Let μ be a positive, finite measure on \mathbb{X} . By assumption **A4**, since $\gamma_0 > 0$, almost surely, for \mathbb{P} -almost any $\omega \in \Omega$, there exists a probability $\nu \in \mathcal{M}_1(\mathbb{X})$ such

that $c_0^\nu d_1^\nu > 0$. By stationarity, with probability 1, there exists ν such that $d_0^\nu > 0$. The definition of d_0^ν yields :

$$d_0^\nu \|M_{0,n}\| \leq \nu(m_{0,n}).$$

Let $\eta \in (\tilde{\eta}, 1)$. Theorem 1, i) implies that, for n large enough,

$$\left| \|\mu M_{0,n}\| - \frac{\mu(h)}{\nu(h)} \nu(m_{0,n}) \right| \leq \eta^n \|\mu M_{0,n}\|,$$

which implies

$$\frac{1}{1 + \eta^n} \frac{\mu(h)}{\nu(h)} \nu(m_{0,n}) \leq \|\mu M_{0,n}\|.$$

This yields

$$\frac{d_0^\nu}{1 + \eta^n} \frac{\mu(h)}{\nu(h)} \|M_{0,n}\| \leq \|\mu M_{0,n}\| \leq \mu(\mathbf{1}) \|M_{0,n}\|.$$

Notice that, almost surely, $0 < \frac{\mu(h)}{\nu(h)}$, $0 < d_0(\nu, \omega)$, and $\mu(\mathbf{1}) < \infty$. Thus, if $n^{-1} \log \|M_{0,n}\|$ converges almost surely, then almost surely, for any non zero, positive measure μ , $n^{-1} \log \|\mu M_{0,n}\|$ converges to the same limit. The \mathbb{P} -almost sure convergence

$$n^{-1} \log \|M_{0,n}\| \xrightarrow{n \rightarrow \infty} \lambda := \inf_{N \geq 1} \frac{1}{N} \mathbb{E} [\log \|M_{0,N}\|]$$

is a classical consequence of Kingman's subadditive ergodic Theorem and the subadditivity property :

$$\log \|M_{0,n+p}\| \leq \log \|M_{0,n}\| + \log \|M_{n,n+p}\|.$$

Note that applying this Kingman's Theorem requires Assumption **A3** to ensure the integrability of $\log^+ \|M_{0,n}\|$. Thus $n^{-1} \log \|\mu M_{0,n}\| \xrightarrow{n \rightarrow \infty} \lambda$ almost surely, for any positive and finite measure μ . \square

3.4. The independent case : proof of Theorem 2. Let us introduce the Markov chain $(\mu_n)_{n \geq 0}$ with state space $\mathcal{M}_1(\mathbb{X})$, defined by $\mu_{n+1} = \mu_n \cdot M_n = \mu_0 \cdot M_{0,n+1}$. The process $((\mu_n, M_n)_{n \geq 0})$, is then clearly also a Markov chain with state space $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$ and transition kernel :

$$Qf(\mu, M) = \int f(\mu \cdot M, N) d\mathcal{P}(N).$$

We denote $\mathbb{P}_{\bar{\chi}}$ the law of the Markov chain $((\mu_n, M_n)_{n \geq 0})$ when (μ_0, M_0) is distributed according to a measure $\bar{\chi}$ on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$. Theorem 2 relies on the study of the invariant measures and the ergodicity properties of the Markov chains $(\mu_n)_{n \geq 0}$ and $(\mu_n, M_n)_{n \geq 0}$. In particular, we show that the limit distribution Λ of the Markov chain (μ_n) is its only invariant distribution. This is stated in the following Proposition.

Proposition 7. *Suppose $(M_n)_{n \geq 0}$ is an i.i.d sequence of elements of \mathcal{K}^+ distributed according to \mathcal{P} , and satisfying the assumptions of Theorem 1. Then*

- i) *For any initial distribution χ on $\mathcal{M}_1(\mathbb{X})$, the Markov chain $(\mu_n)_{n \geq 0}$ converges weakly to Λ .*
- ii) *Λ is the only invariant measure of the Markov chain $(\mu_n)_{n \geq 0}$.*

iii) $\Lambda \otimes \mathcal{P}$ is the only invariant measure of the Markov chain $((\mu_n, M_n))_{n \geq 0}$.

As a consequence, these Markov chains are ergodic.

To prove Proposition 7, we need to define the convolution operation \star between probability measures on $\mathcal{M}_1(\mathcal{K}^+)$ as follows : For any $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{M}_1(\mathcal{K}^+)$, $\mathcal{Q}_1 \star \mathcal{Q}_2$ is the law of $N_1 N_2$, where $(N_1, N_2) \sim \mathcal{Q}_1 \otimes \mathcal{Q}_2$. We note, for any $\mathcal{Q} \in \mathcal{M}_1(\mathcal{K}^+)$, $\mathcal{Q}^{\star n}$ the n -th convolution power of \mathcal{Q} . As an example, if N_0, \dots, N_{n-1} are i.i.d with law \mathcal{Q} , $\mathcal{Q}^{\star n}$ is simply the distribution of $N_{0,n} = N_0 \cdots N_{n-1}$. Given a probability distribution χ on $\mathcal{M}_1(\mathbb{X})$ and \mathcal{Q} on \mathcal{K}^+ , we also note $\chi \star \mathcal{Q}$ the law of $\mu \cdot N$, where $(\mu, N) \sim \chi \otimes \mathcal{Q}$. These operations, previously defined in [BL85] in a finite dimensional context, satisfy some elementary properties, summed up in the following lemma.

Lemma 8. *Let $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ be probability measures on \mathcal{K}^+ and χ be a probability measure on $\mathcal{M}_1(\mathbb{X})$, it holds*

- i) $(\mathcal{Q}_1 \star \mathcal{Q}_2) \star \mathcal{Q}_3 = \mathcal{Q}_1 \star (\mathcal{Q}_2 \star \mathcal{Q}_3)$,
- ii) $(\chi \star \mathcal{Q}_1) \star \mathcal{Q}_2 = \chi \star (\mathcal{Q}_1 \star \mathcal{Q}_2)$,
- iii) *For each $\mathcal{Q} \in \mathcal{M}_1(\mathcal{K}^+)$, $\chi \mapsto \chi \star \mathcal{Q}$ is continuous with respect to the topology of convergence in law on $\mathcal{M}_1(\mathcal{M}_1(\mathbb{X}))$.*

Proof of Lemma 8. Consider $(N_1, N_2, N_3) \sim \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \mathcal{Q}_3$. It holds

$$N_1 N_2 N_3 = (N_1 N_2) N_3 = N_1 (N_2 N_3),$$

with $(N_1 N_2) N_3 \sim (\mathcal{Q}_1 \star \mathcal{Q}_2) \star \mathcal{Q}_3$ and $N_1 (N_2 N_3) \sim \mathcal{Q}_1 \star (\mathcal{Q}_2 \star \mathcal{Q}_3)$. This yields i). Let us prove now point ii). Consider $(\mu, N_1, N_2) \sim \chi \otimes \mathcal{Q}_1 \otimes \mathcal{Q}_2$. It holds

$$\mu \cdot (N_1 N_2) = (\mu \cdot N_1) \cdot N_2,$$

with $\mu \cdot (N_1 N_2) \sim \chi \star (\mathcal{Q}_1 \star \mathcal{Q}_2)$ and $(\mu \cdot N_1) \cdot N_2 \sim (\chi \star \mathcal{Q}_1) \star \mathcal{Q}_2$. This yields ii). Let us move to the proof of iii). Consider a sequence of probability measures (χ_n) on $\mathcal{M}_1(\mathbb{X})$, converging in distribution to χ . Let us show that $(\chi_n \star \mathcal{Q})_{n \geq 0}$ converges in distribution towards $\chi \star \mathcal{Q}$. Let f be a continuous, bounded function on $\mathcal{M}_1(\mathbb{X})$, it holds :

$$\int f(\mu) d(\chi_n \star \mathcal{Q})(\mu) = \int \int f(\mu \cdot N) d\chi_n(\mu) d\mathcal{Q}(N) = \int \chi_n(g_N) d\mathcal{Q}(N),$$

where, for each $N \in \mathcal{K}^+$, the function $g_N : \mu \mapsto f(\mu \cdot N)$ is continuous and bounded. Thus

$$\chi_n(g_N) = \int f(\mu \cdot N) d\chi_n(\mu) \rightarrow \int f(\mu \cdot N) d\chi(\mu) = \chi(g_N).$$

