



**HAL**  
open science

# PI control for the cascade channels modeled by general Saint-Venant equations

Amaury Hayat, Yating Hu, Peipei Shang

► **To cite this version:**

Amaury Hayat, Yating Hu, Peipei Shang. PI control for the cascade channels modeled by general Saint-Venant equations. IEEE Transactions on Automatic Control, 2023, pp.1-14. 10.1109/TAC.2023.3341767 . hal-04355315

**HAL Id: hal-04355315**

**<https://hal.science/hal-04355315v1>**

Submitted on 20 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# PI control for the cascade channels modeled by general Saint-Venant equations

Amaury Hayat, Yating Hu, and Peipei Shang <sup>\*†‡§</sup>

December 20, 2023

## Abstract

The Input-to-State Stability of the non-horizontal cascade channels with different arbitrary cross section, slope and friction modeled by Saint-Venant equations is addressed in this paper. The control input and measured output are both on the collocated boundary. The PI control is proposed to study both the exponential stability and the output regulation of the closed-loop systems with the aid of Lyapunov approach. An explicit quadratic Lyapunov function as a weighted function of a small perturbation of the non-uniform steady-states of different channels is constructed. We show that by a suitable choice of the boundary feedback controls, the local exponential stability and the Input-to-State Stability of the nonlinear Saint-Venant equations for the  $H^2$  norm are guaranteed, then validated with numerical simulations. Meanwhile, the output regulation and the rejection of constant disturbances are realized as well.

## 1 Introduction

Open channel is the most common type of channel found in natural and human society. It could be rivers, irrigation channels, artificial canals, etc. These channels are often organized in networks, either for natural reasons (rivers connected to each other) or practical ones (irrigation channels are designed to irrigate large areas). Therefore, it is of great practical significance to study the movement law of water flow in network channel models.

The Saint-Venant equations are a class of quasilinear hyperbolic partial differential equations that enables to describe the law of gradual unsteady flow in open channels and other shallow-water with free surfaces. The equations consist of the continuity equation based on the conservation law of mass and the motion equation based on the balance law of momentum [14]. Therefore, the Saint-Venant equations constitute a  $2 \times 2$  one-dimensional hyperbolic system. For a long time, the stability of the systems of one-dimensional balance laws has been analyzed widely in the literature. To our knowledge, starting with conservation laws, the first important result of the asymptotic stability of nonlinear  $2 \times 2$  homogeneous systems was given by Greenberg

---

\*The authors would like to thank the National Natural Science Foundation of China (No. 12171368 and 11971119) as well as the PEPS 2022 program of INSMI for their financial support.

†A. H. is with CERMICS, École des Ponts ParisTech, Champs sur Marne, 77455 Marne la Vallée, France (e-mail: amaury.hayat@enpc.fr).

‡Y. H. is with Tongji University, 200092, Shanghai, China (e-mail:1910739@tongji.edu.cn).

§P. S. is with Tongji University, 200092, Shanghai, China (e-mail: peipeishang@hotmail.com).

and Li in [18] by using the characteristic method. Later on, Qin in [37] extended this method to the  $n \times n$  case. However, when the system is highly nonlinear and has source terms, this method is complicated. Another approach was used by Coron et al. in [9] to solve the regulation of irrigation canals without friction or slope. They used a Lyapunov approach where the selection of the Lyapunov function came from the entropy principle. Later on, the same authors introduced a strict Lyapunov function for the boundary control of hyperbolic systems of conservation laws that can be diagonalized with Riemann invariants [10]. They solved the stabilization of the steady-states of linear and nonlinear systems without source terms.

When source terms appear, there are relatively fewer results, and these results are usually obtained under various assumptions on the coefficients or the source terms. Among which, the stability was proved using the method of characteristics in [16, 35] and a Lyapunov approach in [20] under the assumption that the source term is sufficiently small. In [6], Bastin et al. addressed the issue of the exponential stability when the friction and the slope compensates each other. By ignoring the slope, Bastin and Coron obtained the exponential stability in the  $H^2$  norm with nonuniform steady-states in [5]. Recently, [25] improved the method and extended it to more general case of arbitrary friction and space-varying slope. In [21], the exponential stability in the  $C^1$  norm with general source term was obtained. The case of a general density-velocity system was treated in [26]. In the literature, the backstepping method introduced by Krstic et al. in [31] is a powerful tool to study the stabilization of linear hyperbolic systems with source terms, see [12, 27, 30]. In [15], the authors applied this method to deal with the linearized Saint-Venant-Exner equations, a slightly different system. However, this gives rise to more complicated controls and requires a better knowledge of the dynamics inside the channel.

In the real applications, there may be various potential disturbances, either in the dynamics or in the control applications. The Input-to-State Stability (ISS) is the well-known robust stability property that first applied to the analysis and control of dynamical systems [29, 38]. Concerning the ISS applied to the hyperbolic system, the case in the  $L^2$  norm was studied in [17, 42] using a Lyapunov approach and in [28] using a small-gain analysis. The semilinear system with Lipschitz source term was treated in [23] and the nonlinear system in the framework of the  $C^1$  norm in [7]. One can refer to [28] (see also [22]) for an overview of the ISS for PDEs.

When looking at the stabilization of the flow of networks, almost all the works concentrated on the static proportional control and without source terms. In [11, 13, 33], the exponential stabilization is studied under the assumption that there are no source terms. In [6], the authors generalized their result for one channel to the cascade case where the friction and the slope compensates each other. In [1, 2], the authors use a backstepping approach to stabilize a general linear interconnected hyperbolic system. Little research, however, has been carried on in the direction where Proportional-Integral (PI) control is applied to networks. This type of control is useful for its resilience to constant disturbances and is used in many practical situations for the regulation of navigable rivers [34]. Nevertheless, one can note the works by Trinh, Andrieu and Xu [39, 40] on this kind of control for the networks modeled by Saint-Venant equations without source terms.

Our purpose in this paper is to analyze the exponential stability in the  $H^2$  norm as well as the ISS and the regulation of the set-points of the networks modeled by Saint-Venant equations with general source terms. In this general model we considered, the length, the sectional area, the friction and the slope of each reach can be different

along the channels, and can even obey to different models for the same channel. Usually, the static control law may be subject to steady-states regulation errors in case of disturbances or model inaccuracies. In this work, we use PI control combined with Lyapunov approach to study the robust stability of  $N$  cascade channels network. The main difficulty, apart from the source terms and the appearance of the additional disturbances, lies in the complex boundary conditions coupling different channels. On one hand, the equations in one channel coupled together through the boundary, on the other hand, different channels coupled together through the junction points. We show that the existence of a  $L^2$  basic quadratic Lyapunov function for an undisturbed linear hyperbolic system with PI control implies the ISS in the  $H^2$  norm of the associated nonlinear system with disturbances both inside the dynamics and at the boundaries.

Another contribution in this work is that we use less controls to realize the robust stability. Noticing that the system is a  $2N$  quasilinear hyperbolic system, but only has  $N$  controls. In general, it is not obvious at all that a system can be stabilized using fewer boundary controls than equations. Indeed, when one boundary condition is fixed, there exist examples where one cannot stabilize the system using static feedback controls at the other boundaries, see [4, Proposition 5.12], or [19] for a concrete example.

The structure of the paper is as follows. In section 2, we state the problem and give the main result, Theorem 2.1. In section 3, a Lyapunov function for the linearized system is constructed and the exponential stability in the  $L^2$  norm of the linearized system is obtained. In section 4, we show a general result that the exponential stability of the undisturbed linearized system implies the exponential ISS of the nonlinear hyperbolic system with disturbances (Theorem 4.1) and prove Theorem 2.1.

## 2 Description of the cascade channels modeled by Saint-Venant equations and main results

In this section, we shall illustrate the establishment of boundary control strategy for the dynamics described by Saint-Venant equations with disturbances. Here, we consider a cascade case, which is composed of  $N(N \geq 2)$  non-horizontal reaches with different lengths with arbitrary cross section, slope and friction respectively. For each reach, the Saint-Venant equations are defined on  $[0, +\infty) \times [0, L_i]$  respectively as (see [26])

$$\begin{aligned} \partial_t A_i + \partial_x(A_i V_i) &= 0, \\ \partial_t V_i + V_i \partial_x V_i + g \partial_1 G_i(A_i, x) \partial_x A_i & \\ + g \partial_2 G_i(A_i, x) + g(S_{f,i}(A_i, V_i, x) - S_{b,i}(x)) &= 0, \end{aligned} \tag{1}$$

where  $i \in \{1, \dots, N\}$ . For the  $i$ th channel,  $A_i = A_i(t, x)$  is the wet cross section,  $H_i = G_i(A_i, x)$  is the height of the water,  $V_i = V_i(t, x)$  is the horizontal water velocity,  $S_{f,i}$  is the friction and  $S_{b,i}$  is the slope. To be physically acceptable, the wet section has to increase with the height of the water and reciprocally. This is mathematically translated by  $\partial_1 G_i(A_i, x) > 0$  for any  $x \in [0, L_i]$  and  $i \in \{1, \dots, N\}$ . On the other hand,  $\partial_2 G_i(A_i, x)$  represents what would be the variation of the height of the water with respect to the location  $x$  if the wet section remained constant and this term depends on the section profile so we do not assume any sign for it. Also, the friction term has to be increasing with the velocity and decreasing with the wet section (see

[4]).  $\partial_2 S_{f,i} > 0$  and  $\partial_1 S_{f,i} < 0$ . At the boundaries, the conditions are the following

$$\begin{aligned}
A_1(t, 0)V_1(t, 0) &= Q_0, \\
A_{j+1}(t, 0)V_{j+1}(t, 0) &= A_j(t, L_j)V_j(t, L_j), \\
A_i(t, L_i)V_i(t, L_i) &= k_{p,i}(A_i^*(L_i) - A_i(t, L_i)) \\
&\quad + k_{I,i}Z_i(t), \\
\dot{Z}_i(t) &= A_i^*(L_i) - A_i(t, L_i),
\end{aligned} \tag{2}$$

where  $j \in \{1, \dots, N-1\}$ ,  $i \in \{1, \dots, N\}$ ,  $Q_0$  is a constant influx,  $A_i^*(x)$  are steady-states defined by (6)–(7),  $k_{p,i}$  and  $k_{I,i}$  are parameters to be regulated. Let us now assume that there are external disturbances that occur both at the boundaries and inside the dynamics of the system. The disturbances inside the dynamics could be a consequence of rain, unknown external forces, imprecision of the model, while the disturbances at the boundaries could result for instance from an actuation error, an imprecision in the measurements, or an unsteady upstream flow. The system (1)–(2) becomes

$$\begin{aligned}
\partial_t A_i + \partial_x(A_i V_i) &= d_{1,i}^{(1)}(t, x), \\
\partial_t V_i + V_i \partial_x V_i + g \partial_1 G_i(A_i, x) \partial_x A_i \\
+ g \partial_2 G_i(A_i, x) + g(S_{f,i}(A_i, V_i, x) - S_{b,i}(x)) \\
&= d_{1,i}^{(2)}(t, x),
\end{aligned} \tag{3}$$

where  $(d_{1,i}^{(1)}, d_{1,i}^{(2)})_{i \in \{1, \dots, N\}}$  are the disturbances in the dynamics with boundary conditions

$$\begin{aligned}
A_1(t, 0)V_1(t, 0) &= Q_0 + d_{2,0}(t), \\
A_{j+1}(t, 0)V_{j+1}(t, 0) &= A_j(t, L_j)V_j(t, L_j), \\
A_i(t, L_i)V_i(t, L_i) &= k_{p,i}(A_i^*(L_i) - A_i(t, L_i)) \\
&\quad + k_{I,i}Z_i(t) + d_{2,i}(t), \\
\dot{Z}_i(t) &= A_i^*(L_i) - A_i(t, L_i),
\end{aligned} \tag{4}$$

where  $d_{2,0}$  and  $(d_{2,i})_{i \in \{1, \dots, N\}}$  are the boundary disturbances. The steady-states  $(A_i^*(x), V_i^*(x), Z_i^*)$  in the absence of the disturbances satisfy

$$\begin{aligned}
(A_i^* V_i^*)_x &= 0, \\
V_i^* V_{ix}^* + g \partial_1 G_i(A_i^*, x) A_{ix}^* + g \partial_2 G_i(A_i^*, x) \\
+ g(S_{f,i}(A_i^*, V_i^*, x) - S_{b,i}(x)) &= 0, \\
Z_i^* &= \frac{Q_0}{k_{I,i}}.
\end{aligned} \tag{5}$$

This system is well-posed as long as  $g \partial_1 G_i(A_i^*, x) A_i^* - V_i^{*2} \neq 0$  (see [26]) and in this case, the steady-states  $(A_i^*, V_i^*)$  satisfy

$$(A_i^*)_x = \frac{-g A_i^* (\partial_2 G_i(A_i^*, x) + S_{f,i}(A_i^*, V_i^*, x) - S_{b,i}(x))}{g \partial_1 G_i(A_i^*, x) A_i^* - V_i^{*2}}, \tag{6}$$

$$(V_i^*)_x = \frac{g V_i^* (\partial_2 G_i(A_i^*, x) + S_{f,i}(A_i^*, V_i^*, x) - S_{b,i}(x))}{g \partial_1 G_i(A_i^*, x) A_i^* - V_i^{*2}}. \tag{7}$$

In this paper, we consider the physical steady-states:  $A_i^*(x) > 0$  and  $V_i^*(x) > 0$  for all  $x \in [0, L_i]$ . We know that the solutions  $(A_i^*(x), V_i^*(x))$  of (6)–(7) exist for all  $x \in [0, L_i]$  as long as  $g\partial_1 G_i(A_i^*, x)A_i^* - V_i^{*2} \neq 0$  and in particular when the regimes are in the subcritical case, i.e. the following condition holds

$$g\partial_1 G_i(A_i^*, x)A_i^*(x) - V_i^{*2}(x) > 0. \quad (8)$$

The supercritical case where  $g\partial_1 G_i(A_i^*, x)A_i^*(x) - V_i^{*2}(x) < 0$  is less interesting from a control point of view since the wave propagation speeds have the same sign and therefore the system can always be stabilized by boundary controls (see [21]).

