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Flatness of networks of **two** synaptically coupled excitatory-inhibitory neural modules with maximal symmetry*

Florentina Nicolau¹ and Hugues Mounier²

Abstract— We consider networks of **two** synaptically coupled excitatory-inhibitory neural modules with maximal symmetry of the connection strengths, **and for which the nonlinearities are described by a logistic sigmoidal function**. It has been shown that the connection strengths may slowly vary with respect to time and that they can actually be considered as inputs of the network. In the recent publication [13], we considered the case of **two** synaptically coupled subnetworks and studied the problem of determining which connection strengths should be modified (in other words, which connection strengths should be considered as inputs), in order to achieve flatness for the resulting control system when no relation between the connection strengths (in particular, no symmetry) is assumed. In this paper, we consider a similar problem but under the assumption that all interactions (interactions between subnetworks and local interactions within the same subnetwork) are always symmetric.

I. INTRODUCTION

In this paper, we consider population-based networks of N synaptically coupled excitatory-inhibitory neural modules or subnetworks (simply, EI networks) of the following form [18], [19] (see also [1], [4]):

$$\begin{aligned}\tau_e \dot{x}_e^n &= -x_e^n + F_e \left(\sum_{m=1}^N (w_{ee}(n, m)x_e^m - w_{ei}(n, m)x_i^m) \right) \\ \tau_i \dot{x}_i^n &= -x_i^n + F_i \left(\sum_{m=1}^N (w_{ie}(n, m)x_e^m - w_{ii}(n, m)x_i^m) \right),\end{aligned}\tag{1}$$

with $x_e^n, x_i^n \in \mathbb{R}$, and $1 \leq n \leq N$. Each subnetwork is labeled by the discrete index n and contains a pair of mutually coupled local populations of excitatory and inhibitory neurons. The equations describing the dynamics of each subnetwork are usually called Wilson-Cowan equations for cortical dynamics. The states of the system x_e^n and x_i^n are, resp., the proportion of excitatory and inhibitory cells of the n -th subnetwork which become active per unit time. **We will always use lower indices $a \in \{e, i\}$ and $b \in \{e, i\}$ to refer to the excitatory or inhibitory character of the considered objects, and integers $1 \leq n \leq N$ and $1 \leq m \leq N$ as upper indices to label the subnetworks.** The gain functions F_e and F_i are nonlinear, typically sigmoidal functions, τ_e and τ_i are time constants, and $w_{ab}(n, m)$, where $a, b \in \{e, i\}$ and $1 \leq n, m \leq N$, denote the strengths of connections **between x_a^n and x_b^m ($w_{ab}(n, m)$ enters into the dynamics of x_a^n and characterizes the action of x_b^m on x_a^n).** It has been shown, see for instance [3] and the references therein, that (some of) the connection strengths $w_{ab}(n, m)$ may slowly vary with respect to time and can actually be considered as inputs of dynamical system (1). Therefore, activity-dependent synaptic

plasticity is believed to play a crucial role in the development of neural circuits and it is thus possible to control a network of synaptically coupled excitatory-inhibitory modules through the connections strengths. An important question is which connection strengths should be modified (equivalently, which connection strengths should be considered as inputs of the network), in order to achieve a desired property for the resulting control system.

A property that is very useful in applications (e.g., for trajectory tracking, constructive controllability or trajectory generation) is that of flatness (see [5], [6]). In the recent paper [13], we considered for the case of $N = 2$ synaptically coupled subnetworks, the question of which connection strengths among $w_{ab}(n, m)$, for $a, b \in \{e, i\}$, $1 \leq n, m \leq 2$, should be modeled as control variables in order to obtain flatness for the resulting control system when *no relation between the connection strengths* (in particular, no symmetry) is assumed. In this paper, we study a similar problem but under the assumption that *all involved interactions* (that is, interactions between subnetworks and local interactions within the same subnetwork) *are symmetric*. This translates into relations between the connection strengths of the type $w_{ab}(n, m) = w_{ba}(m, n)$, for all $a, b \in \{e, i\}$ and $1 \leq n, m \leq N$. A network of synaptically coupled excitatory-inhibitory neural modules for which all those relations hold will be called *with maximal symmetry*. Contrary to the case when no relation between the connection strengths is assumed (for which the control network becomes, after the application of an invertible static feedback transformation, a control-affine system), in general, control networks with maximal symmetry are nonlinear with respect to the control (and cannot be rendered control-affine via static feedback transformations) which considerably complicates flatness analysis. In general, flatness imposes particular configurations of the interactions between the subnetworks or even for the local interactions within a subnetwork and our goal is to identify network configurations that are flat for **one of** the simplest possible networks with maximal symmetry (i.e., for which $N = 2$).

We will study the following question: for flatness, are there interactions between the subnetworks or local interactions within the same network that are not allowed (translating into zero w 's) or, on the contrary, that necessarily have to take place (translating into nonzero w 's)? In the first case, we will talk about structural conditions, while in the second case we will call them regularity conditions. Of course, flatness of a control network depends on the number of inputs. Similarly to [13], we will show that if enough connection strengths can be considered as inputs (at least 2 for a network

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of 2 synaptically coupled subnetworks) and if we have enough freedom in choosing which connection strengths are modeled as inputs, then the control network can be rendered flat without any restriction on the interactions between the two subnetworks (so without structural conditions). Contrary to [13], there are cases when the network is never flat or for which we are not able to characterize flatness in general (in those cases, we can however state that the control network does not admit any flat output depending on the state and the control variables only, so if the control network is flat, then its flat output necessarily involves derivatives of the control).

