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Every theory is eventually of presheaf type

Christian Espíndola, Kristóf Kanalas

Abstract

We give a detailed and self-contained introduction to the theory of λ -toposes and prove the following: 1) A λ -separable λ -topos (one whose defining site has a certain smallness property) has enough λ -points. 2) Given a κ -site, its classifying λ -topos is of presheaf type (assuming $\kappa \triangleleft \lambda = \lambda^{<\lambda}$).

1 Introduction

The main ideas of this paper have their origin in [Esp19b], [Esp20] and [Esp19a, Section 3.2 and Section 4]. In these notes our intention was to find a detailed and syntax-free, purely categorical presentation.

To do so, we first discuss the fundamental notion of compatibility between extremal epimorphic families and $< \kappa$ -limits. In [Mak90] a κ -regular category is defined as one having $< \kappa$ -limits and effective epi-mono factorizations, such that the transfinite cocomposition of a continuous $< \kappa$ -sequence of effective epis is effective epi. The natural counterpart of this for coherent categories is formulated in terms of covering cotrees: if one builds a cotree on an object, which is locally extremal epimorphic and every branch is continuous, $< \kappa$, then the cotree is globally extremal epimorphic, i.e. the transfinite cocomposition of the branches form an extremal epic family on the root. In the next section we study this notion in detail and prove a completeness theorem.

Then we define a κ -topos as a Grothendieck-topos whose extremal epic families are compatible with $< \kappa$ -limits in the above sense. We lift the classical results concerning classifying toposes to κ -toposes. Finally we reformulate the completeness theorem in terms of κ -toposes whose defining site is small in an appropriate sense, generalizing [MR77, Theorem 6.2.4] (Theorem 5.4), and derive the main result of this paper; that the classifying λ -topos of a κ -site is of presheaf type, assuming $\kappa \triangleleft \lambda = \lambda^{<\lambda}$ (Theorem 5.5).

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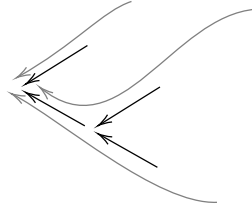
2 A completeness theorem

This section motivates the definition of a κ -Grothendieck-topology, i.e. the correct notion of compatibility between covers and $< \kappa$ -limits. Its content is a completeness theorem: if a κ -lex category \mathcal{C} has $\leq \kappa$ -big Hom-sets then given a $\leq \kappa$ -big collection E of extremal epimorphic families interacting well with $< \kappa$ -limits, there is a jointly conservative set of $\mathcal{C} \rightarrow \mathbf{Set}$ κ -lex E -preserving functors. As an application we give a Rasiowa-Sikorski-like result for sufficiently distributive lattices.

(In Section 5 we will present a more general version of the completeness theorem, based on the same idea.)

Definition 2.1. $\kappa = cf(\kappa) \geq \aleph_0$. \mathcal{C} is κ -lex (i.e. it is a category with $< \kappa$ -limits). A class E of extremal epimorphic families is said to be *compatible with $< \kappa$ -limits* if

1. Each member of E is pullback-stable (i.e. its pullback along any map is extremal epimorphic). E^{pb} denotes the class of extremal epimorphic families obtained this way.
2. Given a rooted cotree (as a diagram in \mathcal{C}), such that on any vertex its predecessors form a member of E^{pb} , every branch has length $< \kappa$ and every branch is continuous (objects sitting at limit points are limits in \mathcal{C}); it follows that the transfinite cocomposition of the branches is extremal epimorphic.



Definition 2.2. We shall name a few kind of diagrams: by *cotree* we will mean the opposite of a rooted tree as a diagram in \mathcal{C} . By continuous (E^{pb}, κ) -cotree we mean that on any vertex its predecessors form a member of E^{pb} , every branch has length $< \kappa$ and every branch is continuous. By a continuous *cofinal* (E^{pb}, κ^+) -cotree we mean that the cotree has height κ and each branch has size κ (hence order type κ^{op}). When E consists of all extremal epimorphic families with $< \lambda$ -many legs, we may write locally covering continuous (λ, κ) -cotree instead of continuous (E, κ) -cotree.

Definition 2.3. A category \mathcal{C} is of *local size* $< \kappa$ if each Hom-set is of cardinality $< \kappa$.

Theorem 2.4. \mathcal{C} is κ -lex of local size $\leq \kappa$, and it has a strict initial object \emptyset . Let E be a set of extremal epimorphic families, such that $|E| \leq \kappa$ and E is compatible with $< \kappa$ -limits. Then given any object $u_0 \in \mathcal{C}$ there is a continuous cofinal (E^{pb}, κ^+) -cotree with root u_0 such that given any branch $u_0 \xleftarrow{p_{0,i_1}} u_{0,i_1} \xleftarrow{p_{0,i_1,i_2}} u_{0,i_1,i_2} \leftarrow \dots$ the colimit of the representable functors $\mathcal{C}(u_0, -) \xrightarrow{p^*} \mathcal{C}(u_{0,i_1}, -) \xrightarrow{p^*} \dots$ is either the terminal copresheaf (iff the branch gets eventually \emptyset) or it preserves the extremal epimorphic families in E . When $u_0 \neq \emptyset$ at least one branch yields a non-terminal copresheaf.

Proof. Let $u_0 \xleftarrow{p_0} u_1 \leftarrow \dots$ be an (at this point not necessarily continuous) κ -chain in \mathcal{C} . First we would like to understand what does it mean for the colimit of $\mathcal{C}(u_0, -) \xrightarrow{p_0^*} \mathcal{C}(u_1, -) \Rightarrow \dots$ to preserve a given extremal epimorphic family $(x_j \rightarrow y)_{j < \gamma}$ (with $\gamma \geq 1$). This is easy: it is mapped to a jointly surjective family iff for any map $u_\alpha \rightarrow y$ there's $u_\beta \rightarrow u_\alpha$ in the sequence such that for some $j < \gamma$ the dashed arrow

$$\begin{array}{ccccc}
 u_\beta & \longrightarrow & u_\alpha & \longrightarrow & y \\
 & & & & \uparrow \\
 & & & & x_0 \quad \dots \quad x_j \quad \dots \\
 & \dashrightarrow & & & \nearrow
 \end{array}$$

exists.

We shall also understand the $\gamma = 0$ case. The empty collection of arrows is extremal epi on y iff y has no proper subobjects. As the initial object is strict, every map out of it is mono, hence this is the same as $y \cong \emptyset$. The colimit of the representables does not preserve the initial object iff one of the u_α 's is initial, in which case all the following u_β 's are initial and hence the colimit is the terminal copresheaf which is allowed. So we assume that the prescribed families are non-empty.

One such issue can certainly be solved at any stage of the construction: if we have started to construct the chain (and we are in a successor step) $u_0 \leftarrow \dots \leftarrow u_\beta$, then given $\alpha \leq \beta$ and a map $u_\alpha \rightarrow y \leftarrow x_j$ just take the pullback

$$\begin{array}{ccc}
 u_{\beta+1} & \longrightarrow & x_j \\
 \downarrow & & \downarrow \\
 u_\beta & \longrightarrow & u_\alpha \longrightarrow y
 \end{array}$$

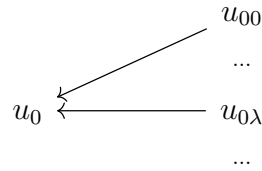
(for any j : hence we have γ -many options to continue the chain, these possibilities will form the cotree that we promised).

As our Hom-sets have size $\leq \kappa$ and we have κ -many prescribed families, there are κ -many tasks concerning a given $u \in \mathcal{C}$. But solving one such issue

will yield a new u_α and hence κ -many new tasks. Luckily we have κ -many steps to arrange everything and this can be done:

For any $u \in \mathcal{C}$ let T_u be the set (or list) of diagrams $u \rightarrow y \leftarrow x_0, x_1, \dots$ (where $(y \leftarrow x_j)_j$ is any member of E), well-ordered in order type κ (technicality: assume $1 \rightarrow 1$ is among the given families, so T_u is non-empty. Allowing repetitions we can assume that T_u has size κ). We have the canonical well-ordering $f : \kappa \times \kappa \rightarrow \kappa$ with the property $f(\alpha, \beta) \geq \beta$. We use it as follows:

Let's fix a table of size $\kappa \times \kappa$ (the second coordinate labels the columns) whose entries are empty at the moment. As the root of the cotree is u_0 we list T_{u_0} in the zeroth column: these tasks will have to be solved. Now, at step 1 we solve $f^{-1}(0)$, whose second coordinate is ≤ 0 , so it is one of the already listed tasks. We do it by forming the pullback of the covering family; this yields the first level of the cotree



To construct the continuation from $u_{0\lambda}$ fill in $T_{u_{0,\lambda}}$ in the first column and solve $f^{-1}(1)$, whose second coordinate is ≤ 1 hence we know which task it is.

The transfinite recursion is then the obvious thing: to define the α -level of the cotree at limit α just take the transfinite cocomposition of the branches. Now do it for $\alpha + 1$: to continue from a given vertex u_{\dots} pick the corresponding table whose columns before α are filled in (with the tasks corresponding to the elements which are above u_{\dots}). Now fill $T_{u_{\dots}}$ into the α^{th} column and solve task number $f^{-1}(\alpha)$.

This defines a continuous cofinal (E^{pb}, κ^+) -cotree. When we go along a branch we see that every task is solved, so the colimit preserves all the given extremal epimorphic families, except the empty union which we cancelled from the list. So every branch yields either the terminal copresheaf or an E -preserving one.

