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On the kernel of vector ε -algorithm and related topics

Claude Brezinski* Michela Redivo-Zaglia† Ahmed Salam ‡

Abstract

The vector ε -algorithm of Wynn is a powerful method for accelerating the convergence of vector sequences. Its kernel is the set of sequences which are transformed into constant sequences whose terms are their limits or antilimits. In 1971, a sufficient condition characterizing sequences in this kernel was given by McLeod. In this paper, we prove that such a condition is not necessary. Moreover, using Clifford algebra, we give a formula for the vector ε_2 -transformation, which is formally the same as in the scalar case, up to operations in a Clifford algebra. Hence, Aitken's Δ^2 process is extended in this way to vectors. Then, we derive the explicit algebraic and geometric expressions of sequences of the kernel of the ε_2 -transformation. We also formulate a conjecture concerning the explicit algebraic expression of kernel of the vector ε -algorithm.

Keywords : vector ε -algorithm, kernels, Shanks transformation, Aitken's Δ^2 process, Clifford algebra.

1 Introduction

Let (S_n) be a sequence of numbers, vectors, matrices or tensors which converges to S . If its convergence is too slow, it can be transformed into another sequence (or a set of sequences), say (T_n) , which, under certain assumptions, converges faster to the same limit, that is $\lim_{n \rightarrow \infty} \|T_n - S\| / \|S_n - S\| = 0$. The mapping $(S_n) \mapsto (T_n)$ is called a *sequence transformation*. Among the many transformations which can be found in the literature, one of the most popular is Shanks' transformation [18] which was first only defined for scalar sequences by

$$e_k(S_n) = \left| \begin{array}{cccc} S_n & S_{n+1} & \cdots & S_{n+k} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{array} \right| / \left| \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{array} \right|, \quad (1)$$

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where Δ is the usual forward difference operator. When $k = 1$, the usual Aitken's Δ^2 process is recovered [1].

This transformation can be recursively implemented by the scalar ε -algorithm of Wynn [23] whose rule is

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + [\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}]^{-1}, \quad (2)$$

with $\varepsilon_{-1}^{(n)} = 0$ and $\varepsilon_0^{(n)} = S_n$ for $n = 0, 1, \dots$, and it holds

$$\varepsilon_{2k}^{(n)} = e_k(S_n), \quad \varepsilon_{2k+1}^{(n)} = [e_k(\Delta S_n)]^{-1}.$$

The transformation $(S_n) \mapsto (\varepsilon_{2k}^{(n)})$, k fixed, is called the ε_{2k} -transformation.

We have the following result

Theorem 1

A necessary and sufficient condition that, for all n , $\varepsilon_{2k}^{(n)} = S$ is that, for all n , the sequence (S_n) satisfies the linear homogeneous difference equation of order k

$$a_0(S_n - S) + a_1(S_{n+1} - S) + \dots + a_k(S_{n+k} - S) = 0,$$

with $a_0 a_k \neq 0$ and $a_0 + a_1 + \dots + a_k \neq 0$.

It does not restrict the generality to assume that the a_i 's sum up to 1. A first proof of this result was given in [4], but its complete proof only appeared in [7, Thm. 1]. The set of sequences satisfying Theorem 1 is called the *kernel* of the scalar ε -algorithm (or of the Shanks' transformation). It is still valid if the sequence (S_n) does not converge, in which case S is called the antilimit of the sequence.

In [24], Wynn extended his algorithm to vector sequences in \mathbb{R}^p by defining the inverse of a non null vector $y \in \mathbb{R}^p \setminus \{0\}$ as $y^{-1} = y/(y, y)$ (called the Samelson inverse) in (2), where $(x, y) = y^T x$ denotes the usual scalar product. Thus the rules of the vector ε -algorithm and of the scalar one are the same up the extension to a vector inverse. However, the vectors $\varepsilon_k^{(n)}$ it produces cannot be represented by ratios of determinants of dimension $k + 1$ as in (1). They can be represented by ratios of much larger determinants [9, 10] or as ratios of designants which generalize determinants in a noncommutative algebra [17].

Using Clifford algebra, McLeod [13] proved the following result for the kernel of the vector ε -algorithm

Theorem 2

Let (S_n) be a vector sequence of \mathbb{R}^p that satisfies the linear homogeneous difference equation of order k

$$a_0(S_n - S) + a_1(S_{n+1} - S) + \dots + a_k(S_{n+k} - S) = 0, \quad \forall n$$

where $a_i \in \mathbb{R}$ with $a_0 a_k \neq 0$ and $a_0 + a_1 + \dots + a_k \neq 0$. Then, for all n , $\varepsilon_{2k}^{(n)} = S$.

In Section 2, we will show that the kernel of the vector ε -algorithm is not only limited to the sequences of Theorem 2, and, hence, that this condition is only sufficient. In Section 3, we recall the main features of a Clifford algebra that will be used in this context. In Section 4, we highlight how

the Clifford algebra's tool can be helpful for obtaining explicit formulas for the ε_2 -transformation. These formulas are formally the same as in the scalar case, up to operations in a Clifford algebra, and hence, they are easier to understand. The extension of Aitken's process to vectors is explained in Section 5. The link with the ε_2 -transformation is established. The kernel of this transformation is explicitly derived both in algebraic and geometric forms. Finally, a conjecture concerning the explicit algebraic form of the vector ε_{2k} -transformation is proposed for any k .

2 On the kernel of the vector ε_{2k} -transformation

Let us first give an example showing that the kernel of the vector ε_2 -transformation contains vector sequences which do not satisfy the condition of Theorem 2.

When $k = 1$, according to Theorem 2, the kernel of the ε_2 -transformation contains the sequences of vectors of \mathbb{R}^p satisfying $\forall n, a_0(S_n - S) + (1 - a_0)(S_{n+1} - S) = 0$ where the real scalar a_0 is different from 0 and 1. Consider now three linearly independent vectors S_n, S_{n+1} and S_{n+2} , and assume they satisfy this difference equation. This is impossible. Indeed, we have $a_0 S_n + (1 - a_0) S_{n+1} = a_0 S_{n+1} + (1 - a_0) S_{n+2}$, that is $a_0 S_n + (1 - 2a_0) S_{n+1} - (1 - a_0) S_{n+2} = 0$, and the three coefficients $a_0, 1 - 2a_0, 1 - a_0$ cannot be simultaneously equal to 0 or 1.

