On the construction of conditional probability densities in the Brownian and compound Poisson filtrations *

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Abstract

In this paper, we construct supermartingales valued in $[0,1]$ as solutions of an appropriate stochastic differential equation on a given reference filtration generated by either a Brownian motion or a compound Poisson process. Then, by means of the results contained in [19], it is possible to construct an associated random time on some extended probability space admitting such a given supermartingale as conditional survival process and we shall check that this construction (with a particular choice of supermartingale) implies that Jacod's equivalence hypothesis, that is, the existence of a family of strictly positive conditional probability densities for the random times with respect to the reference filtration, is satisfied. We use the components of the multiplicative decomposition of the constructed supermartingales to provide explicit expressions for the conditional probability densities of the random times on the Brownian and compound Poisson filtrations.

1 Introduction

In the models of quickest change-point (disorder) detection, one usually starts with the probability space enhanced with a random time and another source of randomness such as a Brownian motion or a compound Poisson process. The quickest detection problems seek to determine stopping times at which the alarms should be sounded to indicate changes in the probabilistic characteristics of continuously observable stochastic processes. These detection times of alarms

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are sought to be as close as possible to the unknown and unobservable random times of change in either the drift rate of the observable Brownian motion or the intensity and the jump distribution of the observable compound Poisson process. In order to solve the problems, the appropriate stochastic differential equations are derived for the (supermartingale) survival conditional probability processes of the random times or the equivalent (submartingale) posterior probability process on the given Brownian or compound Poisson reference filtrations. These processes play the role of sufficient statistics in the appropriate quickest detection problems with rewards containing a linear combination of the false alarm probabilities and the expected linear delay penalties. More precisely, the optimal detection times in these problems represent the first hitting times by either the survival conditional probability or the posterior probability processes of certain boundaries which are found as solutions of the associated free-boundary problems for ordinary or partial (integro-)differential (see, e.g. Shiryaev [30] and the references therein).

In this paper, we proceed in a more natural opposite direction and present a construction of supermartingales valued in the interval $[0,1]$ as solutions of certain stochastic differential equations on the given Brownian or compound Poisson reference filtrations. Then, we apply the results of Jeanblanc and Song [19] to construct the associated random times on the appropriate extended probability spaces. These properties lead to the satisfaction of Jacod's equivalence hypothesis, that is, to the existence of strictly positive conditional densities for the random times with respect to the reference filtrations. Such assumptions are usually satisfied in the classical models of credit risk theory in which the random default times have given strictly positive conditional densities with respect to the reference filtrations reflecting the information observable from the associated models of financial markets (see, e.g. Aksamit and Jeanblanc [2] for further discussions on Jacod's hypothesis and credit risk models). We provide a multiplicative decomposition for the constructed supermartingale and use the resulting components to derive the families of the conditional probability densities of the random times on the Brownian and compound Poisson filtrations.

The paper is organised as follows. We present a general framework for the model in Section 2. Then, we construct the appropriate supermartingales in the case of a reference filtration generated by a Brownian in Section 3, and in the case of a reference filtration generated by a compound Poisson process in Section 4.

2 The framework

We give a model for constructions of supermartingales valued in $[0,1]$ with respect to a reference filtration.

For this purpose, we work on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is a given (reference) filtration. We call an $\mathbb{F}$-conditional density any family of strictly positive $\mathbb{F}$-martingales $(p(u) = (p_t(u))_{t\geq 0}; \forall u \geq 0)$, parameterised by $u \in [0,\infty)$, such that p is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}([0,\infty))$-

measurable and

$$\int_0^\infty p_t(u)\,du = 1,\ \forall t \geq 0\,. \tag{2.1}$$

Actually, there are very few explicit examples of such densities in the literature (see, e.g. [2, Chapter 4], [7] and [20]). In this paper, we will show how to construct such families $(p(u);\forall u \geq 0)$ in two cases: when $\mathbb{F}$ is a Brownian filtration, and when $\mathbb{F}$ is a compound Poisson filtration. In both cases, we start with a nonnegative bounded $\mathbb{F}$-adapted process $\lambda = (\lambda_t)_{t\geq 0}$ satisfying

$$\int_0^t \lambda_s\,ds < \infty,\ \forall t \geq 0\,,\ \int_0^\infty \lambda_s\,ds = \infty,\ (\mathbb{P}\text{-a.s.})\,, \tag{2.2}$$

and we construct a supermartingale $G = (G_t)_{t\geq 0}$ valued in $(0,1]$ such that $G_0 = 1$ and the process $(G_t \exp(\int_0^t \lambda_s ds))_{t\geq 0}$ is a (strictly) positive $\mathbb{F}$-local martingale (see, e.g. [11]). In this case, using the work of [19], on the extended measurable space $(\Omega\times[0,\infty), \mathcal{G}\otimes\mathcal{B}([0,\infty)))$, one can construct a positive random variable τ (in fact, $\tau(\omega',\omega'') = \omega''$) and a probability measure $\mathbb{Q}$ such that

$$\mathbb{Q}(\tau > t\,|\,\mathcal{F}_t) = G_t, \quad \text{and} \quad \mathbb{Q}\big|_{\mathcal{F}_t} = \mathbb{P}\big|_{\mathcal{F}_t},\ \forall t \geq 0\,.$$

In particular, if W is an $(\mathbb{F},\mathbb{P})$-standard Brownian motion, it is also an $(\mathbb{F},\mathbb{Q})$-standard Brownian motion. We recall that Jacod's equivalence hypothesis holds, if there exists a family of $\mathbb{F}$-conditional densities $(p(u);\forall u \geq 0)$ such that

$$\mathbb{Q}(\tau > u\,|\,\mathcal{F}_t) = \int_u^\infty p_t(v)\,dv,\ \forall t \geq 0,\ \forall u \geq 0\,.$$

The family of processes $(p(u);\forall u \geq 0)$ is then called the *conditional density*.