As the  $N$  channels only coupled at the boundary, using a scaling, we can transfer the system (3)–(4) to a new system in which the length of each channel is unit. Indeed, let us introduce for the  $i$ th channel the new variable  $y := x/L_i$  and denote by  $\tilde{A}_i(t, y) := A_i(t, L_i y)$  and  $\tilde{V}_i(t, y) := V_i(t, L_i y)$  the new unknowns. Without ambiguity, we still denote by  $x$  the space variable, then we obtain the new disturbed system

$$\begin{aligned} \partial_t \tilde{A}_i + \frac{1}{L_i} \partial_x (\tilde{A}_i \tilde{V}_i) &= d_{1,i}^{(1)}(t, L_i x) := \tilde{d}_{1,i}^{(1)}(t, x), \\ \partial_t \tilde{V}_i + \frac{1}{L_i} \tilde{V}_i \partial_x \tilde{V}_i + \frac{1}{L_i} g \partial_1 G_i(\tilde{A}_i, L_i x) \partial_x \tilde{A}_i & \\ + g \partial_2 G_i(\tilde{A}_i, L_i x) + g(S_{f,i}(\tilde{A}_i, \tilde{V}_i, L_i x) - S_{b,i}(L_i x)) & \\ = d_{1,i}^{(2)}(t, L_i x) := \tilde{d}_{1,i}^{(2)}(t, x), & \end{aligned} \quad (9)$$

defined on  $[0, +\infty) \times [0, 1]$ . Noticing that if we denote by  $(\tilde{A}_i^*, \tilde{V}_i^*)$  the steady-states of the above new system, then  $\tilde{A}_i^*(x) = A_i^*(L_i x)$  and  $\tilde{V}_i^*(x) = V_i^*(L_i x)$  with  $(A_i^*, V_i^*)$  the steady-states of (1). Thus, the boundary conditions (4) become

$$\begin{aligned} \tilde{A}_1(t, 0) \tilde{V}_1(t, 0) &= Q_0 + d_{2,0}(t), \\ \tilde{A}_{j+1}(t, 0) \tilde{V}_{j+1}(t, 0) &= \tilde{A}_j(t, 1) \tilde{V}_j(t, 1), \\ \tilde{A}_i(t, 1) \tilde{V}_i(t, 1) &= k_{p,i}(\tilde{A}_i^*(1) - \tilde{A}_i(t, 1)) \\ &\quad + k_{I,i} Z_i(t) + d_{2,i}(t), \\ \dot{Z}_i(t) &= \tilde{A}_i^*(1) - \tilde{A}_i(t, 1), \end{aligned} \quad (10)$$

for the new unknowns  $\tilde{A}_i$  and  $\tilde{V}_i$ . Thus, we have transformed the disturbed system (3)–(4) to the new system (9)–(10). In the following, we will focus on the ISS property on the new system and we have the main result as following

**Theorem 2.1.** *Assume that for  $i \in \{1, \dots, N\}$*

$$\begin{aligned} k_{p,i} < 0 \text{ and } k_{I,i} < 0, \text{ or} \\ k_{p,i} > \frac{g\partial_1 G_i(\tilde{A}_i^*(1), L_i) \tilde{A}_i^*(1) - \tilde{V}_i^{*2}(1)}{\tilde{V}_i^*(1)} \text{ and } k_{I,i} > 0, \end{aligned}$$

*then the nonlinear system (9)–(10) is locally (exponentially) Input-to-State Stable. More precisely, there exist  $\gamma > 0$ ,  $\delta > 0$  and  $C > 0$  such that for any  $T > 0$ , any  $\mathbf{c} = (c_i)_{i \in \{0,1,\dots,N\}} \in \mathbb{R}^{N+1}$ , any  $(\tilde{A}_i^0, \tilde{V}_i^0, Z_i^0) \in C^0([0, T]; H^2((0, 1); \mathbb{R}^2)) \times \mathbb{R}$  and any disturbances  $\mathbf{d}_1 = (\tilde{d}_{1,i}^{(1)}, \tilde{d}_{2,i}^{(2)})_{i \in \{1,\dots,N\}} \in C^0([0, T]; H^2((0, 1); \mathbb{R}^{2N}))$  and*

$\mathbf{d}_2 = (d_{2,0}, (d_{2,i})_{i \in \{1, \dots, N\}}) \in H^2((0, T); \mathbb{R}^{N+1})$  satisfying the first-order compatibility conditions<sup>1</sup> associated to (10) with

$$\|(\tilde{A}_i^0 - \tilde{A}_i^*, \tilde{V}_i^0 - \tilde{V}_i^*)\|_{H^2(0,1)} + |Z_i^0 - Z_i^* + \frac{c_i}{k_{I,i}}| < \delta \quad (11)$$

for  $i \in \{1, \dots, N\}$  and

$$\|\mathbf{d}_1(t, x)\|_{C^0([0, T]; H^2(0,1))} + \|\mathbf{d}_2(t) - \mathbf{c}\|_{H^2(0, T)} < \delta,$$

there exists a unique solution  $(\tilde{A}_i, \tilde{V}_i, Z_i)_{i \in \{1, \dots, N\}} \in C^0([0, T]; H^2((0, 1); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R})$  to (9)–(10) with initial condition  $(\tilde{A}_i^0, \tilde{V}_i^0, Z_i^0)_{i \in \{1, \dots, N\}}$  and we have the following ISS estimate

$$\begin{aligned} & \sum_{i=1}^N \left( \|\tilde{A}_i(t, \cdot) - \tilde{A}_i^*, \tilde{V}_i(t, \cdot) - \tilde{V}_i^*\|_{H^2(0,1)} + |Z_i(t) - (Z_i^* - \frac{c_i}{k_{I,i}})| \right) \\ & \leq C e^{-\gamma t} \left( \sum_{i=1}^N \left( \|\tilde{A}_i^0 - \tilde{A}_i^*, \tilde{V}_i^0 - \tilde{V}_i^*\|_{H^2(0,1)} + |Z_i^0 - (Z_i^* - \frac{c_i}{k_{I,i}})| \right) \right) \\ & \quad + C \left( \sum_{k=0}^2 \|e^{-\gamma(t-\tau)} (\mathbf{d}_2 - \mathbf{c})^{(k)}(\tau)\|_{L^2(0,t)} \right) \\ & \quad + C \left( \sum_{\substack{k_1+k_2 \leq 2 \\ k_2 \leq 1}} \|e^{-\gamma(t-\tau)} \partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(\tau, x)\|_{L^2((0,t) \times (0,1))} \right) \end{aligned} \quad (12)$$

for any  $t \in [0, +\infty)$ .

**Remark 2.1** (Constant disturbances). Here  $\mathbf{c}$  can be chosen arbitrarily which means that, thanks to the PI control, the system is robust to any constant disturbances. Hence the result applies when the boundary disturbances  $\mathbf{d}_2$  are not necessarily small but only close to a constant disturbance. This explains that the ISS estimate (12) involves the norm of  $\mathbf{d}_2 - \mathbf{c}$  instead of the norm of  $\mathbf{d}_2$ . In particular, when there are no internal disturbances but only constant boundary disturbances (i.e.  $\mathbf{d}_1 \equiv \mathbf{0}$  and  $\mathbf{d}_2 \equiv \mathbf{c}$  for some  $\mathbf{c} \in \mathbb{R}^{N+1}$ ), the system is exponentially stable. This special case is also studied in [43] for small constant disturbances.

### 3 A Lyapunov function for the linearized cascade system

We start by looking at the exponential stability of the linearized system associated with the undisturbed system (9)–(10). To this aim, we define the perturbation functions  $a_i$ ,  $v_i$  and  $z_i$  with respect to the steady-states  $(\tilde{A}_i^*(x), \tilde{V}_i^*(x), Z_i^*)$  as

$$\begin{aligned} a_i(t, x) &= \tilde{A}_i(t, x) - \tilde{A}_i^*(x), \\ v_i(t, x) &= \tilde{V}_i(t, x) - \tilde{V}_i^*(x), \\ z_i(t) &= Z_i(t) - Z_i^*. \end{aligned} \quad (13)$$

<sup>1</sup>For an explanation of the concept of the compatibility conditions, see [4, Section 4.5.2]

The linearization of the system (9) and the dynamics of the integral terms  $Z_i$  around the steady-states is

$$\begin{aligned}
a_{it} + \frac{1}{L_i} \tilde{V}_i^* a_{ix} + \frac{1}{L_i} \tilde{A}_i^* v_{ix} + \frac{1}{L_i} \tilde{V}_{ix}^* a_i + \frac{1}{L_i} \tilde{A}_{ix}^* v_i &= 0, \\
v_{it} + \frac{1}{L_i} v_i \tilde{V}_{ix}^* + \frac{1}{L_i} \tilde{V}_i^* v_{ix} + \frac{1}{L_i} g \partial_1 G_i(\tilde{A}_i^*, L_i x) a_{ix} \\
+ \frac{1}{L_i} g \partial_{11}^2 G_i(\tilde{A}_i^*, L_i x) \tilde{A}_{ix}^* a_i + g \partial_{21}^2 G_i(\tilde{A}_i^*, L_i x) a_i \\
+ g \partial_1 S_{f,i}(\tilde{A}_i^*, \tilde{V}_i^*, L_i x) a_i + g \partial_2 S_{f,i}(\tilde{A}_i^*, \tilde{V}_i^*, L_i x) v_i &= 0, \\
\dot{z}_i &= -a_i(t, 1).
\end{aligned} \tag{14}$$

The linearized boundary conditions become

$$\tilde{V}_1^*(0) a_1(t, 0) + \tilde{A}_1^*(0) v_1(t, 0) = 0, \tag{15}$$

$$\begin{aligned}
\tilde{V}_{j+1}^*(0) a_{j+1}(t, 0) + \tilde{A}_{j+1}^*(0) v_{j+1}(t, 0) \\
= \tilde{V}_j^*(1) a_j(t, 1) + \tilde{A}_j^*(1) v_j(t, 1),
\end{aligned} \tag{16}$$

$$\begin{aligned}
\tilde{V}_i^*(1) a_i(t, 1) + \tilde{A}_i^*(1) v_i(t, 1) \\
= -k_{p,i} a_i(t, 1) + k_{I,i} z_i(t).
\end{aligned} \tag{17}$$

We consider the linearized system (14), (15)–(17) under an initial condition

$$a_i(0, x) = a_i^0(x), v_i(0, x) = v_i^0(x), z_i(0) = z_i^0 \tag{18}$$

such that

$$(a_i^0(x), v_i^0(x)) \in L^2((0, 1); \mathbb{R}^2), z_i^0 \in \mathbb{R}. \tag{19}$$

The Cauchy problem (14), (15)–(18) is well-posed (see [4]).

We give the following definition of the exponential stability in the  $L^2$  norm.

**Definition 1.** *The system (14), (15)–(18) is exponentially stable for the  $L^2$  norm if there exist  $\nu > 0$  and  $C > 0$  such that for every  $(a_i^0, v_i^0, z_i^0) \in L^2((0, 1); \mathbb{R}^2) \times \mathbb{R}$  ( $i \in \{1, \dots, N\}$ ), the solution to the Cauchy problem (14), (15)–(18) satisfies*

$$\begin{aligned}
&\sum_{i=1}^N \left[ \|(a_i(t, \cdot), v_i(t, \cdot))\|_{L^2(0,1)} + |z_i(t)| \right] \\
&\leq C e^{-\nu t} \sum_{i=1}^N \left[ \|(a_i^0, v_i^0)\|_{L^2(0,1)} + |z_i^0| \right], \quad \forall t \in [0, \infty).
\end{aligned}$$

In this section, we show the following result

**Theorem 3.1.** *If the control parameters satisfy*

$$\begin{aligned}
&k_{p,i} < 0 \text{ and } k_{I,i} < 0, \text{ or} \\
&k_{p,i} > \frac{g \partial_1 G_i(\tilde{A}_i^*(1), L_i) \tilde{A}_i^*(1) - \tilde{V}_i^{*2}(1)}{\tilde{V}_i^*(1)} \text{ and } k_{I,i} > 0,
\end{aligned} \tag{20}$$

*then the system (14), (15)–(18) is exponentially stable in the  $L^2$  norm.*



*Proof.* Observe that the linear system (14), (15)–(17) can be rewritten in Riemann coordinates, let

$$\begin{pmatrix} R_{1i} \\ R_{2i} \end{pmatrix} = \begin{pmatrix} L_i \sqrt{\frac{g\partial_1 G_i(\tilde{A}_i^*, L_i x)}{\tilde{A}_i^*}} & L_i \\ -L_i \sqrt{\frac{g\partial_1 G_i(\tilde{A}_i^*, L_i x)}{\tilde{A}_i^*}} & L_i \end{pmatrix} \begin{pmatrix} a_i \\ v_i \end{pmatrix}. \quad (21)$$