We consider now the sequence (S_n) of vectors in \mathbb{R}^3 defined by

$$S_{3n} = (1, 0, 0)^T, \quad S_{3n+1} = (0, 1, 0)^T, \quad S_{3n+2} = (0, 0, 1)^T, \quad n = 0, 1, \dots$$

Three consecutive vectors of this sequence are obviously linearly independent, and, thus, they do not satisfy the condition of Theorem 2. Let us apply the vector ε -algorithm to this sequence with the Samelson inverse for a vector.

We have, for all n ,

$$\begin{aligned} \Delta S_{3n} &= (-1, 1, 0)^T, & (\Delta S_{3n})^{-1} &= (-1, 1, 0)^T/2, \\ \Delta S_{3n+1} &= (0, -1, 1)^T, & (\Delta S_{3n+1})^{-1} &= (0, -1, 1)^T/2, \\ \Delta S_{3n+2} &= (1, 0, -1)^T, & (\Delta S_{3n+2})^{-1} &= (1, 0, -1)^T/2, \\ \Delta S_{3n+3} &= (-1, 1, 0)^T, & (\Delta S_{3n+3})^{-1} &= (-1, 1, 0)^T/2. \end{aligned}$$

Then

$$\begin{aligned} [(\Delta S_{3n+1})^{-1} - (\Delta S_{3n})^{-1}]^{-1} &= (1, -2, 1)^T/3, \\ [(\Delta S_{3n+2})^{-1} - (\Delta S_{3n+1})^{-1}]^{-1} &= (1, 1, -2)^T/3, \\ [(\Delta S_{3n+3})^{-1} - (\Delta S_{3n+2})^{-1}]^{-1} &= (-2, 1, 1)^T/3, \end{aligned}$$

and we finally obtain

$$\begin{aligned} \varepsilon_2^{(3n)} &= (0, 1, 0)^T + (1, -2, 1)^T/3 = (1, 1, 1)^T/3, \\ \varepsilon_2^{(3n+1)} &= (0, 0, 1)^T + (1, 1, -2)^T/3 = (1, 1, 1)^T/3, \\ \varepsilon_2^{(3n+2)} &= (1, 0, 0)^T + (-2, 1, 1)^T/3 = (1, 1, 1)^T/3, \end{aligned}$$

which shows that our sequence belongs to the kernel of the vector ε_2 -transformation without satisfying the condition of Theorem 2. Thus, this condition is only sufficient, and the kernel of the ε_2 -transformation contains other sequences.

Let us now give an example showing that the kernel of the vector ε_{2k} -transformation contains vector sequences which do not satisfy the condition of Theorem 2.

For an arbitrary integer k , according to McLeod's Theorem (Thm. 2), the kernel of the vector ε_{2k} -transformation contains the sequences (S_n) of vectors of \mathbb{R}^p satisfying

$$a_0(S_n - S) + a_1(S_{n+1} - S) + \cdots + a_k(S_{n+k} - S) = 0, \quad (3)$$

where a_i are real numbers, with $a_0 a_k \neq 0$ and $a_0 + a_1 + \cdots + a_k \neq 0$.

Consider now $2k + 1$ linearly independent vectors $S_n, S_{n+1}, \dots, S_{n+2k}$ (thus we must have $p \geq 2k + 1$) and assume they satisfy the difference equation (3). This is impossible since, otherwise, from (3), $S_n, S_{n+1}, \dots, S_{n+2k}$ would be linearly dependent because the coefficients a_i 's cannot be simultaneously equal to zero.

Let $p = 2k + 1$ and consider the sequence (S_n) of vectors in \mathbb{R}^p defined by

$$S_{pn+l} = e_{l+1}, \quad l = 0, \dots, p - 1,$$

where e_l is the l th canonical vector of \mathbb{R}^p . The p successive vectors of this sequence are obviously linearly independent, and, thus, they do not satisfy (3) of Theorem 2. However, performing the computations in Matlab, the vector ε -algorithm applied to this sequence, gives $\varepsilon_{2k}^{(n)} = (1/p, \dots, 1/p)^T$ for all n .

The difficulties for understanding the vector ε -algorithm come from the fact that it is derived from the scalar ε -algorithm by taking the same rules and extending the inverse of a non null real number to a non null vector, via the Samelson inverse. Thus the algorithm obtained looks like a "black box" for computing vectors without any theoretical support or justification, while the scalar ε -algorithm presents comprehensive features, as for example its connection to Shanks' transformation, Aitken's Δ^2 process or Padé approximants,.... Attempts following the scalar scheme have been made for extending Shanks' transformation to vector sequences. But all of them led to vector generalizations and algorithms, such as the topological ε algorithms, which are different from Wynn's vector ε -algorithm. Such generalizations are based on computations in a linear space $\mathbb{R}^p(+, \cdot)$ having an algebraic structure such that an algebraic inverse of a vector has no meaning. Contrarily, a Clifford algebra associated to $\mathbb{R}^p(+, \cdot)$ is a powerful tool for our aim. It contains $\mathbb{R}^p(+, \cdot)$ and each non null vector $u \in \mathbb{R}^p$ has an inverse u^{-1} with respect to the multiplicative law and this inverse coincides with the Samelson inverse. This property will allow us to overcome the mentioned difficulties and to complete the study of the ε -algorithm

3 Clifford algebra, a powerful tool

In this Section we recall some definitions and properties of a Clifford algebra, that will be useful in the following sections.

Let $\{e_1, \dots, e_p\}$ be an orthonormal basis of the Euclidean real linear space \mathbb{R}^p . There exists a unitary, associative but non-commutative (for $p > 1$) algebra $\mathcal{C}\ell_p(\mathbb{R})$, called the universal real Clifford algebra associated to \mathbb{R}^p (see [2, 14]), for which there exists an \mathbb{R} -linear transformation $\Phi : \mathbb{R}^p \rightarrow \mathcal{C}\ell_p(\mathbb{R})$ such that

$$\forall u = (u_1, \dots, u_p)^T \in \mathbb{R}^p, \quad (\Phi(u))^2 = (u, u)\mathbb{1}, \quad (4)$$

where $\mathbb{1}$ is the unit element of $\mathcal{C}\ell_p(\mathbb{R})$, and (u, v) denotes the usual scalar product of two real vectors u and v .