This yields, by dominated convergence, as $n \rightarrow \infty$,

$$\int \chi_n(g_N) d\mathcal{Q}(N) = \int f(\mu) d(\chi_n \star \mathcal{Q})(\mu) \longrightarrow \int \chi(g_N) d\mathcal{Q}(N) = \int \int f(\mu \cdot N) d\chi(\mu) d\mathcal{Q}(N),$$

which implies iii). □

Proof of Proposition 7. Let f be a continuous and bounded function on $\mathcal{M}_1(\mathbb{X})$, it holds

$$\chi \star \mathcal{P}^{\star n}(f) = \int f(\mu) d(\chi \star \mathcal{P}^{\star n})(\mu) = \int \int f(\mu \cdot M_{0,n}) d\chi(\mu) d\mathcal{P}^{\star n}(M_{0,n}).$$

However, for any $\mu \in \mathcal{M}_1(\mathbb{X})$, Theorem 1, *ii*) states that $(\delta_\mu \star \mathcal{P}^{\star n})_{n \geq 0}$ converges weakly towards Λ . Thus, for any $\mu \in \mathcal{M}_1(\mathbb{X})$, as $n \rightarrow \infty$

$$\int f(\mu \cdot M_{0,n}) d\mathcal{P}^{\star n}(M_{0,n}) = (\delta_\mu \star \mathcal{P}^{\star n})(f) \xrightarrow{n \rightarrow \infty} \Lambda(f).$$

By dominated convergence, this yields

$$\chi \star \mathcal{P}^{\star n}(f) = \int \int f(\mu \cdot M_{0,n}) d\chi(\mu) d\mathcal{P}^{\star n}(M_{0,n}) \xrightarrow{n \rightarrow \infty} \Lambda(f),$$

which proves the weak convergence

$$\chi \star \mathcal{P}^{\star n} \xrightarrow{n \rightarrow \infty} \Lambda$$

in the metric space $\mathcal{M}_1(\mathcal{M}_1(\mathbb{X}))$, for any probability distribution χ . This proves *i*).

Since, by Lemma 8, *iii*), the map $\mu \mapsto \mu \star \mathcal{P}$ is continuous, this proves that Λ is one of its fixed points, namely :

$$\Lambda = \Lambda \star \mathcal{P}.$$

On the other hand, if $\chi \star \mathcal{P} = \chi$, the sequence $(\chi \star \mathcal{P}^{\star n})_{n \geq 0}$ is constant and converges to χ . By unicity of the limit, it holds

$$\chi = \Lambda.$$

This proves that Λ is the only invariant measure of the Markov chain (μ_n) , i.e. *ii*).

Let $(\mu_0, M_0) \sim \Lambda \otimes \mathcal{P}$. Then $\mu_1 = \mu_0 \cdot M_0 \sim \Lambda \star \mathcal{P}$, $M_1 \sim \mathcal{P}$ and M_1 is independent of μ_0 , M_0 and thus μ_1 . Therefore $(\mu_1, M_1) \sim \Lambda \otimes \mathcal{P}$, and $\Lambda \otimes \mathcal{P}$ is thus an invariant measure of the Markov chain $(\mu_n, M_n)_{n \geq 0}$.

Conversely, consider now a probability measure $\bar{\chi}$ on $\mathcal{M}_1(\mathbb{X}) \times \mathcal{K}^+$, suppose it is an invariant measure of the Markov chain $((\mu_n, M_n))_{n \geq 0}$. The definition of the transition kernel Q implies that $\mu_1 = \mu_0 \cdot M_0$, $M_1 \sim \mathcal{P}$ and M_1 is independent of (μ_0, M_0) , and therefore M_1 is independent of μ_1 . However the second term (μ_1, M_1) of the Markov chain is distributed according to $\bar{\chi}Q = \bar{\chi}$ by invariance. Thus $\bar{\chi}$ is of the form $\bar{\chi} = \chi \otimes \mathcal{P}$.

Additionally, if $(\mu_0, M_0) \sim \bar{\chi} = \chi \otimes \mathcal{P}$, then $\mu_1 = \mu_0 \cdot M_0 \sim \chi \star \mathcal{P}$. But by invariance of $\bar{\chi}$, $\mu_1 \sim \chi$, thus

$$\chi \star \mathcal{P} = \chi.$$

By Proposition 7, this implies that $\chi = \Lambda$. Finally, this proves that $\Lambda \otimes \mathcal{P}$ is the only invariant measure of the Markov chain $(\mu_n, M_n)_{n \geq 0}$. By Corollary 5.12 of [Hai], since both the processes $(\mu_n)_{n \geq 0}$ and $(\mu_n, M_n)_{n \geq 0}$ are Markov chains with a unique invariant measure, they are both ergodic. \square

One additional lemma is required before proving Theorem 2.

Lemma 9. *Let $(M_n)_{n \geq 0}$ be an i.i.d sequence of elements of \mathcal{K}^+ with law \mathcal{P} , satisfying the assumptions of Theorem 1. Then, almost surely, for any measure $\mu \in \mathcal{M}_+(\mathbb{X})$, any $n \geq 2$,*

$$(3.11) \quad \gamma_0 \|\mu M_0\| \|M_{1,n}\| \leq \|\mu M_{0,n}\| \leq \|\mu M_0\| \|M_{1,n}\|.$$

As a consequence, the $\sigma((M_n)_{n \geq 0})$ -measurable events $\{\forall \mu \in \mathcal{M}_1(\mathbb{X}), \limsup_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = +\infty\}$, $\{\exists \mu \in \mathcal{M}_1(\mathbb{X}), \limsup_{n \rightarrow \infty} \log \|\mu M_{0,n}\| = +\infty\}$ and $\{\limsup_{n \rightarrow \infty} \log \|M_{1,n}\| = +\infty\}$ coincide up to $\mathcal{P}^{\otimes \mathbb{N}}$ -negligible events. A similar statement holds replacing \limsup by $-\liminf$ in the three events.

Proof of Lemma 9. Let $\mu_0 \in \mathcal{M}_+(\mathbb{X}) - \{0\}$. Then, almost surely, for any $\nu \in \mathcal{M}_1(\mathbb{X})$

$$\mu_0 \cdot M_0 \geq c'_0 \nu.$$

Still noting $\mu_1 = \mu_0 \cdot M_0$, it holds thus $\mu_1 \geq c'_0 \nu$ for any measure $\nu \in \mathcal{M}_1(\mathbb{X})$, and thus

$$\|\mu_1 M_{1,n}\| = \mu_1 M_{1,n} \mathbf{1} \geq c'_0 \|\nu M_1\| = c'_0 \|\nu M_{1,n}\|.$$

By definition of d_1^ν , it holds thus

$$\|\mu_1 M_{1,n}\| \geq c'_0 d_1^\nu \|M_{1,n}\|.$$

Optimizing this inequality in $\nu \in \mathcal{M}_1(\mathbb{X})$, we get

$$\|\mu_1 M_{1,n}\| \geq \gamma_0 \|M_{1,n}\|.$$

Consequently,

$$\gamma_0 \|M_{1,n}\| \leq \|\mu_1 M_{1,n}\| \leq \|M_{1,n}\|,$$

which yields (3.11) when multiplying by $\|\mu_0 M_0\|$. □

Let us prove now Theorem 2.