It remains to prove that when $u_0 \neq \emptyset$ at least one branch yields a non-terminal copresheaf, i.e. that it cannot happen that every branch is becoming constant \emptyset at some point. But this is clear, otherwise (by cutting down each branch at that point) we would get a continuous (E^{pb}, κ) -cotree which is not covering the root. \square

From this we derive the completeness theorem:

Theorem 2.5. *Let \mathcal{C} be κ -lex of local size $\leq \kappa$ with a strict initial object, and let E be a set of extremal epimorphic families, such that it is compatible with $< \kappa$ -limits and $|E| \leq \kappa$. Then given $x \in \mathcal{C}$ and u, v subobjects of x , if for every $M : \mathcal{C} \rightarrow \mathbf{Set}$ κ -lex E -preserving functor we have $Mu \subseteq Mv$ then $u \subseteq v$.*

Proof. Take the above cotree with root $u = u_0$. If a branch h yields a κ -lex E -preserving functor $M_h = \text{colim}(\mathcal{C}(u, -) \xrightarrow{p_0^*} \mathcal{C}(u_{0,h(1)}, -) \Rightarrow \dots)$ then by assumption $M_h u \subseteq M_h v$, in particular $1_u \in M_h(u) \subseteq M_h(x)$ lies in $M_h(v)$, meaning that for some u_λ in the branch we have a lift

$$\begin{array}{ccc} x & \longleftarrow & v \\ \uparrow & & \swarrow \text{---} \\ u & \longleftarrow & u_\lambda \end{array}$$

When the branch becomes eventually \emptyset we also have this lifting. Cutting down each branch at such a point and using that the resulting (E^{pb}, κ) -cotree is covering the root we have an extremal epimorphic family with liftings

$$\begin{array}{ccc} x & \longleftarrow & v \\ \uparrow & & \uparrow \text{---} \\ u & \longleftarrow & u_0 \\ & \swarrow & \dots \\ & & u_\lambda \end{array}$$

Hence each $u_\lambda \rightarrow u$ factors through $v \cap u$ and therefore $v \cap u = u$ so $u \subseteq v$. \square

Definition 2.6. $\lambda = cf(\lambda) \geq \kappa = cf(\kappa) \geq \aleph_0$. A category \mathcal{C} is (λ, κ) -coherent if

- i) it has $< \kappa$ -limits,
- ii) it has pullback-stable effective epi - mono factorization,
- iii) it has pullback-stable $< \lambda$ -unions,
- iv) and $< \lambda$ extremal epimorphic families are compatible with $< \kappa$ -limits (pullback-stability follows from ii) and iii), so the additional requirement is the second clause of Definition 2.1: that locally covering continuous (λ, κ) -cotrees are globally covering).

A functor is (λ, κ) -coherent if it preserves $< \kappa$ -limits, effective epimorphisms and $< \lambda$ -unions.

Remark 2.7. When $\lambda > \kappa$ or $\lambda = \kappa$ and κ has the tree property, then in iv) we have $< \lambda$ -many branches so the union of the images automatically exists. Indeed, using the regularity of λ we see that for each $\alpha < \lambda$ the α -level of the tree is of size $< \lambda$. This solves the $\kappa < \lambda$ case. When $\lambda = \kappa$ and κ has the tree property, by definition the tree has height $< \kappa$ otherwise we had a κ -long branch.

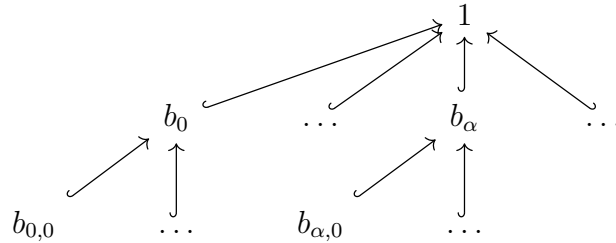
Remark 2.8. When $\kappa = \aleph_0$ iv) is redundant, i.e. (\aleph_0, \aleph_0) -coherent is just coherent, (λ, \aleph_0) -coherent is λ -geometric. Indeed, given a cotree as in iv) with root u_0 and an arbitrary proper subobject $v \subsetneq u_0$ one of the predecessors u_1 of the root does not factor through v . Hence one of the predecessors of u_1 does not factor through v . Since we cannot define an infinite branch we get stuck at a finite stage, meaning that the cocomposition of a branch does not factor through v .

Example 2.9. **Set** is (λ, κ) -coherent for any λ, κ . Given a set X and a locally covering continuous cotree on it, we shall see that the transfinite cocomposition of the branches cover X . But any element $x_0 \in X$ has a preimage in one of the predecessors X_1 . Once we define a compatible family of preimages up to height α , a limit ordinal, this sequence represents an element in the limit, so we've managed to find a preimage $x_\alpha = (x_i)_{i < \alpha} \in X_\alpha = \lim_{i < \alpha} X_i$. This defines a branch whose cocomposition hits x_0 .

As in presheaf categories limits and colimits are pointwise, also **Set**^A is (λ, κ) -coherent for any λ, κ .

Definition 2.10. A λ -complete Boolean-algebra is $(2, \lambda)$ -distributive if for any collection of elements $(b_{i,0})_{i \in I}, (b_{i,1})_{i \in I}$ with $|I| < \lambda$, we have $\bigcap_i (b_{i,0} \cup b_{i,1}) = \bigcup_{h:I \rightarrow 2} \bigcap_i b_{i,h(i)}$. In particular the union exists, though we do not require the existence of such big unions in general.

Example 2.11. Every λ -complete $(2, \lambda)$ -distributive Boolean-algebra is (λ, κ) -coherent if $\kappa < \lambda$ or $\kappa = \lambda$ with the tree property. Given a continuous, locally covering (λ, κ) -cotree of elements



we shall see that intersections along the branches cover the root (which can be assumed to be 1, otherwise just put its complement next to it and put 1 to

the -1^{th} level). This means that for any $0 \neq u \subseteq 1$ there's a branch whose intersection has non-empty intersection with u .

Since $\kappa < \lambda$ or $\kappa = \lambda$ with the tree property, there are $< \lambda$ -many elements in the tree. Let A be the set of these elements. By $(2, \lambda)$ -distributivity $1 = \bigcap_{b \in A} (b \cup \neg b) = \bigcup_{\varepsilon: A \rightarrow \{+, -\}} \bigcap_{b \in A} b^{\varepsilon(b)}$. As $u \neq 0$ it has non-empty intersection with a summand (otherwise $\neg u$ would be a smaller upper bound). Let u' be this intersection. Then it is atomic in the sense that for any $b \in A$ either $u' \subseteq b$ or $u' \subseteq \neg b$.

A member of the union is coming from decorating the tree with signs and then taking intersections of the elements or their complements (depending on the sign). If we want to get non-empty intersection (with u , equivalently with u') we must attach $+$ to 1, given any element with decoration $+$ one of the predecessors is decorated with $+$ and given any chain of $+$ -coloured elements the intersection must be $+$ -coloured as well. So u has non-empty intersection with the intersection of a decorated tree that has a positive branch.

Corollary 2.12 (of Theorem 2.5). *Let \mathcal{L} be a distributive lattice which is a (λ, κ) -coherent category and let $(A_i)_{i < \kappa}$ be a collection of subsets each of size $< \lambda$. Then there's an injective homomorphism of posets $\mathcal{L} \hookrightarrow \mathcal{P}(X)$ to a power set Boolean-algebra, which preserves all $< \kappa$ -meets and which preserves $\bigcup A_i$ for each i .*

Proof. If $a \neq b$ then either $a \cap b \subsetneq b$ or $a \cap b \subsetneq a$, hence they can be separated by an $\mathcal{L} \rightarrow \mathbf{Set}$ functor which preserves all $< \kappa$ -meets, monos, the terminal object, as well as the prescribed unions. In particular it lands in $2 \hookrightarrow \mathbf{Set}$. Putting these together yields the required homomorphism $\mathcal{L} \rightarrow 2^X$. \square

3 Classifying toposes

Definition 3.1. A category \mathcal{E} is a κ -topos if it is a Grothendieck-topos which is (∞, κ) -coherent. A geometric morphism $F_* : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a map of κ -toposes if F^* preserves $< \kappa$ -limits.

Remark 3.2. By the adjoint functor theorem a map of κ -toposes $\mathcal{E}_1 \rightarrow \mathcal{E}_2$ is the same as an $\mathcal{E}_2 \rightarrow \mathcal{E}_1$ functor preserving $< \kappa$ -limits and all colimits. We write $Fun^*(\mathcal{E}_2, \mathcal{E}_1)$ for the category of κ -lex cocontinuous functors and all natural transformations. (It is locally small.)

Example 3.3. By Example 2.9 every presheaf topos is a κ -topos for any κ .

Example 3.4. It is not true that every κ -lex localization of a presheaf topos is a κ -topos. Consider the site $([0, 1], \tau_{sup})$ where $[0, 1]$ is seen as a poset with the usual ordering and $(r_i \leq x)_{i \in I}$ is a cover if $sup_i r_i = x$. $[0, 1]$ has all limits (meets) given by infimum, and τ_{sup} is a Grothendieck-topology.

We claim that the sheafification map $a : \mathbf{Set}^{[0,1]^{op}} \rightarrow Sh([0, 1], \tau_{sup})$ preserves all limits. This follows as on any object $r \in [0, 1]$ there are only two covering sieves: $[0, r)$ and $[0, r]$, hence the $+$ -construction is given by a limit formula (see [MM92, p.134]).