We may identify $\sum_{i=1}^p u_i \Phi(e_i) \in \mathcal{C}\ell_p(\mathbb{R})$ with $\sum_{i=1}^p u_i e_i \in \mathbb{R}^p$ and each $\lambda \mathbb{1} \in \mathcal{C}\ell_p(\mathbb{R})$ with $\lambda \in \mathbb{R}$.

According to this identification, one can consider \mathbb{R}^p and \mathbb{R} as subspaces of $\mathcal{C}\ell_p(\mathbb{R})$. The Clifford algebra $\mathcal{C}\ell_p(\mathbb{R})$ is generated by the elements of the system $\{e_1, \dots, e_p\}$.

For sake of clarity, from now on we denote by $\mathcal{C}\ell_p$ the Clifford algebra, and by \star the multiplication in $\mathcal{C}\ell_p$. Thus relation (4) is rewritten as $u \star u = u^2 = (u, u)$. This algebra is also the real linear space spanned by the products

$$e_{i_1} \star \dots \star e_{i_r}; \quad 1 \leq i_1 < \dots < i_r \leq p, \quad 0 \leq r \leq p,$$

where the empty product ($r = 0$) is the identity element $\mathbb{1}$. Thus, the system

$$\{\mathbb{1}, e_{i_1} \star \dots \star e_{i_r}; \quad 1 \leq i_1 < \dots < i_r \leq p, \quad 1 \leq r \leq p, \}$$

is a basis of $\mathcal{C}\ell_p$ and it is easy to see that its dimension is 2^p .

Let us consider some definitions and the main properties of this algebra when $u = (u_1, \dots, u_p)^T = \sum_{i=1}^p u_i e_i \in \mathbb{R}^p$, and $v = (v_1, \dots, v_p)^T = \sum_{i=1}^p v_i e_i \in \mathbb{R}^p$ are two real vectors.

Property 1

The following properties hold

1.

$$\forall i, j \in \{1, \dots, p\}, i \neq j, \quad e_i \star e_i = 1, \quad e_i \star e_j = -e_j \star e_i, \quad \text{and} \quad (e_i \star e_j)^2 = -1.$$

2.

$$u \star u = (u, u) = \|u\|^2. \tag{5}$$

3. From (4), the product $u \star v$ is given by

$$u \star v = (u, v) + \sum_{i < j} (u_i v_j - u_j v_i) e_i \star e_j, \quad \text{and} \quad u \star v + v \star u = 2(u, v) \in \mathbb{R}^p. \tag{6}$$

4. From (6), we obtain

$$u \star v \star u = 2(u, v)u - \|u\|^2 v \in \mathbb{R}^p.$$

5. Let $u \in \mathbb{R}^p$. From (5), the inverse of u is given by

$$u^{-1} = \frac{u}{\|u\|^2} \in \mathbb{R}^p,$$

and we see that it coincides with the Samelson inverse of a real vector.

4 Expression of the vector $\varepsilon_2^{(n)}$ using Clifford algebra

We will now see that the use of Clifford algebra allows us to overcome the operations that are limited if we restrict ourselves only to the linear space \mathbb{R}^p . Moreover, the expressions obtained are more understandable. To this aim, we focus on the sequence $(\varepsilon_2^{(n)})$ which is the first even step of the vector ε -algorithm. Let us derive it from the rules of the vector ε -algorithm.

Theorem 3

The ε_2 - transformation : $(S_n) \mapsto (\varepsilon_2^{(n)})$ can be explicitly expressed as

$$\varepsilon_2^{(n)} = S_n - \Delta S_n \star (\Delta^2 S_n)^{-1} \star \Delta S_n. \quad (7)$$

Proof: We have

$$\varepsilon_2^{(n)} = S_{n+1} + [\varepsilon_1^{(n+1)} - \varepsilon_1^{(n)}]^{-1} \quad \text{with} \quad \varepsilon_1^{(n)} = [\Delta S_n]^{-1}.$$

Thus

$$\begin{aligned} [(\Delta S_{n+1})^{-1} - (\Delta S_n)^{-1}]^{-1} &= [(\Delta S_n)^{-1} \star (\Delta S_n - \Delta S_{n+1}) \star (\Delta S_{n+1})^{-1}]^{-1}, \\ &= [(\Delta S_{n+1})^{-1} \star (\Delta S_n - \Delta S_{n+1}) \star (\Delta S_n)^{-1}]^{-1}, \end{aligned}$$

and it follows

$$\varepsilon_2^{(n)} = S_{n+1} - \Delta S_n \star [\Delta^2 S_n]^{-1} \star \Delta S_{n+1} = S_{n+1} - \Delta S_{n+1} \star [\Delta^2 S_n]^{-1} \star \Delta S_n. \quad (8)$$

This relation can also be written as

$$\begin{aligned} \varepsilon_2^{(n)} &= S_n + (S_{n+1} - S_n) - \Delta S_{n+1} \star [\Delta^2 S_n]^{-1} \star \Delta S_n \\ &= S_n + \Delta^2 S_n \star [\Delta^2 S_n]^{-1} \star \Delta S_n - \Delta S_{n+1} \star [\Delta^2 S_n]^{-1} \star \Delta S_n \\ &= S_n + (\Delta^2 S_n - \Delta S_{n+1}) \star [\Delta^2 S_n]^{-1} \star \Delta S_n \\ &= S_n + (\Delta S_{n+1} - \Delta S_n - \Delta S_{n+1}) \star [\Delta^2 S_n]^{-1} \star \Delta S_n, \\ &= S_n - \Delta S_n \star (\Delta^2 S_n)^{-1} \star \Delta S_n \end{aligned}$$

and we finally obtain $\varepsilon_2^{(n)} = S_n - \Delta S_n \star (\Delta^2 S_n)^{-1} \star \Delta S_n$. ■

Remark 1 It is worth to notice that the expression (7) is formally the same as that of the scalar case (scalar Aitken's expression, scalar $\varepsilon_2^{(n)}$) up to use the product \star and the respect of the order of the factors.