Remark 2.1 We refer the reader to the seminal papers of Jacod [17], Grorud and Pontier [16], and Amendinger [3], and to the book [2] for more details on Jacod's hypothesis. Note that the knowledge of the conditional density allows to give the decomposition of any martingale in the reference filtration as a semimartingale in the initial (and progressive) enlargement with τ (see, e.g. [2], [21], [8], [4] and [12]-[13]).

3 The case of a Brownian filtration

In this section, we consider the case in which $\mathbb{F}$ is the filtration generated by a standard Brownian motion W. We use the notation of the framework described in Section 2.

3.1 Supermartingales valued in $[0,1]$

Proposition 3.1 *Let us consider the stochastic differential equation*

$$dG_t = -\lambda_t\,G_t\,dt - G_t(1-G_t)\,\rho_t\,dW_t,\ G_0 \in (0,1]\,, \tag{3.1}$$

where λ is a nonnegative $\mathbb{F}$-adapted process such that the condition of (2.2) holds and ρ an $\mathbb{F}$-adapted process satisfying

$$\int_0^t \rho_s^2 \, ds < \infty, \, (\mathbb{P}\text{-}a.s.), \, \forall t \geq 0 \, . \tag{3.2}$$

Then, the equation in (3.1) admits a (pathwise) unique (continuous) solution $G = (G_t)_{t \geq 0}$ which is valued in $[0, 1]$.

Proof: Let us set $T_1 = \inf\{t > 0 \,|\, G_t = 1\}$ and $T_0 = \inf\{t > 0 \,|\, G_t = 0\}$. Obviously, the equality $G_t = 0$ holds, $\forall t \geq T_0$, and we also have $G_t < 1$ on $\{0 < t < T_1\}$. Then, applying Itô's formula (see, e.g. [27, Chapter VI, Theorem 1.2]) to the change of variables $\Phi_t = (1 - G_t)/G_t$, $\forall t \geq 0$, we obtain that the stochastic differential equation of (3.1) is equivalent to the one

$$d\Phi_t = \left(\lambda_t \, (1 + \Phi_t) + \frac{\Phi_t^2}{1 + \Phi_t} \, \rho_t^2 \right) dt + \Phi_t \, \rho_t \, dW_t, \; \Phi_0 \in [0, \infty) \, . \tag{3.3}$$

Assuming that the solution Φ to the equation in (3.3) exists, it is non-negative. Moreover, since the coefficients of the stochastic differential equation in (3.3) are Lipschitz continuous on $[0, \infty)$ and of a linear growth, it follows from the result of [24, Chapter IV, Theorem 4.8] that the equation in (3.3) admits a pathwise unique (strong) solution process $\Phi = (\Phi_t)_{t \geq 0}$, which does not explode at any $t \geq 0$. In this case, because the process G starts at some $G_0 \in (0, 1]$, we may conclude from the structure of the coefficients of the equation in (3.3) for the process $\Phi = (1 - G)/G$ that $T_0 = \infty$ ($\mathbb{P}$-a.s.).

Moreover, by means of the comparison results for pathwise solutions of stochastic differential equations in [25, Theorem 3.2] and [9, Theorem 1], we see that the inequality

$$\Phi_0 \, \exp\left(\int_0^t \rho_s \, dW_s - \frac{1}{2} \int_0^t \rho_s^2 \, ds \right) \leq \Phi_t, \; \forall t \geq 0 \, , \tag{3.4}$$

holds. Hence, due to the assumption in (3.2), we see from (3.4) that the process Φ does not touch 0 after the time 0, when being started at $\Phi_0 \geq 0$. This fact implies that the process $G = 1/(1 + \Phi)$ does not touch 1 after the time 0, so that $T_1 = \infty$ ($\mathbb{P}$-a.s.). □

Remark 3.2 Note that the equation in (3.1) has the same structure as the appropriate stochastic differential equations for the posterior probability processes $\Pi = (\Pi_t)_{t \geq 0}$ of the occurrence of the random change-point (disorder) time defined by $\Pi_t = 1 - G_t$, $\forall t \geq 0$, in the quickest change-point detection problems for Wiener and more general diffusion processes studied in [28, Chapter IV, Section 4] (see also [29, Chapter IV, Section 4], and [26, Chapter VI, Section 22], as well as [14]-[15]). It is shown in the sources mentioned above that the optimal stopping times of alarms in the quickest detection problems are given by the first times at which the processes Π hit boundaries which are determined as solutions to the associated free-boundary problems

for ordinary or partial differential operators. In the case of observable Wiener processes, the optimal hitting boundaries for the processes Π are constant on the allowed infinite observation intervals, but they are time-dependent when the allowed observation time intervals are finite. In the case of observable more general diffusion processes, the optimal hitting boundaries for the processes Π depend on the running values of the observation processes.