Proof of Theorem 2, i). Let us notice first that when μ is a probability measure, $\rho : (\mu, M) \mapsto \log \|\mu M\|$ satisfies the cocycle property

$$(3.12) \quad \rho(\mu_0, M_{0,n}) = \log \|\mu_0 M_{0,n}\| = \sum_{k=0}^{n-1} \log \|(\mu_0 \cdot M_{0,k}) M_k\| = \sum_{k=0}^{n-1} \rho(\mu_k, M_k),$$

By Equation (3.12), it holds

$$\frac{1}{n} \log \|\mu_0 M_{0,n}\| = \frac{1}{n} \sum_{k=0}^{n-1} \rho(\mu_k, M_k).$$

By Birkhoff's Ergodic Theorem, since (μ_k, M_k) is an ergodic Markov chain, with stationary distribution $\Lambda \otimes \mathcal{P}$, this quantity converges $\mathbb{P}_{\Lambda \otimes \mathcal{P}}$ -almost surely and in $\mathcal{L}^1(\mathbb{P}_{\Lambda \otimes \mathcal{P}})$ towards $\int \rho d(\Lambda \otimes \mathcal{P})$ provided ρ is an \mathcal{L}^1 function with respect to $\Lambda \otimes \mathcal{P}$. Let us check now this integrability property. Let $(\mu_0, (M_n)_{n \geq 0}) \sim \Lambda \otimes \mathcal{P}^{\mathbb{N}}$. Then, applying (3.11) with $n = 2$ yields

$$\gamma_0 \|\mu_0 M_0\| \|M_1\| \leq \|\mu_0 M_{0,2}\| \leq \|\mu_0 M_0\| \|M_1\|.$$

Noting $\mu_1 = \mu_0 \cdot M_0$, we get, since $\|\mu_0 M_0\| \neq 0$ almost surely,

$$\gamma_0 \|M_1\| \leq \|\mu_1 M_1\| \leq \|M_1\|,$$

thus

$$|\rho(\mu_1, M_1)| \leq |\log \|M_1\|| + |\log(\gamma_0)|.$$

Note that $(\mu_1, M_1) \sim \Lambda \otimes \mathcal{P}$, since by definition $\mu_1 = \mu_0 \cdot M_0$ and $(\mu_0, (M_n)_{n \in \mathbb{N}}) \sim \Lambda \otimes \mathcal{P}^{\otimes \mathbb{N}}$. Thus, under **A3+** and **A4**, it holds

$$\mathbb{E} [|\rho(\mu_1, M_1)|] = \int |\rho| d(\Lambda \otimes \mathcal{P}) \leq \mathbb{E} |\log \|M_1\|| + \mathbb{E} |\log(\gamma_0)| < \infty.$$

This proves that ρ is integrable with respect to $\Lambda \otimes \mathcal{P}$, thus it holds $n^{-1} \log(\|\mu_0 M_{0,n}\|) \xrightarrow[n \rightarrow \infty]{} \int \rho d\Lambda \otimes \mathcal{P}$ holds in $\mathcal{L}^1(\Lambda \otimes \mathcal{P})$, and $\mathbb{P}_{\Lambda \otimes \mathcal{P}}$ -almost surely. Since by Theorem 1, almost surely, for all μ , it holds $n^{-1} \log \|\mu M_{0,n}\| \xrightarrow[n \rightarrow \infty]{} \lambda$, by unicity of the almost sure limit,

$$\lambda = \int \rho d(\Lambda \otimes \mathcal{P}).$$

□

Proof of Theorem 2, ii). Note now $X_n = \rho(\mu_{n-1}, M_{n-1})$, for $n \geq 1$. Then it holds, for $n \geq 0$, for any probability measure μ_0

$$(\mu_{n+1}, X_{n+1}) = (\mu_n \cdot M_n, \rho(\mu_n, M_n)).$$

Thus, $(\mu_n, X_n)_{n \geq 0}$ is a Markov chain on $\mathcal{M}_1(\mathbb{X}) \times \mathbb{R}$ such that

$$\mathbb{P}[(\mu_{n+1}, X_{n+1}) \in A \times B | (\mu_n, X_n)] = \int \mathbf{1}_A(\mu_n \cdot M) \mathbf{1}_B(\rho(\mu_n, M)) d\mathcal{P}(M).$$

Thus $S_n = \log \|\mu M_{0,n}\| = X_1 + \dots + X_n$ is a Markov random walk associated with (μ_n, X_n) , in the sense of [Als01]. Suppose that $\lambda = 0$. By Theorem 1, it holds

$$n^{-1} S_n = n^{-1} \log \|\mu M_{0,n}\| \rightarrow 0,$$

$\mathbb{P}_{\Lambda \otimes d\mathcal{P}^{\otimes \mathbb{N}}}(\mu, (M_n)_{n \geq 0})$ -almost surely, thus in probability with respect to $\mathbb{P}_{\Lambda \otimes \mathcal{P}^{\otimes \mathbb{N}}}$. Since moreover, (μ_n) is an ergodic Markov chain, the assumptions of [Als01] are satisfied. If there exists a function η such that $\mathbb{P}_{\Lambda \otimes \mathcal{P}}$ -almost surely, for $n \geq 1$,

$$(3.13) \quad X_n = \eta(\mu_n) - \eta(\mu_{n-1}),$$

then taking $n = 1$ shows that we are in the case of Null Homology (NH). In this case, it holds moreover

$$\log \|\mu_0 M_{0,n}\| = X_1 + \dots + X_n = \eta(\mu_n) - \eta(\mu_0).$$

Thus, almost surely, noting a, b the respective infimum and supremum of the support of $\eta(\mu)$, when $\mu \sim \Lambda$, since the sequence (μ_n) is a stationary and ergodic sequence with law Λ , it holds

$$\liminf_{n \rightarrow \infty} \log \|\mu_0 M_{0,n}\| = a - \eta(\mu_0), \quad \text{and} \quad \limsup_{n \rightarrow \infty} \log \|\mu_0 M_{0,n}\| = b - \eta(\mu_0).$$

Thus the almost sure finiteness of these quantities are respectively equivalent to the finiteness of a and b . If Equation (3.13) does not hold, then we are in the setup of Theorem 2 or 3 of [Als01]. These two Theorems imply that the Markov Random Walk (S_n) oscillates : $\limsup S_n = +\infty$ and $\liminf S_n = -\infty$ $\mathbb{P}_{\Lambda \otimes \mathcal{P}}$ -almost surely. However, by Lemma 9, *ii*), this implies that $\mathcal{P}^{\otimes \mathbb{N}}$ -almost surely, for every $\mu \in \mathcal{M}_1(\mathbb{X})$,

$$\limsup \log \|\mu M_{0,n}\| = -\liminf \log \|\mu M_{0,n}\| = +\infty.$$

This concludes the proof. \square

4. SUFFICIENT CONDITIONS UNDER UNIFORM POSITIVITY ASSUMPTIONS

In the finite dimensional case $\mathbb{X} = \{1, \dots, d\}$, that is when studying products of $d \times d$ matrices, similar (and actually, more complete) results are obtained in [Hen97]. They rely on the very mild assumption

$$\mathbf{A5.} \quad \mathbb{P} \left[\bigcup_{k \in \mathbb{N}} \left\{ M_{0,k} \in \mathring{\mathcal{S}} \right\} \right] = 1,$$

where $\mathring{\mathcal{S}}$ refers to the set of $d \times d$ matrices with positive entries. We expect that this approach, based on Hilbert contractions, might be extended in infinite dimensional contexts. This will require to introduce the notion of uniformly positive operators to strengthen the notion of positive matrices, and state an infinite dimensional generalization of **A5**, as we explain in Subsection 4.2.

This section aims at comparing our assumptions both with **A5**, and its natural generalization in infinite dimension.

We did not success in proving that **A5** alone is enough for our assumptions to hold. However we provide mild additional assumptions that, together with **A5**, constitute sufficient conditions for our assumptions (**A2**, **A3**, **A4**) to hold, and thus for Theorem 1 to apply.

4.1. The finite dimensional case. Let us focus in this subsection on the case where \mathbb{X} is finite, let us note $d = |\mathbb{X}|$. Consider a stationary and ergodic sequence $(M_n)_{n \in \mathbb{N}_0}$ of $d \times d$ matrices with nonnegative entries. Checking whether Assumptions **A1**, **A2**, **A3** are satisfied is quite straightforward, since these three assumptions only involve the law of the first matrix of the sequence. Let us see now how the additional Assumption **A5** can help control γ_0 in order to check that Assumptions **A4** holds.