However, $Sh([0, 1], \tau_{sup})$ is not an \aleph_1 -topos. We build a locally covering continuous tree (inside $[0, 1]$) on 1 as follows: the root is 1, given an object r in the tree, if $r = 0$ then it has no predecessors, if $r > 0$ then its predecessors are given by an ω -sequence converging to r , formed by elements that are $< r$. At limit stages we take infimums. This defines a locally covering (the predecessors of any object form a τ_{sup} -cover), continuous cotree such that every branch is countable (there is no uncountable strictly decreasing sequence in $[0, 1]$), and every branch terminates in 0. By the previous paragraph $[0, 1] \xrightarrow{y} \mathbf{Set}^{[0,1]^{op}} \xrightarrow{a} Sh([0, 1], \tau_{sup})$ takes infimums to limits (y preserves all limits), it maps τ_{sup} -covers to extremal epimorphic families, and in particular takes 0 to the initial and 1 to the terminal object. So if $Sh([0, 1], \tau_{sup})$ is an \aleph_1 -topos, then the branches of the ay -image of our tree form an extremal epimorphic family, therefore $\emptyset = *$. This is not the case as τ_{sup} -covers are effective epimorphic, hence ay is conservative (the representables are sheaves).

Definition 3.5. We fix the notation: Let \mathcal{C} be a small κ -lex category and E be a set of arbitrary families in \mathcal{C} (a family is a set of arrows with common codomain). By E^{pb} we denote the closure of E under pullbacks; those families which can be obtained as a pullback of an E -family. E^{tree_κ} is the closure of E under the κ -tree operation: the set of those families which are obtained as transfinite cocompositions of continuous (E, κ) -cotrees (the predecessors on any node form an E -family, every branch is continuous, $< \kappa$). We write $\langle E \rangle_\kappa$ for $(E^{pb})^{tree_\kappa}$.

A (small) κ -site is a pair (\mathcal{C}, E) where \mathcal{C} is κ -lex (small) and E is an arbitrary collection of families containing $id : 1 \rightarrow 1$. $F : (\mathcal{C}, E) \rightarrow (\mathcal{D}, E')$ is a *morphism of κ -sites* if F is a κ -lex $\mathcal{C} \rightarrow \mathcal{D}$ functor which takes E -families (and hence $\langle E \rangle_\kappa$ -families) to $\langle E' \rangle_\kappa$ -families.

Proposition 3.6. *Let (\mathcal{C}, E) be a small κ -site. Then:*

1. $\langle E \rangle_\kappa$ is closed under pullbacks and the κ -tree operation.
2. $\langle E \rangle_\kappa$ is a Grothendieck-topology.
3. $Sh(\mathcal{C}, \langle E \rangle_\kappa)$ is a κ -topos.
4. The full inclusion $Sh(\mathcal{C}, \langle E \rangle_\kappa) \hookrightarrow \mathbf{Set}^{C^{op}}$ is a map of κ -toposes, i.e. the sheafification functor a is κ -lex.

5. Sheafification of the Yoneda-embedding $ay : \mathcal{C} \xrightarrow{y} \text{Set}^{\mathcal{C}^{op}} \xrightarrow{a} \text{Sh}(\mathcal{C}, \tau)$ is κ -lex, E -preserving (maps E -families to extremal epimorphic ones).

Proof. 1. The pullback of a continuous (E^{pb}, κ) -cotree is again a continuous (E^{pb}, κ) -cotree, as the pullback of a pulled back E -family is the pullback of the original E -family along the composite and the pullback of a transfinite cocomposition is the same as the transfinite cocomposition of the pullbacks. Closed under the tree operation: the cotree we build can be seen as pasting some (E^{pb}, κ) -cotrees together, which will be a continuous (E^{pb}, κ) -cotree by the regularity of κ , and as the (transfinite) cocomposition of the transfinite cocompositions is a transfinite cocomposition.

2. It contains the isomorphisms as those are pullbacks of $id : 1 \rightarrow 1$. The rest of the requirements (being closed under pullbacks and being closed under the tree operation when one is building cotrees of height 2) is implied by Claim 1.

3. In any Grothendieck-topos a family $(F \leftarrow F_i)_{i \in I}$ is extremal epimorphic iff the following is satisfied: for any $x \in \mathcal{C}$ and $s \in F(x)$ there's a cover $(x \leftarrow x_j)_{j \in J}$ such that each $s|_{x_j}$ is the image of some $s' \in F_{i(j)}(x_j)$.

Now take a locally covering continuous (∞, κ) -cotree (i.e. every node is covered by an arbitrarily large extremal epimorphic family, each branch is $< \kappa$ and continuous), let F_0 be the root and take an $x_0 \in \mathcal{C}$ and $s^0 \in F_0(x_0)$. x_0 has a cover $(x_0 \leftarrow x_{0,j_1})_{j_1}$, such that the restrictions of s^0 are coming from some F_i 's (there can be many, so fix suitable $i(0, j_1)$'s and preimages $s^{0,j_1} \in F_{i(0,j_1)}(x_{0,j_1})$). Now each x_{0,j_1} has a cover $(x_{0,j_1} \leftarrow x_{0,j_1,j_2})_{j_2}$ such that the restrictions of s^{0,j_1} 's are coming from some predecessors of $F_{i(0,j_1)}$. So we started to build a locally covering continuous $((E^{pb})^{tree}, \kappa)$ -cotree on x_0 , together with a morphism i of cotrees, from the cotree on x_0 to the one on F_0 (meaning: $i(0, j_1, \dots)$ is an initial segment of $i(0, j_1, \dots, j_k)$). It has the property that $s^0|_{x_j}$ has a preimage in $F_{i(\vec{j})}$.

Once we are at a limit stage, for any branch $x_0 \leftarrow x_{0,j_1} \leftarrow x_{0,j_1,j_2} \leftarrow \dots$ its limit $x_{0,j_1,\dots}$ has the property that $s^0|_{x_{0,j_1,\dots}}$ has a preimage in $F_{i(0,j_1,\dots)}(x_{0,j_1,\dots})$, namely the compatible family $[s^0|_{x_{0,j_1,\dots}}, s^{0,j_1}|_{x_{0,j_1,\dots}}, \dots]$ (which is in the limit of $F_0 \leftarrow F_{i(0,j_1)} \leftarrow \dots$ because limits of sheaves are pointwise).

So we managed to define a locally covering continuous $((E^{pb})^{tree}, \kappa)$ -cotree (every branch has length $< \kappa$ because the same holds in the cotree of sheaves and i is a map of cotrees), and the restriction of s^0 to the trans-

finite cocomposition of a branch $x_{\vec{j}}$ has a preimage in the transfinite cocomposition of $F_{i(\vec{j})}$. This family on x_0 is a cover by Claim 1.

4. The poset of covers on an object $x \in \mathcal{C}$ (ordered by refinement) is κ -filtered (one can build a tree whose first level consists of the 0^{th} family and whose $\alpha + 1^{\text{th}}$ level is formed by pulling back the α^{th} family to the leaf). Hence the $+$ -construction is κ -lex.
5. The composite is κ -lex by Claim 4. To check that the E -family $(z \leftarrow z_i)_i$ is preserved we have to see that for any x and map $s : x \rightarrow z$ there's a cover on x s.t. the precompositions $x_j \rightarrow x \rightarrow z$ factor through some $z_{i(j)}$. Simply take the pullback of $(z \leftarrow z_i)_i$ along s .

□

Corollary 3.7. *Let (\mathcal{C}, E) and (\mathcal{D}, E') be small κ -sites and $F : (\mathcal{C}, E) \rightarrow (\mathcal{D}, E')$ be a morphism of κ -sites. Then there is an induced morphism of κ -toposes $F_* : Sh(\mathcal{D}, \langle E' \rangle_\kappa) \rightarrow Sh(\mathcal{C}, \langle E \rangle_\kappa)$ where F_* is $- \circ F^{op}$ and F^* is $Sh(\mathcal{C}) \hookrightarrow \mathbf{Set}^{\mathcal{C}^{op}} \xrightarrow{Lan_{F^{op}}} \mathbf{Set}^{\mathcal{D}^{op}} \xrightarrow{a} Sh(\mathcal{D})$.*

Proof. The geometric morphism exists by [Joh02, Corollary C2.3.4], $Lan_{F^{op}}$ is κ -lex by [Joh02, Example A4.1.10] and a is κ -lex by Claim 4. □

Definition 3.8. Given a small κ -site (\mathcal{C}, E) , we write $Mod(\mathcal{C}) = Mod(\mathcal{C}, E, \kappa)$ for the category of κ -lex E -preserving $\mathcal{C} \rightarrow \mathbf{Set}$ functors and natural transformations. $Mod(\mathcal{C})_{<\mu}$ denotes the full subcategory of models with pointwise cardinality $< \mu$. Similarly, for a κ -topos \mathcal{E} we write $Mod_{\mathcal{E}}(\mathcal{C}) = Mod_{\mathcal{E}}(\mathcal{C}, E, \kappa)$ for the category of $\mathcal{C} \rightarrow \mathcal{E}$ κ -lex E -preserving functors.

Theorem 3.9. *Let (\mathcal{C}, E) be a small κ -site, \mathcal{E} be a κ -topos and $M : \mathcal{C} \rightarrow \mathcal{E}$ be a κ -lex E -preserving functor. Then*

1. *In the left Kan-extension*

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{M} & \mathcal{E} \\
 \downarrow ay & \swarrow \eta & \nearrow Lan_{ay}M \\
 Sh(\mathcal{C}, \langle E \rangle_\kappa) & &
 \end{array}$$

η is an isomorphism, and $Lan_{ay}M$ is κ -lex cocontinuous.

2. $Lan_{ay} : Mod_{\mathcal{E}}(\mathcal{C}, E, \kappa) \rightarrow Fun^*(Sh(\mathcal{C}, \langle E \rangle_\kappa), \mathcal{E})$ is an equivalence of categories, whose quasi-inverse is precomposing with ay .

