Logarithmic Derivations of Adjoint Discriminants

Vladimiro Benedetti, Daniele Faenzi, Simone Marchesi

LOGARITHMIC DERIVATIONS OF ADJOINT DISCRIMINANTS

VLADIMIRO BENEDETTI, DANIELE FAENZI, AND SIMONE MARCHESI

Abstract. We exhibit a relationship between projective duality and the sheaf of logarithmic vector fields along a reduced divisor D of projective space, in that the push-forward of the ideal sheaf of the conormal variety in the point-hyperplane incidence, twisted by the tautological ample line bundle is isomorphic to logarithmic differentials along D.

Then we focus on the adjoint discriminant D of a simple Lie group with Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$ of characteristic zero and study the logarithmic module $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ over $\mathbf{U} = \mathbf{k}[\mathfrak{g}]$. When $\mathfrak{g}$ is simply laced, we show that this module has two direct summands: the G-invariant part, which is free with generators in degrees equal to the exponents of G, and the G-variant part, which is of projective dimension one, presented by the Jacobian matrix of the basic invariants of G and isomorphic to the image of the map $\mathbf{ad} : \mathfrak{g} \otimes \mathbf{U}(-1) \to \mathfrak{g} \otimes \mathbf{U}$ given by the Lie bracket.

When $\mathfrak{g}$ is not simply laced, we give a length-one equivariant graded free resolution of $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ in terms of the exponents of G and of the quasi-minuscule representation of G.

Introduction

Given a reduced hypersurface D of the ℓ-dimensional projective space $\mathbb{P}$, the sheaf of logarithmic derivations $\mathcal{T}_{\mathbb{P}}\langle D\rangle$ is the dual of Deligne's sheaf of 1-forms on $\mathbb{P}$ with logarithmic poles along D. This sheaf was first studied by Saito in connection with discriminants of simple singularities and later on it has found rather diverse applications, for instance in the theory of arrangements and free divisors (starting from [Ter80a, Ter80b, Ter81], for an overview see [OT92, Dim17]), unfolding of singularities (see [BEGvB09, Dam98]) and locally trivial deformations (see [Ser06]).

One of the main questions about the sheaf of logarithmic derivations and its associated graded module of global sections $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ is to know its minimal graded free resolution over the polynomial ring $\mathbf{U} = \mathbf{k}[\mathbb{P}] = \mathbf{k}[x_0, \dots, x_\ell]$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. Ideally, one would like to compute the graded Betti numbers of this module, but, as it turns out, information on the projective dimension or on the number and minimal degree of minimal generators is in general quite hard to acquire. Even freeness of this module is the object of an important open question by Terao (see [Ter81]) to the effect that, when D is a hyperplane arrangement, this depends only on the isomorphism class of the intersection lattice of the hyperplanes. Freeness holds for some important classes of arrangements connected with Weyl groups of simple Lie algebras, see for instance [Yos04]. However, in spite of recent developments (see [DiP23]), even for these arrangements our knowledge of Betti numbers beyond the free range is only conjectural, [AFV16, Conjecture 1].

From a different perspective, a very powerful tool to compute graded free resolutions is provided by the so-called geometric method, that amounts to studying special resolutions of singularities arising from Kempf collapsing as introduced in [Kem76], see [Las78] for example for applications to determinantal varieties. We refer to [Wey03] for a complete presentation of this method and of its many applications. This has been studied with particular emphasis in the framework of projective duality in [GKZ08]. However, to our knowledge this methond has not been applied so far to logarithmic derivations.

The first main result of the present paper outlines a simple and deep connection between logarithmic derivations and projective duality, which in turn affords applications of the geometric method to the computation of Betti numbers of modules of logarithmic derivations.

2020 *Mathematics Subject Classification.* 14B05; 14M17.

Key words and phrases. Logarithmic derivations. Discriminants. Adjoint orbits. Sheaves of logarithmic differentials.

D.F. and V.B. partially supported by FanoHK ANR-20-CE40-0023, SupToPhAG/EIPHI ANR-17-EURE-0002, Région Bourgogne-Franche-Comté, Feder Bourgogne and Bridges ANR-21-CE40-0017. S.M. partially supported by PID2020-113674GB-I00 and by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). .

To explain this connection, we think of D as sitting in the dual space $\check{\mathbb{P}}$ of hyperplanes of $\mathbb{P}$ and consider the projective dual X of D. The *conormal* variety $\mathbb{W}_X$ of X, namely the blow-up of D along its Jacobian scheme, sits in the point-hyperplane incidence $\mathbb{I} \subset \mathbb{P} \times \check{\mathbb{P}}$. Let $\tilde{\rho} : \mathbb{I} \to \check{\mathbb{P}}$ be the projection.

Theorem 1. *Let h be the hyperplane class of X, pulled back to $\mathbb{I}$. Then:*

$$\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) \simeq \tilde{\rho}_* \left(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)\right).$$

This provides a wide generalisation of the approach developed in [FV14, FMV13], where the sheaf of logarithmic derivations of a hyperplane arrangement was computed as Fourier-Mukai transform of $\mathcal{I}_Z(1)$, via the point-hyperplane incidence correspondence $\mathbb{I}$, where Z is the set of points in the dual space $\check{\mathbb{P}}$ corresponding to the arrangement and $\mathcal{I}_Z$ is the ideal sheaf of Z in $\check{\mathbb{P}}$. This result was in some sense at the origin of the study of unexpected curves, carried on by many authors starting with [CHMN18].

However, our main application is to the minimal resolution of logarithmic derivations modules of adjoint discriminants. Let G be a simple affine algebraic group over $\mathbf{k}$ and set $\mathfrak{g}$ for the Lie algebra of G. We let n be the rank of $\mathfrak{g}$ and $e_1 \leq \cdots \leq e_n$ be the exponents of $\mathfrak{g}$. Let $\mathbf{U} = \mathbf{k}[\mathfrak{g}^\vee]$, fix some coordinates $(x_0, \ldots, x_\ell)$ for $\mathbf{k}[\mathfrak{g}^\vee]$ and write $\mathrm{Der}_{\mathbf{U}} = \mathbf{U}\partial_{x_0} \oplus \cdots \oplus \mathbf{U}\partial_{x_\ell}$. The fundamental divisor we study here is the adjoint discriminant Δ, i.e. the discriminant for the adjoint action of G on $\mathfrak{g}$ and set $D = \mathbb{V}(\Delta)$. Our main interest is for the module of logarithmic derivations $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ of D. Let us first state our main result for G of type A, D, E, i.e. $\mathfrak{g}$ is *simply laced*. Let us introduce two main ingredients. The first one is the natural morphism: $\mathbf{ad} : \mathfrak{g} \otimes \mathbf{U}(-1) \to \mathfrak{g} \otimes \mathbf{U}$ given by the dual of the Lie bracket on the degree-1 parts $(\mathfrak{g} \otimes \mathbf{U}(-1))_1 = \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} = (\mathfrak{g} \otimes \mathbf{U})_1$ and extended by $\mathbf{U}$-linearity (here we identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$ via the Killing form). The $\mathbf{U}$-modules $\ker(\mathbf{ad})$ and $\mathbf{A} = \mathrm{Im}(\mathbf{ad})$ are G-equivariant; it will turn out that $\ker(\mathbf{ad}) \simeq \bigoplus_{i=1}^n \mathbf{U}(-e_i - 1)$.

The second ingredient is the *braid arrangement* of type G given by intersecting Δ with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The Weyl group W of G acts on $\mathfrak{h}$ by reflections about the hyperplanes H_α, where α ranges in the set Φ^+ of positive roots in the root system Φ of $\mathfrak{g}$. A fundamental fact is that $\Delta|_{\mathfrak{h}} = \delta$, with $\sqrt{\delta} = (-1)^{|\Phi^+|}\Pi_{\alpha\in\Phi^+}\delta_\alpha$, where δ_α is a linear equation of the hyperplane H_α. The *remarkable formula of Kostant, Macdonald, Shapiro and Steinberg* (KMSS), referring to work of Shapiro and [Kos59, Mac72, Ste59], asserts that the braid arrangement is free with generators sitting in degrees equal to the exponents. This is related to the topology of the complement in $\mathbb{C}^n$ of the arrangement, which is a $K(\pi, 1)$ space of the corresponding Artin-Tits group of type G, see [FN62] and [Bes15] for complex reflection groups. Terao showed in [Ter81] that the Poincaré polynomial of the complement of a free central affine arrangement with generators of degree $(d_1, \ldots, d_n)$ equals $\Pi_{i=1}^n(1+td_i)$, recovering a formula of Orlik and Solomon, [OS80] and extended the KMSS to unitary reflection groups, see [Ter80c]. A free basis of $\mathrm{Der}_{\mathbf{S}}(-\log(\delta))$ is given by the *Saito matrix*, which can be explicitly described as follows (see e.g. [Yos14, Sai04]). Writing $f_1, \ldots, f_n \in \mathbf{S}$ for a free basis of $\mathbf{S}^W$ with $\deg(f_i) = e_i$ for $i = 1, \ldots, n$, one has

$$\mathrm{Der}_{\mathbf{S}}(-\log(\delta)) \simeq \mu_1\mathbf{S} \oplus \cdots \oplus \mu_n\mathbf{S}, \qquad \mu_i = \nabla(f_i).$$

Now, going back to the full Lie algebra $\mathfrak{g}$, the W-invariants $f_1, \ldots, f_n$ lift to G-invariants $F_1, \ldots, F_n \in \mathbf{U}$ via the Chevalley restriction theorem $\mathbf{U}^G \simeq \mathbf{S}^W$. We get a new Saito-type matrix

$$\nu : \bigoplus_{i=1}^n \mathbf{U}(-e_i) \longrightarrow \mathfrak{g} \otimes \mathbf{U}, \qquad \nu_i = \nabla(F_i), \qquad \nu = (\nu_1, \ldots, \nu_n).$$

Our result connects all these objects into the following statement.

Theorem 2. *Let G be a simple algebraic $\mathbf{k}$-group, $\mathfrak{g}$ its lie algebra. Assume $\mathfrak{g}$ is of simply connected type, let $e_1, \ldots, e_n$ be the exponents of $\mathfrak{g}$ and let D be the adjoint discriminant. Then:*

$$\mathrm{Der}_{\mathbf{U}}(-\log(D)) \simeq \bigoplus_{i=1}^n \mathbf{U}(-e_i) \oplus \mathbf{A},$$

where the module $\mathbf{A}$ is the image of $\mathbf{ad} : \mathfrak{g} \otimes \mathbf{U}(-1) \to \mathfrak{g} \otimes \mathbf{U}$ and fits into:

$$0 \to \bigoplus_{i=1}^n \mathbf{U}(-e_i - 1) \xrightarrow{\nu} \mathfrak{g} \otimes \mathbf{U}(-1) \to \mathbf{A} \to 0.$$

In particular, the module $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ is of projective dimension one, with a summand presented by a *rectangular Saito matrix* corresponding to the image of the Lie bracket, and a free summand with the same exponents as the corresponding Weyl arrangement.

Next, let us discuss non-simply laced Dynkin diagrams. In this case, the connection with the KMSS formula is a bit different, as the adjoint discriminant captures long positive roots, rather than all of Φ^+, though the discussion might be related to ideal arrangements as in [ABC$^+$16]. Anyway, we give a minimal graded free equivariant resolution of derivation module even in this case and again this module turns out to have projective dimension one. Let us denote by s the number of long simple roots of $\mathfrak{g}$. The new ingredient is the quasi-minuscule representation $\hat{\mathfrak{g}}$ of $\mathfrak{g}$, namely, the irreducible representation whose highest weight is the highest short root (as opposed to the adjoint representation, whose highest weight is the highest long root). If G is of type C_n, set $j = 1$. If G is the group G_2 set $j = 2$, if G is the group F_4, set $j = 3$ and if G is of type B_n, set $j = n$. Then our last main result is the following.

Theorem 3. *Assume G is simple, not of simply laced type. Then the module $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ admits a G-equivariant graded free resolution:*

$$0 \longrightarrow \begin{array}{c} \bigoplus_{i=1}^{s} \mathbf{U}(-e_i - 1) \\ \oplus \\ \hat{\mathfrak{g}} \otimes \mathbf{U}(-j-1) \end{array} \longrightarrow \begin{array}{c} \bigoplus_{i=1}^{s} \mathbf{U}(-e_i) \oplus \mathfrak{g} \otimes \mathbf{U}(-1) \\ \oplus \\ \hat{\mathfrak{g}} \otimes \mathbf{U}(-j) \end{array} \longrightarrow \mathrm{Der}_{\mathbf{U}}(-\log(D)) \longrightarrow 0.$$

The strategy of the proof of Theorems 2 and 3 goes as follows. We rely Theorem 1 and use the geometric technique mentioned above to $\tilde{\rho}_*(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h))$ via the pushforward of a Koszul complex on $X \times \check{\mathbb{P}}$. This translates into computing the cohomology of twisted exterior powers of the affine tangent sheaf of X. Here, we use the geometry of adjoint varieties and notably the contact structure to identify the affine tangent bundle to the affine cotangent bundle on X, up to a modification of twists. The computation of the cohomology of these bundles turns out to be affordable, on a case-by-case basis, through a quite technical use of the Bott-Borel-Weil Theorem and branching rules for classical groups. However, rather surprisingly, the final result can be stated in a uniform and concise way for all groups, see Theorem 4.18. Of course, a unified proof would be highly desirable but this seems out of reach for the time being. As a final step, for simply laced types, we connect our discussion to the theory of free arrangements to prove that the resolution is minimal and that it induces the decomposition stated in Theorem 2.

The paper is organised as follows. In §1 we develop our study of logarithmic tangent sheaves via push-forward from the normal variety and prove Theorem 1. In §2 we start working out resolutions of the Jacobian ideal and of the logarithmic tangent sheaf, obtained using the Cayley method and push-forward in the spirit of [GKZ08]. We mention a few results here about normalisation of dual varieties notably of Fano varieties, see Proposition 2.2, and relating the geometric method to the Jacobian ideal, see Theorem 2.3. In §3 we review some of the relevant geometry of adjoint varieties and their contact structure and provide a minimal graded free resolution of the normalisation of adjoint discriminants, see Theorem 3.3. In §4, we compute a resolution of the module of logarithmic derivations of adjoint discriminants in Theorem 4.18 both for the simply laced and the non simply laced situation. We also derive part of the results of [FM22] about symmetric determinants in terms of adjoint orbits in case C_n. Then, in §5 we use Terao's results on hyperplane arrangements, we explicitly construct the module of G-invariant logarithmic derivations and finally we deduce Theorem 2.

We would like to thank D. Fratila and M. Yoshinaga for fruitful discussions.

1. Jacobian ideal and logarithmic differential via pushforward

The main goal of this section is to relate, after recalling the necessary definitions, the notion of logarithmic tangent sheaf with the incidence construction that allows to define the notion of dual variety.

1.1. Background and notation.

We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let us denote by $\mathbb{P} = \mathbb{P}^\ell$ the ℓ-dimensional projective space parametrising hyperplanes in $\mathbf{k}^{\ell+1}$ and by $\check{\mathbb{P}} = \check{\mathbb{P}}^\ell$ the dual projective space parametrising hyperplanes of $\mathbb{P}$.

1.1.1. *Logarithmic derivations.* Let $\mathbf{U} = \mathbf{k}[x_0, \dots, x_\ell]$ be the coordinate ring of $\check{\mathbb{P}}^\ell$. We define the module of $\mathbf{U}$-derivations as the free $\mathbf{U}$-module of rank $\ell + 1$:

$$\mathrm{Der}_{\mathbf{U}} = \left\{ \sum_{i=0}^{\ell} \theta_i \partial_i \mid \theta_i \in \mathbf{U} \right\}, \qquad \text{with:} \qquad \partial_i = \frac{\partial}{\partial x_i}, \qquad \text{for } i \in \{0, \dots, \ell\}.$$

Definition 1.1. Let $F \in \mathbf{U}$ be a non-zero square-free homogeneous polynomial of degree d. Then we set:

$$\mathrm{Der}_{\mathbf{U}}(-\log(F)) = \{\theta \in \mathrm{Der}_{\mathbf{U}} \mid \theta(F) \in (F)\}, \qquad \mathrm{Der}_{\mathbf{U}}(-\log(F))_0 = \{\theta \in \mathrm{Der}_{\mathbf{U}} \mid \theta(F) = 0\}.$$

The $\mathbf{U}$-module of *logarithmic derivations* $\mathrm{Der}_{\mathbf{U}}(-\log F)$ and its submodule $\mathrm{Der}_{\mathbf{U}}(-\log(F))_0$ are $\mathbb{Z}$-graded, respectively of rank $\ell + 1$ and ℓ.

The Euler derivation $\epsilon = x_0\partial_0 + \cdots + x_\ell \partial_\ell$ provides a splitting:

$$\mathrm{Der}_{\mathbf{U}}(-\log(F)) = \mathrm{Der}_{\mathbf{U}}(-\log F)_0 \oplus \mathbf{U}\epsilon.$$

This holds since $\mathbf{k}$ has characteristic zero and more generally when working with a field $\mathbf{k}$ such that $\mathrm{char}(\mathbf{k})$ does not divide d. If $D = \mathbb{V}(F) \subset \check{\mathbb{P}}^\ell$ is the vanishing locus of F, we will write $\mathrm{Der}_{\mathbf{U}}(-\log(D)) = \mathrm{Der}_{\mathbf{U}}(-\log(F))$ and $\mathrm{Der}_{\mathbf{U}}(-\log(D))_0 = \mathrm{Der}_{\mathbf{U}}(-\log(F))_0$.

1.1.2. *Logarithmic tangent sheaves.* For this part we refer to [Ser06, §3.4.4]. Useful properties of the logarithmic tangent sheaf can be found for instance in [Dol07]. Let Y be a reduced projective $\mathbf{k}$-scheme and write $\mathcal{T}_Y$ for the tangent sheaf of Y. Assume now that Y lies in $\mathbb{P} = \mathbb{P}^\ell$ with normal sheaf $\mathcal{N}_{Y/\mathbb{P}}$ and consider the composition:

$$\nabla : \mathcal{T}_{\mathbb{P}} \to \mathcal{T}_{\mathbb{P}}|_Y \to \mathcal{N}_{Y/\mathbb{P}}.$$

Definition 1.2. The sheaf of *logarithmic differentials* $\mathcal{T}_{\mathbb{P}}\langle Y\rangle$ or *logarithmic tangent sheaf* is the kernel of ∇ and the equisingular normal sheaf $\mathcal{N}'_{Y/\mathbb{P}}$ is the image of ∇, so we have an exact sequence:

$$0 \to \mathcal{T}_{\mathbb{P}}\langle Y\rangle \to \mathcal{T}_{\mathbb{P}} \to \mathcal{N}'_{Y/\mathbb{P}} \to 0. \tag{1}$$

The *equisingular normal sheaf* $\mathcal{N}'_{Y/\mathbb{P}}$ is supported on Y and its rank along an irreducible component of Y is equal to the codimension of such component.

If Y is a reduced hypersurface $D \subset \check{\mathbb{P}}$ defined by a single homogeneous equation $F \in \mathbf{U}$ of degree d, then:

$$\mathcal{N}_{D/\check{\mathbb{P}}} \simeq \mathcal{O}_D(d), \qquad \mathcal{N}'_{D/\check{\mathbb{P}}} \simeq \mathcal{J}_D(d),$$

where $\mathcal{J}_D$ is the restricted Jacobian ideal sheaf, defined as the restriction to D of the Jacobian ideal sheaf $\mathcal{J}$ generated by the partial derivatives of F, namely:

$$\mathcal{J}(d) = \mathrm{Im}(\nabla(F)) = \mathrm{Im}\left((\partial_0 F, \dots, \partial_\ell F) : \mathcal{O}_{\check{\mathbb{P}}}^{\ell+1}(1) \to \mathcal{O}_{\check{\mathbb{P}}}(d)\right).$$

Rewriting (1), we get the fundamental exact sequence:

$$0 \to \mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle \to \mathcal{T}_{\check{\mathbb{P}}} \to \mathcal{J}_D(d) \to 0, \tag{2}$$

This allows to consider $\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle$ as the sheaf of Jacobian syzygies of F, namely the kernel of $\nabla(F)$, in view of the next exact diagram. Note that commutativity of the top right square follows from the Euler relation, valid in characteristic 0, or more generally when $\mathrm{char}(\mathbf{k})$ does not divide d.

$$\begin{array}{ccccccccc}
 & & & & \mathcal{O}_{\check{\mathbb{P}}}(-1) & \xrightarrow{\cdot d} & \mathcal{O}_{\check{\mathbb{P}}}(-1) & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \cdot F} & & \\
0 & \longrightarrow & \mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) & \longrightarrow & \mathcal{O}_{\check{\mathbb{P}}}^{\ell+1} & \xrightarrow{\nabla F} & \mathcal{J}(d-1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) & \longrightarrow & \mathcal{T}_{\check{\mathbb{P}}}(-1) & \longrightarrow & \mathcal{J}_D(d-1) & \longrightarrow & 0
\end{array} \tag{3}$$

In characteristic zero, or more generally when $\mathrm{char}(\mathbf{k})$ does not divide d, we have the fundamental relationships involving the sheafified module of logarithmic derivations and the graded module of logarithmic differentials, valid for a reduced hypersurface $D \subset \check{\mathbb{P}}$:

$$\mathrm{Der}_{\mathbf{U}}(-\log D)_0 \simeq \bigoplus_{t \in \mathbb{Z}} H^0(\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(t-1)), \qquad \mathrm{Der}_{\mathbf{U}}(-\log D)_0^{\sim} \simeq \mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1).$$

1.1.3. *The affine logarithmic tangent sheaf.* Let Y be a reduced closed subscheme of $\mathbb{P}$ and consider the induced hyperplane bundle $c_1(\mathcal{O}_Y(1))$. The first Chern class of $\mathcal{O}_Y(1)$ provides a non-zero element of $H^1(Y, \Omega_Y)$ and in turn a non-splitting extension

$$0 \to \Omega_Y \to \hat{\Omega}_Y \to \mathcal{O}_Y \to 0,$$

where the middle sheaf $\hat{\Omega}_Y$ defined by the sequence is called the *affine cotangent sheaf* of Y. Its dual is the *affine cotangent sheaf* $\hat{\mathcal{T}}_Y$ of Y, fitting into:

$$0 \to \mathcal{O}_Y \to \hat{\mathcal{T}}_Y \to \mathcal{T}_Y \to 0. \tag{4}$$

Note that the definition of $\hat{\mathcal{T}}_Y$ depends on the inclusion $Y \subset \mathbb{P}$ and that the affine tangent space of the projective space is $\hat{\mathcal{T}}_{\mathbb{P}} = V \otimes \mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O}_{\mathbb{P}}(1)^{\ell+1}$. Moreover, we have an exact sequence

$$0 \to \hat{\mathcal{T}}_Y \to \hat{\mathcal{T}}_{\mathbb{P}}|_Y \to \mathcal{N}_{Y/\mathbb{P}},$$

whose image is again the equisingular normal sheaf $\mathcal{N}'_{Y/\mathbb{P}}$.