Lemma 10. *Consider a random, stationary sequence of $d \times d$ matrices $M_n = (M_n(x, y))_{x, y \in \mathbb{X}}$, with nonnegative entries, satisfying **A2** and **A5**. Then $\mathbb{P}(d\omega)$ -almost surely, for any probability measure ν , it holds $d'_0(\omega) > 0$.*

Proof. The following decomposition holds : for any $n \geq k, x \in \mathbb{X}, \omega \in \Omega$

$$m_{0,n}(x) = \delta_x M_{0,n} \mathbf{1} = \delta_x M_{0,k} m_{k,n} = \sum_{z \in \mathbb{X}} M_{0,k}(x, z) m_{k,n}(z).$$

Thus for any $\nu \in \mathcal{M}_1(\mathbb{X})$,

$$\nu(m_{0,n}) = \nu M_{0,n} \mathbf{1} = \sum_{y, z \in \mathbb{X}} \nu(y) M_{0,k}(y, z) m_{k,n}(z).$$

The fact that ν is a probability measure yields, for any $0 \leq k \leq n$:

$$m_{0,n}(x) \leq \sup_{y,z \in \mathbb{X}} \left(\frac{M_{0,k}(x,z)}{M_{0,k}(y,z)} \right) \sum_{y \in \mathbb{X}} \nu(y) M_{0,k}(y,z) m_{k,n}(z) \leq \sup_{y,z \in \mathbb{X}} \frac{M_{0,k}(x,z)}{M_{0,k}(y,z)} \nu(m_{0,n}),$$

with the convention $\frac{M_{0,k}(x,z)}{M_{0,k}(y,z)} = 0$ as soon as $M_{0,k}(x,z) = 0$ and $\frac{M_{0,k}(x,z)}{M_{0,k}(y,z)} = \infty$ if $M_{0,k}(x,z) \neq 0$ and $M_{0,k}(y,z) = 0$. Thus, for any $n \geq k$,

$$(4.1) \quad \frac{\|M_{0,n}\|}{\nu(m_{0,n})} = \frac{\|m_{0,n}\|_\infty}{\nu(m_{0,n})} \leq \sup_{x,y,z \in \mathbb{X}} \frac{M_{0,k}(x,z)}{M_{0,k}(y,z)}.$$

This yields

$$(4.2) \quad \inf_{x,y,z \in \mathbb{X}} \frac{M_{0,k}(y,z)}{M_{0,k}(x,z)} \leq \inf_{n \geq k} \frac{\nu(m_{0,n})}{\|m_{0,n}\|_\infty},$$

and therefore

$$\sup_{k \in \mathbb{N}} \inf_{x,y,z \in \mathbb{X}} \frac{M_{0,k}(y,z)}{M_{0,k}(x,z)} \leq \liminf_n \frac{\nu(m_{0,n})}{\|M_{0,n}\|}.$$

By Assumption **A5**, $\mathbb{P}(d\omega)$ - almost surely, there exists a random integer k_ω such that $M_{0,k_\omega}(\omega) \in \mathring{\mathcal{S}}$. Since \mathbb{X} is finite, we get, for \mathbb{P} -almost any ω and for any probability measure ν ,

$$0 < \inf_{x,y,z \in \mathbb{X}} \frac{M_{0,k_\omega}(y,z)}{M_{0,k_\omega}(x,z)} < \sup_{k \in \mathbb{N}} \inf_{x,y,z \in \mathbb{X}} \frac{M_{0,k}(y,z)}{M_{0,k}(x,z)} \leq \liminf_n \frac{\nu(m_{0,n})}{\|M_{0,n}\|}.$$

Assumption **A2** implies moreover that for all n , \mathbb{P} -almost any $\omega \in \Omega$ and any probability measure ν , $\frac{\nu(m_{0,n})}{\|M_{0,n}\|} > 0$. Thus, one gets that $\mathbb{P}(d\omega)$ -almost surely, for any $\nu \in \mathcal{M}_1(\mathbb{X})$,

$$d_0^\nu = \inf_{n \in \mathbb{N}} \frac{\nu(m_{0,n})}{\|M_{0,n}\|} > 0.$$

□

This provides nice sufficient conditions for **A4** to hold.

Proposition 11. *Consider a random, stationary sequence of $d \times d$ matrices $M_n = (M_n(x,y))_{x,y \in \mathbb{X}}$, with nonnegative entries, satisfying **A1**, **A3**, **A5**. Assume additionally that*

- *there exists an deterministic integer $K \in \mathbb{N}_0$ such that*

$$(4.3) \quad \mathbb{E} \left[\log \sup_{x,y,z \in \mathbb{X}} \frac{M_{0,K}(x,z)}{M_{0,K}(y,z)} \right] < \infty,$$

- *there exists a measurable map*

$$\omega \in \Omega \mapsto \nu_\omega \in \mathcal{M}_1(\mathbb{X})$$

such that

- $c_0^{\nu_\omega}(\omega) > 0$ for almost any ω ,
- $-\log c_0^{\nu_\omega}(\omega) d\mathbb{P}(\omega) < \infty$,
- $\int [\log \nu_\omega(m_{1,2})]^- d\mathbb{P}(\omega) < \infty$

Then $\gamma_0 > 0$ almost surely, $\mathbb{E}[|\log \gamma_0|] < \infty$, and thus the conclusions of Theorem 1 hold.

In other words, if the assumptions of Proposition 11 are satisfied, then A4 holds, thus so do the conclusions of Theorem 1.

Proof. By definition of γ_0 , for almost any $\omega \in \Omega$ it holds $1 \geq \gamma_0(\omega) \geq c_0^{\nu_\omega}(\omega)d_1^{\nu_\omega}(\omega)$. Notice that $c_0^{\nu_\omega}(\omega) > 0$ $\mathbb{P}(d\omega)$ -a.s. by definition of ν_ω ; on the other hand, Lemma 10 yields $d_1^{\nu_\omega}(\omega) = d_0^{\nu_\omega}(\theta(\omega)) > 0$ $\mathbb{P}(d\omega)$ -a.s. Thus $\gamma_0 > 0$ \mathbb{P} -a.s.

Let us now prove that $\mathbb{E}|\log \gamma_0| < \infty$. By the inequality

$$|\log \gamma_0(\omega)| = -\log(\gamma_0(\omega)) \leq -\log(c_0^{\nu_\omega}(\omega)) - \log(d_1^{\nu_\omega}(\omega))$$

and hypothesis $\int -\log c_0^{\nu_\omega}(\omega) d\mathbb{P}(\omega) < \infty$, it remains to check that $\int -\log d_1^{\nu_\omega}(\omega) d\mathbb{P}(\omega) < \infty$. Inequality (4.1) implies that $\mathbb{P}(d\omega)$ -almost surely, for any $k \geq 1$,

$$\begin{aligned} -\log d_1^{\nu_\omega}(\omega) &= \log \sup_{n \in \mathbb{N}} \frac{\|M_{1,n}\|}{\nu_\omega(m_{1,n})} \leq \max \left(\log \sup_{x,y,z \in \mathbb{X}} \frac{M_{1,k}(x,z)}{M_{1,k}(y,z)}, \max_{1 \leq n \leq k-1} \log \frac{\|M_{1,n}\|}{\nu(m_{1,n})} \right) \\ &\leq \log \sup_{x,y,z \in \mathbb{X}} \frac{M_{1,k+1}(x,z)}{M_{1,k+1}(y,z)} + \sum_{1 \leq n \leq k-1} \log \frac{\|M_{1,n}\|}{\nu_\omega(m_{1,n})}. \end{aligned}$$

In particular, setting $k = K$ and applying condition (4.3), it holds by stationarity

$$\mathbb{E} \left[\log \sup_{x,y,z \in \mathbb{X}} \frac{M_{1,K+1}(x,z)}{M_{1,K+1}(y,z)} \right] < \infty.$$

Consequently, it suffices to prove that $\int \left[\log \frac{\|M_{1,n}\|}{\nu_\omega(m_{1,n})} \right] d\mathbb{P}(\omega) < \infty$ for any $n \geq 1$. Let us decompose this quantity as

$$(4.4) \quad \int \left[\log \frac{\|M_{1,n}\|}{\nu_\omega(m_{1,n})} \right] d\mathbb{P}(\omega) \leq \underbrace{\int (\log \|M_{1,n}(\omega)\|)^+ d\mathbb{P}(\omega)}_{A(n)} + \underbrace{\int [(\log \nu_\omega(m_{1,n}))^-] d\mathbb{P}(\omega)}_{B(n)}.$$