Then the system (14) is transformed into

$$\begin{aligned} \partial_t R_{1i} + \lambda_{1i} \partial_x R_{1i} + \gamma_{1i} R_{1i} + \delta_{1i} R_{2i} &= 0, \\ \partial_t R_{2i} - \lambda_{2i} \partial_x R_{2i} + \gamma_{2i} R_{1i} + \delta_{2i} R_{2i} &= 0, \\ \dot{z}_i &= \frac{R_{2i}(t, 1) - R_{1i}(t, 1)}{2L_i} \sqrt{\frac{\tilde{A}_i^*(1)}{g\partial_1 G_i(\tilde{A}_i^*(1), L_i)}}, \end{aligned} \quad (22)$$

where  $\lambda_{1i}$ ,  $\lambda_{2i}$ ,  $\gamma_{1i}$ ,  $\gamma_{2i}$ ,  $\delta_{1i}$  and  $\delta_{2i}$  are all space-dependent and defined by the following

$$\begin{aligned} \lambda_{1i} &= \frac{1}{L_i} \left( \tilde{V}_i^* + \sqrt{g\partial_1 G_i(\tilde{A}_i^*, L_i x) \tilde{A}_i^*} \right) > 0, \\ \lambda_{2i} &= \frac{1}{L_i} \left( \sqrt{g\partial_1 G_i(\tilde{A}_i^*, L_i x) \tilde{A}_i^*} - \tilde{V}_i^* \right) > 0 \end{aligned} \quad (23)$$

and using the abbreviation  $G_i$  for  $G_i(\tilde{A}_i^*, L_i x)$

$$\begin{aligned} \gamma_{1i} &= \frac{1}{4} \left[ 2g\partial_2 S_{f,i} + 2g(\partial_{21}^2 G_i + \partial_1 S_{f,i}) \sqrt{\frac{\tilde{A}_i^*}{g\partial_1 G_i}} \right. \\ &\quad \left. + 3\lambda_{2i} \frac{\tilde{A}_{ix}^*}{\tilde{A}_i^*} - \lambda_{1i} \frac{L_i \partial_{12}^2 G_i}{\partial_1 G_i} + \lambda_{2i} \frac{\partial_{11}^2 G_i \tilde{A}_{ix}^*}{\partial_1 G_i} \right], \\ \gamma_{2i} &= \frac{1}{4} \left[ 2g\partial_2 S_{f,i} + 2g(\partial_{21}^2 G_i + \partial_1 S_{f,i}) \sqrt{\frac{\tilde{A}_i^*}{g\partial_1 G_i}} \right. \\ &\quad \left. - \lambda_{1i} \frac{\tilde{A}_{ix}^*}{\tilde{A}_i^*} - \lambda_{2i} \frac{L_i \partial_{12}^2 G_i}{\partial_1 G_i} + \lambda_{1i} \frac{\partial_{11}^2 G_i \tilde{A}_{ix}^*}{\partial_1 G_i} \right], \\ \delta_{1i} &= \frac{1}{4} \left[ 2g\partial_2 S_{f,i} - 2g(\partial_{21}^2 G_i + \partial_1 S_{f,i}) \sqrt{\frac{\tilde{A}_i^*}{g\partial_1 G_i}} \right. \\ &\quad \left. + \lambda_{2i} \frac{\tilde{A}_{ix}^*}{\tilde{A}_i^*} + \lambda_{1i} \frac{L_i \partial_{12}^2 G_i}{\partial_1 G_i} - \lambda_{2i} \frac{\partial_{11}^2 G_i \tilde{A}_{ix}^*}{\partial_1 G_i} \right], \\ \delta_{2i} &= \frac{1}{4} \left[ 2g\partial_2 S_{f,i} - 2g(\partial_{21}^2 G_i + \partial_1 S_{f,i}) \sqrt{\frac{\tilde{A}_i^*}{g\partial_1 G_i}} \right. \\ &\quad \left. - 3\lambda_{1i} \frac{\tilde{A}_{ix}^*}{\tilde{A}_i^*} + \lambda_{2i} \frac{L_i \partial_{12}^2 G_i}{\partial_1 G_i} - \lambda_{1i} \frac{\partial_{11}^2 G_i \tilde{A}_{ix}^*}{\partial_1 G_i} \right]. \end{aligned}$$

The boundary conditions (15)–(17) can be expressed as

$$R_{11}(t, 0)\lambda_{11}(0) + R_{21}(t, 0)\lambda_{21}(0) = 0, \quad (24)$$

$$\sqrt{\frac{\tilde{A}_{j+1}^*(0)}{g\partial_1 G_{j+1}(\tilde{A}_{j+1}^*(0), 0)}} \left( R_{1,j+1}(t, 0)\lambda_{1,j+1}(0) + R_{2,j+1}(t, 0)\lambda_{2,j+1}(0) \right) \quad (25)$$

$$= \sqrt{\frac{\tilde{A}_j^*(1)}{g\partial_1 G_j(\tilde{A}_j^*(1), L_i)}} \left( R_{1j}(t, 1)\lambda_{1j}(1) + R_{2j}(t, 1)\lambda_{2j}(1) \right),$$

$$R_{1i}(t, 1)\left(\lambda_{1i}(1) + \frac{1}{L_i}k_{p,i}\right) + R_{2i}(t, 1)\left(\lambda_{2i}(1) - \frac{1}{L_i}k_{p,i}\right)$$

$$= 2k_{L,i}z_i(t) \sqrt{\frac{g\partial_1 G_i(\tilde{A}_i^*(1), L_i)}{\tilde{A}_i^*(1)}}. \quad (26)$$

For each channel, we rely on the construction of a Lyapunov function and since we study the cascade channels, we have to combine the Lyapunov functions by adding proper weights for each channel. Different from [6] or [13], as the extra integral terms need to be processed, we shall consider the items of  $z_i$ . Inspired by [24,26], we consider the Lyapunov function candidate as the following

$$V = \sum_{i=1}^N \alpha_i \int_0^1 \left( f_{1i}(x) e^{2 \int_0^x \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds} R_{1i}^2(t, x) + f_{2i}(x) e^{-2 \int_0^x \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} ds} R_{2i}^2(t, x) \right) dx + \sum_{i=1}^N \theta_i L_i z_i^2(t), \quad (27)$$

where  $\alpha_i > 0$  and  $\theta_i > 0$  are constants to be chosen and  $f_{1i}, f_{2i}$  are given by

$$f_{1i} = \frac{1}{\lambda_{1i}\eta_{i,\varepsilon}}, \quad f_{2i} = \frac{\eta_{i,\varepsilon}}{\lambda_{2i}}, \quad (28)$$

where  $\eta_{i,\varepsilon}$  is the solution to

$$\begin{cases} \eta'_{i,\varepsilon} = \left| \frac{\delta_{1i}\varphi_i}{\lambda_{1i}} + \frac{\gamma_{2i}}{\lambda_{2i}\varphi_i} \eta_{i,\varepsilon}^2 \right| + \varepsilon \\ \eta_{i,\varepsilon}(0) = \frac{\lambda_{2i}(0)}{\lambda_{1i}(0)} + \varepsilon, \end{cases} \quad (29)$$

with  $\varphi_i$  defined by

$$\varphi_i(x) = \exp \left( \int_0^x \left( \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} + \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} \right) ds \right). \quad (30)$$

An important fact (showed in [26]) is that there exists  $\varepsilon_1 > 0$  such that for any  $\varepsilon \in [0, \varepsilon_1)$ ,  $\eta_{i,\varepsilon}$  exists on  $[0, 1]$  and in particular, when  $\varepsilon = 0$

$$\eta_{i,0} = \frac{\lambda_{2i}}{\lambda_{1i}} \varphi_i. \quad (31)$$

By (21), it is not difficult to check that there exist constant  $\beta > 0$  such that

$$\begin{aligned} \frac{1}{\beta} \sum_{i=1}^N \left[ \|(a_i(t, \cdot), v_i(t, \cdot))\|_{L^2(0,1)}^2 + z_i^2(t) \right] &\leq V(t) \\ &\leq \beta \sum_{i=1}^N \left[ \|(a_i(t, \cdot), v_i(t, \cdot))\|_{L^2(0,1)}^2 + z_i^2(t) \right] \end{aligned} \quad (32)$$

for any  $t \in [0, +\infty)$ .

By introducing the compact notations

$$Y_i = \begin{pmatrix} R_{1i} \\ R_{2i} \end{pmatrix} \quad (33)$$

and

$$\mathbf{D}_i(x) = \begin{pmatrix} f_{1i} e^{2 \int_0^x \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds} & 0 \\ 0 & f_{2i} e^{-2 \int_0^x \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} ds} \end{pmatrix}, \quad (34)$$

the equations for  $(R_{1i}, R_{2i})$  in (22) can be rewritten as

$$\partial_t Y_i + \mathbf{A}_i(x) \partial_x Y_i + \mathbf{B}_i(x) Y_i = 0, \quad (35)$$

where

$$\mathbf{A}_i(x) = \begin{pmatrix} \lambda_{1i} & 0 \\ 0 & -\lambda_{2i} \end{pmatrix}, \quad \mathbf{B}_i(x) = \begin{pmatrix} \gamma_{1i} & \delta_{1i} \\ \gamma_{2i} & \delta_{2i} \end{pmatrix} \quad (36)$$

and the simplified form of (27) can be written as

$$V = \sum_{i=1}^N \alpha_i \int_0^1 Y_i^T \mathbf{D}_i(x) Y_i dx + \sum_{i=1}^N \theta_i L_i z_i^2. \quad (37)$$

The time derivative of  $V$  along the  $C^1$  trajectories of (35) is

$$\begin{aligned} \dot{V} &= - \sum_{i=1}^N \alpha_i \int_0^1 (\mathbf{A}_i(x) Y_{ix} + \mathbf{B}_i(x) Y_i)^T \mathbf{D}_i(x) Y_i dx \\ &\quad - \sum_{i=1}^N \alpha_i \int_0^1 Y_i^T \mathbf{D}_i(x) (\mathbf{A}_i(x) Y_{ix} + \mathbf{B}_i(x) Y_i) dx \\ &\quad + \sum_{i=1}^N \theta_i z_i(t) (R_{2i}(t, 1) - R_{1i}(t, 1)) \sqrt{\frac{A_i^*(1)}{g \partial_1 G_i(A_i^*(1), L_i)}}. \end{aligned} \quad (38)$$

According to (34) and (36), we observe that

$$\begin{aligned} \mathbf{M}_i(x) &:= \mathbf{A}_i(x)^T \mathbf{D}_i(x) = \mathbf{D}_i(x) \mathbf{A}_i(x) \\ &= \begin{pmatrix} \lambda_{1i} f_{1i} e^{2 \int_0^x \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds} & 0 \\ 0 & -\lambda_{2i} f_{2i} e^{-2 \int_0^x \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} ds} \end{pmatrix}. \end{aligned}$$

Therefore, using the symmetric matrices  $\mathbf{M}_i(x)$  and (38), we obtain

$$\begin{aligned} \dot{V} = & - \left[ \sum_{i=1}^N \alpha_i \int_0^1 (Y_i^T \mathbf{M}_i(x) Y_i)_x dx + \sum_{i=1}^N \alpha_i \int_0^1 Y_i^T \mathbf{N}_i(x) Y_i dx \right] \\ & + \sum_{i=1}^N \theta_i z_i (R_{2i}(t, 1) - R_{1i}(t, 1)) \sqrt{\frac{A_i^*(1)}{g \partial_1 G_i(A_i^*(1), L_i)}}, \end{aligned}$$

where

$$\mathbf{N}_i(x) = \mathbf{B}_i(x)^T \mathbf{D}_i(x) + \mathbf{D}_i(x) \mathbf{B}_i(x) - \mathbf{M}_i'(x). \quad (39)$$

Under constraints (20), we need to select appropriate  $\alpha_i$  and  $\theta_i$  in order to guarantee that  $\dot{V}$  is negative definite along the  $C^1$  solutions of the system (22), (24)–(26), i.e., the following conditions hold:

(a) The boundary part is nonpositive, namely

$$\begin{aligned} & - \sum_{i=1}^N \alpha_i [Y_i^T \mathbf{M}_i(x) Y_i]_0^1 \\ & + \sum_{i=1}^N \theta_i z_i (R_{2i}(t, 1) - R_{1i}(t, 1)) \sqrt{\frac{A_i^*(1)}{g \partial_1 G_i(A_i^*(1), L_i)}} \\ & \leq 0 \end{aligned} \quad (40)$$

for every  $Y_i(t, 0) \in \mathbb{R}^{2N}$  and  $Y_i(t, 1) \in \mathbb{R}^{2N}$ ;

(b) The matrices  $\mathbf{N}_i(x)$  are positive definite for every  $x \in [0, 1]$ .

Indeed, noticing that once conditions (a) and (b) hold, one can find a sufficiently small positive real number  $\mu$  such that

$$\frac{dV}{dt} \leq -\mu V \quad (41)$$

along the  $C^1$  solutions of the system. Since the  $C^1$  solutions are dense in the set of  $L^2$  solutions, inequality (41) is also satisfied in the sense of distributions for the  $L^2$  solutions (see [4, Section 2.1.3] for more details). Consequently,  $V$  is an exponentially decaying Lyapunov function and this, together with (32), completes the proof of Theorem 3.1.