Definition 1.3. The *affine sheaf of logarithmic differentials* $\hat{\mathcal{T}}_{\mathbb{P}}\langle Y\rangle$ of Y is the kernel of the natural composition:

$$\hat{\mathcal{T}}_{\mathbb{P}} \to \hat{\mathcal{T}}_{\mathbb{P}}|_Y \to \mathcal{N}_{Y/\mathbb{P}}.$$

We have exact sequences:

$$\begin{aligned} &0 \to \hat{\mathcal{T}}_{\mathbb{P}}\langle Y\rangle \to \hat{\mathcal{T}}_{\mathbb{P}} \to \mathcal{N}'_{Y/\mathbb{P}} \to 0, \\ &0 \to \mathcal{O}_{\mathbb{P}} \to \hat{\mathcal{T}}_{\mathbb{P}}\langle Y\rangle \to \mathcal{T}_{\mathbb{P}}\langle Y\rangle \to 0. \end{aligned} \tag{5}$$

When $Y = D$ is a hypersurface of $\check{\mathbb{P}}$, (5) reads as it follows:

$$0 \to \hat{\mathcal{T}}_{\mathbb{P}}\langle D\rangle \to \hat{\mathcal{T}}_{\check{\mathbb{P}}} \to \mathcal{J}_D(d) \to 0. \tag{6}$$

We will recover this sequence via pushforward in Corollary 1.6.
We will now recall the following classical result.

Lemma 1.4. *The affine sheaf of logarithmic differentials of a hypersurface D splits as $\hat{\mathcal{T}}_{\mathbb{P}}\langle D\rangle \simeq \mathcal{T}_{\mathbb{P}}\langle D\rangle \oplus \mathcal{O}_{\mathbb{P}}$.*

Proof. Diagram (3) implies that, applying the functor $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}})$ to its lower horizontal sequence (corresponding to the sequence in (6)), the induced map

$$\mathrm{Ext}^1\left(\mathcal{J}_D(d), \mathcal{O}_{\mathbb{P}}\right) \to \mathrm{Ext}^1\left(\mathcal{T}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}\right) \simeq \mathbf{k}$$

is surjective. This implies that the following map between extension groups vanishes:

$$\mathrm{Ext}^1\left(\mathcal{T}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}\right) \to \mathrm{Ext}^1\left(\mathcal{T}_{\mathbb{P}}\langle D\rangle, \mathcal{O}_{\mathbb{P}}\right). \tag{7}$$

Notice that the extension in (5) is defined between the logarithmic derivations and the Euler one, which implies the following commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{O}_{\mathbb{P}} & = & \mathcal{O}_{\mathbb{P}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \hat{\mathcal{T}}_{\mathbb{P}}\langle D\rangle & \longrightarrow & \hat{\mathcal{T}}_{\mathbb{P}} & \longrightarrow & \mathcal{J}_D(d) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{T}_{\mathbb{P}}\langle D\rangle & \longrightarrow & \mathcal{T}_{\mathbb{P}} & \longrightarrow & \mathcal{J}_D(d) & \longrightarrow & 0 \end{array}$$

This allows us to conclude that the extension class considered in (5) belongs to the image of (7) and, henceforth, it is the zero class. This proves the statement. □

The latter splitting explains the two natural, but at first sight in contrast, ways of defining the logarithmic tangent sheaf, i.e., as a quotient of the affine logarithmic tangent sheaf or as the sheafification of $\mathrm{Der}_{\mathbf{U}}(-\log D)_0\widetilde{\ }$, by definition included in $\mathrm{Der}_{\mathbf{U}}(-\log D)\widetilde{\ }$, whose sheafification gives in turn $\hat{\mathcal{T}}_{\mathbb{P}}\langle D\rangle$.
Finally, notice that if $\mathrm{char}(\mathbf{k})$ does divide d, we have that the Euler derivation belongs to $\mathrm{Der}_{\mathbf{U}}(-\log D)_0\widetilde{\ }$, inducing a different diagram with respect to Diagram 3 from which we cannot have the described splitting.

In any case, $\mathcal{T}_{\mathbb{P}}\langle D\rangle$ can always be defined as the sheafification of the quotient of $\mathrm{Der}_{\mathbf{U}}(-\log D)$ by the Euler derivation, but this no longer identifies with $\mathrm{Der}_{\mathbf{U}}(-\log D)_0$.

1.1.4. *Projective duality.* Consider a reduced closed subscheme X of $\mathbb{P} = \mathbb{P}^\ell$ and write $X_{\mathbf{sm}}$ for its smooth locus. Set $\mathcal{T}_{X,x}$ for the tangent space of X at $x \in X$. Then we define the normal variety $\mathbb{W}_X$ as the Zariski closure in $\mathbb{P} \times \check{\mathbb{P}}$ of:

$$\mathbb{W}_X^\circ = \left\{(x, H) \in X_{\mathbf{sm}} \times \check{\mathbb{P}} \mid H \supset \mathcal{T}_{X,x}\right\}.$$

In particular, if X is smooth, it is possible to define $\mathbb{W}_X$ as $\mathbb{P}(\mathrm{N}_{X/\mathbb{P}}(-1))$.
Observe that $\mathbb{W}_X$ is a reduced closed subscheme of the point-hyperplane incidence variety $\mathbb{I}$:

$$\mathbb{W}_X \subset \mathbb{I} = \left\{(x, H) \in \mathbb{P} \times \check{\mathbb{P}} \mid x \in H\right\}.$$

Denote by h and $\check{h}$, respectively, the hyperplane classes of $\mathbb{P}$ and $\check{\mathbb{P}}$. Denote by $\hat{\pi}$ and $\hat{\rho}$ the canonical projections of the product $\mathbb{P} \times \check{\mathbb{P}}$ in each of its factors. Their restriction to $X \times \check{\mathbb{P}}$ will be denoted respectively by $\overline{\pi}$ and $\overline{\rho}$ and, furthermore, we will denote by π and ρ their respective restrictions on $\mathbb{W}_X$. By abuse of notation, we will denote also by h and $\check{h}$ their respective pullbacks on $\mathbb{P} \times \check{\mathbb{P}}$, as well as their restrictions to the subvarieties described above.

The image $\check{X} := \overline{\rho}(\mathbb{W}_X)$ is known as the *dual variety* of X. The *biduality theorem* asserts that X itself is the dual of $\check{X}$. The crux of the biduality theorem is the fact that $\mathbb{W}_X = \mathbb{W}_{\check{X}}$ as subvarieties of $\mathbb{P} \times \check{\mathbb{P}}$. We refer to [GKZ08]. Finally, we will denote by $\tilde{\pi}$ and $\tilde{\rho}$ the two projections from $\mathbb{I}$, respectively to $\mathbb{P}$ and $\check{\mathbb{P}}$, given once more as the restriction from $\mathbb{P} \times \check{\mathbb{P}}$. We will denote by the same notation the corresponding affine projections. For instance, if $\check{\mathbb{P}} = \mathbb{P}(V^\vee)$ then the projections from $X \times V$ to X and V will be denoted again respectively by $\overline{\pi}$ and $\overline{\rho}$.

1.2. **Logarithmic differentials as direct image sheaves.** We are now in the position to state and prove Theorem 1, the main result of this section.

Theorem 1.5. *Let $X \subset \mathbb{P}$ be a reduced closed subscheme such that all the irreducible components of $D = \check{X}$ are hypersurfaces of $\check{\mathbb{P}}$. Then the logarithmic tangent sheaf $\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle$ of D satisfies*

$$\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) \simeq \tilde{\rho}_* \left(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)\right),$$

where $\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}$ is ideal sheaf of $\mathbb{W}_X$ seen as a subscheme of the flag variety $\mathbb{I}$.

Proof. By [Nob75, Theorem 1], we know that $\mathbb{W}_X$ can be seen as the blowing up of D, which is to say that, locally, it is a monoidal transformation with center the Jacobian ideal defining the singular locus of D. We have therefore a canonical inclusion of the restriction of the Jacobian ideal

$$\mathcal{J}_D(d-1) \hookrightarrow \rho_*(\mathcal{O}_{\mathbb{W}_X}(h)) \simeq \rho_*(\pi^*(\mathcal{O}_X(1))), \tag{8}$$

being, as before, $d = \deg(D)$ and $\mathcal{O}_{\mathbb{W}_X}(h)$ the canonical sheaf induced by the blowing up construction. Finally, the isomorphism $\mathcal{O}_{\mathbb{W}_X}(h) \simeq \pi^*(\mathcal{O}_X(1))$ can be obtained considering the following chain of inclusions

$$\mathbb{W}_X \subset \mathbb{P}(\mathcal{J}_D(d-1)) \subset \mathbb{P}(\mathcal{T}_{\check{\mathbb{P}}}(-1)) = \mathbb{I}. \tag{9}$$

The second inclusion is given by the surjection $\mathcal{T}_{\check{\mathbb{P}}}(-1) \to \mathcal{J}_D(d-1)$ given by the derivatives of the polynomial F defining D (see (2)). The invertible sheaf is given by $\mathcal{O}_{\mathbb{I}}(h) = \mathcal{O}_{\mathbb{I}} \otimes \mathcal{O}_{\mathbb{P}\times\check{\mathbb{P}}}(h)$. Consider the following short exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h) \longrightarrow \mathcal{O}_{\mathbb{I}}(h) \longrightarrow \mathcal{O}_{\mathbb{W}_X}(h) \longrightarrow 0.$$

Taking its pushforward by $\tilde{\rho}$, we get the bottom row of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) & \longrightarrow & \mathcal{T}_{\check{\mathbb{P}}}(-1) & \longrightarrow & \mathcal{J}_D(d-1) & \longrightarrow & 0 \\ & & \downarrow & & \simeq\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\rho}_*(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)) & \longrightarrow & \tilde{\rho}_*(\mathcal{O}_{\mathbb{I}}(h)) & \longrightarrow & \rho_*(\pi^*(\mathcal{O}_X(1))) & & \end{array} \tag{10}$$

Indeed, the middle vertical map is an isomorphism as a direct consequence of the definition of $\mathcal{O}_{\mathbb{W}_X}(h) \simeq \pi^*(\mathcal{O}_X(1))$. Furthermore, notice that the commutativity of the previous diagram (in particular of the right square) follows naturally from the description of the normal variety as a subvariety of the flag $\mathbb{I}$, passing

through the projectivization of the Jacobian ideal, as depicted in (9). Finally, from the snake lemma applied to the previous diagram, we find the desired isomorphism

$$\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) = \tilde{\rho}_*\left(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)\right).$$

□

Let us now state and comment some direct consequences of the previous theorem. The first result will relate the logarithmic sheaf with the ideal sheaf of $\mathbb{W}_X$ seen this time as a subvariety of the product $X \times \check{\mathbb{P}}$. The described relation will be fundamental in Section 2.

Corollary 1.6. *Let X as in Theorem 1.5 and, moreover, assume it to be non degenerate and linearly normal. Then we have the following short exact sequence*

$$0 \longrightarrow \mathcal{O}_{\check{\mathbb{P}}}(-1) \longrightarrow \overline{\rho}_*(\mathcal{I}_{\mathbb{W}_X/(X\times\check{\mathbb{P}})}(h)) \longrightarrow \mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) \longrightarrow 0.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \hat{\pi}^*\mathcal{I}_X(1)\otimes\hat{\rho}^*\mathcal{O}_{\check{\mathbb{P}}} & \longrightarrow & \mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}\times\check{\mathbb{P}}}(-\check{h}) & \longrightarrow & \mathcal{O}_{\mathbb{P}\times\check{\mathbb{P}}}(h) & \longrightarrow & \mathcal{O}_{\mathbb{I}}(h) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{\mathbb{W}_X/(X\times\check{\mathbb{P}})}(h) & \longrightarrow & \mathcal{O}_{X\times\check{\mathbb{P}}}(h) & \longrightarrow & \mathcal{O}_{\mathbb{W}_X}(h) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Being X non degenerate and linearly normal implies that $H^0(\mathcal{I}_X(1)) = H^1(\mathcal{I}_X(1)) = 0$, therefore $\hat{\rho}_*(\mathcal{K}) = 0$ and $R^1\hat{\rho}_*(\mathcal{K}) \simeq \tilde{\rho}_*(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h))$. Combining it with the pushforward by $\hat{\rho}$ of the left vertical sequence in the previous diagram concludes the proof. □

The next result describes more in detail the case where X is 0-dimensional. In particular, we recover a result due to Faenzi and Vallès (see [FV14]).

Corollary 1.7. *Let X be as in Theorem 1.5. Then if $\dim X = 0$, we have that*

$$\mathcal{T}_{\check{\mathbb{P}}}\langle D\rangle(-1) \simeq \tilde{\rho}_*\tilde{\pi}^*\mathcal{I}_X(1).$$

Proof. This result follows directly from the fact that, being $\dim X = 0$, we have $\mathbb{W}_X = (X \times \check{\mathbb{P}}) \cap \mathbb{I}$ and therefore $\tilde{\pi}^*\mathcal{I}_X(1) \simeq \mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)$. □

Remark 1.8. At the level of the Jacobian ideal sheaf, we have $\rho_*(\pi^*(\mathcal{O}_X(1))) \simeq \mathcal{J}_D(d-1)$ if and only if $R^1\tilde{\rho}_*(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)) = 0$. Indeed, this is tantamount to surjectiveness of $\tilde{\rho}_*(\mathcal{O}_{\mathbb{I}}(h)) \to \rho_*\pi^*\mathcal{O}_X(1)$ and thus to the vanishing of the higher direct image sheaf by Diagram (10).

The remaining of this section will be devoted to the discussion on the previous vanishing of the higher direct image. To do so, let us first recall the notions that will be necessary to apply the *Theorem of formal functions* (for more details, see [Har77, III.11]).
Considering a morphism of schemes $f : X \to S$ and a point $s \in S$, we would like to describe the fiber product

$$X_k = X \times_S \operatorname{Spec}\left(\mathcal{O}_{S,s}/\mathfrak{m}_s^k\right).$$

Taking an affine covering, which allows to restrict our description to such case, denote by $\phi : A \to B$ the associate ring map and by $\mathfrak{q}$ the prime ideal of A corresponding to the point s. Therefore X_k will be given by the spectrum of

$$B \otimes_A A_{\mathfrak{q}}/\mathfrak{q}^n A_{\mathfrak{q}} \simeq B \otimes_A \left(A/\mathfrak{q}^n A\right)_{\mathfrak{q}} \simeq \left(B/\phi(\mathfrak{q})^n B\right)_{\phi(\mathfrak{q})}.$$

Moreover if s is a closed point, which means $\mathfrak{q}$ to be maximal, we do not need to localize.
Applied to our case, consider the projection $\tilde{\rho} : \mathbb{I} \to \check{\mathbb{P}}$ and a point of the dual projective space $y \in \check{\mathbb{P}}$. We

will denote both by H_y the correspondent hyperplane in $\mathbb{P}$ and also the fiber $\tilde{\rho}^{-1}(y)$, seen as a subscheme of the incidence variety. Using the introduced notation on the thickened fibers, we have that $\mathbb{I}_1$ is simply the fiber H_y.
Furthermore, the previous considerations on the local description imply the following short exact sequence

$$0 \to \mathcal{I}_{H_y}^k \to \mathcal{O}_{\mathbb{I}} \to \mathcal{O}_{\mathbb{I}_k} \to 0,$$

which means that the *thickened fiber* $\mathbb{I}_k$ is defined by the k-th power of the ideal defining H_y as a subvariety of $\mathbb{I}$. This implies the subsequent short exact sequence

$$0 \to \frac{\mathcal{I}_{H_y}^{k-1}}{\mathcal{I}_{H_y}^k} \to \mathcal{O}_{\mathbb{I}_k} \to \mathcal{O}_{\mathbb{I}_{k-1}} \to 0. \tag{11}$$

Being the ideal of H_y, in the coordinate ring of the incidence variety, defined by a regular sequence, we know by [Har77, II-8.21A(e)] that

$$\mathcal{I}_{H_y}^{k-1}/\mathcal{I}_{H_y}^k \simeq S^{k-1}\left(\mathcal{I}_{H_y}/\mathcal{I}_{H_y}^2\right),$$

where S^{k-1} denotes the $(k-1)$-st symmetric power. Finally, notice that H_y is a smooth complete intersection of n elements in the linear system $|\check{h}|$. This implies that $\mathcal{I}_{H_y}/\mathcal{I}_{H_y}^2$ gives the conormal bundle associated to H_y and, specifically,

$$\frac{\mathcal{I}_{H_y}}{\mathcal{I}_{H_y}^2} \simeq (\mathcal{O}_{\mathbb{I}}(\check{h})^n)_{H_y} \simeq \mathcal{O}_{H_y}^n.$$

Substituting in (11), we obtain that having the vanishing

$$H^1\left(H_y, (\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h))_{H_y}\right) = 0$$

implies all the subsequent ones, on the thickened fibers,

$$H^1\left(\mathbb{I}_k, (\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h))_{\mathbb{I}_k}\right) = 0.$$

We are therefore in the position to state the following result.

Lemma 1.9. *If $H^1\left(H_y, (\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h))_{H_y}\right) = 0$ holds for every point $y \in \check{\mathbb{P}}$, then we have the isomorphism*

$$\rho_*(\pi^*(\mathcal{O}_X(1))) \simeq \mathcal{J}_D(d-1).$$

Proof. By the Theorem of formal functions, all the vanishings induced by $H^1\left(H_y, (\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h))_{H_y}\right) = 0$ imply that $R^1\tilde{\rho}_*(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(1,0)) = 0$. By Diagram 10, this is equivalent to the required isomorphism. □

From a first glance, the cohomological vanishing of the previous lemma seems very restrictive. That is the reason why we will now study the example of plain curves. Nevertheless, such a restrictiveness highlights the choice of considering *adjoint varieties*, for which we will prove the isomorphism always to be true.

Example 1.10. Consider a plane curve $X \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ in the projective plane. Let us fix a point $y \in \check{\mathbb{P}}^2$ and denote, as before, the corresponding line in $\mathbb{P}^2$ by H_y.
First of all, let us consider the restriction to H_y of the short exact sequence defining $\mathbb{W}_X \subset \mathbb{I}$, that gives us

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}or^1(\mathcal{O}_{\mathbb{W}_X}(h), \mathcal{O}_{H_y}) & \longrightarrow & \left(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)\right)_{|H_y} & \longrightarrow & \mathcal{O}_{H_y}(1) & \longrightarrow & \mathcal{O}_{\mathbb{W}_X \cap H_y}(1) \longrightarrow 0 \\ & & & & & \searrow & & \nearrow & \\ & & & & & & \mathcal{I}_{\mathbb{W}_X \cap H_y}(1) & & \\ & & & & 0 & \nearrow & & \searrow & 0 \end{array}$$

We can now divide our studying in the following cases:

- H_y is an irreducible component of W_X, which implies that the line $\tilde{\pi}(H_y)$ is also an irreducible component of X.
- H_y is not an irreducible component of W_X, and neither its projection on X is a component of X.