Proof. The above triangle can be written as

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{M} & \mathcal{E} \\
 \downarrow y & \nearrow \eta_1 & \nearrow Lan_y M \\
 \mathbf{Set}^{C^{op}} & \xrightarrow{=} & \mathbf{Set}^{C^{op}} \\
 \downarrow a & \nearrow \eta_2 & \nearrow i \\
 Sh(\mathcal{C}, \langle E \rangle_\kappa) & &
 \end{array}$$

(To check the universal property: given $F : Sh(\mathcal{C}) \rightarrow \mathcal{E}$ and a natural transformation $\gamma : M \Rightarrow Fay$, there's an induced natural transformation $\tilde{\gamma} : Lan_y M \Rightarrow Fa$ fitting in the picture:

$$\begin{array}{ccccc}
 \mathcal{C} & \xrightarrow{M} & \mathcal{E} & & \\
 \downarrow y & & \downarrow \eta_1 & & \nearrow F \\
 \mathbf{Set}^{C^{op}} & \xrightarrow{=} & \mathbf{Set}^{C^{op}} & \xrightarrow{Lan_y M} & \mathcal{E} \\
 \downarrow a & \nearrow \eta_2 & \downarrow \cong \varepsilon_2 & \nearrow \tilde{\gamma} & \\
 Sh(\mathcal{C}, \langle E \rangle_\kappa) & \xrightarrow{i} & \mathbf{Set}^{C^{op}} & \xrightarrow{a} & Sh(\mathcal{C}, \langle E \rangle_\kappa)
 \end{array}$$

$\gamma =$ (points to the top triangle)
 $id =$ (points to the bottom triangle)

hence the pasting of $\tilde{\gamma}$ and ε_2 yields the unique splitting of γ we were looking for.)

As both i and $Lan_y M$ are κ -lex it follows that $Lan_{ay} M$ is κ -lex.

$Lan_y M$ has a right adjoint $\mathcal{E}(M(-), \bullet)$ which factors through i iff M is E -preserving (easy). In this case it is also a right adjoint for $Lan_y M \circ i$: $\mathcal{E}(Lan_y M(iF), X) \cong \mathbf{Set}^{C^{op}}(iF, i\mathcal{E}(M(-), X)) \cong Sh(\mathcal{C})(F, \mathcal{E}(M(-), X))$. We proved that $Lan_{ay} M$ is κ -lex cocontinuous hence we have the restricted adjunction

$$\begin{array}{ccc}
 Mod_{\mathcal{E}}(\mathcal{C}, E, \kappa) & \xrightarrow{Lan_{ay}} & Fun^*(Sh(\mathcal{C}), \mathcal{E}) \\
 & \perp & \\
 & \xleftarrow{(ay)^*} &
 \end{array}$$

We want to prove $(ay)^*$ to be an equivalence. Then it follows that the adjunction is an adjoint equivalence (whose unit is η in Claim 1.) and the proof will be complete.

In the composite

$$Fun^*(Sh(\mathcal{C}), \mathcal{E}) \xrightarrow{a^*} Fun^*(\mathbf{Set}^{C^{op}}, \mathcal{E}) \xrightarrow{y^*} \mathbf{Lex}_\kappa(\mathcal{C}, \mathcal{E})$$

y^* is an equivalence. a^* is fully faithful: given $\alpha : Fa \Rightarrow Ga$ its unique preimage is $F \xrightarrow{F \circ \varepsilon^{-1}} Fai \xrightarrow{\alpha \circ i} Gai \xrightarrow{G \circ \varepsilon} G$.

A κ -lex cocontinuous functor $N^* : \mathbf{Set}^{C^{op}} \rightarrow \mathcal{E}$ is in the essential image of a^* iff its right adjoint N_* factors through i . \Rightarrow is clear (adjoints compose). To see \Leftarrow assume $N_* = i \circ \tilde{N}_*$. As i preserves and reflects all limits it follows that \tilde{N}_* is continuous, hence it has a left adjoint \tilde{N}^* . By the uniqueness of adjoints $\tilde{N}^* \circ a \cong N^*$. Hence \tilde{N}^* preserves all $< \kappa$ -limits which are in the image of a , that is, all $< \kappa$ -limits as $ai \cong 1_{Sh(\mathcal{C})}$.

But for a κ -lex functor $M : \mathcal{C} \rightarrow \mathbf{Set}$, the left Kan-extension $Lan_y M = (y^*)^{-1}(M)$ satisfies this property iff M was E -preserving, as we claimed before. \square

4 Eventually: enough points \Rightarrow presheaf type

This section proves the first half of the main theorem of these notes: that the classifying λ -topos of a κ -site is of presheaf type. Here we will prove it under the assumption that the classifying λ -topos has enough λ -points. Then in section 5. we shall see that this is automatic when $\lambda^{<\lambda} = \lambda$.

Lemma 4.1. *We have the following assumptions:*

1. $\lambda = cf(\lambda) \geq \kappa = cf(\kappa) \geq \aleph_0$.
2. (\mathcal{C}, E) is a small κ -site, $(\tilde{\mathcal{C}}, \tilde{E})$ is a small λ -site, $\varphi : (\mathcal{C}, E) \rightarrow (\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda)$ is a morphism of κ -sites.
3. $|\mathcal{C}| < \lambda$.
4. For any λ -topos \mathcal{E} , the map $\varphi^* : Mod_\mathcal{E}(\tilde{\mathcal{C}}, \tilde{E}, \lambda) \rightarrow Mod_\mathcal{E}(\mathcal{C}, E, \kappa)$ is an equivalence of categories.
5. There are $(\tilde{N}_i : \tilde{\mathcal{C}} \rightarrow \mathbf{Set})_i$ λ -lex \tilde{E} -preserving functors with $\tilde{N}_i \varphi$ having pointwise size $< \lambda$, such that $\langle Lan_{ay} \tilde{N}_i \rangle_i : Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) \rightarrow \mathbf{Set}^I$ is conservative.

Then there's an equivalence making

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{ev} & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} \\
 \varphi \downarrow & \cong & \nearrow \\
 \tilde{\mathcal{C}} & & \\
 ay \downarrow & \simeq & \\
 Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) & &
 \end{array}$$

commutative.

Proof. We define a functor $\Delta : Mod(\mathcal{C})_{<\lambda}^{op} \rightarrow \tilde{\mathcal{C}}$ by

$$M \xrightarrow{\alpha} N \quad \mapsto \quad \lim_{(x,p) \in f M} \varphi x \xleftarrow{\langle \pi_{(x, \alpha x(p))} \rangle} \lim_{(x,q) \in f N} \varphi x$$

which makes sense as by the regularity of λ : $|f M| < \lambda$. We have a natural transformation:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{ev} & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} \\
 \varphi \downarrow & \eta \rightrightarrows & \uparrow y \\
 \tilde{\mathcal{C}} & \xrightarrow{Lan_y(ay\Delta)} & Mod(\mathcal{C})_{<\lambda}^{op} \\
 ay \downarrow & \cong \uparrow & \\
 Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) & \xleftarrow{ay\Delta} &
 \end{array}$$

whose v -component is defined by

$$\begin{array}{ccc}
 & a\tilde{\mathcal{C}}(-, \varphi v) & \\
 \pi(v, p_0) \nearrow & \uparrow \eta_v & \nwarrow \pi(v, q_0) \\
 & Lan_y(ay\Delta)(ev_v) & \\
 \underbrace{\lim_{(x,p) \in f M} a\tilde{\mathcal{C}}(-, \varphi x)}_{(M, p_0 \in Mv)} & \xleftarrow{\langle \pi_{(x, \alpha x(p))} \rangle} & \underbrace{\lim_{(x,q) \in f N} a\tilde{\mathcal{C}}(-, \varphi x)}_{(N, q_0 \in Nv)} \\
 & \xrightarrow{\alpha} &
 \end{array}$$

(i.e. to compute $Lan_y(ay\Delta)(ev_v)$ we have to write ev_v as the colimit of representables along its category of elements, then apply $ay\Delta$ to this diagram, finally compute its colimit. We have a cocone over this diagram with top $ay\varphi(v)$, and η_v is the induced map).

To check the commutativity of the naturality squares (say at $f : v \rightarrow w$), one has to precompose with a leg of the colimit, then it has the form:

$$\begin{array}{ccc}
& a\tilde{\mathcal{C}}(-, \varphi v) & \xrightarrow{a(\varphi f^*)} & a\tilde{\mathcal{C}}(-, \varphi w) \\
\pi_{(v, p_0)} \curvearrowright & \uparrow \eta_v & & \uparrow \eta_w \curvearrowleft \pi_{(w, Mf(p_0))} \\
& Lan_y(ay\Delta)(ev_v) & \xrightarrow{\quad \quad \quad} & Lan_y(ay\Delta)(ev_w) \\
& \uparrow & & \uparrow \\
\underbrace{\lim_{(x,p) \in f M} a\tilde{\mathcal{C}}(-, \varphi x)}_{(M, p_0 \in Mv)} & \xlongequal{\quad \quad \quad} & \underbrace{\lim_{(x,p) \in f M} a\tilde{\mathcal{C}}(-, \varphi x)}_{(M, Mf(p_0) \in Mw)}
\end{array}$$

which can be drawn as

$$\begin{array}{ccc}
& \lim_{(x,p) \in f M} a\tilde{\mathcal{C}}(-, \varphi x) & \\
\pi_{(v, p_0)} \swarrow & & \searrow \pi_{(w, Mf(p_0))} \\
a\tilde{\mathcal{C}}(-, \varphi v)^{p_0} & \xrightarrow{a(\varphi f^*)} & a\tilde{\mathcal{C}}(-, \varphi w)^{Mf(p_0)}
\end{array}$$

Our goal is to prove that η is an isomorphism and $Lan_y(ay\Delta)$ is an equivalence. We start with the following observation (saying that if M is a sufficiently small model, so that its diagram ΔM is a single formula in $\tilde{\mathcal{C}}$, then its evaluation at a model $\tilde{N}(\Delta M)$ consists of those tuples which enumerate an $M \rightarrow \tilde{N}\varphi$ homomorphism):