We first analyze condition (b). From (39), we get

$$\mathbf{N}_i(x) = \begin{pmatrix} N_{i_1} & N_{i_2} \\ N_{i_3} & N_{i_4} \end{pmatrix}, \quad (42)$$

where

$$\begin{aligned} N_{i_1} &= -(\lambda_{1i} f_{1i})' e^{2 \int_0^x \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds}, \\ N_{i_2} = N_{i_3} &= \gamma_{2i} f_{2i} e^{-2 \int_0^x \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} ds} + \delta_{1i} f_{1i} e^{2 \int_0^x \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds}, \\ N_{i_4} &= (\lambda_{2i} f_{2i})' e^{-2 \int_0^x \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} ds}. \end{aligned}$$

From the definitions of  $f_{1i}$  and  $f_{2i}$  in (28), the diagonal elements of  $\mathbf{N}_i(x)$  are positive. Moreover, we obtain

$$\begin{aligned} \det[\mathbf{N}_i(x)] &= e^{2 \int_0^x \left( \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} - \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} \right) ds} \left( -(\lambda_{1i} f_{1i})' (\lambda_{2i} f_{2i})' \right. \\ &\quad \left. - \left( \frac{\gamma_{2i} f_{2i}}{\varphi_i} + \delta_{1i} f_{1i} \varphi_i \right)^2 \right) > 0 \end{aligned}$$

for any  $\varepsilon \in (0, \varepsilon_1)$ . Thus the matrices  $\mathbf{N}_i(x)$  are positive definite for all  $x \in [0, 1]$  when  $\varepsilon$  is sufficient small.

We now analyze condition (a). We first express the boundary conditions (24)–(26) as follows. From (24),

$$R_{11}(t, 0) = -\frac{R_{21}(t, 0) \lambda_{21}(0)}{\lambda_{11}(0)}. \quad (43)$$

From (26),

$$\begin{aligned} R_{2i}(t, 1) &= \underbrace{\frac{k_{p,i} + L_i \lambda_{1i}(1)}{k_{p,i} - L_i \lambda_{2i}(1)}}_{s_{1i}} R_{1i}(t, 1) \\ &\quad + 2 \underbrace{\sqrt{\frac{g \partial_1 G_i(A_i^*(1), L_i)}{A_i^*(1)} \frac{k_{I,i}}{L_i \lambda_{2i}(1) - k_{p,i}}}}_{s_{2i}} z_i(t). \end{aligned}$$

From (25), one has

$$\begin{aligned} R_{1,j+1}(t, 0) &= c_j \left[ \lambda_{1j}(1) R_{1j}(t, 1) + \lambda_{2j}(1) R_{2j}(t, 1) \right] \\ &\quad - \frac{\lambda_{2,j+1}(0)}{\lambda_{1,j+1}(0)} R_{2,j+1}(t, 0). \end{aligned} \quad (44)$$

Here and hereafter, for simplicity, we denote the coefficient

$$c_j := \frac{1}{\lambda_{1,j+1}(0)} \sqrt{\frac{A_j^*(1)}{g \partial_1 G_j(A_j^*(1), L_j)}} \sqrt{\frac{g \partial_1 G_{j+1}(A_{j+1}^*(0), 0)}{A_{j+1}^*(0)}}.$$

Thus, the boundary part in  $\dot{V}$  given in condition (a) can be rewritten as

$$\begin{aligned} & - \sum_{i=1}^N \alpha_i [Y_i^T \mathbf{M}_i(x) Y_i]_0^1 \\ & + \sum_{i=1}^N \theta_i z_i(t) (R_{2i}(t, 1) - R_{1i}(t, 1)) \sqrt{\frac{A_i^*(1)}{g \partial_1 G_i(A_i^*(1), L_i)}} \\ & = -\alpha_1 R_{21}^2(t, 0) \left( \lambda_{21}(0) f_{21}(0) - \lambda_{11}(0) f_{11}(0) \left( \frac{\lambda_{21}(0)}{\lambda_{11}(0)} \right)^2 \right) \\ & - \begin{pmatrix} R_{1N}(t, 1) & z_N(t) \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \theta \end{pmatrix} \begin{pmatrix} R_{1N}(t, 1) \\ z_N(t) \end{pmatrix} \\ & - \sum_{j=1}^{N-1} \begin{pmatrix} R_{1j}(t, 1) & z_j(t) & R_{2,j+1}(t, 0) \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} \\ \bar{\beta} & \bar{\theta} & \bar{\eta} \\ \bar{\gamma} & \bar{\eta} & \bar{\zeta} \end{pmatrix} \begin{pmatrix} R_{1j}(t, 1) \\ z_j(t) \\ R_{2,j+1}(t, 0) \end{pmatrix} \end{aligned} \quad (45)$$

with

$$\begin{aligned}
\alpha &:= \alpha_N e^{2 \int_0^1 \frac{\gamma_{1N}(s)}{\lambda_{1N}(s)} ds} \left( \lambda_{1N}(1) f_{1N}(1) - s_{1N}^2 \frac{\lambda_{2N}(1) f_{2N}(1)}{\varphi_N^2(1)} \right), \\
\beta &:= -\alpha_N e^{2 \int_0^1 \frac{\gamma_{1N}(s)}{\lambda_{1N}(s)} ds} s_{1N} s_{2N} \frac{\lambda_{2N}(1) f_{2N}(1)}{\varphi_N^2(1)} \\
&\quad - \frac{\theta_N}{2} (s_{1N} - 1) \sqrt{\frac{A_N^*(1)}{g \partial_1 G_N(A_N^*(1), L_N)}}, \\
\theta &:= -\alpha_N e^{2 \int_0^1 \frac{\gamma_{1N}(s)}{\lambda_{1N}(s)} ds} s_{2N}^2 \frac{\lambda_{2N}(1) f_{2N}(1)}{\varphi_N^2(1)} \\
&\quad - \theta_N s_{2N} \sqrt{\frac{A_N^*(1)}{g \partial_1 G_N(A_N^*(1), L_N)}}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\alpha} &:= \alpha_j e^{2 \int_0^1 \frac{\gamma_{1j}(s)}{\lambda_{1j}(s)} ds} \left( \lambda_{1j}(1) f_{1j}(1) - s_{1j}^2 \frac{\lambda_{2j}(1) f_{2j}(1)}{\varphi_j^2(1)} \right) \\
&\quad - \alpha_{j+1} \lambda_{1,j+1}(0) f_{1,j+1}(0) c_j^2 (\lambda_{1j}(1) + \lambda_{2j}(1) s_{1j})^2, \\
\bar{\beta} &:= -\alpha_j e^{2 \int_0^1 \frac{\gamma_{1j}(s)}{\lambda_{1j}(s)} ds} s_{1j} s_{2j} \frac{\lambda_{2j}(1) f_{2j}(1)}{\varphi_j^2(1)} \\
&\quad - \frac{\theta_j}{2} (s_{1j} - 1) \sqrt{\frac{A_j^*(1)}{g \partial_1 G_j(A_j^*(1), L_j)}} \\
&\quad - \alpha_{j+1} \lambda_{1,j+1}(0) f_{1,j+1}(0) c_j^2 (\lambda_{1j}(1) \\
&\quad \quad + \lambda_{2j}(1) s_{1j}) \lambda_{2j}(1) s_{2j}, \\
\bar{\theta} &:= -\alpha_j e^{2 \int_0^1 \frac{\gamma_{1j}(s)}{\lambda_{1j}(s)} ds} s_{2j}^2 \frac{\lambda_{2j}(1) f_{2j}(1)}{\varphi_j^2(1)} \\
&\quad - \alpha_{j+1} \lambda_{1,j+1}(0) f_{1,j+1}(0) c_j^2 \lambda_{2j}^2(1) s_{2j}^2 \\
&\quad - \theta_j s_{2j} \sqrt{\frac{A_j^*(1)}{g \partial_1 G_j(A_j^*(1), L_j)}}, \\
\bar{\gamma} &:= \alpha_{j+1} \lambda_{2,j+1}(0) f_{1,j+1}(0) c_j (\lambda_{1j}(1) + \lambda_{2j}(1) s_{1j}), \\
\bar{\eta} &:= \alpha_{j+1} \lambda_{2,j+1}(0) f_{1,j+1}(0) c_j \lambda_{2j}(1) s_{2j}, \\
\bar{\zeta} &:= \alpha_{j+1} \left( -\lambda_{1,j+1}(0) f_{1,j+1}(0) \left( \frac{\lambda_{2,j+1}(0)}{\lambda_{1,j+1}(0)} \right)^2 \right. \\
&\quad \left. + \lambda_{2,j+1}(0) f_{2,j+1}(0) \right).
\end{aligned} \tag{46}$$

We analyze the sign of (45) from line to line. Firstly, from the definitions of  $f_{1i}$  and  $f_{2i}$  in (28) together with  $\eta_{i,\varepsilon}$  solution to (29), one gets that the first line in (45) is negative. Under condition (20), one has

$$s_{1i}^2 < \left( \frac{\lambda_{1i}(1)}{\lambda_{2i}(1)} \right)^2, \quad i = 1, \dots, N \tag{47}$$

then  $\alpha > 0$  follows immediately. We then calculate  $\alpha\theta - \beta^2$  and obtain

$$\alpha\theta - \beta^2 = \alpha_N^2 F_N(\theta_N/\alpha_N), \quad (48)$$

where

$$\begin{aligned} F_i(x) = & -\frac{x^2}{4}(s_{1i} - 1)^2 \frac{A_i^*(1)}{g\partial_1 G_i(A_i^*(1), L_i)} \\ & -xs_{2i}\lambda_{1i}(1)f_{1i}(1)e^{2\int_0^1 \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds} \sqrt{\frac{A_i^*(1)}{g\partial_1 G_i(A_i^*(1), L_i)}} \\ & +xs_{1i}s_{2i}e^{2\int_0^1 \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds} \sqrt{\frac{A_i^*(1)}{g\partial_1 G_i(A_i^*(1), L_i)}} \frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} \\ & -s_{2i}^2\lambda_{1i}(1)f_{1i}(1)e^{4\int_0^1 \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds} \frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} \end{aligned}$$

for any  $i \in \{1, \dots, N\}$ .

Note that  $F_i$  is a quadratic form whose discriminant is

$$\begin{aligned} \Delta = & \frac{s_{2i}^2 A_i^*(1) e^{4\int_0^1 \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds}}{g\partial_1 G_i(A_i^*(1), L_i)} \left( \frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} - \lambda_{1i}(1)f_{1i}(1) \right) \\ & \cdot \left( s_{1i}^2 \frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} - \lambda_{1i}(1)f_{1i}(1) \right). \end{aligned} \quad (49)$$

From the definitions of  $f_{1i}$ ,  $f_{2i}$  and  $\eta_{i,\varepsilon}$  in (28)–(29) and noticing the property (31) and  $0 < \lambda_{2i} < \lambda_{1i}$ , we have

$$\frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} - \lambda_{1i}(1)f_{1i}(1) = \frac{\eta_{i,\varepsilon}^2(1) - \varphi_i^2(1)}{\varphi_i^2(1)\eta_{i,\varepsilon}(1)} < 0. \quad (50)$$

Moreover, under condition (47), one obtains also that

$$s_{1i}^2 \frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} - \lambda_{1i}(1)f_{1i}(1) < 0, \quad (51)$$

which, together with (49) and (50) gives  $\Delta > 0$ . The axis of the symmetry of the quadratic form is

$$\begin{aligned} & \frac{2s_{2i}e^{2\int_0^1 \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds}}{(s_{1i} - 1)^2} \sqrt{\frac{g\partial_1 G_i(A_i^*(1), L_i)}{A_i^*(1)}} \\ & \cdot \left( s_{1i} \frac{\lambda_{2i}(1)f_{2i}(1)}{\varphi_i^2(1)} - \lambda_{1i}(1)f_{1i}(1) \right). \end{aligned} \quad (52)$$

Under condition (20), we can obtain

$$s_{2i} < 0, \quad s_{1i} < \frac{\lambda_{1i}(1)}{\lambda_{2i}(1)} < \left( \frac{\lambda_{1i}(1)}{\lambda_{2i}(1)} \right)^2, \quad (53)$$

which proves that the term in (52) is positive. Overall, we have proved that under condition (20), there exists  $\theta_i/\alpha_i > 0$  such that  $F_i(\theta_i/\alpha_i) > 0$  for any  $i \in \{1, \dots, N\}$  and in particular from (48), one obtains  $\alpha\theta - \beta^2 > 0$ .

Next, we analyze the last line in (45). This line is related to the connection between the previous reach  $j \in \{1, \dots, N-1\}$  and the next reach  $j+1$  at the

junctions. Therefore, in order to lighten the notations, in the following  $\lambda_{i,j+1}$  stands for  $\lambda_{i,j+1}(0)$  and  $\lambda_{i,j}$  for  $\lambda_{i,j}(1)$ , for  $i = 1, 2$ . We consider the leading principal minors of the  $3 \times 3$  matrix, namely

- (i)  $\bar{\alpha}$ ;
- (ii)  $\bar{\alpha}\bar{\theta} - \bar{\beta}^2$ ;
- (iii)  $\bar{\alpha}\bar{\theta}\bar{\zeta} + 2\bar{\beta}\bar{\eta}\bar{\gamma} - \bar{\gamma}^2\bar{\theta} - \bar{\eta}^2\bar{\alpha} - \bar{\beta}^2\bar{\zeta}$ .

Noticing the expression of  $\bar{\alpha}$  in (46), analyzing similarly to (51), one can easily check that under condition (20),  $\bar{\alpha} > 0$  when  $\alpha_{j+1}/\alpha_j$  is sufficiently small.

After some direct but tedious calculations, we get

$$\bar{\alpha}\bar{\theta} - \bar{\beta}^2 = \alpha_j^2 \left( F_j(\theta_j/\alpha_j) + \frac{\alpha_{j+1}}{\alpha_j} \tilde{F}_j(\theta_j/\alpha_j) \right), \quad (54)$$

where  $\tilde{F}_j$  is an affine function of  $\theta_j/\alpha_j$  and independent of  $(\alpha_i)_{i \in \{1, \dots, N\}}$ . Since there exists  $\theta_j/\alpha_j > 0$  such that  $F_j(\theta_j/\alpha_j) > 0$ , one obtains immediately that, for a given  $j$  and under condition (20), there exist  $\theta_j/\alpha_j > 0$  and sufficiently small  $\alpha_{j+1}/\alpha_j > 0$  such that  $\bar{\alpha}\bar{\theta} - \bar{\beta}^2 > 0$ .