Remark 3.3 This study is easily extended to the case in which $\mathbb{F}$ is still a Brownian filtration, but the process $G = (G_t)_{t\geq 0}$ satisfies the more general that in (3.1) stochastic differential equation

$$dG_t = -G_t\, d\Lambda_t - G_t(1-G_t)\,\rho_t\, dW_t,\; G_0 \in (0,1]\,, \tag{3.5}$$

where $\Lambda = (\Lambda_t)_{t\geq 0}$ is a positive continuous increasing process started at $\Lambda_0 = 0$. This is particularly the case when τ is honest, since, in that case, the dynamics of the process G involves a local time (see, e.g. [2, Proposition 5.19]). However, an honest time satisfies Jacod's hypothesis if and only if it takes countably many values [1, Lemma 4.11]. Therefore, if all supermartingales of the form (3.1) lead to Jacod's hypothesis, this is not the case for supermartingales being the solutions of the stochastic differential equation in (3.5).

We also give another construction of the supermartingale G. Namely, if Y is a positive continuous supermartingale with the Doob-Meyer decomposition $Y = M^Y - A^Y$, where M^Y is a continuous (uniformly integrable) martingale, so that the process $G := Y \wedge 1$ is a supermartingale valued in $(0,1]$. Then, an application of Tanaka's formula for semi-martingales [22, Formula 4.1.15], by virtue of $x \wedge y = x - (x-y)^+$, leads to

$$dG_t = \mathbb{1}_{\{Y_t\leq 1\}}\, dY_t - \frac{1}{2}\, dL^1_t(Y),\; G_0 \in (0,1]\,.$$

Note that any positive supermartingale admits a multiplicative decomposition $Y_t = N_t e^{-\Lambda_t}$, $\forall t \geq 0$. Hence, there exists an $\mathbb{F}$-adapted process $\vartheta = (\vartheta_t)_{t\geq 0}$ satisfying the integrability condition

$$\int_0^t \vartheta_s^2\, ds < \infty,\; (\mathbb{P}\text{-a.s.}),\; \forall t \geq 0\,,$$

and such that

$$dY_t = -Y_t\, d\Lambda_t - Y_t\,\vartheta_t\, dW_t,\; Y_0 \in (0,\infty)\,.$$

Therefore, we see that

$$dG_t = -G_t\left(\mathbb{1}_{\{Y_t\leq 1\}}\, d\Lambda_t + \frac{1}{2}\, d\ell_t\right) - \mathbb{1}_{\{Y_t\leq 1\}}\, G_t(1-G_t)\,\rho_t\, dW_t,\; G_0 \in (0,1]\,,$$

where we have

$$d\ell_t = \frac{1}{G_t}\, dL^1_t(Y) \quad \text{and} \quad \rho_t = \frac{\vartheta_t}{1-G_t},\; \forall t \geq 0\,.$$

continuous and ρ is square integrable.

3.2 The multiplicative decomposition for the process G

As we recalled in Section 2 above, the process $G = (G_t)_{t\geq 0}$ defined in (3.1) when started at $G_0 = 1$ admits a multiplicative decomposition $G_t = e^{-\Lambda_t} N_t$, $\forall t \geq 0$, where $N = (N_t)_{t\geq 0}$ started at $N_0 = 1$ is a continuous local martingale and

$$\Lambda_t = \int_0^t \lambda_s \, ds, \, \forall t \geq 0 \, . \tag{3.6}$$

In this case, the equality $N_t = G_t e^{\Lambda_t}$ holds, $\forall t \geq 0$, and, according to Itô's formula, the process N follows the stochastic differential equation

$$dN_t = -e^{\Lambda_t} \, G_t(1 - G_t) \, \rho_t \, dW_t = -N_t \, (1 - G_t) \, \rho_t \, dW_t, \; N_0 = 1 \, , \tag{3.7}$$

and the explicit expression

$$N_t = \exp\left(- \int_0^t (1 - G_s) \, \rho_s \, dW_s - \frac{1}{2} \int_0^t (1 - G_s)^2 \, \rho_s^2 \, ds \right), \, \forall t \geq 0 \, . \tag{3.8}$$

For further use, we define the processes $L = (L_t)_{t\geq 0}$ and $K = (K_t)_{t\geq 0}$ by the explicit expressions

$$L_t = \exp\left(- \int_0^t \rho_s \, dW_s - \frac{1}{2} \int_0^t \rho_s^2 \, ds + \int_0^t G_s \, \rho_s^2 \, ds \right), \, \forall t \geq 0 \, , \tag{3.9}$$

and

$$K_t = \exp\left(\int_0^t G_s \, \rho_s \, dW_s - \frac{1}{2} \int_0^t G_s^2 \, \rho_s^2 \, ds \right), \, \forall t \geq 0 \, , \tag{3.10}$$

which admit the stochastic differential equation

$$dL_t = -L_t \, \rho_t \, (dW_t - G_t \, \rho_t \, dt), \quad L_0 = 1 \, , \tag{3.11}$$

and

$$dK_t = K_t \, G_t \, \rho_t \, dW_t, \quad K_0 = 1 \, . \tag{3.12}$$

By virtue of the integration-by-parts formula, it is straightforward to see from (3.8) and (3.9)-(3.10) that the equality $K_t L_t = N_t$ holds, $\forall t \geq 0$.