- On the one hand, the inequality $\|M_{1,n}\| \leq \prod_{i=1}^{n-1} \|M_{i,i+1}\|$ readily yields

$$A(n) \leq \sum_{i=1}^{n-1} \mathbb{E} [\log(\|M_{i,i+1}\|)^+] = (n-1) \mathbb{E} [\log(\|M_{0,1}\|)^+] < \infty.$$

- On the other hand, for any $x \in \mathbb{X}$ and $\mathbb{P}(d\omega)$ -a.s.

$$m_{1,n}(x) = \delta_x M_{1,2} m_{2,n} \geq c_1^{\nu_\theta(\omega)}(\omega) m_{1,2}(x) \nu_{\theta(\omega)}(m_{2,n})$$

by definition of $c_1^{\nu_\theta(\omega)}(\omega)$. Consequently, integrating with respect to $\nu_\omega(dx)$, we obtain

$$\nu_\omega(m_{1,n}) = \nu_\omega M_{1,2} m_{2,n} \geq c_1^{\nu_\theta(\omega)}(\omega) \nu_\omega(m_{1,2}) \nu_{\theta(\omega)}(m_{2,n}) \quad \mathbb{P}(d\omega)\text{-a.s.},$$

which yields, by induction

$$\nu_\omega(m_{1,n}) \geq \prod_{k=1}^{n-1} c_k^{\nu_{\theta^k(\omega)}}(\omega) \nu_{\theta^{k-1}(\omega)}(m_{k,k+1}).$$

Consequently, $\mathbb{P}(d\omega)$ -a.s.,

$$(\log \nu_\omega(m_{1,n}))^- \leq \sum_{k=1}^{n-1} -\log c_k^{\nu_{\theta^k(\omega)}}(\omega) + \left[\log \nu_{\theta^{k-1}(\omega)}(m_{k,k+1}) \right]^-.$$

Notice that $c_k^{\nu_{\theta^k(\omega)}}(\omega) = c_0(\nu_{\theta^k(\omega)}, \theta^k(\omega))$ for any $k \geq 1$, therefore $(c_k^{\nu_{\theta^k(\omega)}}(\omega))_{k \geq 1}$ is a stationary sequence. Hence

$$\begin{aligned} B(n) &\leq \sum_{k=1}^{n-1} \int -\log c_k^{\nu_{\theta^k(\omega)}}(\omega) d\mathbb{P}(\omega) + \int \left[\log \nu_{\theta^{k-1}(\omega)}(m_{k,k+1}) \right]^-- d\mathbb{P}(\omega) \\ &= (n-1) \left(\int -\log c_0^{\nu_\omega}(\omega) d\mathbb{P}(\omega) + \int [\log \nu_\omega(m_{1,2})]^- d\mathbb{P}(\omega) \right) < \infty. \end{aligned}$$

Finally, combining these estimates, we get, for any $n \in \mathbb{N}_0$,

$$\begin{aligned} \int \log \frac{\|M_{1,n}\|}{\nu_\omega(m_{1,n})} d\mathbb{P}(\omega) \\ \leq (n-1) \left(\int -\log c_0^{\nu_\omega}(\omega) + [\log \nu_\omega(m_{1,2})]^- d\mathbb{P}(\omega) + \mathbb{E}[(\log \|m_{0,1}\|_\infty)^+] \right) < \infty. \end{aligned}$$

□

4.2. Extension in infinite dimension. When \mathbb{X} is infinite, we need to strengthen the notion of positive matrices as follows.

Definition 12. A positive linear map M on $\mathcal{B}(\mathbb{X})$ is uniformly positive if there exists $K \in \mathbb{R}_+^*$, $h \in \mathcal{B}_+(\mathbb{X})$, such that, for any $f \in \mathcal{B}_+(\mathbb{X})$ there exists $b(f) \in \mathbb{R}_+$, satisfying

$$\frac{1}{K} b(f) h \leq M(f) \leq K b(f) h.$$

When M is uniformly positive, we note for short $M \gg 0$.

Notice that when \mathbb{X} is finite, a matrix of $\mathring{\mathcal{S}}$ is uniformly positive. Moreover, in Hennion's work, assumption **A5** is used as a sufficient condition to obtain projective contraction properties on the product $M_{k,n}$, with respect to a projective distance called the Hilbert distance (once again, see [BK53; Bir57; Lig] for a complement on this distance). In an infinite dimensional setting, this distance can still be defined, and the projective action associated with a positive operator is contracting if and only if the operator is uniformly positive (a proof of this claim is proposed in [Lig]). Uniform positivity is therefore the appropriate infinite dimensional generalization of positivity in our context, and condition **A5** can thus naturally be replaced with the restrictive condition

$$\mathbf{A5}'. \quad \mathbb{P}(\bigcup_{n \in \mathbb{N}} \{M_{0,n} \gg 0\}) = 1.$$

The present subsection aims at comparing our result with the natural extensions of Hennion's work to infinite dimensional settings. For this purpose, the following Lemma extends the idea of Lemma 10 to an infinite dimensional setup, assuming **A5'** instead of **A5**.

Lemma 13. *Consider a random stationary sequence of elements of \mathcal{K}^+ , satisfying **A2** and **A5'**. Then $\mathbb{P}(d\omega)$ -almost surely, for any probability measure ν , it holds $d_0^\nu(\omega) > 0$.*

Proof. For almost any ω and any $k \leq n$, $m_{k,n} \in \mathcal{B}(\mathbb{X})$, it holds,

$$\frac{m_{0,n}(x)}{\nu(m_{0,n})} = \frac{\delta_x M_{0,k} m_{k,n}}{\nu M_{0,k} m_{k,n}} \leq \sup_{y \in \mathbb{X}, f \in \mathcal{B}(\mathbb{X})} \frac{\delta_y M_{0,k} f}{\nu M_{0,k} f}.$$

Taking a supremum in $x \in \mathbb{X}$, we get, for any $k \leq n$,

$$(4.5) \quad \frac{\|M_{0,n}\|}{\nu(m_{0,n})} \leq \sup_{y \in \mathbb{X}, f \in \mathcal{B}(\mathbb{X})} \frac{\delta_y M_{0,k} f}{\nu M_{0,k} f}.$$

By **A5'**, let k_ω be a random integer such that $\mathbb{P}(d\omega)$ -almost surely, $M_{0,k_\omega}(\omega) \gg 0$. Then, almost surely, there is $K \in \mathbb{R}_+^*$, $h \in \mathcal{B}_+(\mathbb{X})$ such that for any $f \in \mathcal{B}_+(\mathbb{X})$, there exists $b(f) \geq 0$, satisfying

$$(4.6) \quad K^{-1}b(f)h \leq M_{0,k_\omega} f \leq Kb(f)h.$$

From (4.6), we deduce $K^{-1}m_{0,k_\omega}(x) \leq h(x)b(\mathbf{1}) \leq Km_{0,k_\omega}(x)$. By **A2**, m_{0,k_ω} is a bounded and positive function, thus so is h . Moreover, $b(\mathbf{1}) > 0$, $\nu(m_{0,k}) \leq K\nu(h)b(\mathbf{1})$, thus $\nu(h) > 0$. Therefore, for any $x \in \mathbb{X}$, any $f \in \mathcal{B}_+(\mathbb{X})$, it holds $\mathbb{P}(d\omega)$ almost surely:

$$(4.7) \quad \frac{\delta_x M_{0,k_\omega} f}{\nu M_{0,k_\omega} f} \leq K^2 \frac{h(x)}{\nu(h)} \leq K^3 \frac{\|M_{0,k_\omega}\|}{b(\mathbf{1})\nu(h)} \leq K^4 \frac{\|M_{0,k_\omega}\|}{\nu(m_{0,k_\omega})}.$$

Finally, combining (4.7) with (4.5), we get for almost any ω and any probability measure ν

$$\limsup_n \frac{\|M_{0,n}\|}{\nu(m_{0,n})} \leq \sup_{n \geq k_\omega} \frac{\|M_{0,n}\|}{\nu(m_{0,n})} \leq K^4 \frac{\|M_{0,k_\omega}\|}{\nu(m_{0,k_\omega})} < \infty.$$

Since moreover almost surely, for all $n \in \mathbb{N}_0$, $\frac{\|M_{0,n}\|}{\nu(m_{0,n})}$ is finite, then almost surely, for any probability measure ν ,

$$\frac{1}{d_0^\nu(\omega)} = \sup_{n \in \mathbb{N}} \frac{\|M_{0,n}\|}{\nu(m_{0,n})} < \infty,$$

thus $d_0^\nu(\omega) > 0$. □

To tackle the integrability of $\log \gamma_0$, Proposition 11 from the previous subsection can clearly be adapted, replacing **A5** by **A5'** and (4.3) by

$$(4.8) \quad \int \log \sup_{x \in \mathbb{X}, f \in \mathcal{B}(\mathbb{X})} \frac{\delta_x M_{0,k} f}{\nu_\omega M_{0,k} f} d\mathbb{P}(\omega) < \infty.$$

This yields a counterpart of Proposition 11 in a infinite dimensional setup.