For the third leading principal minor given by (iii), we decompose it as

$$\begin{aligned} & \bar{\alpha}\bar{\theta}\bar{\zeta} + 2\bar{\beta}\bar{\eta}\bar{\gamma} - \bar{\gamma}^2\bar{\theta} - \bar{\eta}^2\bar{\alpha} - \bar{\beta}^2\bar{\zeta} \\ &= \bar{\zeta}(\bar{\alpha}\bar{\theta} - \bar{\beta}^2) + \bar{\gamma}(\bar{\beta}\bar{\eta} - \bar{\gamma}\bar{\theta}) + \bar{\eta}(\bar{\beta}\bar{\gamma} - \bar{\eta}\bar{\alpha}) \\ &= \alpha_{j+1} \left\{ \left( \lambda_{2,j+1} f_{2,j+1} - \lambda_{1,j+1} f_{1,j+1} \left( \frac{\lambda_{2,j+1}}{\lambda_{1,j+1}} \right)^2 \right) (\bar{\alpha}\bar{\theta} - \bar{\beta}^2) \right. \\ & \quad \left. + \lambda_{2,j+1} f_{1,j+1} c_j \left[ (\lambda_{1j} + \lambda_{2j} s_{1j}) (\bar{\beta}\bar{\eta} - \bar{\gamma}\bar{\theta}) \right. \right. \\ & \quad \left. \left. + \lambda_{2j} s_{2j} (\bar{\beta}\bar{\gamma} - \bar{\eta}\bar{\alpha}) \right] \right\}. \end{aligned} \quad (55)$$

After direct computations, we obtain that

$$\begin{aligned} & (\lambda_{1j} + \lambda_{2j} s_{1j}) (\bar{\beta}\bar{\eta} - \bar{\gamma}\bar{\theta}) + \lambda_{2j} s_{2j} (\bar{\beta}\bar{\gamma} - \bar{\eta}\bar{\alpha}) \\ &= \lambda_{2,j+1} f_{1,j+1} c_j \alpha_j^2 \frac{\alpha_{j+1}}{\alpha_j} s_{2j} \left[ \frac{\theta_j}{\alpha_j} \sqrt{\frac{A_j^*(1)}{g \partial_1 G_j(A_j^*(1), L_j)}} \right. \\ & \quad \left. (\lambda_{1j} + \lambda_{2j} s_{1j}) (\lambda_{1j} + \lambda_{2j}) - e^{2 \int_0^1 \frac{\gamma_{1j}(s)}{\lambda_{1j}(s)} ds} \lambda_{1j} \lambda_{2j} s_{2j} (\lambda_{2j} f_{1j} - \frac{\lambda_{1j} f_{2j}}{\varphi_j^2}) \right]. \end{aligned}$$

Substituting the above equation into (55), we obtain

$$\begin{aligned} & \frac{1}{\alpha_{j+1}} (\bar{\alpha}\bar{\theta}\bar{\zeta} + 2\bar{\beta}\bar{\eta}\bar{\gamma} - \bar{\gamma}^2\bar{\theta} - \bar{\eta}^2\bar{\alpha} - \bar{\beta}^2\bar{\zeta}) \\ &= \alpha_j^2 \left( \lambda_{2,j+1} f_{2,j+1} - \lambda_{1,j+1} f_{1,j+1} \left( \frac{\lambda_{2,j+1}}{\lambda_{1,j+1}} \right)^2 \right) \\ & \quad \cdot \left( F_j(\theta_j/\alpha_j) + \frac{\alpha_{j+1}}{\alpha_j} \widehat{F}_j(\theta_j/\alpha_j) \right), \end{aligned} \quad (56)$$

where  $\widehat{F}_j(x)$  is an affine function of  $\theta_j/\alpha_j$  and independent of  $(\alpha_i)_{i \in \{1, \dots, N\}}$ . From the definitions of  $f_{1i}$ ,  $f_{2i}$ ,  $\eta_{i,\varepsilon}$  and  $\varphi_i$  in (28)–(30), one obtains

$$\lambda_{2,j+1} f_{2,j+1} - \lambda_{1,j+1} f_{1,j+1} \left( \frac{\lambda_{2,j+1}}{\lambda_{1,j+1}} \right)^2 > 0.$$



Since for a given  $j$  and under condition (20), there exists  $\theta_j/\alpha_j > 0$  such that  $F_j(\theta_j/\alpha_j) > 0$ , we can then choose  $\alpha_{j+1}$  sufficiently small given  $\alpha_j$  such that  $F_j(\theta_j/\alpha_j) + (\alpha_{j+1}/\alpha_j)\widehat{F}_j(\theta_j/\alpha_j)$  is positive, which gives that the third leading principal minor (iii) is positive. Thus, we get that the third line of (45) is negative.

Overall, with the Lyapunov function candidate defined in (27), we managed to select appropriate  $\alpha_i$  and  $\theta_i$  under constraints (20) such that condition (a) holds.

The proof of Theorem 3.1 is completed.  $\square$

**Remark 3.1.** *Noticing that one of the key points in the proof relies on the construction of the weighted function (28) that depends on the existence of the solution to ODE (29). We have shown in [26] that for a class of general density-velocity equations, such condition is satisfied. Thus, the proof can be applied also to the general density-velocity equations.*

## 4 Exponential stability of the steady-states and output regulation of the nonlinear system

In fact, the result of Section 3 also holds for the original nonlinear system provided that we use the  $H^2$  norm instead of the  $L^2$  norm. In this section, we give a more general result in an abstract framework: if the linearized hyperbolic system has a basic quadratic Lyapunov function for the  $L^2$  norm, then the nonlinear system admits a basic quadratic Lyapunov function for the  $H^2$  norm and the disturbed system is in addition exponentially ISS in the  $H^2$  norm. Then we show that Theorem 2.1 follows directly. Let a general quasilinear hyperbolic system with disturbances be of the form

$$\begin{aligned} \partial_t \mathbf{u} + A(\mathbf{u}, x) \partial_x \mathbf{u} + B(\mathbf{u}, x) + \mathbf{d}_1(t, x) &= 0, \\ \begin{pmatrix} \mathbf{u}_+(t, 0) \\ \mathbf{u}_-(t, 1) \end{pmatrix} &= \mathcal{H} \left( \begin{pmatrix} \mathbf{u}_+(t, 1) \\ \mathbf{u}_-(t, 0) \end{pmatrix}, Z \right) + \mathbf{d}_2(t), \\ \dot{Z} &= \bar{\mathcal{H}} \left( \begin{pmatrix} \mathbf{u}_+(t, 1) \\ \mathbf{u}_-(t, 0) \end{pmatrix}, Z \right), \end{aligned} \quad (57)$$

where  $\mathbf{u} := (u_1, \dots, u_n)^T$  and  $u_i : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ ,  $Z : [0, +\infty) \rightarrow \mathbb{R}^n$ ,  $\mathbf{d}_1 : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^n$  and  $\mathbf{d}_2 : [0, +\infty) \rightarrow \mathbb{R}^n$ . Let  $\mathcal{M}_{n,n}(\mathbb{R})$  be the set of  $n \times n$  real matrices,  $A : \mathbb{R}^n \times [0, 1] \rightarrow \mathcal{M}_{n,n}(\mathbb{R})$  is of class  $C^2$  such that  $A(\mathbf{0}, x) = \text{diag}\{\lambda_1(x), \dots, \lambda_m(x), -\lambda_{m+1}(x), \dots, -\lambda_n(x)\} := \text{diag}\{\Lambda^+, \Lambda^-\} := \Lambda(x)$ , where  $\lambda_i(x) > 0, \forall x \in [0, 1]$ .  $\mathbf{u}^+ = (u_1, \dots, u_m)^T$  and  $\mathbf{u}^- = (u_{m+1}, \dots, u_n)^T$ .  $B : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$  is of class  $C^2$  such that  $B(\mathbf{0}, x) = \mathbf{0}$ .  $\mathcal{H}$  and  $\bar{\mathcal{H}} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are of class  $C^2$  and such that  $\mathcal{H}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$  and  $\bar{\mathcal{H}}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ .

**Theorem 4.1.** *Assume that there exists a diagonal and positive definite  $C^1$  matrix  $Q(x) := \text{diag}\{Q^+(x), Q^-(x)\}$  with  $Q^+$  a  $m \times m$  matrix function and  $Q^-$  a  $(n-m) \times (n-m)$  matrix function such that the matrix*

$$-(Q\Lambda)' + QM + M^T Q \quad (58)$$

*is positive definite, where  $M = \partial_1 B(\mathbf{0}, x)$ . Assume in addition that there exists a positive diagonal matrix  $\Theta = \text{diag}\{\Theta_1, \dots, \Theta_n\}$  such that the matrix*

$$W = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$$

is positive definite, where

$$\begin{aligned}
W_1 &:= \begin{pmatrix} Q^+(1)\Lambda^+(1) & 0 \\ 0 & -Q^-(0)\Lambda^-(0) \end{pmatrix} \\
&\quad - K_1^T \begin{pmatrix} Q^+(0)\Lambda^+(0) & 0 \\ 0 & -Q^-(1)\Lambda^-(1) \end{pmatrix} K_1 \\
W_2 &:= K_1^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} K_2 - \bar{K}_1^T \Theta^T \\
W_3 &:= K_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} K_1 - \Theta \bar{K}_1 \\
W_4 &:= K_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} K_2 - 2\Theta \bar{K}_2
\end{aligned}$$

with  $K_1 = \partial_1 \mathcal{H}(\mathbf{0}, \mathbf{0})$ ,  $K_2 = \partial_2 \mathcal{H}(\mathbf{0}, \mathbf{0})$ ,  $\bar{K}_1 = \partial_1 \bar{\mathcal{H}}(\mathbf{0}, \mathbf{0})$ ,  $\bar{K}_2 = \partial_2 \bar{\mathcal{H}}(\mathbf{0}, \mathbf{0})$ . Then the system (57) is locally exponentially ISS around  $\mathbf{u} \equiv \mathbf{0}$  for the  $H^2$  norm, i.e. there exist  $\gamma > 0$ ,  $\delta > 0$ ,  $C > 0$  such that for any  $T > 0$ , any  $(\mathbf{u}^0, Z^0) \in H^2((0, 1); \mathbb{R}^n) \times \mathbb{R}^n$ , and any disturbances  $\mathbf{d}_1 \in C^0([0, T]; H^2((0, 1); \mathbb{R}^n))$ ,  $\mathbf{d}_2 \in H^2((0, T); \mathbb{R}^n)$  satisfying the first order compatibility conditions associated to (57) with

$$(\|\mathbf{u}^0\|_{H^2(0,1)} + |Z^0|) < \delta; \quad \text{and}$$

$$\|\mathbf{d}_1(t, x)\|_{C^0([0, T]; H^2(0,1))} + \|\mathbf{d}_2(t)\|_{H^2(0, T)} < \delta, \quad (59)$$

the system (57) with initial condition  $(\mathbf{u}^0, Z^0)$  has a unique solution  $(\mathbf{u}, Z) \in C^0([0, T]; H^2((0, 1); \mathbb{R}^n)) \times C^1([0, T]; \mathbb{R}^n)$  satisfying

$$\begin{aligned}
& (\|\mathbf{u}(t, \cdot)\|_{H^2(0,1)} + |Z(t)|) \\
& \leq C e^{-\gamma t} (\|\mathbf{u}^0\|_{H^2(0,1)} + |Z^0|) \\
& \quad + C \left( \sum_{k=0}^2 \|e^{-\gamma(t-\tau)} \mathbf{d}_2^{(k)}(\tau)\|_{L^2(0,t)} \right) \\
& \quad + C \left( \sum_{\substack{k_1+k_2 \leq 2 \\ k_2 \leq 1}} \|e^{-\gamma(t-\tau)} \partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(\tau, x)\|_{L^2((0,t) \times (0,1))} \right)
\end{aligned}$$

for any  $t \in [0, +\infty)$ .

**Remark 4.1.** Just as in Theorem 2.1,  $\mathbf{d}_2$  can be taken close to a constant disturbance vector  $\mathbf{c}$  rather than close to  $\mathbf{0}$  as long as  $z \rightarrow \mathcal{H}(\mathbf{0}, Z)$  is invertible. Indeed, in this case, one can apply this result with  $\tilde{\mathbf{d}}_2 = \mathbf{d}_2 - \mathbf{c}$  and  $\tilde{\mathcal{H}}(\mathbf{u}, Z) = \mathcal{H}(\mathbf{u}, Z + Z_{eq}) + \mathbf{c}$  where  $Z_{eq}$  is defined by  $\mathcal{H}(\mathbf{0}, Z_{eq}) = -\mathbf{c}$ . To simplify the computations, we consider the case where  $\mathbf{c} = \mathbf{0}$ .

**Remark 4.2.** Concerning the conservativeness of the conditions in Theorem 4.1, so far as we know, it is only sufficient condition to guarantee the existence of quadratic Lyapunov function for general  $n \times n$  system in the  $H^2$  norm (see [4]). But for cascade Saint-Venant equations, when there is no slope or friction, the conditions in Theorem 4.1 are necessary and sufficient to have exponential stability, see [4, Section 2.2].