In the latter case, the support of $\mathcal{T}or^1(\mathcal{O}_{\mathbb{W}_X}, \mathcal{O}_{H_y})$ is at most 0-dimensional, which implies the vanishing of its first and second cohomology group. Therefore, $h^0\left(\mathcal{O}_{\mathbb{W}_X \cap H_y}(1)\right) \geq 3$ is equivalent to $H^1\left(\left(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)\right)_{|H_y}\right) \neq 0$. Observing that $h^0\left(\mathcal{O}_{\mathbb{W}_X \cap H_y}(1)\right)$ is the number of points of X tangent along H_y, we get that for any curve X with (at least) a tritangent line, we cannot hope for the cohomological vanishings required in Lemma 1.9.

Regarding the first case, consider the following short exact sequence, obtained from the inclusions $H_y \subset \mathbb{W}_X \subset \mathbb{I}$,

$$0 \longrightarrow \mathcal{I}_{\mathbb{W}_X/\mathbb{I}} \longrightarrow \mathcal{I}_{H_y/\mathbb{I}} \longrightarrow \mathcal{I}_{H_y/\mathbb{W}_X} \longrightarrow 0.$$

Notice that $\mathcal{I}_{H_y/\mathbb{W}_X}$ is supported on the components of $\mathbb{W}_X$ different from H_y (denote $\mathbb{W}_X = H_y \cup L$), therefore the restriction $\left(\mathcal{I}_{H_y/\mathbb{W}_X}\right)_{|H_y}$ will be given by the 0-dimensional scheme Z defined by the intersection $L \cap H_y$. In particular, if $\text{length}(Z) > 4$, then $H^1\left(\left(\mathcal{I}_{\mathbb{W}_X/\mathbb{I}}(h)\right)_{|H_y}\right) \neq 0$.

2. The geometric method and the Jacobian ideal

Here we describe our use of the *geometric method*, in the terminology of [Wey03], to compute resolutions of sheaves of logarithmic differentials. This approach has been used extensively in [GKZ08] in the framework of projective duality, where the authors use the so-called Cayley method in order to obtain a complex resolving the discriminant, called the discriminant complex. It was already pointed out in [Tev01] that the geometric method could be used to obtain such discriminant complex, but it seems that no application to logarithmic sheaves has been proposed so far. Describing such an application is the main goal of this section.

2.1. **The geometric method and normalisation.** Let $X \subset \mathbb{P} = \mathbb{P}(V)$ be a smooth connected projective variety of dimension m with V a vector space of dimension $\ell + 1$.

2.1.1. *The affine setting.* Write the affine conormal bundle sequence in the smooth case:

$$0 \to \mathcal{N}^\vee_{X/\mathbb{P}}(1) \to V \otimes \mathcal{O}_X \to \hat{\Omega}_X(1) \to 0.$$

The total space $\mathrm{N}^\vee_{X/\mathbb{P}}(1) = \mathrm{Tot}(\mathcal{N}^\vee_{X/\mathbb{P}}(1))$ is a subvariety of the trivial bundle $\mathrm{Tot}(V \otimes \mathcal{O}_X) \simeq X \times V$, of codimension equal to $m+1$, defined in the fibres of the projection $\overline{\rho} : X \times V \to V$ by the equations in $\hat{\Omega}_X(1) \simeq \left(\hat{\mathcal{T}}_X(-1)\right)^\vee$. Then, there exists an exact *Koszul complex* of the form:

$$0 \to \wedge^{m+1}\overline{\pi}^*(\hat{\mathcal{T}}_X(-1)) \to \wedge^m \overline{\pi}^*(\hat{\mathcal{T}}_X(-1)) \to \cdots \to \overline{\pi}^*(\hat{\mathcal{T}}_X(-1)) \to \mathcal{O}_{X\times V} \to \mathcal{O}_{\mathrm{N}^\vee_{X/\mathbb{P}}(1)} \to 0,$$

where $\overline{\pi} : X \times V \to X$ is the obvious projection. Recall that we write $\pi : \mathrm{N}^\vee_{X/\mathbb{P}}(1) \to X$ and $\rho : \mathrm{N}^\vee_{X/\mathbb{P}}(1) \to V$ as the restrictions of π and ρ to $\mathrm{N}^\vee_{X/\mathbb{P}}(1)$.

Then the affine cone $\hat{D} \subset V$ over the dual variety $D = X^\vee$ is the image of ρ. Now let $\mathcal{E}$ be a vector bundle on X. Then the coherent sheaf $\rho_*(\pi^*(\mathcal{E}))$ on V is supported on $\hat{D}$. By computing the pushforward $\overline{\rho}_*$ of the Koszul complex above, Weyman's theorem provides a complex

$$\mathbf{F}^{\mathcal{E}}_\bullet : \cdots \to \mathbf{F}^{\mathcal{E}}_{-1} \to \mathbf{F}^{\mathcal{E}}_0 \to \mathbf{F}^{\mathcal{E}}_1 \to \cdots$$

whose terms, for $u \in \mathbb{Z}$, are of the form:

$$\mathbf{F}^{\mathcal{E}}_u := \bigoplus_{l-p=u} H^l(X, \wedge^p(\hat{\mathcal{T}}_X(-1)) \otimes \mathcal{E}) \otimes \mathcal{O}_V(-p).$$

Here, $\mathcal{O}_V(-p)$ is the trivial line bundle on the affine space V for all $p \in \mathbb{Z}$, however the notation keeps track of the grading in the sense that for all $p, p' \in \mathbb{Z}$, the maps $\mathcal{O}_V(-p) \to \mathcal{O}_V(-p')$ appearing in the above complex are zero or homogeneous of degree $p - p'$, which by convention means that they vanish if $p < p'$.

Remark 2.1. If G is an algebraic group acting on X and both $\mathcal{E}$ and $\mathcal{O}_X(1)$ are G-linearised, then G acts linearly on $\mathbb{P}(V)$ and on $\mathbb{P}(V^\vee)$. In this setup, the complex $\mathbf{F}^{\mathcal{E}}_\bullet$ will be G-equivariant.

2.1.2. *The projective setting.* Working out the previous construction in the projective setting, one considers maps $\pi : \mathbb{P}(\mathrm{N}_{X/\mathbb{P}}(-1)) \to X$ and $\rho : \mathbb{P}(\mathrm{N}_{X/\mathbb{P}}(-1)) \to \mathbb{P}(V^\vee)$ and a projective version of Weyman's complex, for which we use the same notation, which is given by:

$$\mathbf{F}_u^{\mathcal{E}} := \bigoplus_{l-p=u} H^l(X, \wedge^p(\hat{\mathcal{T}}_X(-1)) \otimes \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(-p).$$

Proposition 2.2. *Let $X \subset \mathbb{P}$ be a smooth projective variety whose dual D is a hypersurface.*

i) If $H^l(X, \wedge^p \mathcal{T}_X(-p)) = 0$ for $l - p > 0$, then $\mathbf{F}_{>0}^{\mathcal{O}_X} = 0$ and we have an exact complex:

$$0 \to \mathbf{F}_\bullet^{\mathcal{O}_X} \to \rho_*\pi^*(\mathcal{O}_X) \to 0.$$

Also, $\rho_*\pi^*(\mathcal{O}_X)$ is the normalization of $\hat{D}$.

ii) If moreover $H^l(X, \wedge^l \mathcal{T}_X(-l)) = 0$ for $l > 0$ then $\hat{D}$ is normal with rational singularities.

Furthermore, if X is Fano, namely $\omega_X^\vee$ is ample, then the vanishing required for i) holds.

Proof. This is a direct application of the geometric method. Notice that the morphism $\rho : \mathrm{N}_{X/\mathbb{P}}^\vee(1) \to \hat{D}$ is birational. Indeed, $\hat{D} \subset V$ is the affine cone over $D \subset \check{\mathbb{P}} = \mathbb{P}(V^\vee)$, which is a hypersurface, and thus ρ is an isomorphism over the smooth locus of $\hat{D}$ by the biduality theorem. Then, to obtain the statements about $\mathbf{F}_\bullet^{\mathcal{O}_X}$ and the singularities of $\hat{D}$, just apply Theorem [Wey03, Theorem 5.1.3].

About the last assertion, assume that X is Fano and note that, for all $p \geq 0$, we have:

$$\wedge^p \mathcal{T}_X(-p) \simeq \Omega_X^{m-p}(p) \otimes K_X^\vee. \tag{12}$$

Since we are in characteristic zero and since $\mathcal{O}_X(p) \otimes K_X^\vee$ is ample, Kodaira vanishing gives:

$$H^l(\Omega_X^{m-p} \otimes \mathcal{O}_X(p) \otimes K_X^\vee) = 0, \qquad \text{for } l > p,$$

which, in view of (12), is precisely i). □

2.2. **Resolution of the Jacobian ideal.** Let us come to our main application of the geometric method, namely the resolution of the Jacobian ideal.

Theorem 2.3. *Let $X \subset \mathbb{P}$ be a smooth projective connected variety whose dual $D \subset \check{\mathbb{P}}$ is a degree d hypersurface.*

i) If $H^l(X, \wedge^p \mathcal{T}_X(1-p)) = 0$ for $l - p > 0$ then $\mathbf{F}_{>0}^{\mathcal{O}_X(1)} = 0$ and we have an exact complex:

$$0 \to \mathbf{F}_\bullet^{\mathcal{O}_X(1)} \to \rho_*(\pi^*(\mathcal{O}_X(1))) \to 0.$$

ii) If moreover $H^l(X, \wedge^l \mathcal{T}_X(1-l)) = 0$ for $l > 0$ and $H^0(X, \mathcal{O}_X(1)) = V$ then

$$\mathcal{J}_D(d-1) \simeq \rho_*(\pi^*(\mathcal{O}_X(1))).$$

Proof. From the exact sequence (4), for any integer p with $0 \leq p \leq m+1$, we get a short exact sequence

$$0 \to \wedge^{p-1}\mathcal{T}_X \to \wedge^p \hat{\mathcal{T}}_X \to \wedge^p \mathcal{T}_X \to 0. \tag{13}$$

Then, the vanishing appearing in i) implies that $H^l(X, \wedge^p \hat{\mathcal{T}}_X(1-p)) = 0$ for $l - p > 0$. As in the proof of the previous proposition, the morphism $\rho : \mathrm{N}_{X/\mathbb{P}}^\vee(1) \to \hat{D}$ is birational. By applying Theorem [Wey03, Theorem 5.1.2] we deduce that $\mathbf{F}_u^{\mathcal{O}_X(1)} = 0$ for $u > 0$ and the first statement above.

The hypothesis in the second statement implies moreover that

$$\mathbf{F}_0^{\mathcal{O}_X(1)} = \tilde{\rho}_*(\mathcal{O}_{\mathbb{I}}(h)) = V \otimes \mathcal{O}_{\check{\mathbb{P}}} \simeq \hat{\mathcal{T}}_{\check{\mathbb{P}}}(-1).$$

By construction, the morphism $\mathbf{F}_0^{\mathcal{O}_X(1)} \simeq \hat{\mathcal{T}}_{\check{\mathbb{P}}}(-1) \to \rho_*(\pi^*(\mathcal{O}_X(1)))$ factorises through the surjective morphism $\hat{\mathcal{T}}_{\check{\mathbb{P}}}(-1) \to \mathcal{J}_D(d-1)$ (the commutativity argument is the same as the one in the proof of Theorem 1.5) and the inclusion $\mathcal{J}_D(d-1) \to \rho_*(\pi^*(\mathcal{O}_X(1)))$. The fact that $\mathbf{F}_0^{\mathcal{O}_X(1)} \to \rho_*(\pi^*(\mathcal{O}_X(1)))$ is surjective is a consequence of the first statement above; this implies that the inclusion $\mathcal{J}_D(d-1) \to \rho_*(\pi^*(\mathcal{O}_X(1)))$ is also surjective, and thus an isomorphism. □

Corollary 2.4. *Let us suppose that the hypothesis and the vanishings in both statements of Theorem 2.3 hold. Then there exists a locally free resolution*

$$0 \to \cdots \to \mathbf{F}_{-2}^{\mathcal{O}_X(1)} \to \mathbf{F}_{-1}^{\mathcal{O}_X(1)} \to \hat{\mathcal{T}}_{\check{\mathbb{P}}}(-\log(D)) \to 0.$$

Proof. Combine Theorem 2.3 with Equation (6). □

In the following sections we will apply this result to the special case of hypersurfaces which are dual varieties of adjoint varieties.

3. Adjoint varieties, adjoint discriminants and their normalisation

The goal of this section is to compute a resolution of the structure sheaf of the normalisation of adjoint discriminants via the method developed in the previous section. This is mainly a warm-up for the next section, where we will compute a resolution of the Jacobian ideal of adjoint discriminants. We start by recalling some basic features of adjoint varieties.

Let G be a simple Lie group over $\mathbf{k}$, $V := \mathfrak{g}$ its Lie algebra and X the G-adjoint variety, i.e. the minimal G-orbit in $\mathbb{P}(\mathfrak{g})$. We will identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$ via the Killing form, thus thus identifying canonically V with $V^\vee$. Moreover we will denote by $\hat{\mathfrak{g}}$ the quasi-minuscule representation (i.e. the representation whose highest weight is the highest short root). The dual variety $D = X^\vee \subset \mathbb{P}(\mathfrak{g})$ is a hypersurface, called the adjoint discriminant of G. It is the zero locus of a polynomial Δ of degree equal to the number of long roots of G.

3.1. **Geometry of adjoint varieties.** Here we review some of the properties of adjoint varieties that will be useful to us. An adjoint variety X for a simple algebraic group G is a G-homogeneous projective manifold and thus can be seen as a quotient G/P for a certain parabolic subgroup $P \subset G$. Such parabolic subgroup is associated to a subset I_P of the simple roots $(\alpha_1, \dots, \alpha_n)$; if $I_P = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$ we write $P = P_{i_1,\dots,i_s}$. The subset I_P has a very explicit and elementary description.

We will use the fundamental equivalence between representations of P and G-homogeneous vector bundles on X. According to it, irreducible homogeneous bundles on X are in bijection with P-dominant weights, i.e. combinations $\varpi = \sum_i n_i \varpi_i$ with $n_i \in \mathbb{Z}$ for $i \in I_P$ and $n_i \geq 0$ for $i \notin I_P$, where $\varpi_1, \dots, \varpi_n$ are the fundamental weights with respect to the simple roots $\alpha_1, \dots, \alpha_n$ of G, which we will index according to Bourbaki's convention. If $\mathcal{E}$ is a G-homogeneous vector bundle, then the associated representation of the parabolic group P, restricted to the semisimple part of P, gives rise to a homogeneous bundle which is a direct sum of irreducible bundles. We call this the *semisemplification* of $\mathcal{E}$ and we denote it by $\mathrm{ss}(\mathcal{E})$. We denote by $\mathcal{E}_\varpi$ the irreducible bundle corresponding to the weight ϖ. If ϖ is G-dominant, we denote by V_ϖ the G-representation of highest weight ϖ.

Bott-Borel-Weil Theorem : Let us denote by W the Weyl group of G, and by $\rho = \sum_i \alpha_i$ the half sum of all positive roots. Let $w \in W$ be the unique element such that $w(\varpi + \rho)$ is G-dominant, i.e. $w(\varpi + \rho) = \sum_i m_i \varpi_i$ with $m_i \geq 0$, and let us denote by $l(w)$ the length of w. Bott-Borel-Weil Theorem ([Bot57]) asserts the following two statements. On the one hand, if there exists j such that $m_j = 0$ then $H^u(X, \mathcal{E}_\varpi) = 0$ for all u. On the other hand, if $m_i > 0$ for all i then $H^{l(w)}(X, \mathcal{E}_\varpi) \cong V_{w(\varpi+\rho)-\rho}$ as G-representations and $H^u(X, \mathcal{E}_\varpi) = 0$ for all $u \neq l(w)$.

3.1.1. *Contact structure.* A key feature of adjoint varieties is that they are contact manifolds (see [Bea98] and [BM19]). This means that there exists an exact sequence

$$0 \to \mathcal{F} \to \mathcal{T}_X \xrightarrow{\theta} \mathcal{L} \to 0, \tag{14}$$

where $\mathcal{L}$ is the line bundle defining the embedding $X \subset \mathbb{P}(V)$ and $\mathcal{F}$ is a vector bundle of rank $f = \dim(X) - 1$ on X, equipped with a skew-symmetric self-duality induced by the differential of $\theta \in H^0(X, \Omega_X \otimes \mathcal{L})$ (by abuse of notation, we will denote this duality by $d\theta$):

$$d\theta : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^\vee \otimes \mathcal{L}, \qquad {}^t(d\theta) = -d\theta.$$

In particular, $m = \dim(X) = f + 1$ is odd, and we write $m = 2e + 1$.

Lemma 3.1. *The morphism $d\theta \wedge (\bullet) : \wedge^{p-2}\mathcal{F}^\vee \to \wedge^p \mathcal{F}^\vee \otimes \mathcal{L}$ is an embedding for $p \leq e + 1$.*

Proof. This follows directly from the non-degeneracy of $d\theta$ as an element in $H^0(X, \wedge^2 \mathcal{F}^\vee \otimes \mathcal{L})$. □

From the contact structure we get the following exact sequence:

$$0 \to \wedge^{p-1}\mathcal{F}^\vee \to \Omega^p_X \otimes \mathcal{L} \to \wedge^p \mathcal{F}^\vee \otimes \mathcal{L} \to 0, \tag{15}$$

In all types except A_n, $\mathcal{F}$ is a homogeneous irreducible bundle, thus the semisemplification of $\hat{\Omega}^p_X \otimes \mathcal{L}$ is given by

$$\mathrm{ss}(\hat{\Omega}^p_X \otimes \mathcal{L}) = \wedge^{p-2}\mathcal{F}^\vee \oplus \wedge^{p-1}\mathcal{F}^\vee \oplus \wedge^{p-1}\mathcal{F}^\vee \otimes \mathcal{L} \oplus \wedge^p \mathcal{F}^\vee \otimes \mathcal{L}.$$

In type A_n the terms of the above decomposition are not irreducible (as we will soon see). Let us give a brief introduction to adjoint varieties of classical groups.

3.1.2. *Type A_n. The point-hyperplane incidence variety.* We have $G \simeq \mathrm{SL}_n$ and $\mathfrak{g}$ is the algebra of traceless matrices $\mathfrak{sl}_{n+1}$. The adjoint variety X is $\mathbb{P}(T_{\mathbb{P}^n})$, and $I_P = \{\alpha_1, \alpha_n\}$ so $P = P_{1,n}$. The variety X is identified with the point-hyperplane incidence variety, namely a smooth hyperplane section of $\mathbb{P}^n \times \check{\mathbb{P}}^n$ and its Picard group is generated by two line bundles $\mathcal{O}_X(1,0) \simeq \mathcal{E}_{\varpi_1}$ and $\mathcal{O}_X(0,1) \simeq \mathcal{E}_{\varpi_n}$ obtained by pull-back from the two projections onto $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$. The line bundle $\mathcal{L}$ of (14) is $\mathcal{L} \simeq \mathcal{O}_X(1,1)$. Finally we have $\mathrm{ss}(\mathcal{F}) = \mathcal{E}_{-\varpi_1+\varpi_2+\varpi_n} \oplus \mathcal{E}_{\varpi_1+\varpi_{n-1}-\varpi_n}$.

3.1.3. *Type B_n and D_n. The orthogonal Grassmannian of lines in odd and even dimension.* Let us consider the group $\mathrm{SO}(m)$ acting on a vector space $\mathbf{k}$ and preserving a symmetric form on $\mathbf{k}^m$. In case B_n we have $m = 2n+1$, and in case D_n we have $m = 2n$. The adjoint variety X is the orthogonal Grassmannian of planes $OG(2,m)$ parametrizing isotropic (with respect to the symmetric form) subspaces $\mathbf{k}^2 \subset \mathbf{k}^m$. The variety X is a subvariety of the Grassmannian $G(2,m)$, and the line bundle $\mathcal{L}$ is the restriction of the Plücker line bundle on $G(2,m)$; it gives the embedding $X \subset \mathbb{P}(\mathfrak{so}_m) = \mathbb{P}(\wedge^2\mathbf{k}^m)$. Let us denote by $\mathcal{U}$ the rank two tautological bundle on $OG(2,m)$ which is the restriction of the tautological bundle on $G(2,m)$, and by $\mathcal{U}^\perp/\mathcal{U}$ the rank $m-4$ bundle whose fiber is given by the orthogonal space to the fiber of $\mathcal{U}$ quotiented by the fiber of $\mathcal{U}$ itself. Then $\mathcal{F} = \mathcal{U}^\vee \otimes \mathcal{U}^\perp/\mathcal{U}$, and in term of weights $\mathcal{F} = \mathcal{E}_{\varpi_1-\varpi_2+\varpi_3}$. In type B_n the quasi-minuscule representation is simply given by $\hat{\mathfrak{g}} = \mathbf{k}^{2n+1} = V_{\varpi_1}$.

3.1.4. *Type C_n. The Veronese embedding.* Fix a non-degenerate skew-symmetric 2-form ω on $\mathbf{k}^{2n}$ and consider the group $G = \mathrm{Sp}(2n)$ of linear automorphisms of $\mathbf{k}^{2n}$ preserving ω. In an appropriate basis, we may identify $S^2\mathbf{k}^{2n} \simeq \mathfrak{sp}_{2n}$, the adjoint representation of $\mathrm{Sp}(2n)$. We have a decomposition of irreducible $\mathrm{Sp}(2n)$-representations $\wedge^2\mathbf{k}^{2n} = \mathbf{k} \oplus \wedge^{\langle 2\rangle}\mathbf{k}^{2n}$ where $\wedge^{\langle 2\rangle}\mathbf{k}^{2n} \simeq V_{\varpi_2} \simeq \hat{\mathfrak{g}}$ is the quasi-minuscule representation.