Let us develop this formula in the Clifford algebra. For two vectors $u, v \in \mathbb{R}^p$, we have in the Clifford algebra $u \star v + v \star u = 2(u, v)$. Thus $v \star u \star v = 2(u, v)v - (v, v)u$ (see Property 1, item 3), and (7) becomes

$$\varepsilon_2^{(n)} = S_n - \frac{2(\Delta S_n, \Delta^2 S_n)\Delta S_n - (\Delta S_n, \Delta S_n)\Delta^2 S_n}{(\Delta^2 S_n, \Delta^2 S_n)}. \quad (9)$$

This is an expression for $\varepsilon_2^{(n)}$ only using operations in \mathbb{R}^p .

The vectors $\varepsilon_{2k}^{(n)}$ obtained by the vector ε -algorithm can be expressed as a ratio of determinants of dimension $2k + 1$ as given in [11, Thm. 1]. In particular, we have, for $k = 1$,

$$\varepsilon_2^{(n)} = \frac{\begin{vmatrix} S_n & S_{n+1} & S_{n+2} \\ 0 & \|\Delta S_n\|^2 & 2(\Delta S_n, \Delta S_{n+1}) \\ -\|\Delta S_n\|^2 & 0 & \|\Delta S_{n+1}\|^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & \|\Delta S_n\|^2 & 2(\Delta S_n, \Delta S_{n+1}) \\ -\|\Delta S_n\|^2 & 0 & \|\Delta S_{n+1}\|^2 \end{vmatrix}}.$$

Thus

$$\begin{vmatrix} S_n - \varepsilon_2^{(n)} & S_{n+1} - \varepsilon_2^{(n)} & S_{n+2} - \varepsilon_2^{(n)} \\ 0 & \|\Delta S_n\|^2 & 2(\Delta S_n, \Delta S_{n+1}) \\ -\|\Delta S_n\|^2 & 0 & \|\Delta S_{n+1}\|^2 \end{vmatrix} = 0,$$

and there exists a_0, a_1 and a_2 , which depend on n , such that

$$\begin{aligned} a_0(S_n - \varepsilon_2^{(n)}) + a_1(S_{n+1} - \varepsilon_2^{(n)}) + a_2(S_{n+2} - \varepsilon_2^{(n)}) &= 0 \\ a_1\|\Delta S_n\|^2 + 2a_2(\Delta S_n, \Delta S_{n+1}) &= 0 \\ -a_0\|\Delta S_n\|^2 + a_2\|\Delta S_{n+1}\|^2 &= 0. \end{aligned}$$

It does not restrict the generality to assume that $a_0 + a_1 + a_2 = 1$ since this sum has to be different from 0 to ensure the existence of $\varepsilon_2^{(n)}$. Solving the system we find that

$$a_0 = \|\Delta S_{n+1}\|^2 / \|\Delta^2 S_n\|^2, \quad a_1 = -2(\Delta S_n, \Delta S_{n+1}) / \|\Delta^2 S_n\|^2, \quad a_2 = \|\Delta S_n\|^2 / \|\Delta^2 S_n\|^2.$$

Thus

$$\varepsilon_2^{(n)} = (\|\Delta S_{n+1}\|^2 S_n - 2(\Delta S_n, \Delta S_{n+1}) S_{n+1} + \|\Delta S_n\|^2 S_{n+2}) / \|\Delta^2 S_n\|^2.$$

After some algebraic manipulations, it is easy to see that this expression reduces to (9).

5 The vector Aitken's process

In this Section, we show how to build the vector Aitken's Δ^2 process. The successive steps of this construction are similar to those which have to be followed for constructing many other sequence transformations (see [5, 19, 20, 22]). Namely, we start from an algebraic equation, named the *implicit kernel* of the transformation, satisfied by the sequence (S_n) . Then, this difference equation is solved thus giving these sequences into a closed form named the *explicit kernel* of the transformation. Finally, a sequence transformation transforming all these sequences into a constant sequence is constructed. The drawback of such an approach is that one is confronted with inconsistent systems. The details will be given below.

5.1 Construction of the vector Aitken's Δ^2

To extend Aitken's process to vectors, one naturally tries to adapt the scalar approach to vectors. Thus, let (S_n) be a sequence of vectors in \mathbb{R}^p . A potential Aitken's process could be the

transformation $t_1 : (S_n) \mapsto (t_1^{(n)})$ satisfying formally the system

$$\begin{cases} a_0^{(n)}(S_n - t_1^{(n)}) + a_1^{(n)}(S_{n+1} - t_1^{(n)}) = 0 \\ a_0^{(n)}(S_{n+1} - t_1^{(n)}) + a_1^{(n)}(S_{n+2} - t_1^{(n)}) = 0 \\ a_0^{(n)} + a_1^{(n)} = 1, \text{ with } a_0^{(n)} a_1^{(n)} \neq 0. \end{cases} \quad (10)$$

where $a_0^{(n)} \in \mathbb{R}$, $a_1^{(n)} \in \mathbb{R}$, $t_1^{(n)} \in \mathbb{R}^p$ are unknown. The drawback of this approach, is that for $p \geq 2$, the system (10) is inconsistent. It presents more constraints than free parameters. To overcome this inconvenient, the existing approaches found in the literature project the system, in one way or another, as, for example, in this way

$$\begin{cases} a_0^{(n)}(\Delta S_n, y) + a_1^{(n)}(\Delta S_{n+1}, y) = 0 \\ a_0^{(n)} + a_1^{(n)} = 1, \\ t_1^{(n)} = a_0^{(n)} S_n + a_1^{(n)} S_{n+1}. \end{cases},$$

where y is a chosen vector. These approaches lead to the class of the topological ε -algorithms and all of them are far from the vector ε -algorithm.