3.3 Strictly positive martingales M^u decreasing w.r.t. u

From [19], we deduce that any solution $\Psi^u = (\Psi^u_t)_{t\geq u}$ of the problem

$$\begin{cases} d\Psi^u_t = -\Psi^u_t (e^{-\Lambda_t}/(1 - G_t)) dN_t, \, \forall u \leq t < \infty \, , \\ \Psi^u_u = x \, , \end{cases} \tag{3.13}$$

for any $x \in [0, 1]$ and $u \geq 0$ fixed, is a martingale, increasing with respect to $u \in [0, \infty)$ and is valued in $[0, 1]$. Here, the components in the equation of (3.13) are given by (3.6) and (3.7).

For $x = 1 - G_u$ fixed, one can construct on an extended probability space (see above), a random time τ and a probability measure $\mathbb{Q}$ such that $\mathbb{Q}$ and $\mathbb{P}$ coincide on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{Q}(\tau > u \,|\, \mathcal{F}_t) = \Psi_t^u$, $\forall t \geq 0$, $\forall u \geq 0$ fixed (see [19, Section 3] and [23]). Note that

$$d\Psi_t^u = \Psi_t^u \, G_t \, \rho_t \, dW_t, \, \forall t \geq u \, , \tag{3.14}$$

so that, for any $u \geq 0$ fixed, we have

$$\Psi_t^u = \Psi_u^u \exp\left(\int_u^t G_s \, \rho_s \, dW_s - \frac{1}{2} \int_u^t G_s^2 \, \rho_s^2 \, dW_s \right), \, \forall t \geq u \, , \tag{3.15}$$

or

$$\Psi_t^u = \Psi_u^u \, \frac{K_t}{K_u} = K_t \, \frac{1 - G_u}{K_u}, \, \forall t \geq u \, . \tag{3.16}$$

Then, by integration-by-parts formula, after some easy computations, we get

$$d\left(\frac{1 - G_u}{K_u} \right) = \frac{\lambda_u}{K_u} \, G_u \, du \, , \tag{3.17}$$

and

$$\Psi_t^u = K_t \int_0^u \frac{\lambda_s}{K_s} \, G_s \, ds, \, \forall t \geq u \, . \tag{3.18}$$

It follows that Ψ^u is differentiable w.r.t. u and, for any $t \geq 0$ fixed, we have

$$\frac{d}{du} \Psi_t^u = K_t \, \frac{\lambda_u}{K_u} \, G_u = K_t \, \frac{\lambda_u}{K_u} \, e^{-\Lambda_u} \, K_u \, L_u = K_t \, \lambda_u \, e^{-\Lambda_u} \, L_u, \, \forall t \geq u \, . \tag{3.19}$$

Setting $M_t^u = 1 - \Psi_t^u$, $\forall t \geq u$, $\forall u \geq 0$ fixed, we find

$$\begin{cases} dM_t^u = (1 - M_t^u)(e^{-\Lambda_t}/(1 - G_t)) dN_t, \, \forall u \leq t < \infty \\ M_u^u = G_u \, , \end{cases} \tag{3.20}$$

where the process $G = (G_t)_{t \geq 0}$ is given by (3.5) and the process $N = (N_t)_{t \geq 0}$ admits the stochastic differential of (3.7). Then, we can show that the following assertion holds.

Proposition 3.4 *For any $u \geq 0$ fixed, the following expression holds*

$$M_t^u = N_t \, e^{-\Lambda_t} + K_t \left(\int_u^t \frac{N_s}{K_s} \, \lambda_s \, e^{-\Lambda_s} \, ds \right), \, \forall t \geq u \, , \tag{3.21}$$

where the components are given by the formulas in (3.6) and (3.8)-(3.10).

Proof: For any $u \geq 0$ fixed, after some algebraic computations, we get that the expressions

$$\begin{aligned} dM_t^u &= G_t(1 - G_t) \, \rho_t \, dW_t - (M_t^u - N_t \, e^{-\Lambda_t}) \, G_t \, \rho_t \, dW_t \\ &= -G_t \, (-1 + G_t + M_t^u - G_t) \, \rho_t \, dW_t \\ &= (1 - M_t^u) \, \frac{e^{-\Lambda_t}}{1 - G_t} \, dN_t, \, \forall t \geq u \, , \end{aligned} \tag{3.22}$$

hold. □

For any $0 \leq t < u$, in order to preserve the martingale property of M^u, we define $M_t^u = \mathbb{E}[M_u^u \,|\, \mathcal{F}_t] = \mathbb{E}[G_u \,|\, \mathcal{F}_t] = \mathbb{E}[N_u e^{-\Lambda_u} \,|\, \mathcal{F}_t]$. Furthermore, if λ is deterministic and N a true martingale, we have

$$M_t^u = N_t \, e^{-\Lambda_u} \, , \forall t < u \, . \tag{3.23}$$

3.4 The set of conditional probability density processes $p(u)$

Let us note that, by virtue of (3.19), the family of the conditional probability density processes $(p(u) = (p_t(u))_{t \geq 0}; \forall u \geq 0)$ with respect to Lebesgue measure is defined by

$$p_t(u) = K_t \, \lambda_u \, e^{-\Lambda_u} \, L_u, \, \forall t \geq u \, . \tag{3.24}$$