The problem of deciding which connection strengths have to be considered as inputs in order to achieve flatness for the control network is related to that of constructing flat inputs for a dynamical system [7], [14], [17], but several major differences can be listed and we send the reader to [13] for discussion and comparison.

II. DEFINITIONS AND PROBLEM STATEMENT

A. Flatness and linearization Consider the nonlinear control system $\Xi : \dot{x} = f(x, u)$, where $x \in X \subset \mathbb{R}^s$, $u = (u_1, \dots, u_c) \in \mathbb{R}^c$. The dynamics f are C^∞ -smooth and $\text{rk} \frac{\partial f}{\partial u} = c$. Fix an integer $l \geq -1$ and denote $\bar{u}^l = (u, \dot{u}, \dots, u^{(l)})$. For $l = -1$, the sequence \bar{u}^{-1} is empty.

Definition 1: The system $\Xi : \dot{x} = f(x, u)$ is flat at $(x_0, \bar{u}_0^l) \in X \times \mathbb{R}^{c(l+1)}$, for $l \geq -1$, if there exist a neighborhood \mathcal{O}^l of (x_0, \bar{u}_0^l) and c smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \dots, u^{(l)})$, $1 \leq i \leq c$, defined in \mathcal{O}^l , having the following property: there exist an integer r and smooth functions γ_i , $1 \leq i \leq s$, and δ_j , $1 \leq j \leq c$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \dots, \varphi^{(r-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \dots, \varphi^{(r)})$$

for any C^{l+r} -control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \dots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \dots, \varphi_c)$ and is called a flat output.

If a flat system admits a flat output whose components are functions of x only (resp., of (x, u) only), i.e., we have $\varphi_i = \varphi_i(x)$ (resp., $\varphi_i = \varphi_i(x, u)$) for all $1 \leq i \leq c$, then the system is called x -flat (resp., (x, u) -flat). Flatness is closely related to the notion of feedback linearization. It is well known that systems linearizable via invertible static feedback are flat. The control system Ξ is *locally linearizable by static feedback* if it can be transformed via a local diffeomorphism $\tilde{x} = \phi(x)$ and a local invertible feedback transformation $u = \psi(x, \tilde{u})$ to a linear controllable system $\Lambda : \dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$ (two systems that can be transformed one into each other via transformations of that form are called static feedback equivalent systems). The problem of static feedback linearization was solved by [10] and [9]. In general, a flat system is not linearizable by static feedback, with the exception of the single-input case for which flatness is equivalent to static feedback linearization (thus to the necessary and sufficient conditions of [10]). Flat systems can be seen as a generalization of static feedback linearizable systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see [5], [15] for those notions. In our analysis, we will identify flat excitatory-inhibitory networks that are either static feedback linearizable or become static feedback linearizable after the application of a

one- or two-dimensional dynamic precompensator (actually after prolonging one or two suitably chosen controls, which is the simplest dynamic feedback). Given Ξ and an integer $p \geq 1$, we construct its p -fold prolongation $\tilde{\Xi}^{p,0,\dots,0}$ by, first applying an invertible feedback $u = \psi(x, \tilde{u})$ to get $\tilde{\Xi} : \dot{x} = \tilde{f}(x, \tilde{u})$, where $\tilde{f} = f(x, \psi(x, \tilde{u}))$, and then prolonging the first (feedback modified) control as $\tilde{u}_1^{(p)} = v_1$ (which justifies the notation $\tilde{\Xi}^{p,0,\dots,0}$) and keeping the remaining controls $\tilde{u}_\ell = v_\ell$, $2 \leq \ell \leq c$, unchanged, to obtain

$$\tilde{\Xi}^{p,0,\dots,0} \begin{cases} \dot{x} = \tilde{f}(x, y^1, v_2, \dots, v_c) \\ \dot{y}^j = y^{j+1}, & 1 \leq j \leq p-1, \\ \dot{y}^p = v_1, \end{cases}$$

where $y^1 = \tilde{u}_1$. If there exists an invertible feedback transformation $u = \psi(x, \tilde{u})$ leading to a locally static feedback linearizable p -fold prolongation $\tilde{\Xi}^{p,0,\dots,0}$, then the system Ξ is called *dynamically linearizable via an invertible p -fold prolongation*. In a similar way, we can define the (p_1, p_2) -fold prolongation $\tilde{\Xi}^{p_1, p_2, 0, \dots, 0}$ obtained by prolonging the feedback modified controls \tilde{u}_1 and \tilde{u}_2 as $\tilde{u}_1^{(p_1)} = v_1$ and $\tilde{u}_2^{(p_2)} = v_2$, resp., as well as *dynamically linearizability via an invertible (p_1, p_2) -fold prolongation*.