The adjoint variety is $X = v_2(\mathbb{P}(\mathbf{k}^{2n}))$, the second Veronese embedding of the projective space. In this case, the line bundle appearing in (14) is $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{2n-1}}(2)$ and the dual of (14) reads

$$0 \to \mathcal{O}_{\mathbb{P}^{2n-1}}(-2) \to \Omega_{\mathbb{P}^{2n-1}} \to \mathcal{F}^\vee \to 0,$$

so $\mathcal{F}^\vee \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(1)$ is a null-correlation bundle.

We postpone the description of the exceptional cases and their adjoint varieties to Section 4.4. We just notice that the Betti numbers of these varieties can be found in [CP11], while the exponents of simple Lie groups can be found in [Bou02]. We end this section with a useful lemma.

Lemma 3.2. *Let X be an adjoint variety. Then we have an isomorphism $\hat{\mathcal{T}}_X \otimes \mathcal{L}^\vee \cong \hat{\Omega}_X$.*

Proof. First notice that $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to X. By the Bott-Borel-Weil Theorem and a case by case analysis we deduce that $H^u(X, \mathcal{L}^\vee) = 0$ for any u and any adjoint variety. The dual of the exact sequence defining the contact structure, i.e,

$$0 \to \mathcal{L}^\vee \to \Omega_X \to \mathcal{F}^\vee \to 0$$

yields then, for all u,

$$H^u(X, \mathcal{F} \otimes \mathcal{L}^\vee) = H^u(X, \mathcal{F}^\vee) = H^u(X, \Omega_X).$$

Both the previous short exact sequence defining the contact structure and the one which defines the affine tangent bundle

$$0 \to \mathcal{L}^\vee \to \hat{\mathcal{T}}_X \otimes \mathcal{L}^\vee \to \mathcal{T}_X \otimes \mathcal{L}^\vee \to 0$$

are induced by the element in $H^1(X, \mathcal{F}^\vee) = H^1(X, \Omega_X)$ corresponding to the hyperplane class. Then the inclusion $\mathcal{F}^\vee \cong \mathcal{F} \otimes \mathcal{L}^\vee \to \mathcal{T}_X \otimes \mathcal{L}^\vee$ induces an inclusion $\Omega_X \to \hat{\mathcal{T}}_X \otimes \mathcal{L}^\vee$. We get the following commutative

diagram:

$$\begin{array}{ccccccccc} & & \mathcal{L}^\vee & = & \mathcal{L}^\vee & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \Omega_X & \to & \hat{\mathcal{T}}_X \otimes \mathcal{L}^\vee & \to & \mathcal{O}_X & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathcal{F}^\vee & \to & \mathcal{T}_X \otimes \mathcal{L}^\vee & \to & \mathcal{O}_X & \to & 0 \end{array}$$

The two bottom lines are induced thus once again by the same element in $H^1(X, \mathcal{F}^\vee) = H^1(X, \Omega_X)$, and since the lower line is essentially the contact structure, which is induced by the hyperplane class, the upper line is also induced by the hyperplane class, and it is then the exact sequence defining the affine cotangent bundle. We obtain $\hat{\mathcal{T}}_X \otimes \mathcal{L}^\vee \cong \hat{\Omega}_X$. □

3.2. **The normalisation of adjoint discriminants.** Let D be the dual hypersurface variety of an adjoint variety X. Let us also denote by $D_n \to D$ its normalization. Recall that $m = \dim(X)$ is odd, thus $m = 2e + 1$. Recall that $\mathbf{R} := \mathbf{k}[\mathfrak{g}]^G$ is a polynomial algebra generated by n polynomials $F_1, \dots, F_n$ of degrees $d_1, \dots, d_n$, where n is the rank of G. The exponents $e_1, \dots, e_n$ of G can be defined as $e_i := d_i - 1$ for $i = 1, \dots, n$. Recall that s denotes the number of long simple roots; moreover we order the exponents so that $e_i \leq e_{i+1}$ for $i = 1, \dots, n-1$. Our goal for this section is to prove the following result.

Theorem 3.3. *A sheafified minimal graded free resolution of $\mathcal{O}_{D_n}$ is :*

$$0 \to \bigoplus_{i=1}^{s} \mathcal{O}_{\check{\mathbb{P}}}(-m + e_i - 2) \to \bigoplus_{i=1}^{s} \mathcal{O}_{\check{\mathbb{P}}}(-e_i + 1) \to \mathcal{O}_{D_n} \to 0.$$

We start with the following lemma, which for classical groups is just [Wey03, Exercises 9.3, 9.2, 9.3, 9.4].

Lemma 3.4. *Let X be an adjoint variety and set $m = \dim(X)$. There exists a G-equivariant resolution*

$$0 \to \bigoplus_{p=0}^{m} H^p(\Omega_X^p) \otimes \mathcal{O}_{\check{\mathbb{P}}}(-p-1) \to \bigoplus_{p=0}^{m} H^p(\Omega_X^p) \otimes \mathcal{O}_{\check{\mathbb{P}}}(-p) \to \mathcal{O}_{D_n} \to 0.$$

Proof. We apply Theorem 2.3 to get a resolution of the normalization D_n of D. The terms of the resolution are given by $\mathbf{F}_u^{\mathcal{O}_X}$ for all $u \leq 0$, and these are computed from the cohomology of $\wedge^p(\hat{\mathcal{T}}_X \otimes \mathcal{L}^\vee)$. This computation yields the statement by using Lemma 3.2 and the dual of (13). □

Proof of Theorem 3.3. For all $i \geq 0$, consider the map given by multiplication by the hyperplane class:

$$H^{i-1}(\Omega_X^{i-1}) \to H^i(\Omega_X^i),$$

and denote by K^{i-1} and C^i, respectively, the kernel and cokernel of such map. The non-trivial extension

$$0 \to \Omega_X \to \hat{\Omega}_X \to \mathcal{O}_X \to 0$$

induces a map in cohomology $H^0(\mathcal{O}_X) \to H^1(\Omega_X^1)$ whose image is the hyperplane class corresponding to the embedding $X \subset \mathbb{P}(V)$. Thus, taking duals in (13) we find, for any $p \geq 0$, an exact sequence

$$0 \to \Omega_X^p \to \hat{\Omega}_X^p \to \Omega_X^{p-1} \to 0, \tag{16}$$

whose induced map in cohomology $H^{p-1}(\Omega_X^{p-1}) \to H^p(\Omega_X^p)$ are just given by the multiplication by the hyperplane class. By the Lefschetz' Hyperplane Theorem, we deduce that for all $p \in \mathbb{Z}$, the maps $H^{p-1}(\Omega_X^{p-1}) \otimes \mathcal{O}_{\check{\mathbb{P}}}(-p) \to H^p(\Omega_X^p) \otimes \mathcal{O}_{\check{\mathbb{P}}}(-p)$ appearing in $\mathbf{F}_\bullet^{\mathcal{O}_X}$ have maximal rank.

Therefore we obtain the following minimal resolution:

$$0 \to \bigoplus_{q=e+1}^{m} K^q \otimes \mathcal{O}_{\check{\mathbb{P}}}(-q-1) \to \bigoplus_{q=0}^{e} C^q \otimes \mathcal{O}_{\check{\mathbb{P}}}(-q) \to \mathcal{O}_{D_n} \to 0.$$

The result then follows by noticing, through an explicit comparison of Betti numbers for adjoint varieties and exponents for each simple Lie group, that C^i (and K^{m-i}) is a direct sum of u_i trivial G-representations, where u_i is the cardinality of $\{j \mid \deg(f_j) = i + 2\}$. □

Corollary 3.5. *The adjoint discriminant D is normal if and only if G is of type C_n or G_2.*

Proof. The adjoint varieties in type C_n and G_2 are the only two adjoint varieties whose rational cohomology is completely generated by the hyperplane class. This is reflected in the fact that for the corresponding Lie groups $s = 1$; by Theorem 3.3 and Theorem 2.2 we deduce that the duals of these two adjoint varieties are the only ones which are normal. □

4. Jacobian ideals of adjoint discriminants

In this section we want to compute a locally free resolution of the Jacobian ideal and the tangent logarithmic sheaf of the dual varieties D of adjoint varieties $X \subset \mathbb{P}(\mathfrak{g})$. In order to do so, we will use the method described in §2 to compute a sheafified minimal graded free resolution of $\rho_*(\pi^*\mathcal{L})$. Then we will identify $\rho_*(\pi^*\mathcal{L})$ with the Jacobian ideal sheaf of D. The final goal of this section is to find a G-equivariant locally free resolution of the module of logarithmic differentials. For simply laced groups, that represents the first step to prove Theorem 2. For the non-simply laced case, the resolution we obtain is the one appearing in Theorem 3.

In our setting, one may work indifferently with graded modules of finite type on $\mathbf{U} = \mathbf{k}[\mathfrak{g}^\vee]$ or coherent sheaves on $\mathbb{P}(\mathfrak{g})$. We use the customary identification of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ via the Killing form, hence, to ease the notation, most of the times we will just use $\mathfrak{g}$.

The following result ensures that we can apply Weyman's method in order to compute a locally free resolution of $\rho_*(\pi^*(\mathcal{L}^{\otimes i}))$, for any $i > 0$ and for any adjoint variety X:

Proposition 4.1. *Let X be an adjoint variety and let $i > 0$ be a positive integer. Then Weyman's complex $\mathbf{F}_\bullet^{\mathcal{L}^{\otimes i}}$ is a sheafified graded free resolution of $\rho_*(\pi^*(\mathcal{L}^{\otimes i}))$.*

Proof. By Theorem [Wey03, Theorem 5.1.2], we need to check that $\mathbf{F}_u^{\mathcal{L}^{\otimes i}} = 0$ for any $u > 0$. By definition of this complex and using Lemma 3.2, we get

$$\mathbf{F}_u^{\mathcal{L}^{\otimes i}} \simeq \bigoplus_{l-p=u} H^l(X, \wedge^p \hat{\Omega}_X \otimes \mathcal{L}^{\otimes i}) \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(-p).$$

For any $p \geq 0$ and $i \in \mathbb{Z}$, we tensor (16) by $\mathcal{L}^{\otimes i}$ to obtain an exact sequence

$$0 \to \Omega_X^p \otimes \mathcal{L}^{\otimes i} \to \hat{\Omega}_X^p \otimes \mathcal{L}^{\otimes i} \to \Omega_X^{p-1} \otimes \mathcal{L}^{\otimes i} \to 0.$$

Now, by [Bri09, Theorem 3.18] we have that $H^l(X, \Omega_X^p \otimes \mathcal{L}^{\otimes i}) = 0$ as soon as $i > 0$ and $l > p$. From the above sequence we deduce $\mathbf{F}_u^{\mathcal{L}^{\otimes i}} = 0$ for $u > 0$. The result follows. □

Now we turn to the computation of Weyman's complex $\mathbf{F}_\bullet^{\mathcal{L}}$. We need to perform a certain number of computations on a case-by-case basis, although the final result admits a uniform formulation.

4.1. **Type C_n: Warming up.** In this case $X = v_2(\mathbb{P}^{2n-1}) \subset \mathbb{P}(S^2\mathbf{k}^{2n}) = \mathbb{P}$ is the second Veronese embedding of the projective space and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{2n-1}}(2)$. Recall that $\hat{\mathfrak{sp}}_{2n} = V_{\varpi_2}$.

Proposition 4.2. *Let $X = v_2(\mathbb{P}^{2n-1})$ be the adjoint variety of type C_n and $D = X^\vee$ the adjoint discriminant. We have an isomorphism $\rho_*(\pi^*(\mathcal{L})) \simeq \mathcal{J}_D(d-1)$ and an equivariant sheafified minimal graded free resolution:*

$$0 \to \wedge^2\mathbf{k}^{2n} \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{sp}_{2n})}(-2) \to (\mathbf{k}^{2n} \otimes \mathbf{k}^{2n}) \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{sp}_{2n})}(-1) \to S^2\mathbf{k}^{2n} \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{sp}_{2n})} \to \mathcal{J}_D(d-1) \to 0.$$

Equivalently, we have an equivariant graded free resolution:

$$0 \to (\mathbf{k} \oplus \hat{\mathfrak{sp}}_{2n}) \otimes \mathbf{U}(-2) \to (\mathbf{k} \oplus \hat{\mathfrak{sp}}_{2n} \oplus \mathfrak{sp}_{2n}) \otimes \mathbf{U}(-1) \to \mathrm{Der}_{\mathbf{U}}(-\log(D)) \to 0.$$

Proof. Let us denote by $\mathcal{Q}$ the quotient tautological bundle of X, namely $\mathcal{Q} = \mathcal{T}_{\mathbb{P}^{2n-1}} \otimes \mathcal{O}_{\mathbb{P}^{2n-1}}(-1)$. Then, for all $p \geq 0$ we have a short exact sequence

$$0 \to \wedge^p\mathcal{Q}^\vee(2-p) \to \hat{\Omega}_X^p \otimes \mathcal{L} \to \wedge^{p-1}\mathcal{Q}^\vee(3-p) \to 0.$$

The only non-vanishing cohomology groups of $\wedge^p\mathcal{Q}^\vee(2-p)$ are given by $H^0(\mathbb{P}^{2n-1}, \mathcal{O}(2)) \cong S^2\mathbf{k}^{2n} \cong V$, and $H^0(\mathbb{P}^{2n-1}, \mathcal{Q}^\vee(1)) \cong \wedge^2\mathbf{k}^{2n}$. Thus the only non-vanishing cohomology groups of $\hat{\Omega}_X^p \otimes \mathcal{L}$ are:

$$H^0(\mathbb{P}^{2n-1}, \mathcal{L}) \cong S^2\mathbf{k}^{2n}, \qquad H^0(\mathbb{P}^{2n-1}, \hat{\Omega}_X^1 \otimes \mathcal{L}) \cong \mathbf{k}^{2n} \otimes \mathbf{k}^{2n}, \qquad H^0(\mathbb{P}^{2n-1}, \hat{\Omega}_X^2 \otimes \mathcal{L}) \cong \wedge^2\mathbf{k}^{2n}.$$

The result now follows by noticing that $S^2\mathbf{k}^{2n} \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{sp}_{2n})} \cong \hat{\mathcal{T}}_{\mathbb{P}(\mathfrak{sp}_{2n})}(-1)$. □

Notice that the previous result was already obtained differently in [FM22] as the resolution of the tangent logarithmic sheaf of the discriminant of quadratic forms.

4.2. **Type A_n.** In this case X is the flag variety $F(1,n,n+1)$ parametrizing flags $\mathbf{k}\subset\mathbf{k}^n$ inside a fixed $n+1$ dimensional vector space. It can also be seen as the projectivized of the cotangent bundle of the projective space $\mathbb{P}^n$. The group G is in this case $\mathrm{SL}(n+1)$ and the Weyl group is the group of permutations of $n+1$ elements. The bundle $\mathcal{L}$ is the ample line bundle $\mathcal{O}_X(1,1)$ defining the embedding $X\subset\mathbb{P}(\mathfrak{sl}_{n+1})$, which we also denote by $\mathcal{O}_X(1)$ to ease the notation. Following [Küc95], the weight associated to an irreducible homogeneous bundle over X is given by a sequence of integers

$$\lambda:=[\lambda_1;\lambda_2,\dots,\lambda_n;\lambda_{n+1}]=\sum_i(\lambda_i-\lambda_{i+1})\varpi_i$$

such that $\lambda_2\geq\cdots\geq\lambda_n$. For instance $\mathcal{O}_X(1,0)$ is associated with the weight $[1;0,\dots,0;0]=[1;0^{n-1};0]$, while $\mathcal{O}_X(1,1)$ is associated with $[1;0\dots,0;-1]$. Notice that in this notation λ and $\lambda+c:=[\lambda_1+c;\lambda_2+c,\dots,\lambda_n+c;\lambda_{n+1}+c]$ for any $c\in\mathbb{Z}$, are associated to the same irreducible homogeneous bundle.

Remark 4.3 (Bott-Borel-Weil Theorem for $\mathrm{SL}(n+1)$)**.** The bundle $\mathcal{E}_\lambda$ is globally generated if $\lambda_1\geq\lambda_2$ and $\lambda_n\geq\lambda_{n+1}$; in this case by Bott-Borel-Weil Theorem the space of sections $H^0(X,\mathcal{E}_\lambda)$ is isomorphic to the $\mathrm{SL}(n+1)$-representation given by the plethysm $S_\lambda\mathbf{k}^{n+1}$. More generally, let $\rho:=[n+1,n,\dots,2,1]$ and consider the two following situations: either the integers in $\lambda+\rho$ are all pairwise distinct, either there are two repeated integers. In the latter case, for all $i\geq 0$ we have $H^i(X,\mathcal{E}_\lambda)=0$ by Bott-Borel-Weil theorem. In the former case, denote by w the permutation of $n+1$ elements such that $w(\lambda+\rho)$ is a strictly decreasing sequence of integers. Then, again by Bott-Borel-Weil theorem, for all $i\neq l(w)$ we have $H^i(X,\mathcal{E}_\lambda)=0$, where $l(w)$ is the *length* of the permutation w (i.e. the minimal number of simple permutations needed to obtain w); moreover $H^{l(w)}(X,\mathcal{E}_\lambda)=S_{w(\lambda+\rho)-\rho}\mathbf{k}^{n+1}$.

Notation 4.4. We will fix a $n+1$ dimensional vector space $A\cong\mathbf{k}^{n+1}$. We will denote by $\mathcal{U}_1$, $\mathcal{U}_n$ the tautological bundles of rank 1, n on the flag variety $F(1,n,A)$ parametrizing flags in the vector space A.

Notice that $\hat{\mathcal{T}}_X^\vee\otimes\mathcal{L}\cong\hat{\Omega}_X^\vee=(\mathfrak{sl}(A)\otimes\mathcal{O}_X/\mathrm{N}_{X/\mathbb{P}}^\vee)^\vee$. Then by the geometric method the spectral sequence $H^j(X,\wedge^i\hat{\Omega}_X\otimes\mathcal{L})$ defines a locally free resolution $\{\oplus_i\mathbf{F}^{\mathcal{L}}_{j-i}(-i)\}_{j-i}$ of $\rho_*\pi^*\mathcal{L}$ as soon as $\mathbf{F}^{\mathcal{L}}_k=0$ for $k>0$. We will compute such a resolution by computing the cohomology of the graded pieces of $\hat{\Omega}_X$; by doing so, we will obtain a resolution which a priori is not minimal, but still resolves $\rho_*\pi^*\mathcal{L}$. The semisemplification of $\hat{\mathcal{T}}_X\otimes\mathcal{L}^\vee\cong\hat{\Omega}_X$ is given by

$$\mathrm{ss}(\hat{\Omega}_X)=\mathcal{O}_X\oplus\mathcal{U}_1\otimes(A/\mathcal{U}_n)^\vee\oplus\mathcal{U}_1\otimes(\mathcal{U}_n/\mathcal{U}_1)^\vee\oplus(\mathcal{U}_n/\mathcal{U}_1)\otimes(A/\mathcal{U}_n)^\vee.$$

In the notation of weights, one can write:

$$\mathrm{ss}(\hat{\Omega}_X)=[0;0,\dots,0;0]+[-1;0,\dots,0;1]+[-1;1,0,\dots,0;0]+[0;0,\dots,0,-1;1].$$

The first two terms are line bundles, so they represent the easiest part to deal with. The last two terms have rank $n-1$ and, since we need to compute the cohomology of $\wedge^i\hat{\Omega}_X$, we give the formula to compute the exterior power of their sum:

$$\begin{aligned}\wedge^i(\mathrm{ss}(\hat{\Omega}_X))=&\wedge^i([-1;1,0,\dots,0;0]+[0;0,\dots,0,-1;1])\oplus\\&\oplus\wedge^{i-1}([-1;1,0,\dots,0;0]+[0;0,\dots,0,-1;1])\otimes\mathcal{O}_X\oplus\\&\oplus\wedge^{i-1}([-1;1,0,\dots,0;0]+[0;0,\dots,0,-1;1])\otimes\mathcal{U}_1\otimes(A/\mathcal{U}_n)^\vee\oplus\\&\oplus\wedge^{i-2}([-1;1,0,\dots,0;0]+[0;0,\dots,0,-1;1])\otimes\mathcal{O}_X\otimes\mathcal{U}_1\otimes(A/\mathcal{U}_n)^\vee.\end{aligned}$$

In this decomposition the first factor is equal to $\wedge^i\mathcal{F}^\vee$, the second factor is equal to $\wedge^{i-1}\mathcal{F}^\vee$, the third factor is equal to $\wedge^{i-1}\mathcal{F}^\vee\otimes\mathcal{L}^\vee$ and the fourth factor is equal to $\wedge^{i-2}\mathcal{F}^\vee\otimes\mathcal{L}^\vee$. By tensoring by $\mathcal{L}$ we obtain the semisemplification of $\wedge^i\hat{\Omega}_X\otimes\mathcal{L}$.