We also deduce that

$$1 - M_t^u = \Psi_t^u = K_t \int_0^u \lambda_s \, e^{-\Lambda_s} \, L_s \, ds = \int_0^u p_t(s) \, ds, \, \forall t \geq u \, , \tag{3.25}$$

and, for any $u \geq 0$ fixed, using the martingale property of $p(u)$, we get

$$p_t(u) = \mathbb{E}[K_u \, \lambda_u \, e^{-\Lambda_u} \, L_u \,|\, \mathcal{F}_t] = \mathbb{E}[\lambda_u \, G_u \,|\, \mathcal{F}_t], \, \forall t \leq u \, . \tag{3.26}$$

Finally, we check that the property in (2.1) holds. For this, we note that the equalities

$$\begin{aligned} \int_0^\infty p_t(s) \, ds &= \int_0^t K_t \, \lambda_s \, e^{-\Lambda_s} \, L_s \, ds + \int_t^\infty \mathbb{E}[K_s \, \lambda_s \, e^{-\Lambda_s} \, L_s \,|\, \mathcal{F}_t] \, ds \\ &= \Psi_t^t + \int_t^\infty \mathbb{E}[N_s \, \lambda_s \, e^{-\Lambda_s} \,|\, \mathcal{F}_t] \, ds = \Psi_t^t + \mathbb{E}\bigg[\int_t^\infty \lambda_s \, G_s \, ds \,\bigg|\, \mathcal{F}_t \bigg], \, \forall t \geq 0 \, , \end{aligned} \tag{3.27}$$

hold and, from the stochastic differential equation satisfied by G in (3.1), we get

$$\int_t^\infty \lambda_s \, G_s \, ds = -G_t - \int_t^\infty G_s (1 - G_s) \, \rho_s \, dW_s, \, \forall t \geq 0 \, . \tag{3.28}$$

If λ is a deterministic function, the process $p(u)$ admits the stochastic differential

$$\begin{aligned} d_t p_t(u) &= \lambda_u e^{-\Lambda_u} \left(\mathbb{1}_{\{u<t\}} \, G_t \, K_t \, L_u - \mathbb{1}_{\{u \geq t\}} \, (1 - G_t) \, K_t \, L_t \right) \rho_t \, dW_t \\ &= p_t(u) \left(\mathbb{1}_{\{u<t\}} \, G_t - \mathbb{1}_{\{u \geq t\}} \, (1 - G_t) \right) \rho_t \, dW_t, \quad p_0(u) = 1 \, , \end{aligned} \tag{3.29}$$

so that the representation

$$p_t(u) = \lambda_u e^{-\Lambda_u} \, \exp \bigg(\int_0^t \varphi_s(u) \, dW_s - \frac{1}{2} \int_0^t \varphi_s^2(u) \, ds \bigg), \, \forall t \geq 0 \, , \, \forall u \geq 0 \, , \tag{3.30}$$

holds with

$$\varphi_t(u) = \left(\mathbb{1}_{\{u<t\}} \, G_t - \mathbb{1}_{\{u \geq t\}} \, (1 - G_t) \right) \rho_t = \left(\mathbb{1}_{\{u<t\}} - (1 - G_t) \right) \rho_t, \, \forall t \geq 0 \, , \, \forall u \geq 0 \, . \tag{3.31}$$

4 The case of a compound Poisson filtration

In this section, we consider the case in which $\mathbb{F}$ is the filtration generated by a compound Poisson process. We use the notation of the framework described in Sections 2 and 3.

4.1 The conditional probability process

Suppose that there exists a compound Poisson process $X = (X_t)_{t\geq 0}$ (or a pure jump Lévy process) with a finite intensity Lévy measure $\nu(dx)$ which is a positive σ-finite measure on $\mathcal{B}(\mathbb{R})$ satisfying the conditions

$$\nu(\{0\}) = 0, \quad \int \big(x^2 \wedge 1\big)\, \nu(dx) < \infty \quad \text{and} \quad \int |x|\, \nu(dx) < \infty\,. \tag{4.1}$$

Assume that the process X generates the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which is complete under the probability measure $\mathbb{P}$. Note that, in the compound Poisson case with $\nu(\mathbb{R}) < \infty$, the process X admits the representation $X_t = \sum_{i=1}^{N'_t} \Xi_i$, where $N' = (N'_t)_{t\geq 0}$ is a Poisson process of intensity $\varkappa > 0$ and $(\Xi_i)_{i\in\mathbb{N}}$ is a sequence of independent identically distributed random variables with the distribution $\nu(dx)/\varkappa$, where N' and $(\Xi_i)_{i\in\mathbb{N}}$ are independent under $\mathbb{P}$. We denote by μ the jump measure of the process X defined by

$$\mu\big((0,t]\times A\big) = \sum_{0<s\leq t} \mathbb{1}_{\{\Delta X_s \in A\}}\,,\ \forall t \geq 0\,, \tag{4.2}$$

for any Borel set $A \in \mathcal{B}(\mathbb{R})$, where we set $\Delta X_t = X_t - X_{t-}$, $\forall t \geq 0$ (see, e.g. [18, Chapter II, Section 4]).