The way which allows us to overcome the problem of inconsistency, is to consider the system in Clifford algebra, precisely as follows : the vector Aitken's Δ^2 is the transformation $t_1 : (S_n) \mapsto (t_1^{(n)})$ satisfying formally the system

$$\begin{cases} a_0^{(n)} \star (S_n - t_1^{(n)}) + a_1^{(n)} \star (S_{n+1} - t_1^{(n)}) = 0 \\ a_0^{(n)} \star (S_{n+1} - t_1^{(n)}) + a_1^{(n)} \star (S_{n+2} - t_1^{(n)}) = 0 \\ a_0^{(n)} + a_1^{(n)} = 1, \text{ with } a_0^{(n)} \star a_1^{(n)} \neq 0. \end{cases} \quad (11)$$

where $a_0^{(n)} \in \mathcal{C}\ell_p$, $a_1^{(n)} \in \mathcal{C}\ell_p$, $t_1^{(n)} \in \mathcal{C}\ell_p$ are unknown.

Viewed in this way, the problem of consistency disappear for the system (11). However, an unexpected problem arises from this formulation : the quantity $t_1^{(n)}$ belongs to $\mathcal{C}\ell_p$, and at least at first sight, is not necessarily in \mathbb{R}^p . We will see that $t_1^{(n)}$ is guaranteed to belong to \mathbb{R}^p as we wish. The system (11) can be rewritten as

$$\begin{cases} t_1^{(n)} = S_n + a_1^{(n)} \star \Delta S_n \\ \Delta S_n + a_1^{(n)} \star \Delta^2 S_n = 0 \\ a_0^{(n)} + a_1^{(n)} = 1, \text{ with } a_0^{(n)} \star a_1^{(n)} \neq 0, \end{cases}$$

which gives $a_1^{(n)} = -\Delta S_n \star (\Delta^2 S_n)^{-1}$ and hence $t_1^{(n)} = S_n - \Delta S_n \star (\Delta^2 S_n)^{-1} \star \Delta S_n = \varepsilon_2^{(n)} \in \mathbb{R}^p$. Thus, the vector Aitken's Δ^2 process coincides with the vector ε_2 -transformation. To complete our purpose, we may now define the vector Shanks' transformation as the transformation $t_1 : (S_n) \mapsto (t_1^{(n)})$ satisfying the system (11).

5.2 Kernel of the vector Aitken's Δ^2 : implicit form

The kernel of vector Aitken's Δ^2 process or equivalently of vector ε_2 -transformation is the set of vector sequences (S_n) that are transformed into a constant vector valued sequence (S) . In other

words, the set of vector sequences satisfying for all n the system

$$\begin{cases} a_0^{(n)} \star (S_n - S) + a_1^{(n)} \star (S_{n+1} - S) = 0 \\ a_0^{(n)} \star (S_{n+1} - S) + a_1^{(n)} \star (S_{n+2} - S) = 0 \\ a_0^{(n)} + a_1^{(n)} = 1, \text{ with } a_0^{(n)} \star a_1^{(n)} \neq 0. \end{cases} \quad (12)$$

where $a_0^{(n)} \in \mathcal{C}l_p$, $a_1^{(n)} \in \mathcal{C}l_p$, $S \in \mathbb{R}^p$ are unknown. Writing the system (12) for n and $n + 1$, we get the system

$$\begin{cases} a_0^{(n)} \star (S_n - S) + a_1^{(n)} \star (S_{n+1} - S) = 0 \\ a_0^{(n)} \star (S_{n+1} - S) + a_1^{(n)} \star (S_{n+2} - S) = 0 \\ a_0^{(n+1)} \star (S_{n+1} - S) + a_1^{(n+1)} \star (S_{n+2} - S) = 0 \\ a_0^{(n)} + a_1^{(n)} = 1, a_0^{(n+1)} + a_1^{(n+1)} = 1, \end{cases}$$

from which we deduce

$$\begin{cases} (a_0^{(n+1)} - a_0^{(n)}) \star (S_{n+1} - S) + (a_1^{(n+1)} - a_1^{(n)}) \star (S_{n+2} - S) = 0 \\ a_0^{(n+1)} - a_0^{(n)} + a_1^{(n+1)} - a_1^{(n)} = 0, \end{cases}$$

that is

$$\begin{cases} (a_1^{(n+1)} - a_1^{(n)}) \star (S_{n+2} - S_{n+1}) = 0 \\ a_0^{(n+1)} - a_0^{(n)} + a_1^{(n+1)} - a_1^{(n)} = 0. \end{cases}$$

Assume that $\Delta S_n \neq 0$, $\forall n$. We get $(a_1^{(n+1)} - a_1^{(n)}) = 0$, $\forall n$, and hence $(a_0^{(n+1)} - a_0^{(n)}) = 0$, $\forall n$. Thus we have the *implicit form* of the kernel

Theorem 4

The vector ε_2 -transformation: $(S_n) \mapsto (\varepsilon_2^{(n)})$ is such that $\forall n, \varepsilon_2^{(n)} = S$ if and only if the vector sequence (S_n) satisfies for all n the system

$$\begin{cases} a_0 \star (S_n - S) + a_1 \star (S_{n+1} - S) = 0 \\ a_0 + a_1 = 1, \text{ with } a_0^n \star a_1^n \neq 0. \end{cases} \quad (13)$$

where $a_0 \in \mathcal{C}l_p$, $a_1 \in \mathcal{C}l_p$, $S \in \mathbb{R}^p$.

The condition (13) of Theorem 4 can be expressed in the slightly different form

$$S_n = S + a_0 \star \Delta S_n \quad (14)$$

where $0 \neq a_0 \in \mathcal{C}l_p$, $S \in \mathbb{R}^p$ and $\Delta S_n \neq 0$, $\forall n$.