B. Problem statement Consider a network of two synaptically coupled excitatory-inhibitory neural modules or subnetworks of form (1). Following [18], [19], we adopt the convention that $x_e^n = 0$, $x_i^n = 0$, $1 \leq n \leq 2$, is the resting state, and is assumed to reflect low-level background activity and therefore, small negative values of x_e^n and x_i^n are allowed, they have physiological significance and represent depression or inhibition of resting activity. For simplicity, from now on, we suppose that all time constants are equal to one, i.e., $\tau_e = \tau_i = 1$, and that the gain functions F_e and F_i are equal to the shifted logistic sigmoidal function (which is a classical assumption when mathematically modeling excitatory and inhibitory activity in localized populations of neurons [18], [19]):

$$F(\xi) = F_e(\xi) = F_i(\xi) = \mathcal{S}(\xi) - \mathcal{S}(0), \text{ with } \mathcal{S}(\xi) = \frac{1}{1 + e^{-\xi}}, \quad (2)$$

where $\xi \in \mathbb{R}$, that is, we work with systems of the form

$$\begin{aligned} \dot{x}_e^1 &= -x_e^1 + F(w_{ee}(1,1)x_e^1 + w_{ee}(1,2)x_e^2 \\ &\quad - w_{ei}(1,1)x_i^1 - w_{ei}(1,2)x_i^2) \\ \dot{x}_i^1 &= -x_i^1 + F(w_{ie}(1,1)x_e^1 + w_{ie}(1,2)x_e^2 \\ &\quad - w_{ii}(1,1)x_i^1 - w_{ii}(1,2)x_i^2) \\ \dot{x}_e^2 &= -x_e^2 + F(w_{ee}(2,1)x_e^1 + w_{ee}(2,2)x_e^2 \\ &\quad - w_{ei}(2,1)x_i^1 - w_{ei}(2,2)x_i^2) \\ \dot{x}_i^2 &= -x_i^2 + F(w_{ie}(2,1)x_e^1 + w_{ie}(2,2)x_e^2 \\ &\quad - w_{ii}(2,1)x_i^1 - w_{ii}(2,2)x_i^2), \end{aligned} \quad (3)$$

with F given by (2). Denote by $f = (f_e^1(x, w), f_i^1(x, w), f_e^2(x, w), f_i^2(x, w))$ the right-hand side of (3), i.e., we have

$$\dot{x} = f(x, w), \quad (4)$$

where $x = (x_e^1, x_i^1, x_e^2, x_i^2) \in \mathbb{R}^4$ is the state, and $w = (w_{ee}(1,1), \dots, w_{ii}(2,2))$ denotes the connection strengths. The problem that we are studying in this paper can be formalized as follows:

Problem. Consider the EI network given by (3). For a given integer $1 \leq c \leq 4$, define c controls $u_{ab}^{n,m}$ among

$w_{ab}(n, m)$, for $a, b \in \{e, i\}$ and $1 \leq n, m \leq 2$, leading to a control system $\dot{x} = f(x, u)$, with $\text{rk} \frac{\partial f}{\partial u} = c$, for which the connection strengths not defined as inputs are assumed constant, and study how the subnetworks have to interact –equivalently, which are the conditions that the (constant) configuration strengths have to verify– such that the obtained control system $\dot{x} = f(x, u)$ is flat.

C. *Assumptions* From now on, we assume the following:

(A1) All interactions are perfectly symmetric, that is:

$$w_{ab}(n, m) = w_{ba}(m, n), \forall a, b \in \{e, i\} \text{ and } 1 \leq n, m \leq 2, \quad (5)$$

meaning that two subnetworks act on each other exactly in the same way and, moreover, that the local interactions between the excitatory and the inhibitory populations within the same network are also identical. In particular, if $w_{ab}(n, m) = 0$ (meaning that x_b^m does not directly impact the dynamics of x_a^n), then we necessarily have $w_{ba}(m, n) = 0$, and x_a^n does not act on x_b^m either. A network for which (5) holds will be called *with maximal symmetry*.

(A2) Any connection strength $w_{ab}(n, m)$, for $a, b \in \{e, i\}$ and $1 \leq n, m \leq 2$, may be modeled as an input for the control EI network and the corresponding input will be denoted by $u_{ab}^{n,m}$, and since $w_{ab}(n, m) = w_{ba}(m, n)$, the input $u_{ab}^{n,m}$ acts directly on x_a^n and x_b^m (it is explicitly involved into the expressions of \dot{x}_a^n and \dot{x}_b^m).

(A3) The connection strengths that are not modeled as controls will always be supposed constant (zero or nonzero).

(A4) We work locally, around a nominal point $(x_0, u_0) = (x_0, w_0)$, where u_0 corresponds to nominal values of the connection strengths modeled as inputs, and we always assume that control variables act independently on the system, that is, if c is the number of connection strengths modeled as inputs, then for the obtained control system $\dot{x} = f(x, u)$, we should have $\text{rk} \frac{\partial f}{\partial u}(x, u) = c$, for all (x, u) in a neighborhood of (x_0, u_0) , implying, in particular that $x_0 \neq 0$, see (2)-(4). Define the set of singularities

$$S = \{(x, u) \in \mathbb{R}^4 \times \mathbb{R}^c : \text{rk} \frac{\partial f}{\partial u}(x, u) < c\}. \quad (6)$$

The control network will always be considered around regular points $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^c \setminus S$. Moreover we suppose that the number c of connection strengths that can be modeled as inputs is strictly less than 4 (if $c = 4$, which is the state dimension, the fact that the control variables act independently on the system implies that flatness is trivial).