Lemma 4.5. *In the weight notation we have: for $0\leq i\leq 2n-2$,*

$$\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}=\bigoplus_{\substack{p,q\leq n-1,p+q=i,\\ \max(0,q-p)\leq j\leq\min(q,n-p-1)}}[-q+1;1^j,0^{n-p+q-2j-1},(-1)^{p-q+j};p-1]$$

and for $1 \leq i \leq 2n-1$,

$$\wedge^{i-1}\mathcal{F}^\vee = \bigoplus_{\substack{p,q\leq n-1, p+q=i-1,\\ \max(0,q-p)\leq j\leq \min(q,n-p-1)}} [-q; 1^j, 0^{n-p+q-2j-1}, (-1)^{p-q+j}; p].$$

Proof. In order to obtain the above formulas one needs to combine the formula for computing the exterior power of a direct sum of vector spaces $\wedge^i(U\oplus W) = \bigoplus_{p+q=i} \wedge^p U \otimes \wedge^q W$, and the Littlewood-Richardson rule for tensoring two representations $A_\mu \otimes A_\eta = \bigoplus_\lambda c^\lambda_{\mu\eta} A_\lambda$ (where A_μ, A_η and A_λ are $\mathrm{SL}(A)$-representations of highest weight respectively μ, η and λ, and $c^\lambda_{\mu\eta}$ are the so-called Littlewood-Richardson coefficients, see for instance [Ful97]). One obtains

$$\wedge^i([-1;1,0,\dots,0;0]+[0;0,\dots,0,-1;1]) = \bigoplus_{\substack{p,q\leq n-1, p+q=i,\\ \max(0,q-p)\leq j\leq \min(q,n-p-1)}} [-q; 1^j, 0^{n-p+q-2j-1}, (-1)^{p-q+j}; p],$$

and the result follows. □

Proposition 4.6. *All cohomology groups of* $\wedge^i\mathcal{F}^\vee$ *and* $\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$ *vanish except for the following ones:*

- $H^i(X, \wedge^i\mathcal{F}^\vee) \cong H^i(\Omega^i_X)$ *for* $i \leq e$*, where the isomorphism is induced by the surjection* $\Omega^i_X \to \wedge^i\mathcal{F}^\vee$*;*
- $H^{i-2}(X, \wedge^i\mathcal{F}^\vee\otimes\mathcal{L}) \cong H^{i-2}(\Omega^{i-2}_X)$ *for* $2 \leq i \leq e-1$*, and* $H^0(X, \wedge^0\mathcal{F}^\vee\otimes\mathcal{L}) \cong \mathfrak{sl}_{n+1}$*; moreover the terms* $H^{i-2}(\Omega^{i-2}_X)$ *are the images of the cohomology maps induced by the embeddings* $d\theta\wedge(\bullet) : \wedge^{i-2}\mathcal{F}^\vee \to \wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$*.*

Proof. Let us compute the cohomology of the bundles by using their weight decomposition.

$\wedge^{i-1}\mathcal{F}^\vee$: Let us consider one factor in $\wedge^{i-1}\mathcal{F}^\vee$ equal to $[-q; 1^j, 0^{n-p+q-2j-1}, (-1)^{p-q+j}; p]$. It has cohomology if and only if no integer in the sequence

$$[-q; 1^j, 0^{n-p+q-2j-1}, (-1)^{p-q+j}; p] + [n+1; n, \dots, 2; 1]$$

repeats more than once. Let us look for integers p, q, i, j for which the cohomology does not vanish. Either $q = j$ or $q = n-p+q$ (if $q \geq n-p+q-1$ then in fact $q \geq n+1$, which is impossible). Similarly either $p = p-q+j$ or $p = n-j$. But $q = n-p+q$ is impossible because $p \leq n-1$, which also excludes $p = n-j$. Thus we remain with the only possibility $q = j$ (which implies $p = p-q+j$); for these terms the cohomology is concentrated in degree $j+p-q+j = i-1$. Thus we obtain $H^{i-1}(\wedge^{i-1}\mathcal{F}^\vee) = \mathbf{k}^i$ if and only if $1 \leq i \leq n$ since $q \leq n-p-1$ and q runs from 0 to $i-1$.

$\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$: Let us consider one factor in $\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$ equal to $[-q+1; 1^j, 0^{n-p+q-2j-1}, (-1)^{p-q+j}; p-1]$. It has cohomology if and only if no integer in the sequence

$$[-q+1; 1^j, 0^{n-p+q-2j-1}, (-1)^{p-q+j}; p-1] + [n+1; n, \dots, 2; 1]$$

repeats more than once. Let us look for integers p, q, i, j for which the cohomology does not vanish. One possibility is of course given by $p = q = i = j = 0$, which gives $H^0(\mathcal{L}) = \mathfrak{sl}(A)$. If $q \neq 0$ either $q = j+1$ or $q \geq n-p+q+1$. The latter is impossible since $p \leq n-1$. Similarly if $p \neq 0$ either $p = p-q+j+1$ or $p \geq n-j+1$. The latter is impossible since $j \leq n-p-1$. Thus we remain with the only possibility $q = j+1$ (which implies $p = p-q+j+1$); for these terms the cohomology is concentrated in degree $j+p-q+j = i-2$. Thus we obtain $H^{i-2}(\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}) = \mathbf{k}^{i-1}$ if and only if $2 \leq i \leq n$ since $q \geq 1$, $p \geq 1$, $q-1 = j \leq n-p-1$ and q runs from 1 to $i-1$.

By Serre duality $H^i(X, \wedge^i\mathcal{F}^\vee\otimes\mathcal{L}^\vee)^\vee \cong H^{f+1-i}(X, \wedge^i\mathcal{F}\otimes\mathcal{L}\otimes K_X) \cong H^{f+1-i}(X, \wedge^{f-i}\mathcal{F}^\vee)$. Since $\mathrm{ss}(\Omega^i_X) = \wedge^{i-1}\mathcal{F}^\vee\otimes\mathcal{L}^\vee \oplus \wedge^i\mathcal{F}^\vee$ and $\wedge^{i-1}\mathcal{F}^\vee\otimes\mathcal{L}^\vee$ is acyclic for $i \leq e$ we deduce that the cohomology groups of $\wedge^i\mathcal{F}^\vee$ for $i \leq e$ are induced by the surjection $\Omega^i_X \to \wedge^i\mathcal{F}^\vee$.

In order to show that the terms $H^{i-2}(\Omega^{i-2}_X)$ in the cohomology of $\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$ are induced by $d\theta\wedge(\bullet)$ notice that, by the explicit computations above, for each irreducible bundle in $\mathrm{ss}(\wedge^{i-2}\mathcal{F}^\vee)$, this bundle appears only once in $\mathrm{ss}(\wedge^i\mathcal{F}^\vee\otimes\mathcal{L})$; moreover its cohomology (when it does not vanish) appears both in $H^{i-2}(X, \wedge^{i-2}\mathcal{F}^\vee)$ and $H^{i-2}(X, \wedge^i\mathcal{F}^\vee\otimes\mathcal{L})$. The claim follows. □

4.3. **Type B_n and D_n: orthogonal Grassmannians of planes.** The notation that follows can be found in [Ben18]. Let us consider the group $\mathrm{SO}(m)$, which is of type B_n when $m = 2n+1$ and of type D_n when $m = 2n$. From now on, to uniformize notation we will define $h = 1/2$ (respectively $h = 0$) in type B_n (resp. D_n), so that $m = 2(n+h)$. The adjoint variety X is the orthogonal Grassmannian of planes $OG(2,m)$ parametrizing isotropic subspaces $\mathbf{k}^2 \subset \mathbf{k}^m$ inside a fixed m dimensional vector space endowed with a symmetric 2-form. The bundle $\mathcal{L}$ is the ample line bundle defining the embedding $X \subset \mathbb{P}(\mathfrak{so}_m) = \mathbb{P}(\wedge^2\mathbf{k}^m)$, and we will also denote $\mathcal{L}$ by $\mathcal{O}_X(1)$ to ease notation. Following [Ben18], the weight associated to an irreducible homogeneous bundle over X is given by a sequence made only of integers or half-integers

$$\lambda := [\lambda_1, \lambda_2; \lambda_3, \dots, \lambda_n]$$

such that $\lambda_1 \geq \lambda_2$, $\lambda_3 \geq \cdots \geq \lambda_n$ and: $\lambda_n \geq 0$ in type B_n and $\lambda_{n-1} + \lambda_n \geq 0$ in type D_n. In terms of fundamental weights $\lambda = \sum_{i \leq n-1}(\lambda_i - \lambda_{i-1})\varpi_i + 2\lambda_n\varpi_n$ in type B_n and $\lambda = \sum_{i\leq n-1}(\lambda_i - \lambda_{i-1})\varpi_i + (\lambda_{n-1}+\lambda_n)\varpi_n$ in type D_n.

Remark 4.7 (Bott-Borel-Weil Theorem for $\mathrm{SO}(m)$)**.** The bundle $\mathcal{E}_\lambda$ is globally generated if $\lambda_2 \geq \lambda_3$; in this case by Bott-Borel-Weil Theorem the space of sections $H^0(X, \mathcal{E}_\lambda)$ is isomorphic to the $\mathrm{SO}(m)$-representation V_λ with highest weight λ. More generally, let $\rho := [n-1+h, n-2+h, \dots, 1+h, h]$ and consider the two following situations: either the integers (or half-integers) in $\lambda + \rho$ union $-\lambda - \rho$ are all pairwise distinct except for 0 which can appear twice; or there are two non-zero integers which are the same. In the latter case, for all $i \geq 0$ we have $H^i(X, \mathcal{E}_\lambda) = 0$ by Bott-Borel-Weil theorem. In the former case, consider the Weyl group W of $\mathrm{SO}(m)$ which is a semidirect product of the permutations of n elements of λ and $\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the reflection τ and τ acts by exchanging λ_n into $-\lambda_n$ (respectively exchanging λ_{n-1} into $-\lambda_n$ and λ_n into $-\lambda_{n-1}$) in type B_n (resp. D_n). Denote by $w \in W$ the element such that $w(\lambda+\rho)$ is a strictly decreasing sequence of integers (or half integers) with $\lambda_n \geq h$ in type B_n and $\lambda_{n-1} + \lambda_n \geq 1$ in type D_n. Then, again by Bott-Borel-Weil theorem, for all $i \neq l(w)$ we have $H^i(X, \mathcal{E}_\lambda) = 0$, where $l(w)$ is the *length* of the element w (i.e. the minimal number of simple reflections needed to obtain w); moreover $H^{l(w)}(X, \mathcal{E}_\lambda) = V_{w(\lambda+\rho)-\rho}$.

Notation 4.8. We will denote by $\mathcal{U}$, $\mathcal{U}^\perp/\mathcal{U}$ the tautological bundles of rank 2, $m-4$ on $OG(2,m)$. Here $\mathcal{U}^\perp/\mathcal{U}$ is a subbundle of the quotient tautological bundle whose vectors are orthogonal to elements in $\mathcal{U}$. In the weight notation $\mathcal{U} = [0,-1;0,\dots,0]$, $\mathcal{L} = \det(\mathcal{U}^\vee) = [1,1;0,\dots,0]$ and $\mathcal{U}^\perp/\mathcal{U} = [0,0;1,0,\dots,0]$. Notice that $\mathcal{U}^\perp/\mathcal{U}$ is auto-dual.

The cotangent bundle is given by the extension $0 \to \mathcal{O}_X(-1) \to \Omega_X \to \mathcal{U}\otimes(\mathcal{U}^\perp/\mathcal{U}) \to 0$, from which we see that $\mathcal{F}^\vee = \mathcal{U}\otimes(\mathcal{U}^\perp/\mathcal{U}) = [0,-1;1,0,\dots,0]$. We want to compute the cohomology of $\wedge^p\mathcal{F}^\vee$ and $\wedge^p\mathcal{F}^\vee\otimes\mathcal{L}$, but in order to do so we need to be able to express these bundles as direct sums of $\mathrm{SO}(m)$-homogeneous irreducible bundles. As before, let us denote by $S_\bullet$ the plethysm with weight $\bullet$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ is a partition, we will denote by λ' the *conjugate partition* of λ, i.e. the partition whose Young diagram is obtained by a reflection along the main diagonal of the Young diagram of λ.

Lemma 4.9. *We have the following decomposition for $0 \leq p \leq 2m-8$:*

$$\wedge^p\mathcal{F}^\vee = \bigoplus_{0\leq i\leq j, i+j=p} S_{j,i}\mathcal{U}\otimes S_{2^i,1^{j-i}}\mathcal{U}^\perp/\mathcal{U}.$$

Proof. This is a direct application of the Littlewood-Richardson formula for the exterior power of a tensor product which, if λ' is the conjugate partition of λ and $|\lambda| = \sum_i \lambda_i$, reads as follows

$$\wedge^p(U\otimes W) = \bigoplus_{|\lambda|=p} S_\lambda U \otimes S_{\lambda'}W.$$

□

It is easy to see that $S_{j,i}\mathcal{U} = [-i,-j;0,\dots,0]$ and, in order to do our computation, we thus need to express $S_{2^i,1^{j-i}}\mathcal{U}^\perp/\mathcal{U}$ in terms of weights. Let us come back to $\mathcal{U}^\perp/\mathcal{U}$. This is an irreducible bundle, and as such it corresponds to a certain irreducible representation of the Levi factor of the parabolic subgroup $P_2 \subset G$ defining $X = G/P_2$; this Levi factor is isomorphic to $\mathrm{SL}(3)\times\mathrm{SO}(m-4)$. Moreover, the representation is trivial as a $\mathrm{SL}(3)$-representation, and it is the standard $\mathrm{SO}(m-4)$-representation corresponding to the weight $[1,0,\dots,0]$ (notice that in this case there are $n-2$ entries in the sequence).

As a consequence we want to understand the decomposition in direct sums of irreducible $\mathrm{SO}(m-4)$-representations of the plethysm $S_{2^i,1^{j-i}}[1,0,\dots,0]$. We will do this in two steps. Firstly we will use branching rules from $\mathrm{GL}(m-4)$ to $\mathrm{O}(m-4)$ to decompose $S_{2^i,1^{j-i}}[1,0,\dots,0]$ in $\mathrm{O}(m-4)$-representations, then we will use branching rules from $\mathrm{O}(m-4)$ to $\mathrm{SO}(m-4)$ to obtain the decomposition in $\mathrm{SO}(m-4)$-representations. We will follow [JK21] for notations and results. Finally, we will put everything together and we will compute the cohomology of both $\wedge^p\mathcal{F}^\vee$ and $\wedge^p\mathcal{F}^\vee\otimes\mathcal{L}$.

We will denote representations with respect to $\mathrm{O}(m-4)$ (respectively $\mathrm{SO}(m-4)$, $\mathrm{GL}(m-4)$) with highest weight λ^{O} (resp. λ^{SO}, λ^{GL}) by V^{O}_λ (resp. V^{SO}_λ, V^{GL}_λ). Having already discussed the parametrization of weights for $\mathrm{GL}(m-4)$ and $\mathrm{SO}(m-4)$; let us explicit the corresponding one for $\mathrm{O}(m-4)$-weights: $\mathrm{O}(m-4)$-representations are in 1 : 1-correspondence with decreasing sequences of non-negative integers $\lambda=[\lambda_1,\dots,\lambda_{m-4}]$ such that $\lambda'_1+\lambda'_2\le n-2$.

Lemma 4.10. *We have the following decomposition:*

$$S_{2^i,1^{j-i}}[1,0,\dots,0]^{\mathrm{GL}}=\bigoplus_{0\le\delta\le i,p-\delta\le m-4}[2^{i-\delta},1^{j-i},0^{m-j+\delta-4}]^{\mathrm{O}},$$

where $0\le i\le j$, $i+j=p$, $0\le p\le 2m-8$.

Proof. This is just an application of the branching rule in [JK21, Theorem 4.10], so we will adopt the same notations. It will be sufficient to explain how to recover the statement of this lemma from the cited theorem, and in order to do so we will go back and forth from [JK21] to unravel this branching result.

Indeed, the results in the aforementioned paper allow to compute the Littlewood-Richardson type coefficients $\bar{c}^\lambda_{\eta\mu}$ appearing in the decomposition:

$$S_\lambda[1,0,\dots,0]^{\mathrm{GL}}=\bigoplus_{\mu\in\mathcal{P}^{\mathrm{O}}}\Big(\sum_{\eta\in\mathcal{P}^{(2)}}\bar{c}^\lambda_{\eta\mu}\Big)V^{\mathrm{O}}_\mu.$$

Here we have denoted by $\mathcal{P}^{\mathrm{O}}$ the set of partitions corresponding to weights of $\mathrm{O}(m-4)$ (i.e. $\mu'_1+\mu'_2\le m-4$) and by $\mathcal{P}^{(2)}$ the set of even partitions (or decreasing sequences), i.e. $\eta_u\in 2\mathbb{Z}$ for any u. From now on $\lambda=[2^i,1^{j-i}]$ as in the statement. Let us denote by $LR^\lambda_{\eta\mu}$ the set of Littlewood-Richardson tableaux of shape λ/η with content μ. Then $\bar{c}^\lambda_{\eta\mu}$ is the cardinality of the subset $\overline{LR}^{\lambda'}_{\eta'\mu'}\subset LR^{\lambda'}_{\eta'\mu'}$ defined in [JK21, Section 4.1], where the "prime" of a partition indicates the conjugate partition; so for instance $\lambda'=[i,j,0,\dots,0]$. Let $LR^{\lambda'}_{\eta'\mu'}\neq\emptyset$, then $\eta'\subset\lambda'$ implies that $\eta'=[\delta,\delta,0,\dots,0]$ where $0\le\delta\le i$, and $\mu'=[j-\delta,i-\delta,0,\dots,0]$ (from these we can recover η and μ since the "prime" operation is an involution). Of course, we need to impose that $\mu'_1+\mu'_2=j+i-2\delta\le m-4$, but we will see that this condition is unnecessary. If we denote by $c^{\lambda'}_{\eta'\mu'}$ the cardinality of $LR^{\lambda'}_{\eta'\mu'}$, by the (classical) Littlewood-Richardson rule $c^{\lambda'}_{\eta'\mu'}=1$ if and only if $\eta'=[\delta,\delta,0,\dots,0]$, $0\le\delta\le i$ and $\mu'=[j-\delta,i-\delta,0,\dots,0]$, and $c^{\lambda'}_{\eta'\mu'}=0$ otherwise.

The statement of the lemma will follow if we are able to show that $\overline{LR}^{\lambda'}_{\eta'\mu'}=LR^{\lambda'}_{\eta'\mu'}\neq\emptyset$ if and only if $p-\delta\le m-4$. In the notation of [JK21], let $\eta^{rev}:=[0,\dots,0,2,\dots,2]$ be the reversed sequence of the sequence η. Let us suppose that $S\in LR^{\lambda'}_{\eta'\mu'}\neq\emptyset$, then S is a Young tableaux made of two rows of respectively $j-\delta$ blocks and $i-\delta$ blocks; the numbers in the first row are all equal to one and the numbers in the second row are all equal to two. Thus, always in the notation of the same paper, $a=j-\delta$, $b=i-\delta$, $(s_1,\dots,s_a)=(1,\dots,1)$ and $(t_1,\dots,t_b)=(2,\dots,2)$. Then one defines $r:=m-4-j+\delta$ if $m-4<2j-2\delta$ or $r:=j-\delta$ if $m-4\ge 2j-2\delta$, and $m_1,\dots,m_{j-\delta}$ as follows:

$$m_\iota:=\max\{k\mid\eta^{rev}_k\in X_\iota,\eta^{rev}_k=0\},$$

where X_ι is defined as:

$$X_\iota=\begin{cases}\{\eta^{rev}_\iota,\dots,\eta^{rev}_{2\iota-1}\}\setminus\{\eta^{rev}_{m_{\iota+1}},\dots,\eta^{rev}_{m_p}\} & \text{if } 1\le\iota\le r,\\ \{\eta^{rev}_\iota,\dots,\eta^{rev}_{n-p+\iota}\}\setminus\{\eta^{rev}_{m_{\iota+1}},\dots,\eta^{rev}_{m_p}\} & \text{if } r<\iota\le p.\end{cases}$$

With such definition, one obtains: $m_{j-\delta}=m-4-\delta$, $m_{j-\delta-1}=m-4-\delta-1$, …, $m_1=m-4-j+1$. Set f_u to be the u-th smallest integer in $\{u+1,u+2,\dots,m-4\}\setminus\{m_{u+1},\dots,m_{j-\delta}\}$. Then by [JK21, Theorem 4.10]

$$S\in\overline{LR}^{\lambda'}_{\eta'\mu'}\Leftrightarrow 2=t_u>\eta^{rev}_{f_u}\text{ for }u=1,\dots,b=i-\delta.$$

Since $\eta_k^{rev} = 0$ for $k \leq m-4-\delta$ and $\eta_k^{rev} = 2$ for $k \geq m-3-\delta$, and since the sequence $m_1 - 1, m_2 - 2, \ldots, m_{j-\delta} - j + \delta$ is constant, the above condition is equivalent to

$$f_{i-\delta} < m-4-\delta+1 \Leftrightarrow 2i-2\delta+j-i < m-4-\delta+1 \Leftrightarrow p-\delta \leq m-4.$$

□

Lemma 4.11. *We have the following decomposition:*

$$S_{2^i,1^{j-i}}[1,0,\ldots,0]^{\mathrm{GL}} = \bigoplus_{0\leq\delta\leq i, p-\delta\leq m-4} (R \oplus S \oplus T \oplus U),$$

where $0 \leq i \leq j$, $i+j=p$, $0 \leq p \leq 2m-8$ *and:*

- $R = \bigoplus_{j-\delta<n+h-2}[2^{i-\delta}, 1^{j-i}, 0^{n-j+\delta-2}]^{\mathrm{SO}}$;
- $S = \bigoplus_{j-\delta>n+h-2}[2^{i-\delta}, 1^{m-4-j-i+2\delta}, 0^{j-\delta-n+2}]^{\mathrm{SO}}$;
- $T = \bigoplus_{j-\delta=n+h-2, i<j}([2^{i-\delta}, 1, \ldots, 1, 1]^{\mathrm{SO}} \oplus [2^{i-\delta}, 1, \ldots, 1, -1]^{\mathrm{SO}})$;
- $U = \bigoplus_{j-\delta=n+h-2, i=j}([2, \ldots, 2, 2]^{\mathrm{SO}} \oplus [2, \ldots, 2, -2]^{\mathrm{SO}})$;

Remark 4.12. The terms T and U only appear when $h = 0$, i.e. in type D_n.