Proposition 4.1 *Let us consider the stochastic differential equation*

$$dG_t = -\lambda_t\, G_{t-}\, dt - \int_{\mathbb{R}} \frac{G_{t-}(1-G_{t-})(\Upsilon(x)-1)}{1+(1-G_{t-})(\Upsilon(x)-1)} \big(\mu(dt,dx) - dt\,\nu(dx)\big),\ G_0 \in (0,1]\,, \tag{4.3}$$

where λ is a nonnegative $\mathbb{F}$-adapted process such that the conditions of (2.2) hold, and $\Upsilon(x)$ is a continuous function such that the inequality $\Upsilon(x) > 1$ holds, $\forall x \in \mathbb{R}$, as well as the conditions

$$\int_{\mathbb{R}} \Big(\sqrt{\Upsilon(x)} - 1\Big)^2\, \nu(dx) < \infty \quad \textit{and} \quad \int_{\mathbb{R}} |x|\, \Upsilon(x)\, \nu(dx) < \infty \tag{4.4}$$

are satisfied. (The first condition in (4.4) is used for the equivalent change of probability measure (see, e.g. [18, Chapter III, Theorem 5.34].) Then, the equation in (4.3) admits a (pathwise) unique (piecewise-continuous) solution $G = (G_t)_{t\geq 0}$ which is valued in $[0,1]$.

Proof: Let us define $T_1 = \inf\{t > 0 \,|\, G_t \geq 1\}$ and $T_0 = \inf\{t > 0 \,|\, G_t \leq 0\}$ with the fact, G being a supermartingale the equality $G_t \leq 0$ holds, $\forall t \geq T_0$, as well as $G_t < 1$ on $\{0 < t < T_1\}$.

It is shown by means of Itô's formula (see, e.g. [18, Chapter I, Theorem 4.57]) that the process $\Phi = (\Phi_t)_{t\geq 0}$ defined by $\Phi_t = (1 - G_t)/G_t$, $\forall\, t \geq 0$, admits the stochastic differential

$$\begin{aligned} d\Phi_t &= \lambda_t \,(1 + \Phi_{t-})\, dt \\ &\quad + \Phi_{t-} \int_{\mathbb{R}} \big(\Upsilon(x) - 1\big) \left(\mu(dt, dx) - \frac{1 + \Phi_{t-}}{1 + \Phi_{t-}\Upsilon(x)}\, dt\, \nu(dx)\right),\ \Phi_0 \in [0, \infty). \end{aligned} \tag{4.5}$$

Due to the condition $\Upsilon > 1$, the solution Φ to the equation in (4.5) is non-negative. Since the coefficients of this stochastic differential equation in (4.5) are Lipschitz continuous on $[0, \infty)$ and of a linear growth, it follows from the result of [18, Chapter III, Theorem 2.32] that the equation in (4.5) admits a pathwise unique (strong) solution process $\Phi = (\Phi_t)_{t\geq 0}$ which does not explode at any $t \geq 0$. In this case, because the process G starts at some $G_0 \in (0, 1]$, we may conclude from the structure of the coefficients of the equation in (4.5) for the process $\Phi = (1 - G)/G$ that $T_0 = \infty$ ($\mathbb{P}$-a.s.).

Let us now denote by $(S_n)_{n\in\mathbb{N}}$ the sequence of jumps of the Poisson process N' and put $S_0 = 0$. Then, we see that, between any two jumps of N', the process Φ follows the equation

$$\frac{d\Phi_t}{dt} = \lambda_t \,(1 + \Phi_{t-}) - \int_{\mathbb{R}} \frac{\Phi_{t-}(1 + \Phi_{t-})(\Upsilon(x) - 1)}{1 + \Phi_{t-}\Upsilon(x)}\, \nu(dx),\ \forall t \in (S_{n-1}, S_n]\,, \tag{4.6}$$

such that $t \in [0, T_0 \wedge T_1]$, while, at any jump times of N', we have

$$\Phi_{S_n} = \big(\Upsilon(\Delta X_{S_n}) - 1\big)\, \Phi_{S_n-},\ \forall n \in \mathbb{N}\,. \tag{4.7}$$

Hence, due to the assumptions that $\Upsilon(x) > 1$ holds, $\forall x \in \mathbb{R}$, as well as the conditions in (4.4) are satisfied, we see from (4.6) and (4.7) that the process Φ does not touch 0 after the time 0, when being started at $\Phi_0 \geq 0$, so that $T_1 = \infty$ ($\mathbb{P}$-a.s.). □

Remark 4.2 Note that the equation in (4.3) has the same structure as the appropriate stochastic differential equations for the posterior probability process $\Pi = (\Pi_t)_{t\geq 0}$ of the occurrence of the random change-point (disorder) time defined by $\Pi_t = 1 - G_t$, $\forall\, t \geq 0$, in the quickest change-point detection problems for a compound Poisson process studied in [10], [6], and [5]. It is shown in the sources mentioned above that the optimal stopping times of alarms in the quickest detection problem are given by the first times at which the processes Π hit boundaries which are determined as solutions to the associated free-boundary problems for integro-differential operators. In the case of observable compound Poisson processes, the optimal hitting boundaries for the processes Π are constant on the allowed infinite observation time intervals.