5. APPLICATION TO PRODUCTS OF RANDOM INFINITE LESLIE MATRICES

The previous section focused on products of matrices with positive entries, and more generally, products of uniformly positive operators. This kind of products can be efficiently studied with methods based on projective contractions relatively to the Hilbert metric. The main interest of our techniques, based on Doeblin contractions, is their application to products of operators which are not uniformly positive. The goal of this section is to illustrate how such products can be studied with our theorems. We have chosen to focus here on a quite simple but natural example with no uniform positivity properties : the infinite Leslie Matrices.

5.1. Introduction to Leslie matrices. In this section, we set $\mathbb{X} = \mathbb{N}_0$, thus the operators of \mathcal{K}^+ can be represented as infinite matrices. We choose to consider infinite Leslie matrices, which have the following form : for any $\omega \in \Omega$,

$$(5.1) \quad M(\omega) = \begin{pmatrix} f_0(\omega) & s_0(\omega) & 0 & 0 & \dots \\ f_1(\omega) & 0 & s_1(\omega) & 0 & \dots \\ f_2(\omega) & 0 & 0 & s_2(\omega) & \ddots \\ f_3(\omega) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

where the entries $(f_k(\omega))_{k \in \mathbb{N}_0}$, $(s_k(\omega))_{k \in \mathbb{N}_0}$ are nonnegative real numbers, and $\sup_{x \in \mathbb{X}} s_x(\omega) + f_x(\omega) < \infty$. Notice that such a matrix is not uniformly positive, since there are zeros on every row and every column but the first one. Moreover, if Q is a product of k matrices of this shape, the (x, y) -entry $[Q]_{x,y} = 0$ whenever $y \geq x + k + 1$. This prevents any product of such matrices from being uniformly positive. This example is therefore a typical situation where **A5'** does not hold.

Such matrices appear naturally when studying the dynamics of a population counting individuals according to their age. The coefficients f_x (respectively s_x) represent the mean number of individuals of age 0 (respectively of age $x + 1$) created by an individual of age x , that is the mean size of the offspring of an individual of age x (respectively the survival rate of individuals of age x). Usually, only a finite number of age classes are defined, thus $\mathbb{X} = \llbracket 0, d \rrbracket$, and one considers finite versions of such matrices, called Leslie matrices, see for example [Cas10]. However, it is natural to extend their definition to an infinite number of age classes ($\mathbb{X} = \mathbb{N}_0$) obtaining infinite matrices with this shape. Indeed, several articles already study age-structured populations with an unbounded set of possible ages, see e.g. [BCG20; JK22; Kar90]. Therefore, products of random matrices shaped as in (5.1) model the dynamics of an age structured population evolving in a randomly changing environment which affect their reproductive behavior. This is the kind of matrices we are studying in this section. Let us note from now on

$$s_x^k(\omega) = s_x \circ \theta^k(\omega) \text{ and } f_x^k(\omega) = f_x \circ \theta^k(\omega),$$

so that $(s_x^k, f_x^k)_{x \in \mathbb{X}}$ are the nonzero entries of the random matrix $M_k(\omega) = M \circ \theta^k$. We introduce the quantities

$$d'(\omega) = \sup_{k \in \mathbb{N}_0, x \leq y \in \mathbb{X}} \frac{f_y^k(\omega)}{f_x^k(\omega)} \geq 1,$$

and

$$d''(\omega) = \sup_{x \in \mathbb{X}, k \in \mathbb{N}_0} \frac{s_x^0(\omega) \cdots s_{x+k}^k(\omega)}{s_0^0(\omega) \cdots s_k^k(\omega)} \geq 1,$$

which are useful to estimate γ_0 .

5.2. Ergodic behavior of products of random Leslie Matrices. The following proposition provides sufficient conditions for assumptions **A3** and **A4** to hold in the case of products of infinite Leslie matrices.

Proposition 14. *Consider a random matrix product with $\mathbb{X} = \mathbb{N}_0$ and suppose that for any $\omega \in \Omega$, $M(\omega)$ is of the form of equation (5.1), with $\sup_{x \in \mathbb{X}} s_x(\omega) + f_x(\omega) < \infty$. Suppose that **A1** is satisfied, and $\mathbb{P}(d\omega)$ -almost surely, it holds*

- i) $f_x(\omega) + s_x(\omega) > 0$ for all $x \in \mathbb{X}$;
- ii) $\mathbb{E}[\log^+(\sup_{x \in \mathbb{X}} s_x + f_x)] < \infty$;

then **A2** and **A3** hold. Moreover, if additionally

- iii) $\mathbb{P}(d\omega)$ -almost surely, $\sup_{x \in \mathbb{X}} \frac{s_x}{f_x} < \infty$ and $\mathbb{E}[\log^+(\sup_{x \in \mathbb{X}} \frac{s_x}{f_x})] < \infty$,
- iv) There exists a deterministic real $A > 0$ such that $\mathbb{P}(d\omega)$ -almost surely, $\sup_{x \leq y} \frac{f_y^0}{f_x^0} \leq A$,
- v) $\mathbb{P}(d\omega)$ -almost surely, $d''(\omega) < \infty$ and $\mathbb{E}|\log d''| < \infty$,

then M satisfies also Assumption **A4**, and the conclusions of Theorem 1 hold.

In Proposition 14, we've reduced Assumptions **A2** to **A4** to a series of conditions on the law of the coefficients of the random matrix M_0 , together with finiteness and integrability conditions on d'' . The hardest conditions to check are the ones involving d'' , since checking them requires to consider the joint law of all the $M_{0,n}$ and not only the first marginal. We were not able to find a general sufficient condition for d'' to be finite almost surely and $\log d''$ to be integrable. However, we provide the following quite restrictive sufficient condition.

Remark 2. *Consider a random, stationary sequence of matrices of the form of equation (5.1), and assume that there exists an integer $x_0 \in \mathbb{X}$, such that almost surely, the sequence $(s_x(\omega))_{x \geq x_0}$ is non increasing. Suppose also that almost surely, for all $x \leq x_0$, $s_x > 0$. Then, $\mathbb{P}(d\omega)$ -almost surely*

$$d'' \leq \left(\sup_{i \leq x_0} \sup_{x \leq y \leq x_0} \frac{s_y^i}{s_x^i} \right)^{x_0} < \infty.$$

Moreover, if $\mathbb{E} \left| \log \frac{s_y^i}{s_x^i} \right| < \infty$ for any $x \leq y \leq x_0$, then $\mathbb{E}|\log d''| < \infty$.

In the context of an age structured population, s_x represents the frequency of individuals of age x surviving to the next time step, and thus being replaced by individuals of age $x + 1$. Assuming that $(s_x(\omega))_{x \geq x_0}$ is decreasing implies that the older individuals get, the more they tend to die, which is a reasonable assumption. However this condition is somewhat unsatisfying in a more general setting.

We split the proof of Proposition 14 into several lemmas that involve different groups of assumptions. Notice first that most quantities involved in Assumptions A2 to A4 are explicit in terms of the (f_x, s_x) . Indeed :

Lemma 15. *Consider a product of stationary random Leslie matrices, in the form of equation (5.1). Then A2 and A3 are satisfied if and only if all the following conditions hold simultaneously :*

- $\mathbb{P}(d\omega)$ -almost surely, for each $x \in \mathbb{X}$, $f_x(\omega) > 0$ or $s_x(\omega) > 0$
- $\mathbb{E}[\log^+(\sup_{x \in \mathbb{X}} f_x + s_x)] < \infty$.