The constructed control system will always be nonlinear with respect to the control (because of the nonlinear sigmoidal function F), but in the particular case when all inputs correspond to connection strengths of the type $w_{aa}(n, n)$, it can be transformed via a local invertible static feedback transformation into a control-affine one (this was always the case for the control networks studied in [13]).

Flatness will often imply certain relations or conditions involving the connection strengths (that are not modeled as inputs). Conditions of the type $w_{ab}(n, m) \neq 0$ (or more generally, nonzero-type conditions) will be called *regularity conditions*, they do not restrict the interactions between different subnetworks or between excitatory and inhibitory

populations. On the other hand, conditions of the type $w_{ab}(n, m) = 0$ (or more generally, zero-type conditions) will be called *structural conditions* because they impose certain constraints on the interactions (and the EI network is flat only for certain configurations of the interactions between subnetworks or between excitatory and inhibitory populations). By *interaction configuration* between two subnetworks, we will mean all possible values of the associated connection strengths $w_{ab}(n, m)$, $a, b \in \{e, i\}$, $1 \leq n, m \leq 2$ (respecting the maximal symmetry assumption). We say that there is no interaction if all $w_{ab}(n, m)$ are zero.

III. MAIN RESULTS

In this section we give our main results: we study flatness of networks consisting of two synaptically coupled EI neural subnetworks of form (3) with maximal symmetry, implying that we have 9 candidates for the inputs. If all controls are chosen among connection strengths of the type $w_{aa}(n, n)$, i.e., for fixed $a \in \{e, i\}$ and $1 \leq n \leq 2$, the corresponding input is involved in the expression of \dot{x}_a^n only, and does not appear in the dynamics of any x_b^m , for $(b, m) \neq (a, n)$, then the resulting control system falls into the class of control networks studied in [13], for which we identified and characterized all possible flat configurations. In particular, we showed that the control EI network can always be rendered flat without constraints on the interactions between subnetworks or on the local interactions within the same network, except for the case when only one connection strength is modeled as an input (for which we necessarily need structural conditions). Therefore, from now on, we make the following assumption:

(A5) We suppose that at least one input is defined as $w_{ab}(n, m)$, with $a \neq b$ or $n \neq m$.

Assumption (A5) means that we either act on the coupling of the local excitatory-inhibitory populations within the n -th subnetwork (when $w_{ab}(n, n)$, $a \neq b$, plays the role of the input) or on the interaction between the n -th and the m -th subnetworks, when $n \neq m$.

A. *Control network with three inputs* Assume that we can model three connection strengths as controls. Recall that we work around regular points $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^3 \setminus S$, with S given by (6), and that control variables act independently on the network, i.e., $\text{rk} \frac{\partial f}{\partial u}(x_0, u_0) = c = 3$, with $f = (f_e^1, f_i^1, f_e^2, f_i^2)$ the right hand-side of (3). To make matters more definite, suppose that $\text{rk} \frac{\partial (f_e^1, f_i^1, f_e^2)}{\partial u}(x_0, u_0) = 3$ (all other cases can be treated similarly). The control network is:

- case 1) either of the form

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x, u) & \dot{x}_e^2 &= f_e^2(x, u) \\ \dot{x}_i^1 &= f_i^1(x, u) & \dot{x}_i^2 &= f_i^2(x) \end{aligned}, \quad \text{with } u \in \mathbb{R}^3, \quad (7)$$

where f_i^2 is a function of x only, occurring when no connection strength associated to the inhibitory population of the second subnetwork (i.e., no $w_{ai}(n, 2) = w_{ia}(2, n)$, $a \in \{e, i\}$, $1 \leq n \leq 2$) is modeled as an input;

- case 2) or of the form

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x, u) & \dot{x}_e^2 &= f_e^2(x, u) \\ \dot{x}_i^1 &= f_i^1(x, u) & \dot{x}_i^2 &= f_i^2(x, u) \end{aligned}, \quad \text{with } u \in \mathbb{R}^3, \quad (8)$$

where all f_a^n depend explicitly on u ; which occurs if connection strengths associated to both subnetworks and both excitatory and inhibitory populations (i.e., $w_{ab}(n, m) = w_{ba}(m, n)$ such that all cases $a = e, a = i, n = 1$, and $n = 2$ are present) are necessarily modeled as inputs).

Therefore by applying the invertible (nonlinear) feedback transformation:

$\tilde{u}_\ell = f_a^n(x, u)$, for $(a, n) \in \{(e, 1), (i, 1), (e, 2)\}$, $1 \leq \ell \leq 3$, the three-input network, around $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^3 \setminus S$, can always be transformed, in a neighborhood of (x_0, \tilde{u}_0) with \tilde{u}_0 such that $\tilde{u}_{\ell,0} = f_a^n(x_0, u_0)$, $1 \leq \ell \leq 3$, resp., into:

$$\begin{aligned} \dot{x}_e^1 &= \tilde{u}_1 & \dot{x}_e^2 &= \tilde{u}_3 \\ \dot{x}_i^1 &= \tilde{u}_2 & \dot{x}_i^2 &= \tilde{f}_i^2(x), \end{aligned} \quad (9)$$

for case 1), or into

$$\begin{aligned} \dot{x}_e^1 &= \tilde{u}_1 & \dot{x}_e^2 &= \tilde{u}_3 \\ \dot{x}_i^1 &= \tilde{u}_2 & \dot{x}_i^2 &= \tilde{f}_i^2(x, \tilde{u}), \end{aligned} \quad (10)$$

for case 2), with the function \tilde{f}_i^2 of the feedback modified system depending explicitly (and in a nonlinear way) on at least one control (actually, such that $\frac{\partial^2 \tilde{f}_i^2}{\partial \tilde{u}_\ell^2} \neq 0$, for at least one $1 \leq \ell \leq 3$).