Proof. This is a direct application of the branching rules from $\mathrm{O}(m-4)$ to $\mathrm{SO}(m-4)$ (see [HTW05]). Indeed, these branching rules imply that, if $\mu = [2^{i-\delta}, 1^{j-i}, 0^{m-j+\delta-4}]^{\mathrm{O}} \in \mathcal{P}^{\mathrm{O}}$ with $p-\delta \leq m-4$ then:

- if $2j-2\delta < m-4$ then $V_\mu^{\mathrm{O}} \cong V_\mu^{\mathrm{SO}}$;
- if $2j-2\delta > m-4$ then $V_\mu^{\mathrm{O}} \cong V_\nu^{\mathrm{SO}}$, where $\nu = [2^{i-\delta}, 1^{m-4-p+2\delta}, 0^{p-i-\delta-n+2}]^{\mathrm{SO}}$;
- if $2j-2\delta = m-4$ (which happens only when m is even) and $i < j$ then $V_\mu^{\mathrm{O}} \cong V_\nu^{\mathrm{SO}} \oplus V_{\nu'}^{\mathrm{SO}}$, where $\nu = [2^{i-\delta}, 1, \ldots, 1]^{\mathrm{SO}}$, and $\nu' = [2^{i-\delta}, 1, \ldots, 1, -1]^{\mathrm{SO}}$;
- if $2j-2\delta = m-4$ (which happens only when m is even) and $i = j$ then $V_\mu^{\mathrm{O}} \cong V_\nu^{\mathrm{SO}} \oplus V_{\nu'}^{\mathrm{SO}}$, where $\nu = [2, \ldots, 2]^{\mathrm{SO}}$ and $\nu' = [2, \ldots, 2, -2]^{\mathrm{SO}}$.

□

We are now ready to compute the cohomology of $\wedge^p\mathcal{F}^\vee$ and $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ by applying Bott-Borel-Weil Theorem. Recall that in type B_n the non-vanishing cohomology of Ω_X^p for $p \leq m-4$ is given by $H^p(\Omega_X^p) \cong \mathbf{k}^{\lfloor p/2 \rfloor + 1}$; moreover $\mathfrak{so}_m = V_{\varpi_2}^{\mathrm{SO}(m)} = [1,1,0,\ldots,0]^{\mathrm{SO}(m)}$ and the quasi-minuscule representation is given by the standard representation $V_{\varpi_1}^{\mathrm{SO}(m)} = [1,0,\ldots,0]^{\mathrm{SO}(m)}$.

Proposition 4.13. *Let X be the adjoint variety of type B_n, $m = 2n+1$, $h = 1/2$, $e = m-4$. All cohomology groups of $\wedge^p\mathcal{F}^\vee$ and $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ for $0 \leq p \leq 2m-8$ vanish except for the following ones:*

- *$H^p(X, \wedge^p\mathcal{F}^\vee) \cong H^p(\Omega_X^p)$ for $p \leq e$, where the isomorphism is induced by the surjection $\Omega_X^p \to \wedge^p\mathcal{F}^\vee$;*
- *$H^{p-2}(X, \wedge^p\mathcal{F}^\vee \otimes \mathcal{L}) \cong H^{p-2}(\Omega_X^{p-2})$ for $2 \leq p \leq e-1$, $H^0(X, \wedge^0\mathcal{F}^\vee \otimes \mathcal{L}) \cong \mathfrak{so}_m$ and $H^{n-1}(X, \wedge^n\mathcal{F}^\vee \otimes \mathcal{L}) \cong V_{\varpi_1}^{\mathrm{SO}(m)}$; moreover the terms $H^{p-2}(\Omega_X^{p-2})$ are the images of the cohomology maps induced by the embeddings $d\theta \wedge (\bullet) : \wedge^{p-2}\mathcal{F}^\vee \to \wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$.*

Proof. Let us denote by $\mu := [2^{i-\delta}, 1^{j-i}, 0^{n-j+\delta-2}]^{\mathrm{SO}}$, and by $\nu := [2^{i-\delta}, 1^{m-4-p+2\delta}, 0^{p-i-\delta-n+2}]^{\mathrm{SO}}$. Let moreover p be an integer satisfying $0 \leq p \leq 2m-8$. Putting together the above lemmas we have

$$\wedge^p\mathcal{F}^\vee = \bigoplus_{\substack{0\leq\delta\leq i\leq j\leq m-4 \\ i+j=p \\ p-\delta\leq m-4}} \left(\bigoplus_{2j-2\delta<m-4} [-i,-j;\mu] \oplus \bigoplus_{2j-2\delta>m-4} [-i,-j;\nu] \right),$$

$$\wedge^p\mathcal{F}^\vee \otimes \mathcal{L} = \bigoplus_{\substack{0\leq\delta\leq i\leq j\leq m-4 \\ i+j=p \\ p-\delta\leq m-4}} \left(\bigoplus_{2j-2\delta<m-4} [-i+1,-j+1;\mu] \oplus \bigoplus_{2j-2\delta>m-4} [-i+1,-j+1;\nu] \right).$$

$\wedge^p\mathcal{F}^\vee$ Let us begin with the terms of the form $[-i,-j;\mu]$. Let us compute $[-i,-j;\mu] + \rho =$

$$= [n-i-h, n-j-h-1; n-h, \ldots, n-i+\delta+h, n-i+\delta-h-1, \ldots, n-j+\delta-h, n-j+\delta-2-h, \ldots, h].$$

In order to have non-vanishing cohomology, either $n-i-h = n-i+\delta-h$ or $n-i-h \leq n-j+\delta-1-h$. In the latter case we get $n-j-h-1 \leq -n+i-\delta+h$, which implies $p-\delta \geq m-2$, which is a contradiction. In the former case we get $\delta = 0$ and $n-j-h-1 = n-j+\delta-h-1$. Thus we get that for $\delta = 0$ we have cohomology isomorphic to $\mathbf{k}$ at degree $i-\delta+j-\delta = i+j = p$ for $p \leq m-4 = e$. A similar computation gives that $[-i,-j;\nu]$ has only cohomology isomorphic to $\mathbf{k}$ at degree p for $\delta = 0$ and $p \leq e$. Putting together both contributions, $\wedge^p\mathcal{F}^\vee$ has only cohomology in degree p. To compute the dimension of this cohomology, notice that we have a one dimensional contribution for any couple (i,j) such that $i+j = p$, $i \leq j$. A straightforward computation shows that this dimension is thus $\lfloor p/2 \rfloor + 1$, as expected.

$\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ Let us begin with the terms of the form $[-i+1,-j+1;\mu]$. Let us compute $[-i+1,-j+1;\mu]+\rho =$

$$= [n-i+h, n-j-h; n-h, \dots, n-i+\delta+h, n-i+\delta-h-1, \dots, n-j+\delta-h, n-j+\delta-2-h, \dots, h].$$

In order to have non-vanishing cohomology, either $i = 0$ or $n-i+h = n-i+\delta-h$ or $n-i+h \leq n-j+\delta-1-h$. In the last case we get $n-j-h \leq -n+i-\delta+h$, which implies $p-\delta \geq m-3$, which is a contradiction. In the first case we get $\delta = 0$ and there are two possibilities: either $j = 0$, $p = 0$, the cohomology is in degree 0 and it is isomorphic to $\mathfrak{so}_m$; either $j = p = n$, the cohomology is in degree $n-1$ and it is isomorphic to $V^{\mathrm{SO}(m)}_{\varpi_1}$. In the middle case we get $\delta = 1$, $i \geq 1$ and we have cohomology isomorphic to $\mathbf{k}$ at degree $i-\delta+j-\delta = i+j-2 = p-2$ for $p \leq e-1$. A similar computation gives that $[-i+1,-j+1;\nu]$ has only cohomology isomorphic to $\mathbf{k}$ at degree $p-2$ for $\delta = 1$ and $p \leq e-1$. Putting together both contributions, $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ has only cohomology in degree 0 (when $p = 0$), $n-1$ (when $p = n$) or p. Apart from the terms $\mathfrak{so}_m$ and $V^{\mathrm{SO}(m)}_{\varpi_1}$, notice that we have a one dimensional contribution for any couple (i,j) such that $i+j = p$, $1 \leq i \leq j$. A straightforward computation shows that this dimension is thus $\lfloor p/2 \rfloor$, as expected.

The argument about the cohomology groups induced by $\Omega^i_X \to \wedge^i\mathcal{F}^\vee \to 0$ and $d\theta \wedge (\bullet)$ are the same as in the proof of Proposition 4.6. □

Let us now compute the cohomology of $\wedge^p\mathcal{F}^\vee$ and $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ by applying Bott-Borel-Weil Theorem in type D_n. Recall that in type D_n the non-vanishing cohomology of Ω^p_X for $p \leq e = m-4$ is given by $H^p(\Omega^p_X) \cong \mathbf{k}^{\lfloor p/2 \rfloor + 1 + \delta_{p \geq n-2}}$; moreover $\mathfrak{so}_m = V^{\mathrm{SO}(m)}_{\varpi_2} = [1,1,0,\dots,0]^{\mathrm{SO}(m)}$.

Proposition 4.14. *Let X be the adjoint variety of type D_n, $m = 2n$, $h = 0$, $e = m-4$. All cohomology groups of $\wedge^p\mathcal{F}^\vee$ and $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ for $0 \leq p \leq 2m-8$ vanish except for the following ones:*

- *$H^p(X, \wedge^p\mathcal{F}^\vee) \cong H^p(\Omega^p_X)$ for $p \leq e$, where the isomorphism is induced by $\Omega^p_X \twoheadrightarrow \wedge^p\mathcal{F}^\vee$;*
- *$H^{p-2}(X, \wedge^p\mathcal{F}^\vee \otimes \mathcal{L}) \cong H^{p-2}(\Omega^{p-2}_X)$ for $2 \leq i \leq e-1$, and $H^0(X, \wedge^0\mathcal{F}^\vee \otimes \mathcal{L}) \cong \mathfrak{so}_m$; moreover the terms $H^{p-2}(\Omega^{p-2}_X)$ are the images of the cohomology maps induced $d\theta \wedge (\bullet) : \wedge^{p-2}\mathcal{F}^\vee \hookrightarrow \wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$.*

Proof. Let us denote by $\mu := [2^{i-\delta}, 1^{j-i}, 0^{n-j+\delta-2}]^{\mathrm{SO}}$, by $\nu := [2^{i-\delta}, 1^{m-4-p+2\delta}, 0^{p-i-\delta-n+2}]^{\mathrm{SO}}$, by $\alpha^\pm := [2^{i-\delta}, 1, \dots, 1, \pm 1]^{\mathrm{SO}}$ and by $\beta^\pm := [2, \dots, 2, \pm 2]^{\mathrm{SO}}$. Let moreover $0 \leq p \leq 2m-8$. Putting together the above lemmas we have

$$\wedge^p \mathcal{F}^\vee = \bigoplus_{\substack{0 \leq \delta \leq i \leq j \leq m-4 \\ i+j=p \\ p-\delta \leq m-4}} \left(\bigoplus_{2j-2\delta < m-4} [-i,-j;\mu] \oplus \bigoplus_{2j-2\delta > m-4} [-i,-j;\nu] \oplus \right.$$

$$\left. \oplus \bigoplus_{2j-2\delta = m-4, i<j} ([-i,-j;\alpha^+] \oplus [-i,-j;\alpha^-]) \oplus \bigoplus_{2j-2\delta = m-4, i=j} ([-i,-j;\beta^+] \oplus [-i,-j;\beta^-]) \right),$$

and $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ has a similar expression. The cohomology of the terms $[-i,-j;\mu]$ and $[-i,-j;\nu]$ for $\wedge^p\mathcal{F}^\vee$ and $[-i+1,-j+1;\mu]$ and $[-i+1,-j+1;\nu]$ for $\wedge^p\mathcal{F}^\vee \otimes \mathcal{L}$ are dealt with exactly in the same way as in the proof of Proposition 4.13 by choosing $h = 0$; the only difference will be that the term $V^{\mathrm{SO}(m)}_{\varpi_1}$ does not appear in this case. So, in order not to repeat tedious computations, in the following we deal only with the cohomology of the remaining terms.

$\wedge^p\mathcal{F}^\vee$ proceeding as in the proof of Proposition 4.13 one shows that $[-i,-j;\alpha^+]$ and $[-i,-j;\alpha^-]$ have cohomology isomorphic to $\mathbf{k}$ only for $\delta = 0$ in degree p. Similarly, one shows that $[-i,-j;\beta^+]$

and $[-i,-j;\beta^-]$ have cohomology isomorphic to $\mathbf{k}$ only for $\delta=0$ in degree p. Since $\delta=0$, these contributions only appear for $j=n-2$ and $p\geq n-2$. Putting them together with the contributions coming from the terms $[-i,-j;\mu]$ and $[-i,-j;\nu]$ one obtains that the dimension of the cohomology of $\wedge^p\mathcal{F}^\vee$ in degree p is equal to: $\lfloor p/2\rfloor+1$ if $p<n-2$ and $\lfloor p/2\rfloor+2$ if $p\geq n-2$, as expected.

$\wedge^p\mathcal{F}^\vee\otimes\mathcal{L}$ The cohomology of this bundle does not present any novelty with respect to the above computations and the computations in Proposition 4.13, so we leave the verification of the statement of this proposition to the reader.

The argument about the cohomology groups induced by $\Omega^i_X\to\wedge^i\mathcal{F}^\vee\to 0$ and $d\theta\wedge(\bullet)$ are the same as in the proof of Proposition 4.6. □

4.4. **Exceptional types.** There are only five simple exceptional Lie groups: G_2, F_4, E_6, E_7, E_8. For each of these cases - except G_2, which can be done by hand - we have written a Python script using [vLCL92] in order to compute the relevant cohomology groups through the combinatorics of Bott-Borel-Weil Theorem.

4.4.1. *Type G_2.* In this case $X\subset\mathbb{P}(\mathfrak{g}_2)\cong\mathbb{P}(V_{\varpi_1})$ is a Fano fivefold of index 3. It is isomorphic to the quotient G/P_1 with $G=G_2$. We have $\mathcal{L}=\mathcal{E}_{\varpi_1}$, $\mathcal{F}=\mathcal{E}_{-\varpi_1+3\varpi_2}$ and $\mathcal{F}^\vee=\mathcal{E}_{\alpha_1}=\mathcal{E}_{-2\varpi_1+3\varpi_2}$. The quasi-minuscule representation is $\hat{\mathfrak{g}}_2=V_{\varpi_2}$.

4.4.2. *Type F_4.* In this case $X\subset\mathbb{P}(\mathfrak{f}_4)\cong\mathbb{P}(V_{\varpi_1})$ is a Fano 15-fold of index 8. It is isomorphic to the quotient G/P_1 with $G=F_4$. We have $\mathcal{L}=\mathcal{E}_{\varpi_1}$, $\mathcal{F}=\mathcal{E}_{-\varpi_1+\varpi_2}$ and $\mathcal{F}^\vee=\mathcal{E}_{\alpha_1}=\mathcal{E}_{-2\varpi_1+\varpi_2}$. Recall that the quasi-minuscule representation in this case is $\hat{\mathfrak{f}}_4=V_{\varpi_4}$.

Proposition 4.15. *Let X be the adjoint variety of type G_2 or F_4. All cohomology groups of $\wedge^i\mathcal{F}^\vee$ and $\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$ vanish except for the following ones:*

- *$H^i(X,\wedge^i\mathcal{F}^\vee)\cong H^i(\Omega^i_X)$ for $i\leq e$, where the isomorphism is induced by the surjection $\Omega^i_X\to\wedge^i\mathcal{F}^\vee$;*
- *$H^{i-2}(X,\wedge^i\mathcal{F}^\vee\otimes\mathcal{L})\cong H^{i-2}(\Omega^{i-2}_X)$ for $2\leq i\leq e-1$, $H^0(X,\wedge^0\mathcal{F}^\vee\otimes\mathcal{L})\cong\mathfrak{g}$ and $H^1(X,\wedge^2\mathcal{F}^\vee\otimes\mathcal{L})\cong\hat{\mathfrak{g}}$; moreover the terms $H^{i-2}(\Omega^{i-2}_X)$ are the images of the cohomology maps induced by the embeddings $d\theta\wedge(\bullet):\wedge^{i-2}\mathcal{F}^\vee\to\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$.*

4.4.3. *Type E_6.* In this case $X\subset\mathbb{P}(\mathfrak{e}_6)\cong\mathbb{P}(V_{\varpi_2})$ has dimension 21 and index 11. It is isomorphic to the quotient G/P_2 with $G=E_6$. The following bundles can be interpreted as homogeneous bundles as follows: $\mathcal{L}=\mathcal{E}_{\varpi_2}$, $\mathcal{F}=\mathcal{E}_{-\varpi_2+\varpi_4}$ and $\mathcal{F}^\vee=\mathcal{E}_{\alpha_2}=\mathcal{E}_{-2\varpi_2+\varpi_4}$.

4.4.4. *Type E_7.* In this case $X\subset\mathbb{P}(\mathfrak{e}_7)\cong\mathbb{P}(V_{\varpi_1})$ has dimension 33 and Fano index 17. We have $X\simeq E_7/P_1$, $\mathcal{L}=\mathcal{E}_{\varpi_1}$, $\mathcal{F}=\mathcal{E}_{-\varpi_1+\varpi_3}$ and $\mathcal{F}^\vee=\mathcal{E}_{\alpha_1}=\mathcal{E}_{-2\varpi_1+\varpi_3}$.

4.4.5. *Type E_8.* The E_8-adjoint variety $X\subset\mathbb{P}(\mathfrak{e}_8)\cong\mathbb{P}(V_{\varpi_8})$ has dimension 57 and Fano index 29. We have quotient $X\simeq E_8/P_8$, $\mathcal{L}=\mathcal{E}_{\varpi_8}$, $\mathcal{F}=\mathcal{E}_{-\varpi_8+\varpi_7}$ and $\mathcal{F}^\vee=\mathcal{E}_{\alpha_8}=\mathcal{E}_{-2\varpi_8+\varpi_7}$.