**Remark 4.3.** *With a special choice of  $Q(x)$ , conditions in Theorem 4.1 become LMI conditions in [32]. The way we construct the weighted functions for our Lyapunov function is to find explicitly the matrix function  $Q(x)$  and the constraints (20) on the tuning parameters  $k_{p,i}$  and  $k_{I,i}$  are correspondingly requirements for  $Q(0)$  and  $Q(1)$  (see [32, Proposition 2.1] for the LMI condition). However, so far as we know, the LMI condition is theoretical and one can only find the conditions numerically as shown in [32, 36]. And the advantage of our work is that we manage to construct explicitly the weighted functions that results in the explicit conditions (20) for the tuning parameters for the cascade Saint-Venant equations.*

*Proof of Theorem 4.1.* The well-posedness is guaranteed by [41], we focus on the ISS estimate. The proof is inspired from the proof of Theorems 6.6 and 6.10 in [4], adapted in order to deal with the integral terms and the ISS. We consider the augmented system with state  $(\mathbf{u}, \partial_t \mathbf{u}, \partial_t^2 \mathbf{u})$  where the dynamics of  $\partial_t \mathbf{u}$  and  $\partial_t^2 \mathbf{u}$  can be obtained by taking partial time derivatives of the system and the boundary conditions in (57). Let us set the following modified Lyapunov function candidate  $V(\mathbf{u}, Z)$ :

$$V(\mathbf{u}, Z) = V_1(\mathbf{u}, Z) + \sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(t, \cdot)\|_{L^2(0,1)}^2, \quad (60)$$

where

$$V_1(\mathbf{u}, Z) = \sum_{i=0}^2 V_0(E(\mathbf{u}, x) \partial_t^i \mathbf{u}) + \sum_{i=0}^2 \left| \Theta \frac{d^i Z}{dt^i} \right|^2$$

and  $V_0 : L^2((0, 1); \mathbb{R}^n) \rightarrow \mathbb{R}$  is defined by

$$V_0(\mathbf{u}) = \int_0^1 \mathbf{u}^T Q(x) \mathbf{u} dx \quad (61)$$

for any  $\mathbf{u} \in L^2((0, 1); \mathbb{R}^n)$ . Here the  $C^2$  class function matrix  $E(\mathbf{u}, x)$  diagonalizes  $A(\mathbf{u}, x)$  at least in a neighborhood of  $\mathbf{u} = \mathbf{0}$ , i.e.,  $E(\mathbf{u}, x)A(\mathbf{u}, x) = D(\mathbf{u}, x)E(\mathbf{u}, x)$  with  $D(\mathbf{u}, x)$  a diagonal matrix whose diagonal entries are the eigenvalues  $\lambda_i (i = 1, \dots, n)$  of the matrix  $A(\mathbf{u}, x)$ . Moreover,  $E(\mathbf{0}, x) = I_d$  for any  $x \in [0, 1]$  (see [4, Chapter 6] for the existence of  $E(\mathbf{u}, x)$  and more details). Finally,  $\Theta$  and  $Q$  are the matrices given in Theorem 4.1.

Differentiating  $V_0(E(\mathbf{u}, x)\mathbf{u})$  with respect to the time  $t$  along the  $H^3$  trajectories of (57), and noticing the properties of the matrix function  $E(\mathbf{u}, x)$ , one obtains

$$\begin{aligned} \frac{dV_0(E(\mathbf{u}, x)\mathbf{u})}{dt} &= \int_0^1 2\mathbf{u}^T E(\mathbf{u}, x)^T Q [E(\mathbf{u}, x)\mathbf{u}]_t dx \\ &= - \int_0^1 [\mathbf{u}^T E(\mathbf{u}, x)^T Q D E(\mathbf{u}, x)\mathbf{u}]_x dx \\ &\quad + \int_0^1 \mathbf{u}^T [E(\mathbf{u}, x)^T Q D E(\mathbf{u}, x)]_x \mathbf{u} dx \\ &\quad - 2 \int_0^1 \mathbf{u}^T E(\mathbf{u}, x)^T Q E(\mathbf{u}, x) B(\mathbf{u}, x) dx \\ &\quad - 2 \int_0^1 \mathbf{u}^T E(\mathbf{u}, x)^T Q E(\mathbf{u}, x) \mathbf{d}_1(t, x) dx \\ &\quad + 2 \int_0^1 \mathbf{u}^T E(\mathbf{u}, x)^T Q [E(\mathbf{u}, x)]_t \mathbf{u} dx. \end{aligned} \quad (62)$$

Integrating by parts and using Cauchy-Schwarz inequality, noticing the properties of the matrix function  $E(\mathbf{u}, x)$ , (62) becomes

$$\begin{aligned}
& \frac{dV_0(E(\mathbf{u}, x)\mathbf{u})}{dt} \\
& \leq - [\mathbf{u}^T Q \Lambda \mathbf{u}]_0^1 \\
& \quad + \int_0^1 \mathbf{u}^T E(\mathbf{u}, x)^T ((Q\Lambda)' - QM - M^T Q) E(\mathbf{u}, x) \mathbf{u} dx \\
& \quad + O(\|\mathbf{u}\|_{C^0} (|\mathbf{u}(0)| + |\mathbf{u}(1)|)^2) + O(\|\mathbf{u}\|_{C^1} \|\mathbf{u}\|_{L^2}^2) \\
& \quad + C(1 + O(\|\mathbf{u}\|_{C^0})) \|\mathbf{u}\|_{L^2} \|\mathbf{d}_1\|_{L^2}. \tag{63}
\end{aligned}$$

Here and hereafter, we omit the variable  $t$  when there is no ambiguity and we denote by  $C$  various constants independent of  $\mathbf{u}$ ,  $Z$ ,  $\mathbf{d}_1$  and  $\mathbf{d}_2$ . The notation  $O(s)$  means that there exist  $\epsilon > 0$  and  $C > 0$  such that

$$(s \leq \epsilon) \implies (|O(s)| \leq Cs).$$

Using the boundary conditions of (57), we have

$$\begin{aligned}
& [\mathbf{u}^T Q \Lambda \mathbf{u}]_0^1 \\
& = \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix}^T \begin{pmatrix} Q^+(1)\Lambda^+(1) & 0 \\ 0 & -Q^-(0)\Lambda^-(0) \end{pmatrix} \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix} \\
& \quad - \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix}^T K_1^T \begin{pmatrix} Q^+(0)\Lambda^+(0) & 0 \\ 0 & -Q^-(1)\Lambda^-(1) \end{pmatrix} K_1 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix} \\
& \quad + \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix}^T K_1^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} K_2 Z \\
& \quad + Z^T K_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} K_1 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix} \\
& \quad + \mathbf{d}_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} \\
& \quad \cdot \left( \mathbf{d}_2 + 2(1 + O(\|\mathbf{u}\|_{C^0} + |Z|)) \left[ K_1 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix} + K_2 Z \right] \right) \\
& = \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \\ Z \end{pmatrix}^T W_0 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \\ Z \end{pmatrix} \\
& \quad + \mathbf{d}_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} \\
& \quad \cdot \left( \mathbf{d}_2 + 2(1 + O(\|\mathbf{u}\|_{C^0} + |Z|)) \left[ K_1 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix} + K_2 Z \right] \right) \\
& \quad + O((\|\mathbf{u}\|_{C^0} + |Z|)(|\mathbf{u}(1)| + |\mathbf{u}(0)| + |Z|)^2), \tag{64}
\end{aligned}$$

where

$$W_0 := \begin{pmatrix} W_{01} & W_{02} \\ W_{03} & W_{04} \end{pmatrix}$$

with

$$\begin{aligned} W_{01} &= W_1, W_{02} = W_2 + \bar{K}_1^T \Theta^T, \\ W_{03} &= W_3 + \Theta \bar{K}_1, W_{04} = W_4 + 2\Theta K_2, \end{aligned}$$

and  $W_1, W_2, W_3, W_4$  are given in Theorem 4.1.

Overall, combining (63) and (64), we get

$$\begin{aligned} & \frac{dV_0(E(\mathbf{u}, x)\mathbf{u})}{dt} \\ & \leq - \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \\ Z \end{pmatrix}^T W_0 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \\ Z \end{pmatrix} \\ & \quad + \int_0^1 \mathbf{u}^T E(\mathbf{u}, x)^T ((Q\Lambda)' - QM - M^T Q) E(\mathbf{u}, x) \mathbf{u} dx \\ & \quad + O(\|\mathbf{u}\|_{C^1} \|\mathbf{u}\|_{L^2}^2) + C(1 + O(\|\mathbf{u}\|_{C^0})) \|\mathbf{u}\|_{L^2} \|\mathbf{d}_1\|_{L^2} \\ & \quad + O((|\mathbf{u}(1)| + |\mathbf{u}(0)| + |Z|)^2 (\|\mathbf{u}\|_{C^0} + |Z|)) \\ & \quad + \mathbf{d}_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} \\ & \quad \cdot \left( \mathbf{d}_2 + 2(1 + O(\|\mathbf{u}\|_{C^0} + |Z|)) \left[ K_1 \begin{pmatrix} \mathbf{u}_+(1) \\ \mathbf{u}_-(0) \end{pmatrix} + K_2 Z \right] \right). \end{aligned} \quad (65)$$

Similarly, taking the time derivative of  $V_0(E(\mathbf{u}, x)\partial_t \mathbf{u})$  and  $V_0(E(\mathbf{u}, x)\partial_t^2 \mathbf{u})$  along the augmented system solutions of (57), one obtains

$$\begin{aligned} & \frac{dV_0(E(\mathbf{u}, x)\partial_t \mathbf{u})}{dt} \\ & \leq - \begin{pmatrix} \partial_t \mathbf{u}_+(1) \\ \partial_t \mathbf{u}_-(0) \\ \dot{Z} \end{pmatrix}^T W_0 \begin{pmatrix} \partial_t \mathbf{u}_+(1) \\ \partial_t \mathbf{u}_-(0) \\ \dot{Z} \end{pmatrix} \\ & \quad + \int_0^1 \partial_t \mathbf{u}^T E(\mathbf{u}, x)^T ((Q\Lambda)' - QM - M^T Q) E(\mathbf{u}, x) \partial_t \mathbf{u} dx \\ & \quad + O(\|\mathbf{u}\|_{C^1} \|\partial_t \mathbf{u}\|_{L^2}^2) + C(1 + O(\|\mathbf{u}\|_{C^0})) \|\partial_t \mathbf{u}\|_{L^2} \|\partial_t \mathbf{d}_1\|_{L^2} \\ & \quad + O((|\partial_t \mathbf{u}(1)| + |\partial_t \mathbf{u}(0)| + |\dot{Z}|)^2 (\|\mathbf{u}\|_{C^0} + |Z|)) \\ & \quad + \dot{\mathbf{d}}_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} \\ & \quad \cdot \left( \dot{\mathbf{d}}_2 + 2(1 + O(\|\mathbf{u}\|_{C^0} + |Z|)) \left[ K_1 \begin{pmatrix} \partial_t \mathbf{u}_+(1) \\ \partial_t \mathbf{u}_-(0) \end{pmatrix} + K_2 \dot{Z} \right] \right) \end{aligned} \quad (66)$$

and

$$\begin{aligned}
& \frac{dV_0(E(\mathbf{u}, x)\partial_t^2\mathbf{u})}{dt} \\
& \leq - \begin{pmatrix} \partial_t^2\mathbf{u}_+(1) \\ \partial_t^2\mathbf{u}_-(0) \\ \ddot{Z} \end{pmatrix}^T W_0 \begin{pmatrix} \partial_t^2\mathbf{u}_+(1) \\ \partial_t^2\mathbf{u}_-(0) \\ \ddot{Z} \end{pmatrix} \\
& + \int_0^1 \partial_t^2\mathbf{u}^T E(\mathbf{u}, x)^T ((Q\Lambda)' - QM - M^T Q) E(\mathbf{u}, x) \partial_t^2\mathbf{u} dx \\
& + O(\|\mathbf{u}\|_{C^1} (\|\mathbf{u}\|_{C^1} + \|\partial_{tx}\mathbf{u}\|_{L^2}) \|\partial_t^2\mathbf{u}\|_{L^2} + \|\partial_t^2\mathbf{u}\|_{L^2}^2) \\
& + C(1 + O(\|\mathbf{u}\|_{C^0})) \|\partial_t^2\mathbf{u}\|_{L^2} \|\partial_t^2\mathbf{d}_1\|_{L^2} \\
& + O(|\partial_t^2\mathbf{u}(1)| + |\partial_t^2\mathbf{u}(0)| + |\ddot{Z}| \\
& \quad + |\partial_t\mathbf{u}(1)| + |\partial_t\mathbf{u}(0)| + |\dot{Z}|)^2 (\|\mathbf{u}\|_{C^1} + |Z| + |\dot{Z}|) \\
& + \ddot{\mathbf{d}}_2^T \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} \\
& \cdot \left( \ddot{\mathbf{d}}_2 + 2(1 + O(\|\mathbf{u}\|_{C^0} + |Z|)) \left[ K_1 \begin{pmatrix} \partial_t^2\mathbf{u}_+(1) \\ \partial_t^2\mathbf{u}_-(0) \end{pmatrix} + K_2 \ddot{Z} \right] \right) \\
& + |\ddot{\mathbf{d}}_2| O((\|\mathbf{u}\|_{C^1} + |\partial_t\mathbf{u}(1)| + |\partial_t\mathbf{u}(0)| + |Z| + |\dot{Z}|)^2). \tag{67}
\end{aligned}$$