It can be easily shown that control network (7) is flat at x_0 if and only if $(\frac{\partial f_i^2}{\partial x_1^1}, \frac{\partial f_i^2}{\partial x_1^2}, \frac{\partial f_i^2}{\partial x_2^2})(x_0) \neq (0, 0, 0)$, equivalently $(w_{ie}(2, 1), w_{ii}(2, 1), w_{ie}(2, 2)) \neq (0, 0, 0)$, and we actually recover [13, Proposition 2] characterizing flatness for three-inputs EI networks when no relation between the connection strengths is assumed. This is due to the fact that after applying suitable feedback transformations, both network (7) and that of [13, Proposition 2]) can be transformed into the equivalent control-affine one (9). First of all, notice that there are no structural conditions (i.e., there are no connections strengths that have to be zero or to satisfy a zero-type relation) for flatness of the above three-input EI network and that only a regularity condition is needed (translating into a certain connection strength that has to be nonzero, which is the case for generic systems). The regularity condition means that either the first subnetwork directly affects the second one (more precisely, its inhibitory population) or that there are interactions between the excitatory and the inhibitory populations within the second subnetwork. Finally, observe that when the regularity condition is satisfied, control network (7) is actually static feedback linearizable. The next result identifies flat network configurations of form (8) (actually, of the feedback equivalent form (10), for which the conditions are easier to formulate and use simpler notations).

Proposition 1: Consider the three-input control EI network given by (8) around $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^3 \setminus S$ and its feedback equivalent network (10) around (x_0, \tilde{u}_0) .

(i) *If there exists $1 \leq \ell^* \leq 3$ such that $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_{\ell^*}} \equiv 0$ and $\frac{\partial \tilde{f}_i^2}{\partial x_{a^*}^{n^*}}(x_0, \tilde{u}_0) \neq 0$, where $x_{a^*}^{n^*}$ is such that $\dot{x}_{a^*}^{n^*} = \tilde{u}_{\ell^*}$, then system (10) is flat at (x_0, \tilde{u}_0) , implying that control EI network (8) is flat at (x_0, u_0) as well. Moreover, system (8) is dynamically linearizable via an invertible either 1-fold or (1,1)-fold prolongation.*

(ii) *If $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_\ell} \neq 0$, for all $1 \leq \ell \leq 3$, then system (10) is not x -flat thus (8) is not x -flat either.*

Sketch of proof: (i) Without loss of generality, suppose that $\ell^* = 3$, i.e., $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_3} \equiv 0$ and $\frac{\partial \tilde{f}_i^2}{\partial x_e^2}(x_0, u_0) \neq 0$, then it is obvious that (x_e^1, x_i^1, x_i^2) is a flat output of (10) (and therefore of (8) as well). Moreover, it is easy to see that the control EI network (10) is never static feedback linearizable (the necessary and sufficient conditions of [10] are not verified), however its (1,1)-prolongation given by:

$$\begin{aligned} \dot{x}_e^1 &= y_1 & \dot{x}_e^2 &= v_3 \\ \dot{x}_i^1 &= y_2 & \dot{x}_i^2 &= \tilde{f}_i^2(x, y_1^1, y_2^1), \\ \dot{y}_1^1 &= v_1 \\ \dot{y}_2^1 &= v_2, \end{aligned}$$

where (x, y_1^1, y_2^1) , with $y_1^1 = \tilde{u}_1$, $y_2^1 = \tilde{u}_2$, is the state of the prolonged system and $v = (v_1, v_2, v_3)$, with $v_\ell = \dot{\tilde{u}}_\ell$, $\ell = 1, 2$, and $v_3 = \tilde{u}_3$, is the new control, is locally static feedback linearizable around (x_0, \tilde{u}_0) . Observe also that in the case when \tilde{f}_i^2 depends on one control only (say \tilde{u}_1) only a 1-fold prolongation of \tilde{u}_1 is needed to dynamically linearize the original system. Suppose now system (10) is flat.