There is a natural isomorphism

$$\begin{array}{ccc}
Mod(\mathcal{C})_{<\lambda}^{op} & \xrightarrow{\Delta} & \tilde{\mathcal{C}} \\
\downarrow y & \swarrow \delta & \downarrow ev \\
\mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} & \xrightarrow{(\varphi^*)^*} & \mathbf{Set}^{Mod(\tilde{\mathcal{C}})_{-\circ\varphi <\lambda}}
\end{array}$$

given by

$$\begin{array}{ccccccc}
ev_{\Delta M} & \cong & \lim_{(x,p) \in f M} ev_{\varphi x} & \cong & \lim_{f M} Nat(\mathcal{C}(-, x), - \circ \varphi) & \cong & \\
\uparrow ev_{\Delta \alpha} & & \uparrow \langle \pi_{(x, \alpha_x p)} \rangle & & \uparrow \langle \pi_{(x, \alpha_x p)} \rangle & & \\
ev_{\Delta N} & \cong & \lim_{(x,q) \in f N} ev_{\varphi x} & \cong & \lim_{f N} Nat(\mathcal{C}(-, x), - \circ \varphi) & \cong & \\
& & & & \cong & Nat(\text{colim}_{f M} \mathcal{C}(-, x), - \circ \varphi) & \cong & Nat(M, - \circ \varphi) \\
& & & & \uparrow (i_{(x, \alpha_x p)})^* & & \uparrow \alpha^* & \\
& & & & \cong & Nat(\text{colim}_{f N} \mathcal{C}(-, x), - \circ \varphi) & \cong & Nat(N, - \circ \varphi)
\end{array}$$

Now we will prove that η is an isomorphism. For this it suffices to prove that

$$\begin{array}{ccc}
& \varphi v & \\
\pi_{(v, p_0)} \curvearrowright & & \curvearrowleft \pi_{(v, q_0)} \\
\underbrace{\lim_{(x,p) \in f M} \varphi x}_{(M, p_0 \in Mv)} & \xleftarrow{\langle \pi_{(x, \alpha_x(p))} \rangle} & \xrightarrow{\alpha} \underbrace{\lim_{(x,q) \in f N} \varphi x}_{(N, q_0 \in Nv)}
\end{array}$$

is mapped to a colimit by ay . By Theorem 3.9 and by Assumption 5, it is enough that it is mapped to a colimit by any $\tilde{N} : \tilde{\mathcal{C}} \rightarrow \mathbf{Set}$ λ -lex \tilde{E} -preserving functor with $|\tilde{N}\varphi| < \lambda$ (as in this case the $\text{colim} \rightarrow ay\varphi(v)$ map is taken to an iso by each (cocontinuous) $Lan_{ay}\tilde{N}$, and those are jointly conservative).

But using the isomorphism δ constructed above we have

$$\begin{array}{ccc}
\tilde{N}\varphi v & \xrightarrow{\cong} & Nat(\mathcal{C}(-, v), \tilde{N}\varphi) \\
\uparrow \pi_{(v, p_0)} & & \uparrow i_{(v, p_0)}^* \\
\underbrace{\tilde{N}(\Delta M) = \lim_{(x,p) \in f M} \tilde{N}\varphi x}_{(M, p_0 \in Mv)} & \xrightarrow[\cong]{(\delta_M)_{\tilde{N}}} & \underbrace{Nat(M, \tilde{N}\varphi)}_{(M, p_0 \in Mv)}
\end{array}$$

and the composite $Nat(M, \tilde{N}\varphi) \rightarrow \tilde{N}\varphi v$ takes β to $\beta_v(p_0)$. So this is the canonical colimit $ev_v \cong \text{colim}_{(M, p_0) \in f ev_v} Mod(\mathcal{C})_{<\lambda}(M, -)$, evaluated at $\tilde{N}\varphi$.

It remains to check that $Lan_y(\text{ay}\Delta)$ is an equivalence. Its proposed quasi-inverse is $((\varphi^*)^*)^{-1} \circ Lan_{\text{ay}ev}$, fitting in the diagram

$$\begin{array}{ccccc}
\mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} & \xrightarrow{(\varphi^*)^*} & \mathbf{Set}^{Mod(\tilde{\mathcal{C}})_{-\circ\varphi<\lambda}} & & \\
\uparrow y & \swarrow \delta & \uparrow ev & \swarrow \gamma^{-1} & \\
Mod(\mathcal{C})_{<\lambda}^{op} & \xrightarrow{\Delta} & \tilde{\mathcal{C}} & \xrightarrow{\text{ay}} & Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) \\
& & & \swarrow Lan_{\text{ay}ev} & \\
& & & & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}}
\end{array}$$

Since both $Lan_y(\text{ay}\Delta)$ and $((\varphi^*)^*)^{-1} \circ Lan_{\text{ay}ev}$ are cocontinuous, this iso extends to an isomorphism between identity on $\mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}}$ and $((\varphi^*)^*)^{-1} \circ Lan_{\text{ay}ev} \circ Lan_y(\text{ay}\Delta)$.

To check the other composite, observe the diagram

$$\begin{array}{ccccccc}
& & & \mathcal{C} & & & \\
& & & \swarrow \varphi & & \searrow \varphi & \\
& & \tilde{\mathcal{C}} & & & & \tilde{\mathcal{C}} \\
& \swarrow \text{ay} & \downarrow ev & \cong & \downarrow ev & \swarrow \eta & \downarrow \text{ay} \\
Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) & \xrightarrow{Lan_{\text{ay}ev}} & \mathbf{Set}^{Mod(\tilde{\mathcal{C}})_{-\circ\varphi<\lambda}} & \xrightarrow{((\varphi^*)^*)^{-1}} & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} & \xrightarrow{Lan_y(\text{ay}\Delta)} & Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda)
\end{array}$$

Assume that $Lan_y(\text{ay}\Delta)$ preserves $< \lambda$ -limits. Then both ay and $Lan_y(\text{ay}\Delta) \circ ((\varphi^*)^*)^{-1} \circ Lan_{\text{ay}ev} \circ \text{ay}$ are $\tilde{\mathcal{C}} \rightarrow Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda)$ λ -lex \tilde{E} -preserving functors, and there's an iso between their φ -restrictions. By Assumption 4. φ^* is fully faithful, so there's an iso between the $\tilde{\mathcal{C}} \rightarrow Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda)$ functors. So identity on $Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda)$ and $Lan_y(\text{ay}\Delta) \circ ((\varphi^*)^*)^{-1} \circ Lan_{\text{ay}ev}$ are two λ -topos maps whose ay -restrictions are isomorphic. Then Theorem 3.9 Claim 2. completes the proof.

It remains to check that $Lan_y(\text{ay}\Delta)$ preserves $< \lambda$ -limits. For any $\tilde{N} : \tilde{\mathcal{C}} \rightarrow \mathbf{Set}$ λ -lex \tilde{E} -preserving with $|\tilde{N}\varphi| < \lambda$ we have

$$\begin{array}{ccccc}
\mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} & & & & \\
\uparrow y & \swarrow Lan_y(\text{ay}\Delta) & & & \\
Mod(\mathcal{C})_{<\lambda}^{op} & \xrightarrow{\text{ay}\Delta} & Sh(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) & \xrightarrow{\tilde{N}^*} & \mathbf{Set} \\
& & \cong & & \\
& & Mod(\mathcal{C})_{<\lambda}(-, \tilde{N}\varphi) & &
\end{array}$$

($\tilde{N}^* = Lan_{\text{ay}}\tilde{N}$ is the inverse image of the induced geometric morphism). As left adjoints preserve left Kan-extensions, $\tilde{N}^* \circ Lan_y(\text{ay}\Delta)$ is the left Kan-extension

of a representable, hence representable, in particular λ -lex. So the composite $\mathbf{Set}^{\text{Mod}(\mathcal{C})_{<\lambda}} \xrightarrow{\text{Lan}_y(ay\Delta)} \text{Sh}(\tilde{\mathcal{C}}, \langle \tilde{E} \rangle_\lambda) \xrightarrow{\tilde{N}_i^*} \mathbf{Set}^I$ is λ -lex, the second map is λ -lex, conservative, therefore the first map is λ -lex (the second map inverts the connecting homomorphism going from the image of the limit to the limit of the images). \square

Now we would like to identify $\tilde{\mathcal{C}}$ from the previous lemma. Assume that we can find $\varphi : \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ which is the free completion of \mathcal{C} under $<\lambda$ -limits, in the sense that $\varphi^* : \mathbf{Lex}_\lambda(\tilde{\mathcal{C}}, \mathcal{E}) \rightarrow \mathbf{Lex}_\kappa(\mathcal{C}, \mathcal{E})$ is an equivalence for any λ -lex (or at least for complete) \mathcal{E} . Then it restricts to an equivalence between the full subcategories of $\varphi[E]$ -preserving and E -preserving functors.

We have a good candidate: assume it exists, then apply the previous lemma with trivial E (only the identities are contained). We get an equivalence: $\mathbf{Set}^{\tilde{\mathcal{C}}^{op}} \simeq \mathbf{Set}^{\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})_{<\lambda}}$. So our guess is that $y : \mathcal{C} \rightarrow (\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})_{<\lambda})^{op}$ will do the job.

The following is [AR94, Proposition 1.45 (ii)]:

Proposition 4.2. *\mathcal{C} is κ -lex, small. Take $y : \mathcal{C} \hookrightarrow \mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})^{op}$. Then y is κ -lex and $y^* : \mathbf{LEX}_\infty(\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})^{op}, \mathcal{E}) \rightarrow \mathbf{Lex}_\kappa(\mathcal{C}, \mathcal{E})$ is an equivalence for any complete category \mathcal{E} .*

Proof. y preserves $<\kappa$ -limits: we need that $\mathcal{C}(\text{lim } x_i, -) \xleftarrow{\pi_j^*} \mathcal{C}(x_j, -)$ is a colimit diagram in $\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})$. A cocone with top M is a compatible family of elements $(a_i \in Mx_i)_i$ in the M -image of the base diagram. This corresponds to a unique element $(a_i)_i$ in $M(\text{lim } x_i)$ as M preserves $<\kappa$ -limits, which yields the unique induced arrow $\mathcal{C}(\text{lim } x_i, -) \xrightarrow{1 \mapsto (a_i)_i} M$.