Next, using the equation for  $Z$  in (57), we get

$$\begin{aligned}
& \frac{d}{dt} \left( \sum_{i=0}^2 \left| \Theta \frac{d^i Z}{dt^i} \right|^2 \right) \\
& = 2 \sum_{i=0}^2 \left( \frac{d^i Z}{dt^i} \right)^T \Theta \frac{d^{i+1} Z}{dt^{i+1}} \\
& = 2 \sum_{i=0}^2 \frac{d^i Z^T}{dt^i} \Theta \left( \bar{K}_1 \begin{pmatrix} \partial_t^i \mathbf{u}_+(1) \\ \partial_t^i \mathbf{u}_-(0) \end{pmatrix} + \bar{K}_2 \frac{d^i Z}{dt^i} \right) \\
& + O\left( \sum_{i=0}^2 (|\partial_t^i \mathbf{u}(1)| + |\partial_t^i \mathbf{u}(0)| + \left| \frac{d^i Z}{dt^i} \right|)^2 (\|\mathbf{u}\|_{C^0} + |Z|) \right). \tag{68}
\end{aligned}$$

Combining (65)–(68) and noticing that  $-(Q\Lambda)' + QM + M^T Q$  is positive definite,

there exists  $\gamma > 0$  such that

$$\begin{aligned}
& \frac{dV_1(\mathbf{u}, Z)}{dt} \\
& \leq - \sum_{i=0}^2 \left( \begin{array}{c} \partial_t^i \mathbf{u}_+(1) \\ \partial_t^i \mathbf{u}_-(0) \\ \frac{d^i Z}{dt^i} \end{array} \right)^T W \left( \begin{array}{c} \partial_t^i \mathbf{u}_+(1) \\ \partial_t^i \mathbf{u}_-(0) \\ \frac{d^i Z}{dt^i} \end{array} \right) - \gamma \sum_{i=0}^2 V_0(E(\mathbf{u}, x) \partial_t^i \mathbf{u}) \\
& \quad + O\left( (\|\mathbf{u}\|_{C^1} + |Z| + |\dot{Z}|) \left( \sum_{i=0}^2 |\partial_t^i \mathbf{u}(1)| + |\partial_t^i \mathbf{u}(0)| + \left| \frac{d^i Z}{dt^i} \right| \right)^2 \right) \\
& \quad + O(\|\mathbf{u}\|_{C^1} \sum_{i=0}^2 \|\partial_t^i \mathbf{u}\|_{L^2}^2) \\
& \quad + C(1 + O(\|\mathbf{u}\|_{C^1})) \sum_{i=0}^2 \|\partial_t^i \mathbf{u}\|_{L^2} \sum_{i=0}^2 \|\partial_t^i \mathbf{d}_1\|_{L^2} \\
& \quad + O(\|\mathbf{u}\|_{C^1} (\|\mathbf{u}\|_{C^1} + \|\partial_{tx}^2 \mathbf{u}\|_{L^2})) \|\partial_t^2 \mathbf{u}\|_{L^2} \\
& \quad + \sum_{i=0}^2 \left[ \frac{d^i \mathbf{d}_2^T}{dt^i} \begin{pmatrix} -Q^+(0)\Lambda^+(0) & 0 \\ 0 & Q^-(1)\Lambda^-(1) \end{pmatrix} \right. \\
& \quad \cdot \left. \left( \frac{d^i \mathbf{d}_2}{dt^i} + 2(1 + O(\|\mathbf{u}\|_{C^0} + |Z|)) \left[ K_1 \begin{pmatrix} \partial_t^i \mathbf{u}_+(1) \\ \partial_t^i \mathbf{u}_-(0) \end{pmatrix} + K_2 \frac{d^i Z}{dt^i} \right] \right) \right] \\
& \quad + |\ddot{\mathbf{d}}_2| O((\|\mathbf{u}\|_{C^1} + |\partial_t \mathbf{u}(1)| + |\partial_t \mathbf{u}(0)| + |Z| + |\dot{Z}|)^2),
\end{aligned}$$

where  $W$  is the matrix defined in Theorem 4.1. Since  $W$  is positive definite, then using Young's inequality on the crossed term involving  $\mathbf{d}_1$ , there exists constant  $\nu > 0$  such that

$$\begin{aligned}
& \frac{dV_1(\mathbf{u}, Z)}{dt} \\
& \leq -\nu \sum_{i=0}^2 \left| \left( \partial_t^i \mathbf{u}_+^T(1), \partial_t^i \mathbf{u}_-^T(0), \frac{d^i Z^T}{dt^i} \right) \right|^2 - \frac{2\gamma}{3} \sum_{i=0}^2 V_0(E(\mathbf{u}, x) \partial_t^i \mathbf{u}) \\
& \quad + C(1 + O(\|\mathbf{u}\|_{C^1} + |Z| + |\dot{Z}|)) \left( \sum_{k=0}^2 |\mathbf{d}_2^{(k)}|^2 + \sum_{k=0}^2 \|\partial_t^k \mathbf{d}_1\|_{L^2}^2 \right) \\
& \quad + O(\|\mathbf{u}\|_{C^1} \sum_{i=0}^2 \|\partial_t^i \mathbf{u}\|_{L^2}^2) \\
& \quad + O(\|\mathbf{u}\|_{C^1} (\|\mathbf{u}\|_{C^1} + \|\partial_{tx}^2 \mathbf{u}\|_{L^2})) \|\partial_t^2 \mathbf{u}\|_{L^2} \\
& \quad + O\left( (\|\mathbf{u}\|_{C^1} + |Z| + |\dot{Z}|) \left( \sum_{i=0}^2 |\partial_t^i \mathbf{u}(1)| + |\partial_t^i \mathbf{u}(0)| + \left| \frac{d^i Z}{dt^i} \right| \right)^2 \right) \\
& \quad + C|\ddot{\mathbf{d}}_2| (|\partial_t \mathbf{u}(1)| + |\partial_t \mathbf{u}(0)|)^2. \tag{69}
\end{aligned}$$

Note that, from (57)

$$(|\partial_t \mathbf{u}(1)| + |\partial_t \mathbf{u}(0)|) \leq C(\|\mathbf{u}\|_{C^1} + \|\mathbf{d}_1\|_{C^0}). \tag{70}$$

Hence, using Young's inequality again

$$\begin{aligned} & |\ddot{\mathbf{d}}_2| (|\partial_t \mathbf{u}(1)| + |\partial_t \mathbf{u}(0)|)^2 \\ & \leq C |\dot{\mathbf{d}}_2|^2 (\|\mathbf{u}\|_{C^1} + \|\mathbf{d}_1\|_{C^0}) \\ & \quad + (|\partial_t \mathbf{u}(1)| + |\partial_t \mathbf{u}(0)|)^2 (\|\mathbf{u}\|_{C^1} + \|\mathbf{d}_1\|_{C^0}). \end{aligned} \quad (71)$$

Note also that, from (57),

$$\|\mathbf{u}_{tx}\|_{L^2} \leq C(\|\mathbf{u}\|_{H^2} + \|\partial_x \mathbf{d}_1\|_{L^2})(1 + O(\|\mathbf{u}\|_{C^0})). \quad (72)$$

Besides, from (57), one has

$$|\dot{Z}| \leq C(|Z| + \|\mathbf{u}\|_{C^0}). \quad (73)$$

Using Young's inequality on the crossed term involving  $\mathbf{d}_1$  and then Sobolev inequality that  $\|\mathbf{u}\|_{C^1} \leq C\|\mathbf{u}\|_{H^2}$ , there exist  $\delta > 0$  and  $\eta > 0$  such that if  $\|\mathbf{u}\|_{H^2} + |Z| \leq \eta$  and (59) holds, then

$$\begin{aligned} & \frac{dV_1(\mathbf{u}, Z)}{dt} \\ & \leq -\frac{\nu}{2} \sum_{i=0}^2 \left| (\partial_t^i \mathbf{u}_+(1), \partial_t^i \mathbf{u}_-(0), \frac{d^i Z}{dt^i}) \right|^2 - \frac{\gamma}{2} \sum_{i=0}^2 V_0(E(\mathbf{u}, x) \partial_t^i \mathbf{u}) \\ & \quad + C \sum_{k=0}^2 |\mathbf{d}_2^{(k)}|^2 + C(\|\partial_x \mathbf{d}_1\|_{L^2}^2 + \sum_{k=0}^2 \|\partial_t^k \mathbf{d}_1\|_{L^2}^2) \\ & \leq -\frac{1}{2} \min\left(\frac{\nu}{\max_i(\Theta_i)}, \gamma\right) V_1(\mathbf{u}, Z) + C \sum_{k=0}^2 |\mathbf{d}_2^{(k)}|^2 \\ & \quad + C\left(\|\partial_t^2 \mathbf{d}_1\|_{L^2}^2 + \sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1\|_{L^2}^2\right). \end{aligned} \quad (74)$$

On the other hand, using Cauchy-Schwarz inequality, one has

$$\frac{d}{dt} \sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1\|_{L^2}^2 \leq 2 \sum_{\substack{k_1+k_2 \leq 2 \\ k_2 \leq 1}} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1\|_{L^2}^2,$$

which, together with (74) and the definition of  $V(\mathbf{u}, Z)$  gives

$$\frac{dV}{dt} \leq -\mu V + C \sum_{k=0}^2 |\mathbf{d}_2^{(k)}|^2 + C \sum_{\substack{k_1+k_2 \leq 2 \\ k_2 \leq 1}} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1\|_{L^2}^2, \quad (75)$$

where  $\mu = 1/2 \min(\nu/\max_i(\Theta_i), \gamma)$ . Using Gronwall's inequality, the equivalence of  $V$  and the square of  $\|\mathbf{u}\|_{H^2} + |Z| + \sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1\|_{L^2}$ , one obtains

$$\begin{aligned} & (\|\mathbf{u}\|_{H^2(0,1)} + |Z|) \\ & \leq C e^{-\frac{\mu}{2}t} \left( \|\mathbf{u}^0\|_{H^2(0,1)} + |Z^0| + \sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(0, \cdot)\|_{L^2(0,1)} \right) \\ & \quad + C \sum_{k=0}^2 \|e^{-\frac{\mu}{2}(t-\tau)} \mathbf{d}_2^{(k)}(\tau)\|_{L^2(0,t)} \\ & \quad + C \sum_{\substack{k_1+k_2 \leq 2 \\ k_2 \leq 1}} \|e^{-\frac{\mu}{2}(t-\tau)} \partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(\tau, x)\|_{L^2((0,t) \times (0,1))}. \end{aligned} \quad (76)$$



Using Sobolev inequality again, (76) gives the exponential ISS with fading memory in Theorem 4.1 for the  $H^2$  norm for any trajectory with  $H^3$  initial condition satisfying the corresponding assumptions of Theorem 4.1. Finally, since this estimate only involves the  $H^2$  norm of  $\mathbf{u}$ , the result also holds for the  $H^2$  trajectories by the density argument (see for instance [8, Lemma 4.2] for a similar argument).  $\square$

**Remark 4.4.** *Compared to the undisturbed case,  $V_1(\mathbf{u}, Z)$  is not anymore equivalent to the square of the  $H^2((0, 1); \mathbb{R}^n) \times \mathbb{R}^n$  norm of  $(\mathbf{u}, Z)$ . Therefore, we add the term  $\sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(t, \cdot)\|_{L^2(0,1)}^2$  so that the Lyapunov function candidate  $V(\mathbf{u}, Z)$  defined in (60) becomes equivalent to the square of  $\|\mathbf{u}(t, \cdot)\|_{H^2(0,1)} + |Z(t)| + \sum_{k_1+k_2 \leq 1} \|\partial_t^{k_1} \partial_x^{k_2} \mathbf{d}_1(t, \cdot)\|_{L^2(0,1)}$  which allows us to derive the ISS estimate.*

**Remark 4.5.** *One can note that for the Saint-Venant equations, the existence of  $Q$  and  $\Theta$  such that the matrices  $-(Q\Lambda)' + QM + M^T Q$  and  $W$  are positive definite is exactly the condition of Theorem 3.1 for the linearized system around the null steady-state to have a basic quadratic Lyapunov function for the  $L^2$  norm.*

We now show that Theorem 2.1 is an application of Theorem 4.1.

*Proof of Theorem 2.1.* This is a direct consequence of Theorem 4.1. Indeed, first observe that we can assume without loss of generality that  $\mathbf{c} = 0$ . Then for the original nonlinear system, we define  $a_i = \tilde{A}_i - \tilde{A}_i^*$ ,  $v_i = \tilde{V}_i - \tilde{V}_i^*$ ,  $z_i = Z_i - Z_i^*$  as previously in (13) and use the same change of variables as in (21). After that, we change the order of the unknowns such that  $A(\mathbf{0}, x) = \{\lambda_{11}, \lambda_{12}, \dots, \lambda_{1N}, -\lambda_{21}, -\lambda_{22}, \dots, -\lambda_{2N}\}$ . Then, the system (9)–(10) is rewritten in the form of (57). Choosing  $Q(x) = \text{diag}\{g_{11}, \dots, g_{1N}, g_{21}, \dots, g_{2N}\}$ , where for each  $i \in \{1, \dots, N\}$ ,  $g_{1i} = \alpha_i f_{1i} e^{2 \int_0^x \frac{\gamma_{1i}(s)}{\lambda_{1i}(s)} ds}$ ,  $g_{2i} = \alpha_i f_{2i} e^{-2 \int_0^x \frac{\delta_{2i}(s)}{\lambda_{2i}(s)} ds}$  with  $f_{1i}$  and  $f_{2i}$  defined in (28) and  $\Theta = \text{diag}\{\theta_1 L_1, \dots, \theta_N L_N\}$  with  $\theta_i$  and  $\alpha_i$  chosen by the way proposed in the proof of Theorem 3.1, the result follows directly.  $\square$

## 5 Numerical Simulations

In this section, we make numerical simulations for the cascade nonlinear network of three channels with different lengths. In Fig 1 and Fig 2, the curves of different colors represent the corresponding values of different channels. Fig 1 represents the height at the output of each channel, and Fig 2 represents the main result (12) for each channel. We consider the case where the cross-section of the river channel is a rectangular shape with a width of 1m, then  $A_i(t, x) = H_i(t, x)$ ,  $\partial_1 G_i(A_i, x) = 1$  ( $i = 1, 2, 3$ ). The specific data is as follows:

- The lengths of channels are  $L_1 = 400m$ ,  $L_2 = 500m$ ,  $L_3 = 600m$  and  $H_1^*(0) = 1.5m$ ,  $H_2^*(0) = 1.6m$ ,  $H_3^*(0) = 2m$  respectively, the constant discharge  $Q_0 = 1m^3/s$  and the acceleration of gravity is  $9.81m/s^2$ . These values are typically reasonable values for navigable river [3].
- We consider the steep slope,  $S_{f,i} = \frac{k_{f,i} V_i^2}{H_i}$ ,  $S_{b,i} = 1.5 \frac{k_{f,i} V_i^{*2}(0)}{H_i^*(0)}$ , the selection of the friction coefficients is  $k_{f,1} = 0.01$ ,  $k_{f,2} = 0.02$ ,  $k_{f,3} = 0.03$ .