(ii) *To show that (10) is never x -flat, one can first prove that a system of the form $\dot{x} = f(x, u)$, $x \in \mathbb{R}^4$, $u \in \mathbb{R}^3$, is x -flat if and only if it can be transformed via a local diffeomorphism $\tilde{x} = \phi(x)$ and an invertible static feedback transformation $u = \psi(x, \tilde{u})$ into $\dot{\tilde{x}}_\ell = \tilde{u}_\ell$, for $1 \leq \ell \leq 3$, and $\dot{\tilde{x}}_4 = f_4(\tilde{x}, \tilde{u}_1, \tilde{u}_2)$. This form actually falls into the class of nonlinear control systems of codimension 1 (with one input less than the number of states) considered in [12], where a geometric characterization of that class is provided. It is now enough to show that control network (10) with \tilde{f}_i^2 depending on all three inputs does not verify the necessary and sufficient condition of [12].* \square

If the function \tilde{f}_i^2 of the feedback modified system (10) depends on at most two controls among u_ℓ , $1 \leq \ell \leq 3$, (equivalently, if each function f_a^n of (8) involves at most two controls¹ $u_{ab}^{n,m}$), then control EI network (10) (equivalently, (8)) is flat without structural conditions, only one regularity condition is needed. Contrary to the case of control network (7) (and, of all results of [13] as well), notice that the nonlinear feedback transformations introduce singularities for flatness involving the state and the control (at points (x, u) such that $\frac{\partial \tilde{f}_i^2}{\partial x_{a^*}^{n^*}}(x, u) = 0$). Notice that Proposition 1(i) gives only sufficient conditions for flatness of a network of form (10) and does not describe all possible flat configurations. If \tilde{f}_i^2 explicitly depends on all controls (see Proposition 1(ii)), then the picture is even more complicated that that of Proposition 1(i). Indeed, even in the case of control systems nonlinear with respect to the control with four states and three controls, the (general) problem of flatness is still open. We can however show that the system is never x -flat. The only possibility for \tilde{f}_i^2 to involve all three controls is that the inputs are necessarily defined as $w_{ei}(1, 2)$, $w_{ei}(2, 2)$ and $w_{ii}(1, 2)$. So in that case, if the control network is flat, then its flat outputs necessarily

¹Under the assumption that $\text{rk} \frac{\partial (f_e^1, f_i^1, f_e^2)}{\partial u} = 3$, all functions f_a^n of (8) depend on at most two controls if at least one input is not of the form $w_{ai}(n, 2) = w_{ia}(2, n)$, with $a \in \{e, i\}$, $1 \leq n \leq 2$.

depend on u or its time-derivatives and in general, flatness would imply additional structural conditions.

Conclusion (three inputs): For both cases 1) and 2) leading, resp., to (7) and (8), if we have enough freedom in choosing which connection strengths are modeled as inputs, then we can always identify EI network configurations that are flat (actually x -flat) without structural conditions (i.e., there are no connections that have to be zero or to satisfy a zero-type relation) and for flatness of the proposed control EI network only a regularity condition is needed.

B. Control network with two inputs Assume that two connection strengths are modeled as controls leading to a control network $\dot{x} = f(x, u)$, considered around a nominal point (x_0, u_0) such that $\text{rk} \frac{\partial f}{\partial u}(x_0, u_0) = 2$, that is, $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^2 \setminus S$, with S given by (6). According to Assumption (A5), at least one input is defined as $w_{ab}(n, m)$, with $a \neq b$ or $n \neq m$, and let us suppose that $\text{rk} \frac{\partial(f_e^1, f_i^1)}{\partial u}(x_0, u_0) = 2$ (all remaining cases are similar and we do not treat them here). The control network is:

- case 1) either of the form

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x, u) & \dot{x}_e^2 &= f_e^2(x) \\ \dot{x}_i^1 &= f_i^1(x, u) & \dot{x}_i^2 &= f_i^2(x) \end{aligned}, \quad \text{with } u \in \mathbb{R}^2, \quad (11)$$

where f_e^2 and f_i^2 are functions of x only, occurring when only connection strengths among $w_{ab}(1, 1) = w_{ba}(1, 1)$, $a \in \{e, i\}$ are modeled as inputs;

- case 2) or of the form

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x, u) & \dot{x}_e^2 &= f_e^2(x, u) \\ \dot{x}_i^1 &= f_i^1(x, u) & \dot{x}_i^2 &= f_i^2(x) \end{aligned}, \quad \text{with } u \in \mathbb{R}^2, \quad (12)$$

where f_i^2 is a function of the state only, occurring when no connection strength among $w_{ai}(n, 2) = w_{ia}(2, n)$, $a \in \{e, i\}$, $1 \leq n \leq 2$, is modeled as an input;

- case 3) or of the form

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x, u) & \dot{x}_e^2 &= f_e^2(x, u) \\ \dot{x}_i^1 &= f_i^1(x, u) & \dot{x}_i^2 &= f_i^2(x, u) \end{aligned}, \quad \text{with } u \in \mathbb{R}^2, \quad (13)$$

where all f_a^n depend explicitly on u ; this occurs if connection strengths associated to both subnetworks and both excitatory and inhibitory populations (i.e., $w_{ab}(n, m) = w_{ba}(m, n)$ such that all cases $a = e$, $a = i$, $n = 1$, and $n = 2$ are present) are necessarily modeled as inputs.