5.3 Kernel of the vector Aitken's Δ^2 : algebraic explicit form

The difference equation (13) (or (14)) has to be solved for obtaining the *explicit form* of the kernel. We have the following result

Corollary 1

The kernel of the vector Aitken's Δ^2 process is the set of sequences satisfying, for all n ,

$$S_n = S + \Delta S_0 \star (\Delta^2 S_0)^{-1} \star [\Delta S_1 \star (\Delta S_0)^{-1}]^n \star \Delta S_0. \quad (15)$$

Proof : From Theorem 4, the kernel of the vector Δ^2 process or of the vector ε_2 -transformation is the set of sequences (S_n) satisfying, $\forall n, S_n = S + a_0 \star \Delta S_n$, which, by applying the operator Δ to both sides, gives $\Delta S_n = a_0 \star \Delta^2 S_n$. Thus, for any value of n , $a_0 = \Delta S_n \star (\Delta^2 S_n)^{-1}$. But we also have $\Delta S_n = a_0 \star (\Delta S_{n+1} - \Delta S_n)$, that is $\Delta S_{n+1} = a_0^{-1} \star (1 + a_0) \star \Delta S_n$. Setting $\gamma = a_0^{-1} \star (1 + a_0)$, we have $\Delta S_{n+1} = \gamma \star \Delta S_n$, and it follows by induction that $\Delta S_n = \gamma^n \star \Delta S_0$. For $n = 1$, we have $\Delta S_1 = a_0^{-1} \star (1 + a_0) \star \Delta S_0 = \gamma \star \Delta S_0$, and we get $\gamma = \Delta S_1 \star (\Delta S_0)^{-1}$. Finally $\Delta S_n = [\Delta S_1 \star (\Delta S_0)^{-1}]^n \star \Delta S_0$, $a_0 = \Delta S_0 \star (\Delta^2 S_0)^{-1}$, and the result follows. ■

Remark 2

Notice that we also have $1 + a_0 = \Delta S_1 (\Delta^2 S_0)^{-1}$, and $\gamma = a_0^{-1} (1 + a_0) = a_0^{-1} + 1 = \Delta^2 S_0 (\Delta S_0)^{-1} + \Delta S_0 (\Delta S_0)^{-1} = \Delta S_1 (\Delta S_0)^{-1}$.

In the scalar case, the usual form of the kernel is recovered since, due to the commutativity, the kernel is the set of sequences such that, for all n ,

$$S_n = S + \Delta S_0 (\Delta^2 S_0)^{-1} \Delta S_0 [\Delta S_1 \star (\Delta S_0)^{-1}]^n = S + \alpha \lambda^n,$$

with $\alpha = \Delta S_0 (\Delta^2 S_0)^{-1} \Delta S_0 = (\Delta S_0)^2 / \Delta^2 S_0$ and $\lambda = \Delta S_1 \star (\Delta S_0)^{-1} = \Delta S_1 / \Delta S_0$.

In fact, the result in Corollary 1 can be improved as follows

Theorem 5

The kernel of vector Aitken's Δ^2 process is the set of vector sequences satisfying,

$$\forall n, S_n = S + \alpha \star \lambda^n, \tag{16}$$

where $\alpha = S_0 - S = e_0$ is an arbitrary non null vector of \mathbb{R}^p and λ any arbitrary bivector $\lambda = \alpha^{-1} \star e_1$, with $e_1 = S_1 - S$ any arbitrary non null vector of \mathbb{R}^p .

Proof: From Corollary 5, we have $S_n = S + \Delta S_0 \star (\Delta^2 S_0)^{-1} \star [\Delta S_1 \star (\Delta S_0)^{-1}]^n \star \Delta S_0$. By induction, we have $(\Delta S_0)^{-1} [\Delta S_1 \star (\Delta S_0)^{-1}]^n \star \Delta S_0 = [(\Delta S_0)^{-1} \star \Delta S_1]^n$. Hence, we obtain

$$S_n = S + \Delta S_0 \star (\Delta^2 S_0)^{-1} \star \Delta S_0 \star [(\Delta S_0)^{-1} \star \Delta S_1]^n = S + \alpha \star \lambda^n, \tag{17}$$

with $\alpha = \Delta S_0 \star (\Delta^2 S_0)^{-1} \star \Delta S_0$ and $\lambda = (\Delta S_0)^{-1} \star \Delta S_1$. Taking $n = 0$ in equation (17), we get also $\alpha = e_0$ and then for $n = 1$, we obtain $\lambda = e_0^{-1} \star e_1$. ■

Thus, using the tool of Clifford algebra, we formally obtain the same expression for the kernel as in the scalar case. It is also interesting to notice that α is a vector of \mathbb{R}^p , but not λ^n . However, the computation of $\alpha \star \lambda^n$ can be simply performed in the linear space $(\mathbb{R}^p, +, \cdot)$

5.4 Expression of the vector Aitken's Δ^2 with designants

In the scalar case, the Aitken's Δ^2 can be expressed as a ratio of two determinants as given in equation (1). As we saw, the vector Aitken's Δ^2 is algebraically obtained by immersing the linear space \mathbb{R}^p into the Clifford algebra $\mathcal{C}\ell_p$. Since determinants do not exist in a non-commutative ring, a determinantal formula similar to the scalar case does not exist. However, such a formula holds

with designants introduced by Heyting in [12]. They play, in a non-commutative ring, the same role as determinants in a commutative ring.

Thus, let $a_{11}, a_{12}, a_{21}, a_{22}$ be coefficients in $\mathcal{C}\ell_p$, with a_{11} invertible. The right designant of order 2, is defined as

$$\begin{vmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{vmatrix}_r = a_{22} - a_{12} \star a_{11}^{-1} \star a_{21},$$

which is formally the expression of a Schur complement in a ring of matrices. This is not surprising since the construction of designants is based on an elimination process. In [17], using Clifford algebra and designants, the following expression of the vector $\varepsilon_2^{(n)}$ was derived

$$\varepsilon_2^{(n)} = \begin{vmatrix} \Delta S_n & S_n \\ \Delta S_{n+1} & S_{n+1} \end{vmatrix}_r \begin{vmatrix} \Delta S_n & 1 \\ \Delta S_{n+1} & 1 \end{vmatrix}_r^{-1}.$$

Using properties of designants and the powerful tool of Clifford algebra, we derive below another proof for the explicit form of the kernel of the vector Aitken's Δ^2 process. Thus, $\varepsilon_2^{(n)} = S, \forall n$ is equivalent to

$$\begin{vmatrix} \Delta S_n & S_n - S \\ \Delta S_{n+1} & S_{n+1} - S \end{vmatrix}_r = 0, \forall n.$$