We claim that for an $M : \mathcal{C} \rightarrow \mathcal{E}$ κ -lex functor (for complete \mathcal{E}), the right Kan-extension

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{M} & \mathcal{E} \\ \downarrow y & \nearrow \varepsilon & \nearrow \text{Ran}_y M \\ & \cong & \end{array}$$

$\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})^{op}$

is continuous. This follows as $\mathcal{E} \ni x \mapsto \mathcal{E}(x, M(-)) \in \mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})^{op}$ is a left adjoint to it.

So y^* is essentially surjective. It is also fully faithful: given continuous functors $\tilde{M}, \tilde{N} : \mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})^{op} \rightarrow \mathcal{E}$, they are the right Kan-extensions of $\tilde{M}y$, resp. $\tilde{N}y$ (with identity as ε), hence any natural transformation $\alpha : \tilde{M}y \Rightarrow \tilde{N}y$ induces a unique $\tilde{\alpha} : \tilde{M} \rightarrow \tilde{N}$ for which $\tilde{\alpha} \circ y = \alpha$. \square

Definition 4.3. $\kappa = cf(\kappa)$ is *sharply smaller* than $\lambda = cf(\lambda)$ (written as $\kappa \triangleleft \lambda$) if for any set X with $|X| < \lambda$, the poset $\mathcal{P}_\kappa(X)$ of $< \kappa$ subsets contains a cofinal set of size $< \lambda$.

Proposition 4.4. \mathcal{C} is κ -lex, small, $\kappa = cf(\kappa) \triangleleft \lambda = cf(\lambda)$. Write $\mathcal{C} \xrightarrow{\varphi} \tilde{\mathcal{C}} \xrightarrow{j} \mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})^{op}$ for the factorization of y through the full subcategory spanned by κ -cofiltered $< \lambda$ limits of representables. Then $\tilde{\mathcal{C}}$ is λ -lex, φ is κ -lex and $\varphi^* : \mathbf{Lex}_\lambda(\tilde{\mathcal{C}}, \mathcal{E}) \rightarrow \mathbf{Lex}_\kappa(\mathcal{C}, \mathcal{E})$ is an equivalence for any complete category \mathcal{E} .

Proof. We claim that $\tilde{\mathcal{C}}$ is the full subcategory spanned by $< \lambda$ limits of representables. Indeed, take a $< \lambda$ diagram $\mathcal{I} \rightarrow \mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})$. Its colimit is the κ -directed colimit indexed by $\mathcal{P}_\kappa(\text{Arr}(\mathcal{I}))$, of the colimits of $< \kappa$ subdiagrams and the induced maps between them. These $< \kappa$ -colimits are representable (see the first paragraph of the previous proof) and by assumption the diagram has a cofinal subdiagram of size $< \lambda$. It follows that $\tilde{\mathcal{C}}$ is λ -lex and φ is κ -lex.

φ^* is essentially surjective: given $M : \mathcal{C} \rightarrow \mathcal{E}$ κ -lex $\text{Ran}_y M \circ j$ gives a preimage. It is faithful: any natural transformation between $\tilde{M}, \tilde{N} : \tilde{\mathcal{C}} \rightarrow \mathcal{E}$ λ -lex maps is uniquely determined by its φ -restriction; the components at the $< \lambda$ -limits are the induced morphisms.

φ^* is full: to simplify notation note that since y is injective on objects, ε in the right Kan-extension can be assumed to be an identity. Take $\tilde{M}, \tilde{N} : \tilde{\mathcal{C}} \rightarrow \mathcal{E}$ λ -lex and a natural transformation $\alpha : \tilde{M}\varphi \Rightarrow \tilde{N}\varphi$. Then $\text{Ran}_y(\tilde{M}\varphi) \circ j$ is a map whose φ -restriction equals $\tilde{M}\varphi$ and similarly for \tilde{N} , moreover α has a preimage between these functors. So it suffices to prove that if \tilde{M}_1 and \tilde{M}_2 are $\tilde{\mathcal{C}} \rightarrow \mathcal{E}$ λ -lex functors such that $\tilde{M}_1\varphi = \tilde{M}_2\varphi$ then there's a natural isomorphism $\gamma : \tilde{M}_1 \Rightarrow \tilde{M}_2$ such that $\gamma|_{\mathcal{C}}$ is identity. But given an arrow $f : x \rightarrow y$ in $\tilde{\mathcal{C}}^{op}$ its domain and codomain are both κ -filtered colimits of representables, which is also a colimit in $\mathbf{Set}^{\mathcal{C}}$ where representables are tiny, so the restrictions of our arrow factor through some leg of the colimit with top y . It follows that once \tilde{M}_1 decides to which limit object x and y would be sent, the image of f is uniquely determined. The comparison maps between the limits chosen by M_1 and those chosen by M_2 give the required isomorphism. \square

Now we prove that "if a theory eventually has enough models then it is eventually of presheaf type":

Theorem 4.5. *We have the following assumptions:*

- $\aleph_0 \leq \kappa = cf(\kappa) \triangleleft \lambda = cf(\lambda)$
- $(\mathcal{C}, \mathcal{E})$ is a κ -site, $|\mathcal{C}|, |\mathcal{E}| < \lambda$.

Write $\mathcal{C} \xrightarrow{\varphi} \tilde{\mathcal{C}}$ for $\mathcal{C} \xrightarrow{y} (\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})_{<\lambda})^{op}$. Then:

- The map $\varphi^* : Mod_{\mathcal{E}}(\tilde{\mathcal{C}}, \varphi[E], \lambda) \rightarrow Mod_{\mathcal{E}}(\mathcal{C}, E, \kappa)$ is an equivalence for any λ -topos \mathcal{E} .
- TFAE:
 1. $Sh(\tilde{\mathcal{C}}, \langle \varphi[E] \rangle_{\lambda})$ has enough λ -points.
 2. In the diagram

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{ev} & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} \\
 \varphi \downarrow & \nearrow \text{Ran}_{\varphi}(ev) \cong & \nearrow \\
 \tilde{\mathcal{C}} & & \\
 ay \downarrow & \nearrow \cong \nearrow & \nearrow \\
 Sh(\tilde{\mathcal{C}}, \langle \varphi[E] \rangle_{\lambda}) & & \nearrow Lan_{ay} Ran_{\varphi}(ev)
 \end{array}$$

$Lan_{ay} Ran_{\varphi}(ev)$ is an equivalence.

Proof. The first claim follows from Proposition 4.4. All we need is that κ -filtered $< \lambda$ colimits of representables in $\mathbf{Lex}_{\kappa}(\mathcal{C}, \mathbf{Set})$ (equivalently in $\mathbf{Set}^{\mathcal{C}}$) are precisely those functors which have pointwise cardinality $< \lambda$. ” \subseteq ” follows as $\mathcal{C}(x, -)$ has pointwise size $< \lambda$ and a $< \lambda$ colimit of pointwise $< \lambda$ functors is a quotient of the pointwise disjoint union whose cardinality is $< \lambda$ as λ was regular. ” \supseteq ” follows as (again by regularity) if M is pointwise $< \lambda$ then $|\int M| < \lambda$. The equivalence $\varphi^* : \mathbf{Lex}_{\lambda}(\tilde{\mathcal{C}}, \mathcal{E}) \rightarrow \mathbf{Lex}_{\kappa}(\mathcal{C}, \mathcal{E})$ restricts to an equivalence between $\varphi[E]$ -, resp. E -preserving functors.

(Note also that φ is a morphism of κ -sites, as since φ is κ -lex the $\langle E \rangle_{\kappa}$ -families are taken to $\langle \varphi[E] \rangle_{\kappa} \subseteq \langle \varphi[E] \rangle_{\lambda}$ -families.)

In the second claim the implication 2 \Rightarrow 1 is easy: $Nat(Mod(\mathcal{C})_{<\lambda}(M, -), \bullet)$ for $M \in Mod(\mathcal{C})_{<\lambda}$ is a jointly conservative set of λ -points. To prove the converse we have to check that $\varphi : (\mathcal{C}, E) \rightarrow (\tilde{\mathcal{C}}, \varphi[E])$ satisfies Assumption 5. in Lemma 4.1. That is, $Sh(\tilde{\mathcal{C}}, \langle \varphi[E] \rangle_{\lambda})$ has enough λ -points whose restriction to \mathcal{C} is pointwise $< \lambda$. (This is enough, as if there is an equivalence $\mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} \rightarrow Sh(\tilde{\mathcal{C}})$ making the triangle commute up to isomorphism, then the same holds for its quasi-inverse, in which case it must be $Lan_{ay} Ran_{\varphi}(ev)$.)