Proposition 4.16. *Let X be the adjoint variety of type E_6, E_7, E_8. All cohomology groups of $\wedge^i\mathcal{F}^\vee$ and $\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$ vanish except for the following ones:*

- *$H^i(X,\wedge^i\mathcal{F}^\vee)\cong H^i(\Omega^i_X)$ for $i\leq e$, where the isomorphism is induced by the surjection $\Omega^i_X\to\wedge^i\mathcal{F}^\vee$;*
- *$H^{i-2}(X,\wedge^i\mathcal{F}^\vee\otimes\mathcal{L})\cong H^{i-2}(\Omega^{i-2}_X)$ for $2\leq i\leq e-1$, and $H^0(X,\wedge^0\mathcal{F}^\vee\otimes\mathcal{L})\cong\mathfrak{g}$; moreover the terms $H^{i-2}(\Omega^{i-2}_X)$ are the images of the cohomology maps induced by the embeddings $d\theta\wedge(\bullet):\wedge^{i-2}\mathcal{F}^\vee\to\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$.*

Proof of Propositions 4.15, 4.16. We used a Python script in order to compute the cohomology groups of the relevant homogeneous bundles through the Bott-Borel-Weil Theorem. The argument about the cohomology groups induced by $\Omega^i_X\to\wedge^i\mathcal{F}^\vee\to 0$ and $d\theta\wedge(\bullet)$ are the same as in the proof of Proposition 4.6. □

4.5. **Free resolution of the Jacobian ideal of adjoint discriminant.** From the exact sequence (15), we can compute the cohomology of $\Omega^i_X\otimes\mathcal{L}$ from the cohomology of $\wedge^{i-1}\mathcal{F}^\vee$ and $\wedge^i\mathcal{F}^\vee\otimes\mathcal{L}$. Recall that by K^{i-1} and C^i we denote respectively the kernel and cokernel of the multiplication by the hyperplane class $H^{i-1}(\Omega^{i-1}_X)\to H^i(\Omega^i_X)$. Recall moreover that $H^{i-1}(X,\wedge^{i-1}\mathcal{F}^\vee)\cong H^{i-1}(\Omega^{i-1}_X)$ and that the factor $H^{i-2}(\Omega^{i-2}_X)$ inside $H^{i-2}(X,\wedge^i\mathcal{F}^\vee\otimes\mathcal{L})$ is induced by the contact form $d\theta$ (Propositions 4.6, 4.13, 4.14, 4.15, 4.16). We can restrict the connecting homomorphism $H^{i-2}(X,\wedge^i\mathcal{F}^\vee\otimes\mathcal{L})\to H^{i-1}(X,\wedge^{i-1}\mathcal{F}^\vee)$ induced by the exact sequence (15) to obtain a morphism

$$\eta_i:H^{i-2}(\Omega^{i-2}_X)\to H^{i-1}(\Omega^{i-1}_X).$$

Lemma 4.17. *The morphism η_i is, modulo non-zero scalar, the multiplication by the hyperplane class $[c_1(\mathcal{L})]$. In particular it has maximal rank.*

Proof. The statement about the rank is a consequence of Lefschetz' Hyperplane Theorem. Moreover, since the maps η_i for each i are induced by the map η_2 by definition of $d\theta \wedge (\bullet)$, it is sufficient to prove the statement for η_2. In this case the exact sequence (15) is just a twist of

$$0 \to \mathcal{L}^\vee \to \Omega_X \to \mathcal{F}^\vee \to 0.$$

This extension is non-trivial and induced by the element in $H^1(X, \mathcal{F} \otimes \mathcal{L}^\vee) \cong H^1(X, \mathcal{F}^\vee) \cong H^1(\Omega_X)$ corresponding to $\mathcal{L}$ (see for instance [BM19]). Thus the image of η_2 is $c_1[\mathcal{L}]$ and as a consequence η_i is the multiplication by the hyperplane class. □

We are now ready to obtain a uniform formula for a locally free resolution of the Jacobian ideal of discriminants of adjoint varieties. Let G be a simple Lie group of rank n and let $\epsilon = 0$ (respectively $\epsilon = 1$) if G is not-simply laced (resp. G is simply laced). Let $\mathfrak{g}$ be the Lie algebra of G and let $\hat{\mathfrak{g}}$ be the quasi-minuscule representation (in the simply laced case $\hat{\mathfrak{g}} \cong \mathfrak{g}$). Let $j = 0$ if G is simply laced, $j = 1$ if $G = C_n$, $j = 2$ if $G = G_2$, $j = 3$ if $G = F_4$ and $j = n$ if $G = B_n$. Let X be the adjoint variety of G. Recall that $(e_1 \leq \cdots \leq e_n)$ are the exponents of $\mathfrak{g}$ and s is the number of long simple roots of $\mathfrak{g}$.

Theorem 4.18. *There exists a G-equivariant locally free resolution:*

$$0 \to \begin{array}{c} \bigoplus_{i=1}^s \mathcal{O}(-e_i - 1) \\ \oplus \\ \delta_{0,\epsilon}\hat{\mathfrak{g}} \otimes \mathcal{O}(-j-1) \end{array} \to \begin{array}{c} \bigoplus_{i=1}^s \mathcal{O}(-e_i) \\ \oplus \\ \delta_{0,\epsilon}\hat{\mathfrak{g}} \otimes \mathcal{O}(-j) \oplus \mathfrak{g} \otimes \mathcal{O}(-1) \end{array} \to \mathfrak{g} \otimes \mathcal{O} \to \mathcal{J}_D(d-1) \to 0$$

We believe j to have a Lie-theoretical meaning, but at the moment this is not clear to us.

Proof. The result is a consequence of Theorem 2.3. In order to apply it, we use the cohomology computations in Propositions 4.6, 4.13, 4.14, 4.15, 4.16. Then by Lemma 4.17 one obtains

$$0 \to \begin{array}{c} \bigoplus_{p \leq e} C^p \otimes \mathcal{O}(-p-2) \\ \oplus \\ \delta_{0,\epsilon}\hat{\mathfrak{g}} \otimes \mathcal{O}(-j-1) \end{array} \to \begin{array}{c} \bigoplus_{p \leq e} C^p \otimes \mathcal{O}(-p-1) \\ \oplus \\ \delta_{0,\epsilon}\hat{\mathfrak{g}} \otimes \mathcal{O}(-j) \oplus \mathfrak{g} \otimes \mathcal{O}(-1) \end{array} \to \mathfrak{g} \otimes \mathcal{O} \to \rho_*\pi^*\mathcal{L} \to 0.$$

The result follows by noticing, as in the proof of Theorem 3.3, that C^i is a direct sum of u_i trivial G-representations, where u_i is the cardinality of $\{j \mid \deg(f_j) = i+2\}$. □

Let us end this section by focusing on the G-equivariant morphism. $\mathfrak{g} \otimes \mathcal{O}(-1) \to \mathfrak{g} \otimes \mathcal{O}$. By taking global sections this morphism is given by a trivial G-factor of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ (if needed, use the Killing form to identify $\mathfrak{g} \cong \mathfrak{g}^\vee$). The following result should come with no surprise.

Proposition 4.19. *The morphism $\mathfrak{g} \otimes \mathcal{O}(-1) \to \mathfrak{g} \otimes \mathcal{O}$ appearing in the resolution of Theorem 2.3 is given by the Lie bracket in $\mathfrak{g}$.*

Proof. Following the proof of Theorem 2.3, the term $\mathfrak{g} \otimes \mathcal{O}$ comes from the term $H^0(\mathcal{L})$ resolution and the term $\mathfrak{g} \otimes \mathcal{O}(-1)$ comes from the term $H^0(\hat{\Omega}_X \otimes \mathcal{L})$ in Weyman's resolution. Thus, the morphism $\eta : \mathfrak{g} \otimes \mathcal{O}(-1) \to \mathfrak{g} \otimes \mathcal{O}$ is the pushforward through the second projection from $X \times \mathfrak{g}$ of the morphism $\hat{\Omega}_X \otimes \mathcal{L}(-1) \to \mathcal{L}$ in the Koszul complex of Weyman's resolution (here we are denoting by (-1) the trivial twist on the affine space $\mathfrak{g}$). By Lemma 3.2 we can identify $\hat{\Omega}_X \otimes \mathcal{L} \cong \hat{\mathcal{T}}_X$, so we obtain $\eta' : \hat{\mathcal{T}}_X(-1) \to \hat{\Omega}_X \otimes \mathcal{L}(-1) \to \mathcal{L}$. Notice that the isomorphism $\hat{\Omega}_X \otimes \mathcal{L} \cong \hat{\mathcal{T}}_X$ is given by the contact structure (see the proof of Lemma 3.2). Indeed, if $\theta' : \hat{\mathcal{T}}_X \to \mathcal{L}$ is the map induced by the contact structure $\theta : \mathcal{T}_X \to \mathcal{L}$ then $d\theta' : \wedge^2\hat{\mathcal{T}}_X \to \mathcal{L}$, and the map $d\theta'$ induces the isomorphism $\hat{\mathcal{T}}_X \cong \hat{\Omega}_X \otimes \mathcal{L}$. Thus we get $d\theta' \in H^0(\hat{\mathcal{T}}_X \otimes \mathcal{L}) \subset H^0(\hat{\mathcal{T}}_{\mathbb{P}} \otimes \mathcal{L}) \subset \wedge^2\mathfrak{g} \otimes \mathfrak{g}$, where in the last equality we have used the Killing isomorphism. Since the contact structure is defined from the Kostant-Kirillov form (see [Bea98]), $d\theta' \in \wedge^2\mathfrak{g} \otimes \mathfrak{g}$ is the Lie bracket (modulo scalar).

Finally notice that $H^0(\hat{\mathcal{T}}_X)$ is an extension of a trivial factor with $\mathfrak{g}$, where the latter comes from $H^0(\mathcal{T}_X) \cong \mathfrak{g}$. Since the map $\hat{\Omega}_X \otimes \mathcal{L} \to \mathcal{L}$ in the Koszul complex is just the contraction (after $X \times \mathfrak{g}$ is identified with the total space of $\hat{\Omega}_{\mathbb{P}} \otimes \mathcal{L}$), we obtain that the restriction η of $H^0(\eta') : H^0(\hat{\mathcal{T}}_X)(-1) \to H^0(\mathcal{L})$ to $\mathfrak{g}(-1) \subset H^0(\hat{\mathcal{T}}_X)$ is the Lie bracket. □

5. Logarithmic derivations for simply laced adjoint discriminants

The main goal of this section is to prove Theorem 2. The idea is to interpret the results of the previous section in terms of derivations, inspired on the treatment of invariant derivations of [OT92]. We use the notation of the introduction, so G a simple Lie group with a Lie algebra $\mathfrak{g}$ of *simply laced type*. Choosing a maximal torus $T \subset G$ we get a Weyl group $W := N_G(T)/Z_G(T)$, acting linearly on the Lie algebra $\mathfrak{h}$ of T. We write $\mathbf{U} := \mathbf{k}[\mathfrak{g}^\vee]$, $\mathbf{S} := \mathbf{k}[\mathfrak{h}^\vee]$ and recall that Chevalley's restriction theorem gives $\mathbf{R} := \mathbf{U}^G \cong \mathbf{S}^W$, i.e., in terms of GIT, we have $\mathfrak{g}/\!\!/G \simeq \mathfrak{h}/W \simeq \mathbf{k}^n$ and $\mathfrak{h}$ is a slice for the action of G on $\mathfrak{g}$. Put $n = \dim(\mathfrak{h})$.

5.1. **G-variant and G-invariant logarithmic derivations.** Let $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ be the $\mathbf{U}$-module of logarithmic derivations of the discriminant locus $\mathbb{V}(\Delta) = D$, where $X \subset \mathbb{P}(\mathfrak{g})$ is the G-adjoint variety. Let us use the following notation:

$$\mathrm{Der}_{\mathbf{U}}(\Delta) := \mathrm{Der}_{\mathbf{U}}(-\log(D)) = \{\eta \in \mathrm{Der}_{\mathbf{U}} \mid \eta(\Delta) \in (\Delta) \subset \mathbf{U}\}.$$

Let us write $\delta := \Delta|_{\mathfrak{h}}$. Since $\mathfrak{g}$ is of simply laced type, δ is the squared equation of the Weyl arrangement associated to W. We define $\mathrm{Der}_{\mathbf{S}}^W$ as the W-invariant $\mathbf{S}$-derivations and $\mathrm{Der}_{\mathbf{U}}^G$ as the G-invariant $\mathbf{U}$-derivations. Since $\mathbf{R} = \mathbf{U}^G$, for any $\eta \in \mathrm{Der}_{\mathbf{U}}^G$, we have $\eta(\mathbf{R}) \subset \mathbf{R}$, so that η defines an element $\pi_G(\eta) \in \mathrm{Der}_{\mathbf{R}}$. Similarly, since $\mathbf{R} = \mathbf{S}^W$, any $\eta \in \mathrm{Der}_{\mathbf{S}}^W$, defines an element $\pi_W(\eta) \in \mathrm{Der}_{\mathbf{R}}$. The maps $\pi_G : \mathrm{Der}_{\mathbf{U}}^G \to \mathrm{Der}_{\mathbf{R}}$ and $\pi_W : \mathrm{Der}_{\mathbf{S}}^W \to \mathrm{Der}_{\mathbf{R}}$ are morphisms of $\mathbf{R}$-modules. We call $\mathrm{Der}_{\mathbf{U}} / \mathrm{Der}_{\mathbf{U}}^G$ the module of G-variant derivations.

Next, we follow [Ter81] and, as in the introduction, we write $\sqrt{\delta} = \Pi_{\alpha\in\Phi^+}\delta_\alpha$, so:

$$\mathrm{Der}_{\mathbf{R}}(\delta) := \{\eta \in \mathrm{Der}_{\mathbf{R}} \mid \eta(\delta) \in (\delta) \subset \mathbf{R}\} \cong \mathrm{Der}_{\mathbf{R}}(\sqrt{\delta}).$$

Since we are in characteristic zero and since $\mathrm{Frac}(\mathbf{S})$ is finite algebraic over $\mathrm{Frac}(\mathbf{R})$, for any $\eta \in \mathrm{Der}_{\mathbf{R}} \subset \mathrm{Der}_{\mathrm{Frac}(\mathbf{R})}$, there exists a unique derivation $\eta_{\mathbf{S}} \in \mathrm{Der}^W_{\mathrm{Frac}(\mathbf{S})}$ such that $\eta_{\mathbf{S}}|_{\mathbf{R}} = \eta$. Set:

$$\mathrm{Der}^0_{\mathbf{R}} := \{\eta \in \mathrm{Der}_{\mathrm{Frac}(\mathbf{R})} \mid \eta_{\mathbf{S}}(\mathbf{S}) \subset \mathbf{S}\} = \{\eta \in \mathrm{Der}_{\mathrm{Frac}(\mathbf{R})} \mid \eta_{\mathbf{S}} \in \mathrm{Der}_{\mathbf{S}} \subset \mathrm{Der}_{\mathrm{Frac}(\mathbf{S})}\}.$$

Theorem 5.1 ([Ter81])**.** *The morphism $\pi_W : \mathrm{Der}_{\mathbf{S}}^W \to \mathrm{Der}^0_{\mathbf{R}}$ is an isomorphism of $\mathbf{R}$-modules. Moreover $\mathrm{Der}^0_{\mathbf{R}} \cong \mathrm{Der}_{\mathbf{R}}(\delta)$ is a free $\mathbf{R}$-module. Finally $\mathrm{Der}_{\mathbf{S}}(\delta) \cong \mathbf{S} \otimes_{\mathbf{R}} \mathrm{Der}_{\mathbf{S}}^W$ is a free $\mathbf{S}$-module.*

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$ be a Cartan decomposition of $\mathfrak{g}$. Let $x_1, \dots, x_{\ell+1}$ be a Killing-orthonormal basis of $\mathfrak{g}^\vee$ such that $x_1, \dots, x_n$ is a basis of $\mathfrak{h}^\vee$ and for any $n+1 \le i \le \ell+1$, there exists a root α such that $x_i \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ (here we are identifying $\mathfrak{g}$ and $\mathfrak{g}^\vee$ via the Killing form). Notice that $\mathbf{S} = \mathbf{U}/(x_i)_{i=n+1}^{\ell+1}$. We will denote by $\mathbf{I} := (x_i)_{i=n+1}^{\ell+1}$ the ideal defining $\mathbf{S}$. Any homogeneous element $\eta \in \mathrm{Der}_{\mathbf{U}}$ of degree d can be written as

$$\eta = \sum_{i=1}^{\ell+1} \eta_i \frac{\partial}{\partial x_i}, \qquad \text{with } \eta_i := \eta(x_i) \in \mathbf{U}_d.$$

Let us also denote by $(g_{i,j})_{1\le i,j\le \ell+1}$ the matrix associated to the linear action of $g \in G$ on $\mathfrak{g}^\vee$ in the basis $x_1, \dots, x_{\ell+1}$, i.e. $g(x_i) = \sum_{j=1}^{\ell+1} g_{i,j} x_j$. We will need the following lemma.

Lemma 5.2. *Let $\eta \in \mathrm{Der}_{\mathbf{U}}^G$ be a G-invariant derivation. Then for any $g \in G$, $g(\eta_i) = \sum_{j=1}^{\ell+1} g_{i,j}\eta_j$.*

Proof. The action of $g \in G$ on $\mathrm{Der}_{\mathbf{U}}$ is given by:

$$(g \cdot \eta)(f) := g \cdot (\eta(g^{-1} \cdot f)).$$

Since

$$g \cdot \left(\frac{\partial}{\partial x_i}(g^{-1}) \cdot x_j\right) = g \cdot \left(\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{\ell+1} g^{-1}_{j,k} x_k\right)\right) = g^{-1}_{j,i},$$

we deduce that $g \cdot \frac{\partial}{\partial x_i} = \sum_{j=1}^{\ell+1} g^{-1}_{j,i} \frac{\partial}{\partial x_j}$. Let now $\eta \in \mathrm{Der}_{\mathbf{U}}$ be G-invariant, which means that $g \cdot \eta = \eta$ for any $g \in G$. This means that for any $g \in G$,

$$\sum_{1\le i,j\le \ell+1} g(\eta_i) g^{-1}_{j,i} \frac{\partial}{\partial x_j} = \sum_{j=1}^{\ell+1} \eta_j \frac{\partial}{\partial x_j},$$

hence the claim. □

Lemma 5.3. *Let $\eta \in \mathrm{Der}_{\mathbf{U}}^G$ be a G-invariant derivation.*

- *If $i \leq n$ then $\frac{\partial f}{\partial x_i}|_{\mathfrak{h}} = \frac{\partial f|_{\mathfrak{h}}}{\partial x_i}$ for any $f \in \mathbf{U}$;*
- *if $i \geq n+1$ then $\eta_i|_{\mathfrak{h}} = 0$.*

Proof. The first equality is due to the fact that restricting to $\mathfrak{h}$ amounts to working modulo $\mathbf{I}$. For the second equality, let us fix a root α such that $y_i := x_i + x_{i'} \in \mathfrak{g}_\alpha$ and $y_i' := x_i - x_{i'} \in \mathfrak{g}_{-\alpha}$, where i, i' satisfy $x_i, x_{i'} \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Moreover let $t \in T$ be any point in a maximal torus $T \subset G$ stabilizing $\mathfrak{h}$. Then, since the basis $\{\{x_j\}_{j\leq n}, \{y_i\}_{i\geq n+1}\}$ is compatible with the Cartan decomposition defined by T, T acts diagonally on $\mathfrak{g}^\vee$ in this basis. More precisely $t \cdot y_i = \exp(\alpha(t))y_i$. Since the action of G on η_i is the same as the action of G on x_i we also get that $t \cdot (\eta_i \pm \eta_{i'}) = \exp(\alpha(t))(\eta_i \pm \eta_{i'})$. Since T acts as the identity on $\mathfrak{h}$ we deduce that $t \cdot (\eta_i|_{\mathfrak{h}}) = \eta_i|_{\mathfrak{h}}$. Since T acts diagonally we also deduce that $t \cdot ((\eta_i \pm \eta_{i'})|_{\mathfrak{h}}) = (t \cdot (\eta_i \pm \eta_{i'}))|_{\mathfrak{h}}$. By choosing $t \in T$ such that $\alpha(t) \neq 0$ we deduce that $(\eta_i \pm \eta_{i'})|_{\mathfrak{h}} = 0$, i.e. that $\eta_i|_{\mathfrak{h}} = 0$ for any $i \geq n+1$. □

Recall that $(F_1, \ldots, F_n)$ is a basis of homogeneous G-invariant polynomials in $\mathbf{U}$, i.e. $\mathbf{R} \cong \mathbf{k}[F_1, \ldots, F_n]$. Write $f_1, \ldots, f_n$ for their restrictions to $\mathbf{S}$. For $i = 1, \ldots, n$, let us define $\mu_i \in \mathrm{Der}_{\mathbf{S}}$ and $\nu_i \in \mathrm{Der}_{\mathbf{U}}$ by

$$\mu_i := \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_j}, \qquad \nu_i := \sum_{j=1}^{\ell+1} \frac{\partial F_i}{\partial x_j} \frac{\partial}{\partial x_j}.$$

Lemma 5.4. *The derivations ν_i for $i = 1, \ldots, n$ are G-invariant.*

Proof. Let us compute $g \cdot \nu_i$. We obtain

$$(g \cdot \nu_i)(x_k) = \sum_j g\left(\frac{\partial F_i}{\partial x_j}\right) g\left(\frac{\partial (g^{-1} \cdot x_k)}{\partial x_j}\right) = \sum_j g\left(\frac{\partial F_i}{\partial x_j}\right) g^{-1}_{k,j}.$$

By the G-invariance of F_i we also deduce that

$$g\left(\frac{\partial F_i}{\partial x_j}\right) = \left(g \cdot \frac{\partial}{\partial x_j}\right)(g(F_i)) = \left(\sum_h g_{j,h} \frac{\partial}{\partial x_h}\right)(F_i) = \sum_h g_{j,h} \frac{\partial F_i}{\partial x_h}.$$

Putting the two expressions together gives

$$(g \cdot \nu_i)(x_k) = \sum_{j,h} g^{-1}_{k,j} g_{j,h} \frac{\partial F_i}{\partial x_h} = \sum_h \delta_{k,h} \frac{\partial F_i}{\partial x_h} = \frac{\partial F_i}{\partial x_k} = \nu_i(x_k).$$

We have thus shown that $g \cdot \nu_i = \nu_i$ for any $g \in G$. □

The same proof shows that $\mu_i \in \mathrm{Der}_{\mathbf{S}}^W$. Actually $\mu_1, \ldots, \mu_n$ is an $\mathbf{R}$-basis of $\mathrm{Der}_{\mathbf{S}}^W$ (see [Yos14]).