So

$$\varepsilon_2^{(n)} = S \iff (S_{n+1} - S) - (S_n - S) \star (\Delta S_n)^{-1} \Delta S_{n+1} = 0 \iff (S_{n+1} - S) \star (\Delta S_{n+1})^{-1} = (S_n - S) \star (\Delta S_n)^{-1},$$

from which, it is straightforward that

$$S_n = S + \lambda^n \star e_0 \quad \text{with } e_0 \in \mathbb{R}^p, \lambda \in \mathcal{C}\ell_p.$$

This formula can also be rewritten as

$$S_n = S + (e_1 \star e_0^{-1})^n \star e_0 \quad \text{with } e_0 \in \mathbb{R}^p, e_1 \in \mathbb{R}^p.$$

Notice that Theorem 5 gives $S_n = S + e_0 \star (e_0^{-1} \star e_1)^n$. In fact the two expressions are the same since

$$e_0 \star (e_0^{-1} \star e_1)^n = (e_1 \star e_0^{-1})^n \star e_0, \quad \forall e_0 \in \mathbb{R}^p, \forall e_1 \in \mathbb{R}^p.$$

How can one see that $w_n = (e_1 \star e_0^{-1})^n \star e_0 \in \mathbb{R}^p$ and compute it in \mathbb{R}^p ? We have seen that, for $u \in \mathbb{R}^p, v \in \mathbb{R}^p, u \star u = (u, u) = \|u\|_2^2, u^{-1} = u/\|u\|^2 \in \mathbb{R}^d, u \star v \star u = 2(u, v)u - \|u\|^2 v \in \mathbb{R}^d$. Thus we obtain

$$u \star v \star (\alpha u + \beta v) \in \text{span}\{u, v\}$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$, that is the stability

$$u \star v \star \text{span}\{u, v\} \subset \text{span}\{u, v\}.$$

Then, we get iteratively

$$(u \star v)^n \star \text{span}\{u, v\} \subset \text{span}\{u, v\}.$$

Thus, $w_n = (e_1 \star e_0^{-1})^n \star e_0 \in \mathbb{R}^d$. Let us summarize this in a slightly more general way

Lemma 1 *Let u, v be any vectors of \mathbb{R}^p and x any vector of $\text{span}\{u, v\}$. Then*

$$\forall n \in \mathbb{N}, (v \star u)^n \star x \in \text{span}\{u, v\}, x \star (u \star v)^n \in \text{span}\{u, v\}, \text{ and } (v \star u)^n \star x = x \star (u \star v)^n.$$

Proof : The property is true for $n = 0$ since $(v \star u)^0 \star x = x = x \star (u \star v)^0 \in \text{span}\{u, v\}$.

Assume now $(v \star u)^j \star x \in \text{span}\{u, v\}$, $x \star (u \star v)^j \in \text{span}\{u, v\}$, and $(v \star u)^j \star x = x \star (u \star v)^j$, for $j \in \{1, \dots, n\}$. For $n + 1$, we obtain $(v \star u)^{n+1} \star x = (v \star u)^n \star (v \star u) \star x$. By the induction hypothesis, $y = (v \star u) \star x \in \text{span}\{u, v\}$ and $(v \star u)^n \star y \in \text{span}\{u, v\}$. It follows

$$(v \star u)^{n+1} \star x = (v \star u)^n \star y = y \star (u \star v)^n = (v \star u) \star x \star (u \star v)^n = x \star (u \star v) \star (u \star v)^n = x \star (u \star v)^{n+1},$$

which ends the proof. ■

5.5 Kernel of the vector Aitken's Δ^2 : geometric explicit form

Let us now establish a geometric explicit form of vector Aitken's Δ^2 process from its algebraic explicit form. To do this, we need what follows.

5.5.1 Rotation's expression in $\mathcal{C}l_p$

We have the

Lemma 2 *Let u, v be two vectors of \mathbb{R}^p with $\|u\|_2 = \|v\|_2 = 1$ and consider the map $R : \mathbb{R}^p \rightarrow \mathbb{R}^p$ defined by*

$$\forall x \in \mathbb{R}^p, R(x) = (v \star u) \star x \star (u \star v). \quad (18)$$

R is the rotation in the plane $\text{span}\{u, v\}$, of angle ψ with angle $\angle(u, v) = \psi/2$.

Proof: It is straightforward that R is a linear map. Firstly, let us check that R is an isometry. In fact, $\forall x \in \mathbb{R}^p$, $\|R(x)\|_2^2 = (v \star u) \star x \star \underbrace{(u \star v) \star (v \star u)}_{=1} \star x \star (u \star v)$. Thus $\|R(x)\|_2^2 = (v \star u) \star x \star x \star (u \star v) = \|x\|_2^2 \star v \star u \star u \star v = \|x\|_2^2 \star \|u\|_2^2 \star \|v\|_2^2 = \|x\|_2^2$.

On the one hand, for $x \in \text{span}\{u, v\}^\perp$, we have $R(x) = v \star (u \star x \star u) \star v = v \star (-x) \star v = -(-x) = x$, where $\text{span}\{u, v\}^\perp$ denotes the orthogonal of $\text{span}\{u, v\}$. On the other hand, $R(u) = (v \star u) \star u \star (u \star v) = v \star u \star v = 2(u, v)v - u$, which shows that $R(u)$ is obtained by a rotation of angle ψ on the plane $\text{span}\{u, v\}$, where the angle $\angle(u, v) = \psi/2$. In fact, $(u, v) = \cos(\psi/2)$. Hence $2(v, R(u)) = v \star R(u) + R(u) \star v = v \star (v \star u \star v) + (v \star u \star v) \star v = u \star v + v \star u = 2(u, v)$. Thus $\angle(v, Ru) = \angle(u, v) = \psi/2$. It follows that $\angle(u, Ru) = \angle(u, v) + \angle(v, Ru) = \psi/2 + \psi/2 = \psi$. In a similar way, $R(v) = (v \star u) \star v \star (u \star v) \in \text{span}\{u, v\}$. To see now that $R(v)$ is also obtained from v by a rotation of angle ψ in the plane $\text{span}\{u, v\}$, we have to show that $\angle(v, R(v)) = \psi$. In fact, we saw that $\angle(v, R(u)) = \psi/2$. Now, $2(R(u), R(v)) = R(u) \star R(v) + R(v) \star R(u) = (v \star u \star v) \star (v \star u \star v \star u \star v) + (v \star u \star v \star u \star v) \star (v \star u \star v) = u \star v + v \star u = 2(u, v)$. Hence $\angle(R(u), R(v)) = \psi/2$. It follows then $\angle(v, R(v)) = \angle(v, R(u)) + \angle(R(u), R(v)) = \psi/2 + \psi/2 = \psi$. Thus, $R(u)$ (respectively $R(v)$) is obtained from u (respectively from v) by a rotation in $\text{span}\{u, v\}$ of angle ψ with $\angle(u, v) = \psi/2$. ■