By applying the invertible (nonlinear) feedback transformation (recall that $\text{rk} \frac{\partial(f_e^1, f_i^1)}{\partial u}(x_0, u_0) = 2$):

$$\tilde{u}_\ell = f_a^1(x, u), \quad \text{for } a = e, i, \quad \text{and } 1 \leq \ell \leq 2,$$

the control network, around $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^2 \setminus S$, can always be transformed, in a neighborhood of (x_0, \tilde{u}_0) with \tilde{u}_0 such that $\tilde{u}_{\ell,0} = f_a^1(x_0, u_0)$, $a = e, i$, and $1 \leq \ell \leq 2$, can always be transformed either into:

$$\begin{aligned} \dot{x}_e^1 &= \tilde{u}_1 & \dot{x}_e^2 &= f_e^2(x) \\ \dot{x}_i^1 &= \tilde{u}_2 & \dot{x}_i^2 &= f_i^2(x), \end{aligned} \quad (14)$$

for case 1), or into:

$$\begin{aligned} \dot{x}_e^1 &= \tilde{u}_1 & \dot{x}_e^2 &= \tilde{f}_e^2(x, \tilde{u}) \\ \dot{x}_i^1 &= \tilde{u}_2 & \dot{x}_i^2 &= \tilde{f}_i^2(x), \end{aligned} \quad (15)$$

for case 2), with the function \tilde{f}_e^2 of the feedback modified system depending explicitly, in a nonlinear way, on at least one control (actually, such that $\frac{\partial^2 \tilde{f}_e^2}{\partial \tilde{u}_\ell^2} \neq 0$, for at least one

$1 \leq \ell \leq 2$) and \tilde{f}_i^2 depending on x only and remaining unchanged with respect to (12), i.e., $\tilde{f}_i^2(x) = f_i^2(x)$, or into:

$$\begin{aligned} \dot{x}_e^1 &= \tilde{u}_1 & \dot{x}_e^2 &= \tilde{f}_e^2(x, \tilde{u}) \\ \dot{x}_i^1 &= \tilde{u}_2 & \dot{x}_i^2 &= \tilde{f}_i^2(x, \tilde{u}), \end{aligned} \quad (16)$$

for case 3), with each function \tilde{f}_e^2 and \tilde{f}_i^2 of the feedback modified system depending explicitly on one (and only one) control among \tilde{u}_1 and \tilde{u}_2 . More precisely, form (16) is such that for $1 \leq \ell', \ell'' \leq 2$ with $\ell' \neq \ell''$, we have $\tilde{f}_e^2(x, \tilde{u}) = \tilde{f}_e^2(x, \tilde{u}_{\ell'})$ with $\frac{\partial^2 \tilde{f}_e^2}{\partial \tilde{u}_{\ell'}^2} \neq 0$ (but $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_{\ell''}} \equiv 0$), and $\tilde{f}_i^2(x, \tilde{u}) = \tilde{f}_i^2(x, \tilde{u}_{\ell''})$ with $\frac{\partial^2 \tilde{f}_i^2}{\partial \tilde{u}_{\ell''}^2} \neq 0$ (but $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_{\ell'}} \equiv 0$).

Similarly to Section III-A case 1), here as well flatness in case 1) reduces to static feedback linearizability and we actually recover [13, Proposition 3]. Thus all remarks made in the aforementioned paper apply or can be adapted for (11), in particular we can render the EI network flat without structural conditions. Let us now turn to case 2). Like for Section III-A case 2), we will identify flat network configurations of form (12) when at least one input is not of the form $w_{ae}(n, 2) = w_{ea}(2, n)$, with $a \in \{e, i\}$, $1 \leq n \leq 2$, leading to the feedback equivalent form (15) with the function \tilde{f}_e^2 depending on one control only (say \tilde{u}_1). Therefore, consider (15) and suppose that $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_1} \neq 0$ (implying actually that $\frac{\partial^2 \tilde{f}_e^2}{\partial \tilde{u}_1^2} \neq 0$) and $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_2} \equiv 0$. We will distinguish three mutually exclusive subcases: (A) $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_1} \neq 0$ but $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_1} \equiv 0$, (B) $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_1} \equiv 0$ but $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_1} \neq 0$, and (C) $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_1} \neq 0$ and $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_1} \neq 0$. Notice that both cases (A) and (B) necessarily impose structural conditions on the interactions between subnetworks.

Proposition 2: Consider the two-input control EI network given by (12) around $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}^2 \setminus S$, and its feedback equivalent network (15) around (x_0, \tilde{u}_0) . Suppose that \tilde{f}_e^2 depends on one and only one control, say \tilde{u}_1 (i.e., $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_1} \neq 0$ and $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_2} \equiv 0$). The following statements hold:

- (i) If \tilde{f}_e^2 and \tilde{f}_i^2 are in case (A), with $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_1}(x_0, \tilde{u}_0) \neq 0$, and if additionally $(\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_1}, \frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_2})(x_0) \neq (0, 0)$, then system (15) is flat at (x_0, \tilde{u}_0) with either $\varphi = (x_e^2, x_i^2)$ or $\varphi = (x_e^1, x_i^2)$ a flat output, implying that control EI network (12) is flat at (x_0, u_0) as well. Moreover, system (12) is dynamically linearizable via an invertible 1-fold prolongation.
- (ii) If \tilde{f}_e^2 and \tilde{f}_i^2 are in case (B), with $\frac{\partial \tilde{f}_i^2}{\partial \tilde{u}_1}(x_0, \tilde{u}_0) \neq 0$, and if additionally $\frac{\partial \tilde{f}_e^2}{\partial \tilde{u}_2}(x_0, \tilde{u}_0) \neq 0$, then system (15) is flat at (x_0, \tilde{u}_0) with $\varphi = (x_e^1, x_e^2)$ a flat output implying that control EI network (12) is flat at (x_0, u_0) as well. Moreover, system (12) is dynamically linearizable via an invertible 2-fold prolongation.
- (iii) If \tilde{f}_e^2 and \tilde{f}_i^2 are in case (C), then (15) is neither x -flat nor (x, u) -flat and thus system (12) is not either.