Assume that every $\mathcal{C} \rightarrow \mathbf{Set}$ κ -lex E -preserving functor is the λ -filtered colimit of pointwise $< \lambda$ such maps. Now pick an arrow $f : x \rightarrow y$ in $Sh(\tilde{\mathcal{C}})$ which is not an isomorphism and choose $\tilde{N}^* : Sh(\tilde{\mathcal{C}}) \rightarrow \mathbf{Set}$ which keeps it non-iso. By the assumption we can write $\tilde{N}^* \circ ay \circ \varphi$ as the λ -filtered colimit of M_i 's, each being pointwise $< \lambda$. Since φ^* was an equivalence (whose quasi-inverse

hence preserves this colimit), we can write $\widetilde{N}^* \circ ay$ as the λ -filtered colimit of \widetilde{M}_i 's with $\widetilde{M}_i \circ \varphi \cong M_i$. This colimit is pointwise as $Mod(\widetilde{\mathcal{C}}, \varphi[E], \lambda)$ is closed under λ -filtered colimits in $\mathbf{Set}^{\widetilde{\mathcal{C}}}$. Now the quasi-inverse of $(ay)^*$ maps this to a λ -filtered colimit of \widetilde{M}_i^* 's with top \widetilde{N}^* such that $\widetilde{M}_i^* \circ ay \circ \varphi \cong M_i$. This colimit is also pointwise: in fact, the pointwise colimit \widehat{N}^* is λ -lex cocontinuous, so there's an induced map $\widetilde{N}^* \Rightarrow \widehat{N}^*$, inverted by the equivalence $(ay)^*$. It follows that $\widetilde{M}_i^*(f)$ is non-iso for some i .

It remains to check that in $Mod(\mathcal{C}, E, \kappa)$ every object is a λ -filtered colimit of pointwise $< \lambda$ models. So start with a model $M : \mathcal{C} \rightarrow \mathbf{Set}$. In $\mathbf{Set}^{\mathcal{C}}$ it is the λ -filtered union of its pointwise $< \lambda$ subfunctors, as the subfunctor generated by a collection of subsets $A(x) \subseteq M(x)$ is the closure under $M(f)$ -images for $f \in \mathcal{C}$ but $|\mathcal{C}| < \lambda$. A filtered union is a filtered colimit, because this holds in \mathbf{Set} and in presheaf categories colimits are pointwise. So it suffices to prove that every pointwise $< \lambda$ subfunctor of M is contained in a pointwise $< \lambda$ subfunctor which is κ -lex E -preserving.

We will prove the following: i) every $< \lambda$ subfunctor is contained in a $< \lambda$ subfunctor which is lex and E -preserving, ii) it is contained in a $< \lambda$ subfunctor which preserves $< \kappa$ products. This is sufficient as we can build a κ -chain out of A , in odd steps applying i), in even steps applying ii), in limit steps taking the colimit, then the union of this chain is $< \lambda$, E -preserving, and preserves both finite limits and $< \kappa$ -products, hence κ -lex.

i) This is easy: one has to build an ω -chain out of A , in each step applying the following three closure operators:

- close under $M(f)$ -images for $f \in \mathcal{C}$,
- for each family $(u_i \rightarrow x)_i$ in E , and each $a \in A_n(x)$ choose one arbitrary preimage in one $M(u_j)$ and add it to $A_n(u_j)$,
- for each finite diagram $\mathcal{I} \rightarrow \mathcal{C}$ and each compatible family formed by elements in A_n ($(< \lambda)^{< \omega} = (< \lambda)$) add the corresponding element to $A_n(\lim_{\mathcal{I}})$.

Each of these add $< \lambda$ elements so A_ω satisfies our requirements.

ii) By [AR94, Remark 1.21] we can write M as a κ -directed (as opposed to just κ -filtered) colimit of representables (indexed by P). By [AR94, Theorem 2.11] $\kappa \triangleleft \lambda$ implies that the κ -directed $< \lambda$ subsets of P form a λ -directed poset Q . Hence we can write M as a λ -directed colimit of pointwise $< \lambda$ and κ -lex functors (indexed by Q), as representables are pointwise $< \lambda$ and therefore their $< \lambda$ colimits are $< \lambda$. We can take the pointwise image-factorization of each $M_q \Rightarrow M$ hence M is the λ -directed union of these images which are easily proved to preserve $< \kappa$ products. \square

In the next section we will prove that when $\lambda^{<\lambda} = \lambda$, the equivalent properties 1) \Leftrightarrow 2) hold.

5 A λ -separable λ -topos has enough λ -points

Lemma 5.1. *Let (\mathcal{C}, E) be a λ -site with $|E| \leq \lambda$, \mathcal{C} being of local size $\leq \lambda$, $\lambda = cf(\lambda)$, and take a family $(f_i : u_i \rightarrow x_0)_i$. If for every λ -lex E -preserving $M : \mathcal{C} \rightarrow \mathbf{Set}$ functor the M -image of the family is jointly surjective then the ay -image is extremal epimorphic.*

Proof. We repeat the proof of the completeness theorem from the first section. Write E^- for the subset of E consisting of non-empty covers (containing at least $1 \rightarrow 1$). By T_{x_i} we denote a list containing all diagrams consisting of some E^- -family and a map from x_i to the common codomain. As there are $\geq 1, \leq \lambda$ such diagrams we can assume that the list has size λ .

Now start filling a $\lambda \times \lambda$ -big table with the tasks we have to solve. We shall use the canonical well-ordering $h : \lambda \times \lambda \rightarrow \lambda$ with the property $h(\alpha, \beta) \geq \beta$. In the 0^{th} column fill in T_{x_0} . Then solve task number $h^{-1}(0)$, whose second coordinate is ≤ 0 so it is defined. By solving we mean: take the pullback of the given family along the given arrow. So now we see an $(E^-)^{pb}$ -covering family on x_0 .

Inductively we build a tree of height λ , where all branches are cofinal. In a successor step $(\alpha + 1)$, where the α^{th} object of the branch is $x_{0, i_1, \dots}$ take the table whose columns are filled in with tasks concerning the preceding objects of the branch, fill $T_{x_{0, i_1, \dots}}$ to the α^{th} -column, and solve task number $h^{-1}(\alpha)$, which is defined. Solving means: this is a diagram formed by an arrow from a preceding object x' above $x_{0, i_1, \dots}$ to the codomain of some E^- -family, precompose with the $x_{0, i_1, \dots} \rightarrow x'$ arrow of the branch, then take the pullback of the family along this composite. In limit steps take the limits (cocompositions) of the branches.

As we deleted the empty covers from E , no branch dies out and we get a locally $(E^-)^{pb}$ -covering continuous cotree of height λ , and when we take the colimit of the representables $\mathcal{C}(x_0, -) \rightarrow \mathcal{C}(x_{0, i_1}, -) \rightarrow \dots$ along a (cofinal) branch we get a $\mathcal{C} \rightarrow \mathbf{Set}$ λ -lex functor which maps each E^- -family to a jointly surjective one.

If a branch is not preserving an empty cover $\{z \ \emptyset\}$, it means that from some element x_i of the branch there exists a morphism $x_i \rightarrow z$. If the branch is E -preserving then by assumption the colimit takes $(f_i : u_i \rightarrow x_0)_i$ to a jointly surjective family, in particular $[1_{x_0}]$ is hit, meaning that for some x_i in the branch the map $x_i \rightarrow x$ factors through some f_i . Cut down each branch at such a point, then we see an $\langle E^- \rangle_\lambda$ -family on x_0 , such that each leg either factors through some f_i or the domain admits a map to some object over which \emptyset is

a cover. ay turns these objects initial, so now each leg factors through some $ay(f_i)$, the inductively built family is mapped to an (extremal) epimorphic one, consequently $(ay(f_i) : ay(u_i) \rightarrow ay(x_0))_i$ is epimorphic. \square

By [MR77, Lemma 6.1.4.] this is sufficient. For the reader's convenience we repeat the proof.

Lemma 5.2. *Let $Sh(\mathcal{C}, \langle E \rangle_\omega)$ be any Grothendieck-topos. Take a natural transformation between sheafified representables $\alpha : a\mathcal{C}(-, x) \Rightarrow a\mathcal{C}(-, x')$. Then there is a cover $(f_i : u_i \rightarrow x)_i$ such that each composite $a\mathcal{C}(-, u_i) \xrightarrow{a((f_i)_*)} a\mathcal{C}(-, x) \xrightarrow{\alpha} a\mathcal{C}(-, x')$ is a sheafified post-composition $a((g_i)_*)$ for some $g_i : u_i \rightarrow x'$.*

Proof. A consequence of the $+$ -construction is that given an arbitrary presheaf $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ and a natural transformation $\beta : \mathcal{C}(-, x) \Rightarrow F^+$, there is a cover $(f_i : u_i \rightarrow x)_i$ such that each $\mathcal{C}(-, u_i) \xrightarrow{(f_i)_*} \mathcal{C}(-, x) \xrightarrow{\beta} F^+$ factors through the canonical map $F \Rightarrow F^+$. If we start with $F^{++} = aF$ instead of F^+ , the same remains true as covering trees of height 2 compose to a cover.

Apply this for the composite $\mathcal{C}(-, x) \Rightarrow a\mathcal{C}(-, x) \xrightarrow{\alpha} a\mathcal{C}(-, x')$. We get a cover $(f_i : u_i \rightarrow x)_i$ such that for each i there's a commutative diagram

$$\begin{array}{ccccc} \mathcal{C}(-, u_i) & \xrightarrow{(f_i)_*} & \mathcal{C}(-, x) & \xrightarrow{\quad} & a\mathcal{C}(-, x) & \xrightarrow{\alpha} & a\mathcal{C}(-, x') \\ & & & \searrow & & & \uparrow \\ & & & & & & \mathcal{C}(-, x') \end{array}$$

$(g_i)_*$

whose image under sheafification is exactly what we were looking for. \square

Corollary 5.3. *Let $Sh(\mathcal{C}, \langle E \rangle_\omega)$ be any Grothendieck-topos and take a monomorphism $\iota : F \Rightarrow a\mathcal{C}(-, x)$. Then there is a cover of F with sheafified representables $a\mathcal{C}(-, u_i) \xrightarrow{\beta_i} F$ such that each $\iota \circ \beta_i$ is a sheafified post-composition.*

Proof. Take any cover of F with sheafified representables $a\mathcal{C}(-, v_i) \Rightarrow F$ then apply the previous lemma for the composites $a\mathcal{C}(-, v_i) \Rightarrow a\mathcal{C}(-, x)$. \square

Theorem 5.4. *Let (\mathcal{C}, E) be a λ -site with $|E| \leq \lambda$, \mathcal{C} being of local size $\leq \lambda$, $\lambda = cf(\lambda)$. Then $Sh(\mathcal{C}, \langle E \rangle_\lambda)$ has enough λ -points.*

Proof. We have to prove that for an arbitrary proper mono $\iota : F \Rightarrow G$ there is some $M^* : Sh(\mathcal{C}, \langle E \rangle_\lambda) \rightarrow \mathbf{Set}$ λ -lex cocontinuous functor which keeps it

proper. It suffices to prove this for $G = a\mathcal{C}(-, x)$, as we can find a cover $(a\mathcal{C}(-, x_j) \Rightarrow G)_j$, then form the pullback:

$$\begin{array}{ccc}
 G & \longleftarrow & \bigsqcup_j a\mathcal{C}(-, x_j) \\
 \uparrow \iota & & \uparrow \bigsqcup_j \iota_j \\
 F & \longleftarrow & \bigsqcup_j F_j
 \end{array}
 \quad pb$$

finally note that ι is proper iff at least one ι_j is proper.