Proposition 5.5. *There exists a morphism of $\mathbf{R}$-modules $\pi : \mathrm{Der}_{\mathbf{U}}^G \to \mathrm{Der}_{\mathbf{S}}^W$ making the following diagram commutative:*

$$\begin{array}{ccc} \mathrm{Der}_{\mathbf{U}}^G & \xrightarrow[\cong]{\pi_G} & \mathrm{Der}_{\mathbf{R}}(\Delta) \\ \pi \downarrow & & \downarrow \cong \\ \mathrm{Der}_{\mathbf{S}}^W & \xrightarrow[\pi_W]{\cong} & \mathrm{Der}_{\mathbf{R}}(\delta) \end{array}.$$

Proof. Let us explicitly define π. Let $\eta \in \mathrm{Der}_{\mathbf{U}}^G$ be a G-invariant derivation such that $\eta = \sum_i \eta_i \frac{\partial}{\partial x_i}$. For any polynomial $f \in \mathbf{k}[\mathfrak{g}^\vee]$, we can define $\pi(\eta)(f|_{\mathfrak{h}}) := \eta(f)|_{\mathfrak{h}}$. This is well defined because if $f|_{\mathfrak{h}} = 0$ then $f \in \mathbf{I}$ and, by the above lemmas,

$$\eta(f)|_{\mathfrak{h}} = \sum_i \eta_i|_{\mathfrak{h}} \frac{\partial f}{\partial x_i}|_{\mathfrak{h}} = \sum_{i\leq n} \eta_i|_{\mathfrak{h}} \frac{\partial f}{\partial x_i}|_{\mathfrak{h}} = \sum_{i\leq n} \eta_i \frac{\partial f|_{\mathfrak{h}}}{\partial x_i} = 0.$$

Even though a priori π is defined as a morphism $\pi : \mathrm{Der}_{\mathbf{U}}^G \to \mathrm{Der}_{\mathbf{S}}$, its image is W-invariant. Chevalley's Theorem tells us that there exists an isomorphism between $\mathrm{Der}_{\mathbf{R}}(\Delta)$ and $\mathrm{Der}_{\mathbf{R}}(\delta)$ given by restriction, so, modulo this isomorphism, $\pi_W \circ \pi = \pi_G$. As a consequence we deduce $\mathrm{Der}_{\mathbf{U}}^G \subset \mathrm{Der}_{\mathbf{U}}(\Delta)$.

Having this set up, need to show that π is an isomorphism to conclude. Let us begin with the injectivity. Let $\theta \in \mathrm{Der}_{\mathbf{S}}^W$ and $\eta, \eta' \in \mathrm{Der}_{\mathbf{U}}^G$ such that $\pi(\eta) = \pi(\eta') = \theta$ and denote their difference by $\xi := \eta - \eta' \in \mathrm{Der}_{\mathbf{U}}^G$.

By linearity $\pi(\xi)=0$. Composing with π_W we deduce that, for any $f\in\mathbf{R}$, $\xi(f)|_{\mathfrak{h}}=0$. Since $\xi(f)\in\mathbf{R}$, by Chevalley's Theorem we have that $\xi(f)=0$ for any $f\in\mathbf{R}$.

Now, let $\mathfrak{h}'$ be any Cartan subalgebra of $\mathfrak{g}$. For such subalgebra, we can proceed as before: find a basis $x'_1,\dots,x'_{\ell+1}$, write $\xi=\sum_i \xi'_i\frac{\partial}{\partial x'_i}$, construct $\pi':\mathrm{Der}_{\mathbf{U}}^G\to\mathrm{Der}_{\mathbf{S}'}^{W'}$ given by restriction. However, since for any $f\in\mathbf{R}$, $\xi(f)|_{\mathfrak{h}'}=0|_{\mathfrak{h}'}=0$, we have that $\pi_{W'}\circ\pi'(\xi)=0$. $\pi'(\xi)$ is the unique extension of $\pi_{W'}\circ\pi'(\xi)\in\mathrm{Der}_{\mathbf{R}}(\delta)$ to $\mathrm{Der}_{\mathbf{S}'}^{W'}$ (the extension is unique because $\mathrm{Frac}(\mathbf{S})$ is a finite extension of $\mathrm{Frac}(\mathbf{R})$ and we are in characteristic zero). Clearly 0 extends 0, so $\pi'(\xi)=0$.

To resume, we have shown that, for any Cartan subalgebra $\mathfrak{h}'\subset\mathfrak{g}$ and for any $f\in\mathbf{U}$, we have $\xi(f)|_{\mathfrak{h}'}=0$. Since Cartan subalgebras cover a dense subset of $\mathfrak{g}$, the equality above implies that $\xi(f)=0$ for any $f\in\mathbf{U}$, i.e. that $\xi=0$. Thus π is injective.

For the surjectivity take a basis of $\mathrm{Der}_{\mathbf{S}}^W$ given by $\mu_1,\dots,\mu_n$. By construction $\pi(\nu_i)=\mu_i$, and we have already shown that $\nu_i\in\mathrm{Der}_{\mathbf{U}}^G$ for $i=1,\dots,n$. Thus the $\mathbf{R}$-morphism π is surjective and thus an isomorphism. □

Proposition 5.6. *The morphism π_G is an isomorphism of $\mathbf{R}$-modules and there is a G-equivariant inclusion $\mathrm{Der}_{\mathbf{U}}(\Delta)^G:=\mathrm{Der}_{\mathbf{U}}^G\otimes_{\mathbf{R}}\mathbf{U}\subset\mathrm{Der}_{\mathbf{U}}(\Delta)$, where $\mathrm{Der}_{\mathbf{U}}(\Delta)^G$ is a free $\mathbf{U}$-module of rank n.*

Proof. The morphism π_G is an isomorphism between $\mathrm{Der}_{\mathbf{U}}^G$ and $\mathrm{Der}_{\mathbf{R}}(\Delta)$ by the lemma above. Since $\mathrm{Der}_{\mathbf{U}}^G\cong\mathrm{Der}_{\mathbf{R}}^0\cong\mathrm{Der}_{\mathbf{R}}(\Delta)$ as $\mathbf{R}$-modules and $\mathrm{Der}_{\mathbf{R}}(\Delta)$ is a free $\mathbf{R}$-module of rank n we deduce that $\mathrm{Der}_{\mathbf{U}}^G\otimes_{\mathbf{R}}\mathbf{U}$ is a free $\mathbf{U}$-module of rank equal to n. □

5.2. **Minimality of the resolution for simply laced types.** In the following we use the identification of the sheafification of the $\mathbf{U}$-module $\mathrm{Der}_{\mathbf{U}}(-\log(D))$ with the kernel of the differential surjective morphism $\mathfrak{g}\otimes\mathcal{O}\to\mathcal{J}_D(d-1)$ defining the Jacobian ideal.

Theorem 5.7. *Let G be a simply laced group. Then the resolution in Theorem 4.18 is minimal.*

Proof. In the simply laced types the above resolution takes the following form:

$$0\to\bigoplus_{1\le i\le n}\mathbf{U}(-e_i-1)\to\bigoplus_{1\le i\le n}\mathbf{U}(-e_i)\oplus\mathfrak{g}\otimes\mathbf{U}(-1)\to\mathrm{Der}_{\mathbf{U}}(-\log(D))\to 0.$$

This resolution is G-equivariant for the natural action of G on $\mathfrak{g}$ and for the trivial action of G on $\mathbf{U}(j)$ for all j's and $\dim(\bigoplus_{1\le i\le n}\mathbf{U}(-e_i))=\mathrm{rank}(G)=n$. Thus the G-invariant $\mathbf{U}$-module $\mathrm{Der}_{\mathbf{U}}(\Delta)^G$ is generated by the image of $\bigoplus_{1\le i\le n}\mathbf{U}(-e_i)$ inside $\mathrm{Der}_{\mathbf{U}}(-\log(D))$. Moreover since $\mathrm{Der}_{\mathbf{U}}(\Delta)^G$ is free of rank n, the restriction morphism $\bigoplus_{1\le i\le n}\mathbf{U}(-e_i)\to\mathrm{Der}_{\mathbf{U}}(-\log(D))$ is injective and induces an isomorphism $\bigoplus_{1\le i\le n}\mathbf{U}(-e_i)\to\mathrm{Der}_{\mathbf{U}}(\Delta)^G$. As a consequence in the resolution there cannot be a non-zero constant morphism $\mathbf{U}(-j)\to\mathbf{U}(-j)$. This shows that the resolution is minimal. □

The module of *G-variant logarithmic derivations* of Δ is $\mathrm{Der}_{\mathbf{U}}(-\log(D))_1:=\mathrm{Der}_{\mathbf{U}}(\Delta)/\mathrm{Der}_{\mathbf{U}}(\Delta)^G$. From the proof of Theorem 5.7, we get the following result.

Corollary 5.8. *Assume G is of simply laced type. A free resolution of the G-variant logarithmic derivation $\mathbf{U}$-module $\mathrm{Der}_{\mathbf{U}}(-\log(D))_1$ of Δ is:*

$$0\to\bigoplus_{i=1}^{n}\mathbf{U}(-e_i-1)\to\mathfrak{g}\otimes\mathbf{U}(-1)\to\mathrm{Der}_{\mathbf{U}}(-\log(D))_1\to 0.$$

Remark 5.9. We believe that the result can be generalised to the non-simply laced case, and even further to the case of *complex* reflection groups and graded Lie algebras.

5.3. **The adjoint image module.** The restriction to $\mathfrak{g}\otimes\mathcal{O}(-1)$ of the middle map in Theorem 4.18 induces a morphism

$$\mathbf{ad}:\mathfrak{g}\otimes\mathbf{U}(-1)\to\mathfrak{g}\otimes\mathbf{U}.$$

We want to understand what the kernel of this morphism is.

Lemma 5.10. *In the simply laced case, for $1\le i\le n$, the derivation $\nu_i\in\mathfrak{g}\otimes\mathbf{U}(-1)$ lies in $\ker(\mathbf{ad})$.*

Proof. By Lemma 4.19, the restriction of the middle map in Theorem 4.18 to $\mathfrak{g}\otimes\mathcal{O}(-1)$ is the Lie bracket of $\mathfrak{g}$, i.e. the differential at the identity of the adjoint action of G on $\mathfrak{g}$. The induced morphism $\mathfrak{g}\otimes\mathbf{U}(-1)\to\mathfrak{g}\otimes\mathbf{U}$ is thus the differential of the action of G on $\mathfrak{g}\otimes\mathbf{U}$, which can be identified with $\mathrm{Der}_{\mathbf{U}}$. On a point $\nu\in\mathrm{Der}_{\mathbf{U}}$, the action $\psi_\nu : G\to\mathrm{Der}_{\mathbf{U}}$ is given by $g\mapsto g\cdot\nu$. Since ν_i is a G-invariant derivation for $1\le i\le n$, we get that $g\cdot\nu_i=\nu_i$ for all $g\in G$; thus the differential of ψ_{ν_i}, along any tangent direction at the identity of G, vanishes. This shows that ν_i is in the kernel of $\mathbf{ad}$. □

We conclude this section by proving Theorem 2. For$G$ of simply laced type, we show that the module of G-variant logarithmic derivations $\mathrm{Der}_{\mathbf{U}}(-\log(D))_1$ is a direct summand of the module of logarithmic derivations and that furthermore it coincides with the adjoint image module $\mathbf{A}=\mathrm{Im}(\mathbf{ad})$.

Theorem 5.11. *Let G be a simply laced group. Then* $\mathrm{Der}_{\mathbf{U}}(-\log(D))_1\simeq\mathbf{A}$ *and*

$$\mathrm{Der}_{\mathbf{U}}(-\log(D))\simeq\mathbf{A}\oplus\mathrm{Der}_{\mathbf{U}}(\Delta)^G.$$

Proof. The resolution obtained in Theorem 5.7 induces the following commutative diagram, constructed starting from the central vertical exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\mathbf{ad} & \longrightarrow & \mathfrak{g}\otimes\mathbf{U}(-1) & \xrightarrow{\mathbf{ad}} & \mathfrak{g}\otimes\mathbf{U} & \longrightarrow & \mathrm{coker}(\mathbf{ad}) \longrightarrow 0\\
 & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & \bigoplus_{i=1}^n \mathbf{U}(-e_i-1) & \xrightarrow{(\alpha,\beta)} & \bigoplus_{i=1}^n \mathbf{U}(-e_i)\oplus\mathfrak{g}\otimes\mathbf{U}(-1) & \xrightarrow{\varphi} & \mathfrak{g}\otimes\mathbf{U} & \longrightarrow & J_D(d-1)\longrightarrow 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow & & & & \\
 & & \bigoplus_{i=1}^n \mathbf{U}(-e_i) & \xrightarrow{\simeq} & \bigoplus_{i=1}^n \mathbf{U}(-e_i) & & & &
\end{array}$$

where φ denotes the composition of the surjection

$$\bigoplus_{i=1}^n\mathbf{U}(-e_i)\oplus\mathfrak{g}\otimes\mathbf{U}(-1)\longrightarrow\mathrm{Der}_{\mathbf{U}}(-\log(D))$$

with the natural injection $\mathrm{Der}_{\mathbf{U}}(-\log(D))\hookrightarrow\mathfrak{g}\otimes\mathbf{U}$. Applying the snake lemma to the previous diagram induces the following exact sequence

$$0\to\ker(\mathbf{ad})\to\bigoplus_{i=1}^n\mathbf{U}(-e_i-1)\xrightarrow{\beta}\bigoplus_{i=1}^n\mathbf{U}(-e_i)\to\mathrm{coker}(\mathbf{ad})\to J_D(d-1)\to 0.$$

Furthermore, in light of Lemma 5.10 and the first injection of the previous sequence, we have $\ker(\mathbf{ad})\simeq\bigoplus_{i=1}^n\mathbf{U}(-e_i-1)$ and hence $\beta=0$. This implies that

$$\mathrm{Der}_{\mathbf{U}}(-\log(D))=\mathrm{Im}(\varphi)\simeq\mathbf{A}\oplus\bigoplus_{i=1}^n\mathbf{U}(-e_i)\simeq\mathrm{Der}_{\mathbf{U}}(-\log(D))_1\oplus\mathrm{Der}_{\mathbf{U}}(\Delta)^G.$$

□

References

[ABC+16] Takuro Abe, Mohamed Barakat, Michael Cuntz, Torsten Hoge, and Hiroaki Terao, *The freeness of ideal subarrangements of Weyl arrangements*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 6, 1339–1348.

[AFV16] Takuro Abe, Daniele Faenzi, and Jean Vallès, *Logarithmic bundles of deformed Weyl arrangements of type A_2*, Bull. Soc. Math. France **144** (2016), no. 4, 745–761.

[Bea98] Arnaud Beauville, *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv. **73** (1998), no. 4, 566–583.

[Ben18] Vladimiro Benedetti, *Manifolds of low dimension with trivial canonical bundle in Grassmannians*, Math. Z. **290** (2018), no. 1-2, 251–287.

[Bes15] David Bessis, *Finite complex reflection arrangements are $K(\pi,1)$*, Ann. of Math. (2) **181** (2015), no. 3, 809–904.

[Bot57] Raoul Bott, *Homogeneous vector bundles*, Ann. of Math. **(2)** (1957), 203–248.

[Bou02] Nicolas Bourbaki, *Lie groups and lie algebras (translated from the 1968 french original) IV-V-VI*, Elements of Mathematics, Springer, 2002.

[Bri09] Michel Brion, *Vanishing theorems for Dolbeault cohomology of log homogeneous varieties*, Tohoku Math. J. (2) **61** (2009), no. 3, 365–392.

[BEGvB09] Ragnar-Olaf Buchweitz, Wolfgang Ebeling, and Hans-Christian Graf von Bothmer, *Low-dimensional singularities with free divisors as discriminants*, J. Algebraic Geom. **18** (2009), no. 2, 371–406.

[BM19] Jarosław Buczyński and Giovanni Moreno, *Complex contact manifolds, varieties of minimal rational tangents, and exterior differential systems*, Geometry of Lagrangian Grassmannians and nonlinear PDEs, Banach Center Publ., vol. 117, Polish Acad. Sci. Inst. Math., Warsaw, 2019, pp. 145–176.

[CP11] Pierre-Emmanuel Chaput and Nicolas Perrin, *On the quantum cohomology of adjoint varieties*, Proc. Lond. Math. Soc. (3) **103** (2011), no. 2, 294–330.

[CHMN18] David Cook, II, Brian Harbourne, Juan Carlos Migliore, and Uwe Nagel, *Line arrangements and configurations of points with an unexpected geometric property*, Compos. Math. **154** (2018), no. 10, 2150–2194.

[Dam98] James Damon, *On the legacy of free divisors: discriminants and Morse-type singularities*, Amer. J. Math. **120** (1998), no. 3, 453–492.

[Dim17] Alexandru Dimca, *Hyperplane arrangements*, Universitext, Springer, Cham, 2017, An introduction.

[DiP23] Michael DiPasquale, *A homological characterization for freeness of multi-arrangements*, Math. Ann. **385** (2023), no. 1-2, 745–786.

[Dol07] Igor V. Dolgachev, *Logarithmic sheaves attached to arrangements of hyperplanes*, J. Math. Kyoto Univ. **47** (2007), no. 1, 35–64.

[FM22] Daniele Faenzi and Simone Marchesi, *On stability of logarithmic tangent sheaves: symmetric and generic determinants*, Int. Math. Res. Not. IMRN (2022), no. 23, 18589–18631.

[FMV13] Daniele Faenzi, Daniel Matei, and Jean Vallès, *Hyperplane arrangements of Torelli type*, Compos. Math. **149** (2013), no. 2, 309–332.

[FV14] Daniele Faenzi and Jean Vallès, *Logarithmic bundles and line arrangements, an approach via the standard construction*, J. Lond. Math. Soc. (2) **90** (2014), no. 3, 675–694.

[FN62] Ralph Fox and Lee Neuwirth, *The braid groups*, Math. Scand. **10** (1962), 119–126.

[Ful97] William Fulton, *Young tableaux*, Cambridge University Press, Cambridge, UK, 1997.

[GKZ08] Izrail M. Gelfand, Mikhail M. Kapranov, and Andei V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008, Reprint of the 1994 edition.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[HTW05] Roger Howe, Eng-Chye Tan, and Jeb F. Willenbring, *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626.

[JK21] Il-Seung Jang and Jae-Hoon Kwon, *Flagged Littlewood-Richardson tableaux and branching rule for classical groups*, J. Combin. Theory Ser. A **181** (2021), Paper No. 105419, 51.

[Kem76] George R. Kempf, *On the collapsing of homogeneous bundles*, Invent. Math. **37** (1976), no. 3, 229–239.

[Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[Küc95] Oliver Küchle, *On Fano 4-fold of index 1 and homogeneous vector bundles over Grassmannians*, Math. Z. **218** (1995), no. 4, 563–575.

[Las78] Alain Lascoux, *Syzygies des variétés déterminantales*, Adv. in Math. **30** (1978), no. 3, 202–237.

[Mac72] Ian G. Macdonald, *The Poincaré series of a Coxeter group*, Math. Ann. **199** (1972), 161–174.

[Nob75] Augusto Nobile, *Some properties of the Nash blowing-up*, Pacific J. Math. **60** (1975), no. 1, 297–305.

[OS80] Peter Orlik and Louis Solomon, *Unitary reflection groups and cohomology*, Invent. Math. **59** (1980), no. 1, 77–94.

[OT92] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.

[Sai04] Kyoji Saito, *Uniformization of the orbifold of a finite reflection group*, Frobenius manifolds, Aspects Math., vol. E36, Friedr. Vieweg, Wiesbaden, 2004, pp. 265–320.

[Ser06] Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006.

[Ste59] Robert Steinberg, *Finite reflection groups*, Trans. Amer. Math. Soc. **91** (1959), 493–504.

[Ter80a] Hiroaki Terao, *Arrangements of hyperplanes and their freeness. I*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 293–312.

[Ter80b] ———, *Arrangements of hyperplanes and their freeness. II. The Coxeter equality*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 313–320.

[Ter80c] ———, *Free arrangements of hyperplanes and unitary reflection groups*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 8, 389–392.

[Ter81] ———, *Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula*, Invent. Math. **63** (1981), no. 1, 159–179.

[Tev01] Jenia Tevelev, *Projectively dual varieties*, Journal of Mathematical Sciences **117** (2001), 4585–4732.

[vLCL92] Marc A. A. van Leeuwen, Arej M. Cohen, and Bert Lisser, *LiE, a package for lie group computations*, Computer Algebra Nederland, Amsterdam, 1992.

[Wey03] Jerzy Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003.

[Yos04] Masahiko Yoshinaga, *Characterization of a free arrangement and conjecture of Edelman and Reiner*, Invent. Math. **157** (2004), no. 2, 449–454.

[Yos14] ———, *Freeness of hyperplane arrangements and related topics*, Annales de la Faculté des sciences de Toulouse **(6), 23** (2014), 483–512.

Vladimiro Benedetti. Université Côte d'Azur, CNRS – Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice Cedex 2, France. Email: vladimiro.benedetti@univ-cotedazur.fr

Daniele Faenzi. Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université de Bourgogne, 9 Avenue Alain Savary, F-21000, Dijon, France. Email: daniele.faenzi@u-bourgogne.fr

Simone Marchesi. Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain. Centre de Recerca Matemàtica Edifici C, Campus Bellaterra, 08193 Bellaterra, Spain. Email: marchesi@ub.edu