Sketch of proof: The structural conditions (A) and (B) translate into triangular forms which together with the regularity conditions of (i) and (ii), immediately guarantee that the control network is flat and that it admits the flat outputs given by conditions of (i) and (ii). Moreover, it is clear

that it becomes static feedback linearizable via at most 2-fold prolongation of \tilde{u}_1 . If the functions \tilde{f}_e^2 and \tilde{f}_i^2 are in case (C), then one can show that the system is neither x -flat nor (x, u) -flat by applying, for instance, the results of [8], [11] according to which a two-input (x, u) -flat systems can be statically linearized after an at most $\dim(x)$ -fold prolongation of a suitably chosen input. \square

Notice that for all flat configurations of items (i) and (ii) of the above proposition, we need a structural condition (either condition (A) or condition (B)). If the control EI network satisfies (C) and if it is possible for the system to be flat (so with flat outputs necessarily depending on the derivatives of u), then even if (C) does not contain any structural conditions, flatness will necessarily imply that some interactions are restricted. Let us now consider the case when the connection strengths that are modeled as inputs lead to the control network (13).

Proposition 3: The two-input control EI network given by (13), or equivalently the feedback modified system (16), is not flat.

Sketch of proof: One can use the ruled-manifold criterion [5], [16] for the feedback modified system, for which \tilde{f}_e^2 and \tilde{f}_i^2 are such that their second derivatives with respect to \tilde{u} are nonzero. \square

Conclusion (two inputs): If only two connection strengths can be modeled as inputs, the only flat network configuration that we are able to identify without constraining the interactions between the subnetworks (i.e., without structural conditions) are those for which the control network is of form (11) and can be transformed into a control-affine system by feedback transformations. If the connection strengths modeled as inputs yield a network of form (12), flat configurations can be identified, they are in a triangular form, and they involve a structural condition. We do not however completely describe flatness of networks of form (12) (i.e., there may be flat control networks of form (12), for which neither condition (i) nor condition (ii) of Proposition 2 is satisfied, but if such a flat network exists, then flatness will require structural conditions). If the connection strengths modeled as inputs are such that the network is of form (13), then the control network is never flat.

C. Control network with one input We consider next the case when among all connection strengths, only one can be modeled as an input. According to (A5), the input is defined as $w_{ab}(n, m)$, with $a \neq b$ or $n \neq m$. Consider the network around a point $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R} \setminus S$, i.e., for which $\text{rk} \frac{\partial f}{\partial u}(x_0, u_0) = 1$. Suppose that $u = u_{ei}^{1,1} = w_{ei}(1, 1) = w_{ie}(1, 1)$ (all remaining cases are similar and we do not treat them here), implying in particular that x_0 is such that $(x_{e,0}^1, x_{i,0}^1) \neq (0, 0)$ and assume additionally that $x_{e,0}^1 \neq 0$. The control network is of the form:

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x, u) & \dot{x}_e^2 &= f_e^2(x) \\ \dot{x}_i^1 &= f_i^1(x, u) & \dot{x}_i^2 &= f_i^2(x), \end{aligned} \quad (17)$$

and by applying the invertible (nonlinear) feedback transformation $\tilde{u}_1 = f_e^1(x, u)$, it can be transformed into:

$$\begin{aligned} \dot{x}_e^1 &= \tilde{u}_1 & \dot{x}_e^2 &= f_e^2(x) \\ \dot{x}_i^1 &= \tilde{f}_i^1(x, \tilde{u}_1) & \dot{x}_i^2 &= f_i^2(x), \end{aligned} \quad (18)$$

with \tilde{f}_i^1 depending explicitly on \tilde{u}_1 and such that $\frac{\partial^2 \tilde{f}_i^1}{\partial \tilde{u}_1^2} \neq 0$.

Proposition 4: The single-input control EI network given by (17), or equivalently the feedback modified system (18), is not flat.

Sketch of proof: By [2], the single-input control network is flat if and only if it is static feedback linearizable, that is, satisfies the necessary and sufficient conditions of [10]. Consider the feedback modified system (18). Since $\frac{\partial^2 \tilde{f}_i^1}{\partial \tilde{u}_1^2} \neq 0$, it can be easily shown that the structural condition (the involutivity of some distributions) of [10] is not verified, implying that (18) is not static feedback linearizable thus not flat, and consequently (17) is not flat either. \square

Conclusion ($N = 2$, one input): If the input is defined following Assumption (A5), then the single-input control network is never flat. If we drop Assumption (A5), and we define the control by $u = u_{aa}^{n,n} = w_{aa}(n, n)$, with $a = e$ or $a = i$, then we recover the results of [13] according to which it is possible to render the control network flat but in order to do so, we always need structural conditions for the interactions between the subnetworks and even for the local interactions within the same subnetwork.

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