Proof. This lemma is straightforward after noticing that for any $x \in \mathbb{X}, \omega \in \Omega$,

$$m_{0,1}(x, \omega) = f_x(\omega) + s_x(\omega).$$

□

Moreover, in this model, the behavior of $c_0^y(\omega)$ is quite clear.

Lemma 16. *Consider a product of stationary, random Leslie matrices and assume that A2 holds. Then $\mathbb{P}(d\omega)$ -almost surely, for any probability measure $\nu \neq \delta_0 \in \mathcal{M}_1(\mathbb{X})$, it holds $c_0^y(\omega) = 0$, thus $\mathbb{P}(d\omega)$ almost surely,*

$$\gamma_0(\omega) = c_0^{\delta_0}(\omega) d_1^{\delta_0}(\omega).$$

Moreover,

$$c_0^{\delta_0}(\omega) = \inf_{x \in \mathbb{X}} \frac{f_x(\omega)}{f_x(\omega) + s_x(\omega)} = \inf_{x \in \mathbb{X}} \left(1 + \frac{s_x(\omega)}{f_x(\omega)} \right)^{-1}.$$

Proof. Notice that for any $x \in \mathbb{X}, \omega \in \Omega$,

$$\delta_x M_{0,1} = f_x(\omega) \delta_0 + s_x(\omega) \delta_{x+1}.$$

Thus, let ν be a probability measure on \mathbb{X} . Suppose that $\nu \neq \delta_0$. Then, there exists $k > 0$ such that $\nu(k) > 0$. Consider a real $c \geq 0$, such that for all $x \in \mathbb{X}, \delta_x M_{0,1} \geq m_{0,1}(x) c \nu$. Then in particular

$$0 = f_k(\omega) \delta_0(\mathbf{1}_k) + s_k(\omega) \delta_{k+1}(\mathbf{1}_k) \geq c m_{0,1}(k) \nu(k),$$

By A2, almost surely, $m_{0,1}(k) > 0$, which implies that $c = 0$. □

Let us try now to control the random variable $d_0^{\delta_0}$.

Lemma 17. *Consider a product of stationary random Leslie matrices, of the form of equation (5.1). Then,*

$$d_0^{\delta_0}(\omega) \geq \frac{1}{d'(\omega) d''(\omega)}$$

.

Proof. Let $n \in \mathbb{N}_0$, $x \in \mathbb{X}$, $\omega \in \Omega$, it holds

$$m_{0,n}(x) = \delta_x M_0 \cdots M_{n-1} \mathbb{1} = \sum_{i_0, i_1, \dots, i_n \in \mathbb{N}_0} \delta_x(i_0) M_0(i_0, i_1) \cdots M_{n-1}(i_{n-1}, i_n).$$

Thus

$$m_{0,n}(x) = \sum_{i_1, \dots, i_n \in \mathbb{N}_0} M_0(x, i_1) \cdots M_{n-1}(i_{n-1}, i_n).$$

Let us rearrange this sum according to the first index $k \leq n$ such that $i_k = 0$:

$$\begin{aligned} m_{0,n}(x) &= \sum_{k=1}^n \sum_{i_1, \dots, i_{k-1} > 0} M_0(x, i_1) \cdots M_{k-1}(i_{k-1}, 0) \sum_{i_{k+1}, \dots, i_n \in \mathbb{N}_0} M_k(0, i_{k+1}) \cdots M_{n-1}(i_{n-1}, i_n) \\ &\quad + \sum_{i_1, \dots, i_n > 0} M_0(x, i_1) \cdots M_{n-1}(i_{n-1}, i_n). \end{aligned}$$

Notice that

$$\sum_{i_{k+1}, \dots, i_n \in \mathbb{N}_0} M_k(0, i_{k+1}) \cdots M_{n-1}(i_{n-1}, i_n) = m_{k,n}(0).$$

Moreover, the matrices M_k are shaped according to (5.1). Thus for any $i \geq 0, j > 0$, in order for $M_k(i, j)$ to be non zero, one must have $j = i + 1$. Thus :

$$m_{0,n}(x) = \sum_{k=1}^n M_0(x, x+1) \cdots M_{k-1}(x+k-1, 0) m_{k,n}(0) + M_0(x, x+1) \cdots M_{n-1}(x+n-1, x+n).$$

Therefore

$$m_{0,n}(x) = \sum_{i=0}^{n-1} s_x^0 s_{x+1}^1 \cdots s_{x+i-1}^{i-1} f_{x+i}^i m_{i+1,n}(0) + s_x^0 \cdots s_{x+n-1}^{n-1}.$$

This is true in particular for $x = 0$:

$$m_{0,n}(0) = \sum_{i=0}^{n-1} s_0^0 s_1^1 \cdots s_{i-1}^{i-1} f_i^i m_{i+1,n}(0) + s_0^0 \cdots s_{n-1}^{n-1}.$$

By definition of d', d'' , it holds, for any $k \in \mathbb{N}_0$ and any $x \in \mathbb{N}_0$,

$$f_{x+i}^i \leq d' f_i^i,$$

and

$$s_x^0 s_{x+1}^1 \cdots s_{x+i-1}^{i-1} \leq d'' s_0^0 s_1^1 \cdots s_{i-1}^{i-1}.$$

Therefore, controlling independently each term of the sum yields

$$m_{0,n}(x) \leq d' d'' \sum_{i=0}^{n-1} s_0^0 s_1^1 \cdots s_{i-1}^{i-1} f_i^i m_{i+1,n}(0) + d'' s_0^0 \cdots s_{n-1}^{n-1} \leq d' d'' m_{0,n}(0),$$

thus

$$\frac{1}{d'(\omega) d''(\omega)} \leq d_0^{\delta_0}(\omega) = \inf_{n \in \mathbb{N}_0} \frac{m_{0,n}(0)}{\|m_{0,n}\|_\infty}.$$

□

Let us study separately $d'(\omega)$ and $d''(\omega)$.

Lemma 18. *Consider a random product of matrices of the form of equation (5.1), satisfying **A1**. Then the random variable d' is $\mathbb{P}(d\omega)$ -almost surely finite if and only if there exists $A > 0$ such that*

$$\mathbb{P} \left[\sup_{x \leq y} \frac{f_y^0}{f_x^0} \leq A \right] = 1.$$

In this case $d'(\omega) \leq A$, $\mathbb{P}(d\omega)$ -almost surely.

Proof. Notice that

$$d'(\omega) = \sup_{k \in \mathbb{N}_0} \sup_{x \leq y} \frac{f_y^k}{f_x^k} = \sup_{k \in \mathbb{N}_0} X \circ \theta^k(\omega).$$

where

$$X(\omega) = \sup_{x \leq y} \frac{f_y(\omega)}{f_x(\omega)} = \sup_{x \leq y} \frac{f_y^0}{f_x^0}.$$

Since θ is an ergodic mapping, $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is $\mathbb{P}(d\omega)$ -almost surely equal to the supremum of the support of X . In particular $\sup_{k \in \mathbb{N}_0} X \circ \theta^k$ is finite almost surely if and only if the support of X is bounded. \square

Putting these lemmas together allows to prove Proposition 14.