4.2 The multiplicative decomposition for the process G

In this setting, the process $G = (G_t)_{t\geq 0}$ defined in (4.3) when started at $G_0 = 1$ also admits a multiplicative decomposition $G_t = e^{-\Lambda_t} N_t$, $\forall\, t \geq 0$, where $N = (N_t)_{t\geq 0}$ started at $N_0 = 1$ is

a piecewise-continuous local martingale and $\Lambda = (\Lambda_t)_{t\geq 0}$, given by (3.6) is continuous, hence it is predictable. In this case, the equality $N_t = G_t e^{\Lambda_t}$ holds, $\forall t \geq 0$, and, according to Itô's formula, similar to the formulas in (3.7)-(3.8) above, the process N admits the stochastic differentials

$$\begin{aligned} dN_t &= -e^{\Lambda_t}\, G_{t-}(1-G_{t-}) \int_{\mathbb{R}} \frac{\Upsilon(x)-1}{1+(1-G_{t-})(\Upsilon(x)-1)} \big(\mu(dt,dx) - dt\,\nu(dx)\big) \\ &= -N_{t-}\,(1-G_{t-}) \int_{\mathbb{R}} \frac{\Upsilon(x)-1}{1+(1-G_{t-})(\Upsilon(x)-1)} \big(\mu(dt,dx) - dt\,\nu(dx)\big),\ N_0 = 1\,, \end{aligned} \tag{4.8}$$

and the explicit expression

$$\begin{aligned} N_t = \exp\bigg(&- \int_0^t \int_{\mathbb{R}} \ln\big(1+(1-G_{t-})(\Upsilon(x)-1)\big)\,\mu(ds,dx) \\ &+ \int_0^t \int_{\mathbb{R}} \frac{(1-G_{s-})(\Upsilon(x)-1)}{1+(1-G_{s-})(\Upsilon(x)-1)}\, ds\,\nu(dx)\bigg),\ \forall t \geq 0\,. \end{aligned} \tag{4.9}$$

Similar to the formulas in (3.9)-(3.12) above, we define the processes $L = (L_t)_{t\geq 0}$ and $K = (K_t)_{t\geq 0}$ by the explicit expressions

$$\begin{aligned} L_t = \exp\bigg(&- \int_0^t \int_{\mathbb{R}} \ln\big(\Upsilon(x)\big)\,\mu(ds,dx) \\ &+ \int_0^t \int_{\mathbb{R}} \frac{\Upsilon(x)-1}{1+(1-G_{s-})(\Upsilon(x)-1)}\, ds\,\nu(dx)\bigg),\ \forall t \geq 0\,, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} K_t = \exp\bigg(&\int_0^t \int_{\mathbb{R}} \ln\bigg(\frac{\Upsilon(x)}{1+(1-G_{s-})(\Upsilon(x)-1)}\bigg)\,\mu(ds,dx) \\ &- \int_0^t \int_{\mathbb{R}} \frac{G_{s-}(\Upsilon(x)-1)}{1+(1-G_{s-})(\Upsilon(x)-1)}\, ds\,\nu(dx)\bigg),\ \forall t \geq 0\,, \end{aligned} \tag{4.11}$$

which admit the stochastic differentials

$$dL_t = L_{t-} \int_{\mathbb{R}} \bigg(\frac{1}{\Upsilon(x)} - 1\bigg)\bigg(\mu(dt,dx) - \frac{\Upsilon(x)}{1+(1-G_{t-})(\Upsilon(x)-1)}\, dt\,\nu(dx)\bigg),\quad L_0 = 1\,, \tag{4.12}$$

and

$$dK_t = K_{t-} \int_{\mathbb{R}} \frac{G_{t-}(\Upsilon(x)-1)}{1+(1-G_{t-})(\Upsilon(x)-1)} \big(\mu(dt,dx) - dt\,\nu(dx)\big),\quad K_0 = 1\,. \tag{4.13}$$

By virtue of the integration-by-parts formula, it is straightforward to see from (4.9) and (4.10)-(4.11) that the equality $K_t L_t = N_t$ holds, $\forall t \geq 0$.

4.3 Strictly positive martingales M^u decreasing w.r.t. u

In this setting, from [19], we also deduce that any solution $\Psi^u = (\Psi^u_t)_{t\geq u}$ of the problem

$$\left\{ \begin{array}{l} d\Psi^u_t = -\Psi^u_{t-}(e^{-\Lambda_t}/(1-G_{t-}))dN_t,\, \forall u \leq t < \infty\,, \\ \Psi^u_u = x\,, \end{array} \right. \tag{4.14}$$

for any $x \in [0,1]$ and $u \geq 0$ fixed, is a martingale, increasing with respect to $u \in [0,\infty)$ and is valued in $[0,1]$. Here, the components in the equation of (3.13) are given by (3.6) and (4.8). For $x = 1-G_u$ fixed, one can construct on an extended probability space (see above), a random time τ and a probability measure $\mathbb{Q}$ such that $\mathbb{Q}$ and $\mathbb{P}$ coincide on the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{Q}(\tau > u \,|\, \mathcal{F}_t) = \Psi^u_t$, $\forall t \geq 0$, $\forall u \geq 0$ fixed (see [19, Section 3] and [23]). Note that