5.5.2 Geometric explicit form of the vector Aitken's Δ^2 kernel

The kernel of vector Aitken's Δ^2 process, expressed in terms of rotations, is as follows

Theorem 6 *The kernel of vector Aitken's Δ^2 process is the set of vector sequences (S_n) given by*

$$\forall n, S_n = S + \delta^n R^n(e_0), \quad (19)$$

where R is the rotation in the plane $\text{span}\{e_0, e_1\}$, with $e_0 = S_0 - S$, $e_1 = S_1 - S_0$ and moving \hat{e}_0 into \hat{e}_1 with $\hat{e}_0 = e_0/\|e_0\|_2$ and $\hat{e}_1 = e_1/\|e_1\|_2$.

Proof: We showed that the kernel of vector Aitken's Δ^2 process is the set of vector sequences (S_n) given by $S_n = S + (e_1 \star e_0^{-1})^n \star e_0$ for any arbitrary vectors $e_0 = S_0 - S$ and $e_1 = S_1 - S$. Setting $\alpha = \|e_0\|_2$, $\beta = \|e_1\|_2$, $\hat{e}_0 = e_0/\alpha$, $\hat{e}_1 = e_1/\beta$, we get $S_n = S + \alpha\delta^n(\hat{e}_1 \star \hat{e}_0)^n \star \hat{e}_0$, with $\delta = \beta/\alpha$. Let ψ be the angle $\psi = \angle(\hat{e}_0, \hat{e}_1)$ and \hat{v} in $\text{span}\{\hat{e}_0, \hat{e}_1\}$ with $\|\hat{v}\|_2 = 1$ and $\angle(\hat{e}_0, \hat{v}) = \psi/2$. Then, the rotation R in $\text{span}\{\hat{e}_0, \hat{e}_1\}$ of angle ψ is explicitly given by $R(x) = \hat{v} \star \hat{e}_0 \star x \star \hat{e}_0 \star \hat{v}$. Thus $\hat{e}_1 = R(\hat{e}_0) = \hat{v} \star \hat{e}_0 \star \hat{e}_0 \star \hat{v} = \hat{v} \star \hat{e}_0 \star \hat{v}$. Hence $(\hat{e}_1 \star \hat{e}_0)^n = (\hat{v} \star \hat{e}_0 \star \hat{v} \star \hat{e}_0)^n = (\hat{v} \star \hat{e}_0)^{2n}$. From Lemma 1, we have $(\hat{v} \star \hat{e}_0)^n \star \hat{e}_0 = \hat{e}_0 \star (\hat{e}_0 \star \hat{v})^n$. It follows that $(\hat{e}_1 \star \hat{e}_0)^n \star \hat{e}_0 = (\hat{v} \star \hat{e}_0)^{2n} \star \hat{e}_0 = (\hat{v} \star \hat{e}_0)^n \star (\hat{v} \star \hat{e}_0)^n \star \hat{e}_0 = (\hat{v} \star \hat{e}_0)^n \star \hat{e}_0 \star (\hat{e}_0 \star \hat{v})^n = R^n(\hat{e}_0)$ and finally we obtain $S_n = S + \alpha\delta^n R^n(\hat{e}_0) = S + \delta^n R^n(\alpha\hat{e}_0) = S + \delta^n R^n(e_0)$. ■

In [21], Steele et al. experimentally observed the result of Theorem 6 and gave a proof based on a geometric construction using concepts of elementary geometry, typically the similarity and congruence between triangles.

5.6 Kernel of the vector ε -algorithm : implicit form

The following result proved by Salam [16, Thm. 12] by using designants describes the implicit form of the kernel of ε -algorithm

Theorem 7

The kernel of the ε_{2k} -transformation: $(S_n) \mapsto (\varepsilon_{2k}^{(n)})$ is the set of sequences satisfying, for all n ,

$$S_n = S + a_0 \star \Delta S_n + \cdots + a_{k-1} \star \Delta S_{n+k-1}, \quad (20)$$

where the coefficients a_i are in the Clifford algebra.

The relation (20) is the *implicit form* of the kernel of the ε_{2k} -transformation. It can also be written under the form

$$\alpha_0 \star (S_n - S) + \cdots + \alpha_k \star (S_{n+k} - S) = 0,$$

with $\alpha_i \in \mathcal{C}\ell_p$, $\alpha_0 \star \alpha_k \neq 0$, and $\alpha_0 + \cdots + \alpha_k = \mathbb{1}$. It is a necessary and sufficient condition.

5.7 Conjecture on the explicit form of the kernel of the vector ε -algorithm

We conjecture that the explicit form of vector ε -algorithm kernel is given by

Conjecture 1

The kernel of the transformation $\varepsilon_{2k} : (S_n) \mapsto (\varepsilon_{2k}^{(n)})$ is the set of sequences satisfying, for all n ,

$$S_n = S + \alpha_1 \star \lambda_1^n + \cdots + \alpha_k \star \lambda_k^n,$$

where the coefficients α_i , λ_i are in \mathcal{C}^p , and such that $\alpha_i \star \lambda_i^n \in \mathbb{R}^p$ for $i = 1, \dots, k$ and for all n .

Intensive computations using Matlab confirm the conjecture for $k = 2, 4, 6$. Theoretical investigations are actually under consideration.

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