By the previous corollary we have

$$\begin{array}{ccc}
 a\mathcal{C}(-, u_i) & \xrightarrow{a((g_i)_*)} & a\mathcal{C}(-, x) \\
 \searrow \beta_i & & \uparrow \iota \\
 \dots & \xrightarrow{\quad} & F
 \end{array}$$

and since ι is proper $(ay(g_i))_i$ cannot be effective epimorphic. Therefore by Lemma 5.1 there is some $M : \mathcal{C} \rightarrow \mathbf{Set}$ λ -lex E -preserving for which $(M(g_i))_i$ is not jointly surjective. By Theorem 3.9 its left Kan-extension $M^* = Lan_{ay} M$ is λ -lex cocontinuous and $M^* \circ ay \cong M$. But since $M^*(\beta_i)$'s form a jointly surjective family and $M^*(ay(g_i))$'s do not, it follows that $M^*(\iota)$ is proper. \square

Theorem 5.5. *We have the following assumptions:*

- $\aleph_0 \leq \kappa = cf(\kappa) \triangleleft \lambda = cf(\lambda), 2^{<\lambda} = \lambda$
- (\mathcal{C}, E) is a κ -site, $|\mathcal{C}|, |E| < \lambda$

Then using the notation of Theorem 4.5, in the diagram

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{ev} & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} \\
 \varphi \downarrow & \nearrow Ran_\varphi(ev) \cong & \uparrow \\
 \tilde{\mathcal{C}} & & \\
 ay \downarrow & \nearrow Lan_{ay} Ran_\varphi(ev) \cong & \\
 Sh(\tilde{\mathcal{C}}, \langle \varphi[E] \rangle_\lambda) & &
 \end{array}$$

$Lan_{ay} Ran_\varphi(ev)$ is an equivalence.

Proof. All we have to check is that $\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})_{<\lambda}$ has local size $\leq \lambda$. But there are at most as many $F \Rightarrow G$ natural transformations as $\bigsqcup_{x \in \mathcal{C}} F(x) \rightarrow \bigsqcup_{x \in \mathcal{C}} G(x)$ maps, and since λ is regular this is $(\mu_2)^{\mu_1}$ for some $\mu_1, \mu_2 < \lambda$. This is $\leq \lambda$ by the assumption $\mu < \lambda \Rightarrow 2^\mu \leq \lambda$. \square

Remark 5.6. To ensure that $\tilde{\mathcal{C}}$ has local size $\leq \lambda$ we assumed $2^{<\lambda} = \lambda$. To get that $\tilde{\mathcal{C}}$ has size $\leq \lambda$ we would need $\lambda^{<\lambda} = \lambda$. However, these two assumptions are equivalent for a regular cardinal λ , see [Jec02, Exercise 5.21].

As a corollary we get:

Theorem 5.7. *Assume*

- $\aleph_0 \leq \kappa = cf(\kappa) < \lambda = \lambda^{<\lambda}$
- (\mathcal{C}, E) is a κ -site, $|\mathcal{C}|, |E| < \lambda$

Then in $\mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}}$ the closure of $\{ev_x : x \in \mathcal{C}\}$ under κ -cofiltered limits of size $< \lambda$ form a generating set.

Another consequence is:

Theorem 5.8. *Assume:*

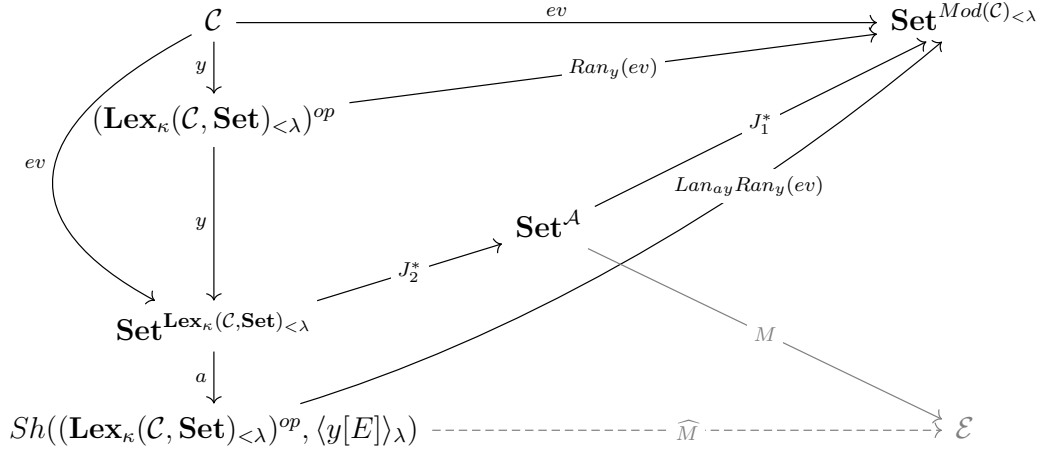
- $\aleph_0 \leq \kappa = cf(\kappa) < \lambda = \lambda^{<\lambda}$
- (\mathcal{C}, E) is a κ -site, $|\mathcal{C}|, |E| < \lambda$

Take a full subcategory \mathcal{A} of $\mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})_{<\lambda}$ containing $Mod(\mathcal{C})_{<\lambda}$ and write $I : Mod(\mathcal{C})_{<\lambda} \xrightarrow{J_1} \mathcal{A} \xrightarrow{J_2} \mathbf{Lex}_\kappa(\mathcal{C}, \mathbf{Set})_{<\lambda}$ for the full embeddings. Let \mathcal{E} be a λ -topos and $\mathbf{Set}^{\mathcal{A}} \xrightarrow{M} \mathcal{E}$ be a λ -lex cocontinuous functor such that for any $(u_i \rightarrow x)_i$ family in E , the family $M((ev_{u_i}|_{\mathcal{A}} \rightarrow ev_x|_{\mathcal{A}})_i)$ is extremal epimorphic. Then a λ -lex cocontinuous extension

$$\begin{array}{ccc}
 \mathbf{Set}^{\mathcal{A}} & \xrightarrow{J_1^*} & \mathbf{Set}^{Mod(\mathcal{C})_{<\lambda}} \\
 \downarrow M & \searrow \cong & \\
 \mathcal{E} & \xleftarrow{\widehat{M}} &
 \end{array}$$

exists.

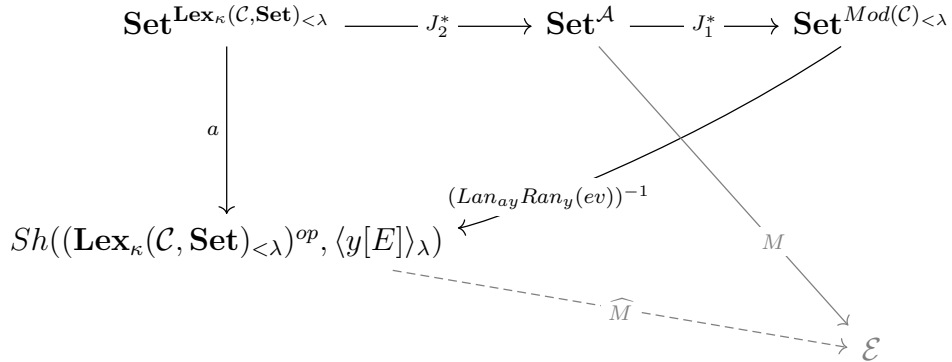
Proof. Unwinding everything that we've been hiding yields



First note that all black triangles commute. In fact, $I^* \circ y$ is a λ -lex functor whose y -restriction is isomorphic to ev , therefore $I^* \circ y \cong Ran_y(ev)$. Also $(Lan_{ay}Ran_y(ev) \circ a) \circ y$ is isomorphic to $Ran_y(ev)$, and since both I^* and $Lan_{ay}Ran_y(ev) \circ a$ are cocontinuous this extends to an isomorphism between $Lan_{ay}Ran_y(ev) \circ a$ and I^* .

By assumption $M \circ J_2^* \circ y$ is λ -lex $y[E]$ -preserving, hence there is a λ -lex cocontinuous \widehat{M} with $\widehat{M} \circ a \circ y \cong M \circ J_2^* \circ y$. As $\widehat{M} \circ a$ and $M \circ J_2^*$ are cocontinuous it follows that $\widehat{M} \circ a \cong M \circ J_2^*$.

So now we have a commutative diagram



and if we write $\widetilde{M} = \widehat{M} \circ (Lan_{ay}Ran_y(ev))^{-1}$ we have $\widetilde{M} \circ J_1^* \circ J_2^* \cong M \circ J_2^*$ and therefore $\widetilde{M} \circ J_1^* \cong M$ since $J_2^* \circ Lan_{J_2} \cong 1_{\mathbf{Set}^A}$. \square

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