Proof of Proposition 14. The assumptions of Proposition 14 contain the conditions mentioned in Lemma 15. Hence, this lemma allows to check **A2** and **A3**. Moreover, Lemma 16 guarantees that

$$c_0^{\delta_0}(\omega) = \inf_{x \in \mathbb{X}} \left(1 + \frac{s_x(\omega)}{f_x(\omega)} \right)^{-1} = \left(1 + \sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)} \right)^{-1} > 0, \mathbb{P}(d\omega) - \text{almost surely.}$$

Moreover, because of Lemma 17, it holds

$$d_0^{\delta_0}(\omega) \geq \frac{1}{d'(\omega)d''(\omega)}.$$

The assumptions of Lemma 18 are satisfied here, therefore $d'(\omega) < \infty$, $\mathbb{P}(d\omega)$ -almost surely. Moreover, we have assumed that $d''(\omega) < \infty$, $\mathbb{P}(d\omega)$ -almost surely. Thus $\mathbb{P}(d\omega)$ -almost surely, $d_0^{\delta_0}(\omega) > 0$, and by stationarity, $d_1^{\delta_0}(\omega) > 0$ almost surely. Thus, $\mathbb{P}(d\omega)$ -almost surely,

$$\gamma_0(\omega) = \sup_{\nu \in \mathcal{M}_1(\mathbb{X})} c_0^{\nu}(\omega) d_1^{\nu}(\omega) = c_0^{\delta_0}(\omega) d_1^{\delta_0}(\omega) > 0.$$

Moreover,

$$\mathbb{E} |\log \gamma_0| \leq \int -\log c_0^{\delta_0}(\omega) d\mathbb{P}(\omega) - \int \log d_1^{\delta_0}(\omega) d\mathbb{P}(\omega).$$

On the one hand,

$$\int -\log c_0^{\delta_0}(\omega) d\mathbb{P}(\omega) = \int \log \left(1 + \sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)} \right) d\mathbb{P}(\omega).$$

Notice that for any positive real variable X , $\log(1 + X)$ is integrable as soon as $\log(X)^+$ is integrable. Since we've assumed that

$$\int \log \left(\sup_{x \in \mathbb{X}} \frac{s_x(\omega)}{f_x(\omega)} \right)^+ d\mathbb{P}(\omega) < \infty,$$

then

$$\int -\log c_0^{\delta_0}(\omega) d\mathbb{P}(\omega) < \infty.$$

On the other hand,

$$\int -\log d_1^{\delta_0}(\omega) d\mathbb{P}(\omega) = \int \log d_0^{\delta_0}(\omega) d\mathbb{P}(\omega) \leq \mathbb{E} \log d' + \mathbb{E} \log d''.$$

Since $\sup_{x \leq y} \frac{f_y^0}{f_x^0} \leq A$ almost surely, then by stationarity, almost surely,

$$1 \leq d' = \sup_{k \in \mathbb{N}_0} \sup_{y \geq x} \frac{f_y^k}{f_x^k} \leq A.$$

Thus $\log d'$ is bounded and integrable. We have assumed additionally that $\log d''$ was integrable. This is enough to conclude to the integrability of $|\log \gamma_0|$, which proves assumption **A4**. \square

5.3. A situation where $\gamma_0 = 0$. It was not clear to us how strong an assumption **A4** is, or whether it was hard to find a system breaking it while satisfying all the other assumptions. We shall present here an example of an infinite Leslie matrix, such that $\gamma_0 = 0$ even if all other assumptions are satisfied. This example is in a deterministic environment, that is $|\mathcal{E}| = 1$, $\Omega = \mathcal{E}^{\mathbb{N}_0}$, $|\Omega| = 1$. The random matrix $M(\omega)$ is therefore constant, and $M_{0,n} = M^n$. Let us set :

$$(5.2) \quad M = \begin{pmatrix} cm(0) & (1-c)m(0) & 0 & 0 & \dots \\ cm(1) & 0 & (1-c)m(1) & 0 & \dots \\ cm(2) & 0 & 0 & (1-c)m(2) & \ddots \\ cm(3) & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

where $c \in (0, 1)$, and $m(x) = cm(x) + (1-c)m(x)$ is the mean offspring size of a type x individual. Such a model satisfies **A2** and **A3**, as soon as $x \mapsto m(x)$ is bounded and positive, since $c > 0$. The ergodicity and integrability properties are trivially satisfied since this model is in a constant environment. Moreover, Lemma 16 applies, therefore $\gamma_0 = c_0^{\delta_0}(\omega) d_1^{\delta_0}(\omega) = c d_1^{\delta_0}(\omega)$. We note from now on $d = d_1^{\delta_0}(\omega)$. Let us prove that we can tune the parameters $x \mapsto m(x)$ and c in such a way that $d = 0$.

Consider a sequence of integers $(\varepsilon_x)_{x \in \mathbb{N}_0} \in \{0, 1\}^{\mathbb{N}_0}$, such that :

- There are arbitrarily long subsequences of consecutive 1 in the sequence (ε_x) .
- Noting $S_x = \sum_{k=0}^{x-1} \varepsilon_k$, $\frac{S_x}{x} \rightarrow 0$ as $x \rightarrow \infty$.
- There exists $\alpha < 1$ such that for all $x \in \mathbb{N}_0$, $\frac{S_x}{x} \leq \alpha$.

Let a be a real number such that $a > 1$. Then, we set, for any $x \in \mathbb{X}$,

$$m(x) = 1 + (a - 1)\varepsilon_x.$$

Defined as such, m is a positive and bounded function, thus Assumptions **A2** and **A3** are satisfied. This yields that for any sequence (x_i) ,

$$\prod_{i=0}^{n-1} m(x_i) = a^{\sum_{i=0}^{n-1} \varepsilon_{x_i}}.$$

Since

$$m_{0,n}(x) = \sum_{\substack{x_0=x, \dots, x_n \in \mathbb{N} \\ x_{i+1} \in \{x_i+1, 0\}}} c^{N(x_0, \dots, x_n)} (1-c)^{n-N(x_0, \dots, x_n)} \prod_{i=0}^{n-1} m(x_i) \geq (1-c)^n \prod_{i=0}^{n-1} m(x+i),$$

where $N(x_0, \dots, x_n) = |\{1 \leq i \leq n \mid x_i = 0\}|$. Then

$$m_{0,n}(x) \geq (1-c)^n a^{\sum_{i=x}^{x+n-1} \varepsilon_i},$$

In particular, x can be chosen such that $\varepsilon_x = \dots = \varepsilon_{x+n-1} = 1$, which implies that

$$\|m_{0,n}\|_\infty \geq m_{0,n}(x) \geq (a(1-c))^n.$$

On the other hand

$$m_{0,n}(0) \leq 2^n \sup_{\substack{x_0=0, \dots, x_n \in \mathbb{N}_0 \\ x_{i+1} \in \{x_i+1, 0\}}} \prod_{i=0}^{n-1} m(x_i) \leq 2^n a^{\sup_{\substack{x_0=0, \dots, x_n \in \mathbb{N}_0 \\ x_{i+1} \in \{x_i+1, 0\}}} \sum_{i=0}^{n-1} \varepsilon_{x_i}}.$$

A sequence $(x_i)_{0 \leq i \leq n}$ of integers such that $x_0 = 0$ and for each i , $x_{i+1} \in \{x_i + 1, 0\}$ is entirely determined by the sequence $(T_k)_k$ of the lengths of its excursions away from zero. By convention, if there are only p excursions away from zero, we set T_p such that $T_0 + \dots + T_p = n$ and $T_{p+1} = \dots = T_n = 0$. The $(x_{T_0+\dots+T_{i-1}})_{i \leq p}$ are the only zero terms in the sequence (x_1, \dots, x_n) , and $T_0 + \dots + T_n \leq n - 1$. Thus

$$\sup_{\substack{x_0=0, \dots, x_n \in \mathbb{N} \\ x_{i+1} \in \{x_i+1, 0\}}} \sum_{i=0}^{n-1} \varepsilon_{x_i} \leq \sup_{T_0+\dots+T_n=n} \sum_{i=0}^n S_{T_i} \leq \alpha \sum_{i=0}^n T_i \leq \alpha n,$$

and

$$m_{0,n}(0) \leq (2a^\alpha)^n.$$

Hence

$$\frac{\|m_{0,n}\|_\infty}{m_{0,n}(0)} \geq \left(\frac{a(1-c)}{2a^\alpha} \right)^n = \left(\frac{a^{1-\alpha}(1-c)}{2} \right)^n \xrightarrow{n \rightarrow \infty} \infty,$$

whenever $\frac{a^{1-\alpha}(1-c)}{2} > 1$. Thus for any values of $\alpha, c \in (0, 1)$, if a is large enough, then $d_0^{\delta_0}(\omega) = 0$. Since $|\Omega| = 1$, for each $n \in \mathbb{N}_0$, $m_{1,n+1} = m_{0,n}$, thus $d_0^{\delta_0}(\omega) = d_1^{\delta_0}(\omega) = d = 0$.

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