- The tuning parameters are  $k_{p,1} = -10$ ,  $k_{I,1} = -10$ ,  $k_{p,2} = 100$ ,  $k_{I,2} = 1$ ,  $k_{p,3} = -0.5$ ,  $k_{I,3} = -0.9$ . The initial perturbations are chosen to be sinusoidal with an amplitude of 0.15 m in height and the disturbances in the dynamics  $d_{1,i}^{(1)} = 0.0001(\sin(t) + \sin(x))$ ,  $d_{1,i}^{(2)} = 0.0001(\sin(t) + \sin(x))$ , the boundary disturbances  $d_{2,0} = 0.001 \sin(t)$ ,  $d_{2,i} = 0.001 \sin(t)$  for  $i = 1, 2, 3$ .

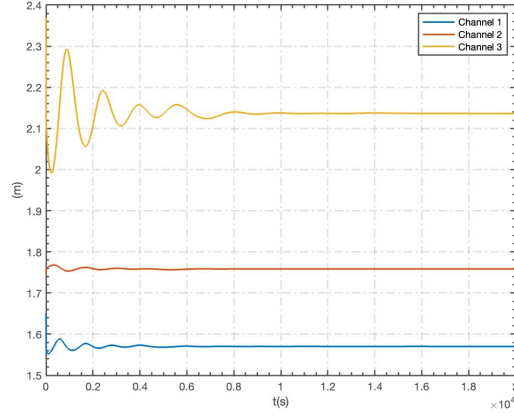


Figure 1: Output measurements  $H_i(t, L_i)$  with the PI control

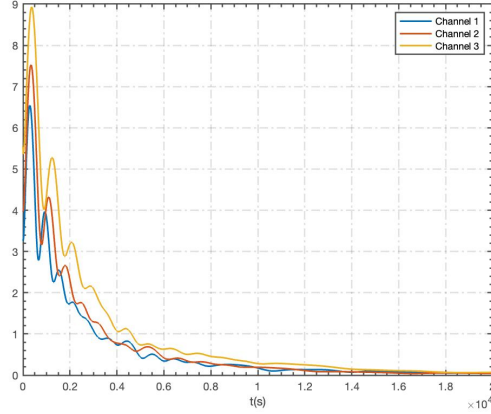


Figure 2: Convergence of  $\|H_i(t, \cdot) - H_i^*, V_i(t, \cdot) - V_i^*\|_{H^2(0, L_i)} + |Z_i(t) - Z_i^*|$ .

We can remark that even if the theorem only gives a local result, it looks like in practice the basin of attraction is significant. Indeed, adding perturbations with an amplitude up to 15% of the nominal steady-state still seem to fall in the stability area. Moreover, as could be expected, the controller takes reasonable values from a physical perspective.

## 6 Conclusion

In this paper, we have addressed the boundary Input-to-State Stability of a network of channels in cascade, using Proportional-Integral controllers. For each channel, the control input and measured output are on the downstream boundary and the channels are coupling through the boundaries. We give exponential stability and ISS conditions using a quadratic Lyapunov function. Interestingly, this control works even without having direct information on the cross-section, the slope, friction model, or even the steady-state, except at the boundaries. This is very useful from an application point of view. Besides, the system is robust to constant boundary disturbances, as could be expected with a PI control. As a side result we show (see Theorem 4.1) that for any 1D quasilinear hyperbolic system with a PI control, if the linearized system admits a basic quadratic Lyapunov function for the  $L^2$  norm, then the nonlinear system admits a basic quadratic Lyapunov function for the  $H^2$  norm and is (locally) exponentially ISS with respect to internal and boundary disturbances. This goes beyond the framework considered in the paper and could be applied to many other systems especially for the general density-velocity systems, see [26].

## References

- [1] Jean Auriol. Output feedback stabilization of an underactuated cascade network of interconnected linear pde systems using a backstepping approach. *Automatica*, 117:108964, 2020.
- [2] Jean Auriol and Delphine Bresch Pietri. Robust state-feedback stabilization of an underactuated network of interconnected  $n+m$  hyperbolic pde systems. *Automatica*, 136:110040, 2022.
- [3] Georges Bastin and Jean-Michel Coron. On boundary feedback stabilization of non-uniform linear  $2 \times 2$  hyperbolic systems over a bounded interval. *Systems & Control Letters*, 60(11):900–906, 2011.
- [4] Georges Bastin and Jean-Michel Coron. *Stability and Boundary Stabilisation of 1-D Hyperbolic Systems*. Number 88 in *Progress in Nonlinear Differential Equations and Their Applications*. Springer International, 2016.
- [5] Georges Bastin and Jean-Michel Coron. A quadratic Lyapunov function for hyperbolic density-velocity systems with nonuniform steady states. *Systems & Control Letters*, 104:66–71, 2017.
- [6] Georges Bastin, Jean-Michel Coron, and Brigitte d’Andréa Novel. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel. *Netw. Heterog. Media*, 4(2):177–187, 2009.
- [7] Georges Bastin, Jean-Michel Coron, and Amaury Hayat. Input-to-state stability in sup norms for hyperbolic systems with boundary disturbances. *Nonlinear Analysis*, 208:112300, 2021.
- [8] Georges Bastin, Jean-Michel Coron, Amaury Hayat, and Peipei Shang. Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation. *Math. Models Methods Appl. Sci.*, 29(2):271–316, 2019.

- [9] Jean-Michel Coron, Brigitte d’Andréa Novel, and Georges Bastin. A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations. In CD-Rom Proceedings, Paper F1008-5, ECC99, Karlsruhe, Germany, pages 3178–3183, 1999.
- [10] Jean-Michel Coron, Brigitte d’Andréa Novel, and Georges Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. IEEE Transactions on Automatic Control, 52(1):2–11, 2007.
- [11] Jean-Michel Coron, Jonathan de Halleux, Georges Bastin, and Brigitte d’Andréa Novel. On boundary control design for quasilinear hyperbolic systems with entropies as Lyapunov functions. In Proceedings of the 41st IEEE, Conference on Decision and Control, Las Vegas, Nevada USA, December 2002, pages 3010–3014, 2002.
- [12] Jean-Michel Coron, Long Hu, Guillaume Olive, and Peipei Shang. Boundary stabilization in finite time of one-dimensional linear hyperbolic balance laws with coefficients depending on time and space. J. Differential Equations, 271:1109–1170, 2021.
- [13] Jonathan de Halleux, Christophe Prieur, Jean-Michel Coron, Brigitte d’Andréa Novel, and Georges Bastin. Boundary feedback control in networks of open channels. Automatica. A Journal of IFAC, the International Federation of Automatic Control, 39(8):1365–1376, 2003.
- [14] Barré de Saint-Venant. Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l’introduction des marées dans leur lit. Comptes Rendus de l’Académie des Sciences, 53:147–154, 1871.
- [15] Ababacar Diagne, Mamadou Diagne, Shuxia Tang, and Miroslav Krstic. Backstepping stabilization of the linearized *saint-venant-exner* model. Automatica J. IFAC, 76:345–354, 2017.
- [16] Valérie Dos Santos and Christophe Prieur. Boundary control of open channels with numerical and experimental validations. IEEE transactions on Control systems technology, 16(6):1252–1264, 2008.
- [17] Francesco Ferrante and Christophe Prieur. Boundary control design for conservation laws in the presence of measurement disturbances. Mathematics of Control, Signals, and Systems, 33(1):49–77, 2021.
- [18] James. M. Greenberg and Tatsien Li. The effect of boundary damping for the quasilinear wave equation. J. Differential Equations, 52(1):66–75, 1984.
- [19] Martin Gugat and Stephan Gerster. On the limits of stabilizability for networks of strings. Systems Control Lett., 131:104494, 10, 2019.
- [20] Martin Gugat, Vincent Perrollaz, and Lionel Rosier. Boundary stabilization of quasilinear hyperbolic systems of balance laws: Exponential decay for small source terms. Journal of Evolution Equations, 18(3):1471–1500, 2018.
- [21] Amaury Hayat. On boundary stability of inhomogeneous  $2 \times 2$  1-D hyperbolic systems for the  $C^1$  norm. ESAIM Control Optim. Calc. Var., 25:Paper No. 82, 31, 2019.

- [22] Amaury Hayat. Boundary stabilization of 1d hyperbolic systems. Annual Reviews in Control, 52:222–242, 2021.
- [23] Amaury Hayat. Global exponential stability and input-to-state stability of semi-linear hyperbolic systems for the  $l_2$  norm. Systems & Control Letters, 148:104848, 2021.
- [24] Amaury Hayat. PI controllers for the general saint-venant equations. arXiv preprint arXiv:2108.02703, 2021.
- [25] Amaury Hayat and Peipei Shang. A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope. Automatica J. IFAC, 100:52–60, 2019.
- [26] Amaury Hayat and Peipei Shang. Exponential stability of density-velocity systems with boundary conditions and source term for the  $H^2$  norm. J. Math. Pures Appl. (9), 153:187–212, 2021.
- [27] Long Hu, Florent Di Meglio, Rafael Vazquez, and Miroslav Krstic. Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. IEEE Trans. Automat. Control, 61(11):3301–3314, 2016.
- [28] Iasson Karafyllis and Miroslav Krstic. Input-to-state stability for PDEs. Springer, 2019.
- [29] Christopher M. Kellett and Peter M. Dower. Input-to-state stability, integral input-to-state stability, and  $\mathcal{L}_2$ -gain properties: qualitative equivalences and interconnected systems. IEEE Trans. Automat. Control, 61(1):3–17, 2016.
- [30] Miroslav Krstic and Andrey Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. Systems Control Lett., 57(9):750–758, 2008.
- [31] Miroslav Krstic and Andrey Smyshlyaev. Boundary Control of PDEs: A Course on Backstepping Designs, volume 16 of Advances in Design and Control. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.
- [32] Pierre-Olivier Lamare, Antoine Girard, and Christophe Prieur. An optimisation approach for stability analysis and controller synthesis of linear hyperbolic systems. ESAIM Control Optim. Calc. Var., 22(4):1236–1263, 2016.
- [33] Günter Leugering and E. J. P. Georg Schmidt. On the modelling and stabilization of flows in networks of open canals. SIAM Journal on Control and Optimization, 41(1):164–180, 2002.
- [34] Xavier Litrico and Vincent Fromion. Modeling and control of hydrosystems. Springer Science & Business Media, 2009.
- [35] Christophe Prieur, Joseph Winkin, and Georges Bastin. Robust boundary control of systems of conservation laws. Mathematics of Control, Signals, and Systems, 20(2):173–197, 2008.
- [36] Christophe Prieur and Joseph J. Winkin. Boundary feedback control of linear hyperbolic systems: application to the Saint-Venant–Exner equations. Automatica J. IFAC, 89:44–51, 2018.

- [37] Tie Hu Qin. Global smooth solutions of dissipative boundary value problems for first order quasilinear hyperbolic systems. Chinese Ann. Math. Ser. B, 6(3):289–298, 1985.
- [38] Eduardo D. Sontag. Smooth stabilization implies coprime factorization. IEEE Trans. Automat. Control, 34(4):435–443, 1989.
- [39] Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu. Boundary PI controllers for a star-shaped network of 2x2 systems governed by hyperbolic partial differential equations. IFAC PapersOnLine, 50(1):7070–7075, 2017.
- [40] Ngoc-Tu Trinh, Vincent Andrieu, and Cheng-Zhong Xu. Output regulation for a cascaded network of  $2 \times 2$  hyperbolic systems with PI controller. Automatica J. IFAC, 91:270–278, 2018.
- [41] Zhiqiang Wang. Exact controllability for nonautonomous first order quasilinear hyperbolic systems. Chinese Ann. Math. Ser. B, 27(6):643–656, 2006.
- [42] Gediyon Yemane Weldegiyorgis and Mapundi Kondwani Banda. An analysis of the input-to-state-stabilisation of linear hyperbolic systems of balance laws with boundary disturbances. arXiv preprint arXiv:2006.02492, 2020.
- [43] Liguozhang, Christophe Prieur, and Junfei Qiao. Local proportional-integral boundary feedback stabilization for quasilinear hyperbolic systems of balance laws. SIAM Journal on Control and Optimization, 58(4):2143–2170, 2020.