$$d\Psi^u_t = \Psi^u_{t-} \int_{\mathbb{R}} \frac{G_{t-}(\Upsilon(x)-1)}{1+(1-G_{t-})(\Upsilon(x)-1)} \big(\mu(dt,dx) - dt\,\nu(dx)\big),\, \forall t \geq u\,, \tag{4.15}$$

so that, for any $u \geq 0$ fixed, we have

$$\begin{aligned} \Psi^u_t = \Psi^u_u \exp\Bigg(& \int_u^t \int_{\mathbb{R}} \ln\left(\frac{\Upsilon(x)}{1+(1-G_{s-})(\Upsilon(x)-1)}\right) \mu(ds,dx) \\ & - \int_u^t \int_{\mathbb{R}} \frac{G_{s-}(\Upsilon(x)-1)}{1+(1-G_{s-})(\Upsilon(x)-1)}\, ds\,\nu(dx)\Bigg),\, \forall t \geq u\,, \end{aligned} \tag{4.16}$$

and thus, the expressions in (3.16)-(3.19) hold with $K = (K_t)_{t\geq 0}$ and $L = (L_t)_{t\geq 0}$ given by (4.10) and (4.11). Similarly, setting $M^u_t = 1 - \Psi^u_t$, $\forall t \geq u$, $\forall u \geq 0$ fixed, we find

$$\left\{ \begin{array}{l} dM^u_t = (1-M^u_{t-})(e^{-\Lambda_t}/(1-G_{t-}))dN_t,\, \forall u \leq t < \infty \\ M^u_u = G_u\,, \end{array} \right. \tag{4.17}$$

where the process $G = (G_t)_{t\geq 0}$ is given by (4.3) and the process $N = (N_t)_{t\geq 0}$ admits the stochastic differential of (4.8). Then, we can show that the following assertion holds, which is proved by means of arguments similar to the ones used in Proposition 3.4 above.

Proposition 4.3 *For any $u \geq 0$ fixed, the same expression as in (3.21) holds, where the components are given by the formulas in (3.6) and (4.9)-(4.11).*

4.4 The set of conditional probability density processes $p(u)$

Let us finally recall the family of the conditional probability density processes $(p(u) = (p_t(u))_{t\geq 0}; \forall u \geq 0)$ which are also defined as in (3.24). It is shown by means of standard arguments that the expressions of (3.25)-(3.27) hold in this setting, while the expression in (3.28) takes the form

$$\int_t^\infty \lambda_s\, G_{s-}\, ds = -G_t - \int_t^\infty \int_{\mathbb{R}} \frac{G_{s-}(1-G_{s-})(\Upsilon(x)-1)}{1+(1-G_{s-})(\Upsilon(x)-1)} \big(\mu(ds,dx) - ds\,\nu(dx)\big),\, \forall t \geq 0\,. \tag{4.18}$$

We also note that if λ is a deterministic function, the process $p(u)$ admits the stochastic differential

$$\begin{aligned} dp_t(u) &= \lambda_u e^{-\Lambda_u} \left(\mathbb{1}_{\{u<t\}} G_{t-} K_{t-} L_u - \mathbb{1}_{\{u\geq t\}} (1-G_{t-}) K_{t-} L_{t-} \right) \\ &\quad \times \int_{\mathbb{R}} \frac{\Upsilon(x)-1}{1+(1-G_{t-})(\Upsilon(x)-1)} \left(\mu(dt,dx) - dt\, \nu(dx) \right) \\ &= p_{t-}(u) \left(\mathbb{1}_{\{u<t\}} G_{t-} - \mathbb{1}_{\{u\geq t\}} (1-G_{t-}) \right) \\ &\quad \times \int_{\mathbb{R}} \frac{\Upsilon(x)-1}{1+(1-G_{t-})(\Upsilon(x)-1)} \left(\mu(dt,dx) - dt\, \nu(dx) \right), \quad p_0(u) = 1\,, \end{aligned} \tag{4.19}$$

so that the representation

$$\begin{aligned} &p_t(u) = \lambda_u e^{-\Lambda_u} \\ &\times \exp\left(\int_0^t \int_{\mathbb{R}} \ln\left(1+\xi_{s-}(u,x)\right) \mu(ds,dx) - \int_0^t \int_{\mathbb{R}} \xi_{s-}(u,x)\, ds\, \nu(dx) \right), \ \forall t \geq 0\,, \ \forall u \geq 0\,, \end{aligned} \tag{4.20}$$

holds with

$$\begin{aligned} \xi_{t-}(u,x) &= \left(\mathbb{1}_{\{u<t\}} G_{t-} - \mathbb{1}_{\{u\geq t\}} (1-G_{t-}) \right) \frac{\Upsilon(x)-1}{1+(1-G_{t-})(\Upsilon(x)-1)} \\ &= \left(\mathbb{1}_{\{u<t\}} - (1-G_{t-}) \right) \frac{\Upsilon(x)-1}{1+(1-G_{t-})(\Upsilon(x)-1)}, \ \forall t \geq 0\,, \ \forall u \geq 0\,. \end{aligned} \tag{4.21}$$

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