



**HAL**  
open science

# Lattice paths and branched continued fractions III. Generalizations of the Laguerre, rook and Lah polynomials

Bishal Deb, Alexander Dyachenko, Mathias Pétréolle, Alan D. Sokal

► **To cite this version:**

Bishal Deb, Alexander Dyachenko, Mathias Pétréolle, Alan D. Sokal. Lattice paths and branched continued fractions III. Generalizations of the Laguerre, rook and Lah polynomials. 2023. hal-04354289

**HAL Id: hal-04354289**

**<https://hal.science/hal-04354289v1>**

Preprint submitted on 19 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Lattice paths and branched continued fractions

## III. Generalizations of the Laguerre, rook and Lah polynomials

Bishal Deb<sup>1,2</sup>, Alexander Dyachenko<sup>3</sup>, Mathias Pétréolle<sup>1</sup> and Alan D. Sokal<sup>1,4</sup>

<sup>1</sup>Department of Mathematics, University College London, London WC1E 6BT, UK

<sup>2</sup>Sorbonne Université and Université Paris Cité, CNRS, Laboratoire de Probabilités,  
Statistique et Modélisation, 75005 Paris, France

<sup>3</sup>Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow 125047, Russia

<sup>4</sup>Department of Physics, New York University, New York, NY 10003, USA

bishal@gonitsora.com, diachenko@sfedu.ru,  
mathias.petreolle@gmail.com, sokal@nyu.edu

November 28, 2023

### Abstract

We introduce a triangular array  $\widehat{L}^{(\alpha)}$  of 5-variable homogeneous polynomials that enumerate Laguerre digraphs (digraphs in which each vertex has out-degree 0 or 1 and in-degree 0 or 1) with separate weights for peaks, valleys, double ascents, double descents, and loops. These polynomials generalize the classical Laguerre polynomials as well as the rook and Lah polynomials. We show that this triangular array is totally positive and that the sequence of its row-generating polynomials is Hankel-totally positive, under suitable restrictions on the values given to the indeterminates. This implies, in particular, the coefficientwise Hankel-total positivity of the monic unsigned univariate Laguerre polynomials. Our proof uses the method of production matrices as applied to exponential Riordan arrays. Our main technical lemma concerns the total positivity of a large class of quadridiagonal production matrices; it generalizes the tridiagonal comparison theorem. In some cases these polynomials are given by a branched continued fraction. Our constructions are motivated in part by recurrences for the multiple orthogonal polynomials associated to weights based on modified Bessel functions of the first kind  $I_\alpha$ .

**Key Words:** Laguerre polynomial, rook polynomial, Lah polynomial, Laguerre digraph, production matrix, quadridiagonal matrix, exponential Riordan array, Hankel matrix, total positivity, Hankel-total positivity, continued fraction, branched continued fraction, multiple orthogonal polynomial, modified Bessel function.

**Mathematics Subject Classification (MSC 2020) codes:** 05A15 (Primary); 05A05, 05A19, 05A20, 05C30, 05C38, 15B48, 30B70, 33C10, 33C45, 42C05 (Secondary).

# Contents

<b>1</b>	<b>Introduction and statement of main results</b>	<b>3</b>
1.1	Univariate Laguerre polynomials . . . . .	4
1.2	Combinatorial interpretation in terms of Laguerre digraphs . . . . .	6
1.3	A first multivariate generalization . . . . .	8
1.4	A second multivariate generalization . . . . .	10
1.5	Total positivity . . . . .	11
1.6	Statement of main results: Univariate case . . . . .	12
1.7	Statement of main results: Multivariate case . . . . .	14
1.8	Plan of this paper . . . . .	16
<b>2</b>	<b>Preliminaries</b>	<b>17</b>
2.1	Partially ordered commutative rings and total positivity . . . . .	17
2.2	Production matrices . . . . .	21
2.3	Production matrices and total positivity . . . . .	23
2.4	Binomial row-generating matrices . . . . .	26
2.5	Exponential Riordan arrays . . . . .	28
<b>3</b>	<b>Production matrices: Univariate case</b>	<b>31</b>
3.1	Coefficient matrix $L^{(\alpha)}$ : Proof of Proposition 1.2(a) . . . . .	31
3.2	Binomial row-generating matrix $L^{(\alpha)}B_x$ : Proof of Proposition 1.2(b) . . . . .	33
3.3	Total positivity of the coefficient matrix $L^{(-1+\lambda)}$ . . . . .	34
<b>4</b>	<b>Multivariate Laguerre polynomials:</b>	
	<b>Exponential generating functions and production matrices</b>	<b>34</b>
4.1	Exponential generating functions . . . . .	35
4.2	Production matrices: Proof of Proposition 1.5(a,b) . . . . .	37
<b>5</b>	<b>Total positivity of the production matrices: Univariate case</b>	<b>39</b>
5.1	The tridiagonal matrix $P^\circ$ : Proof of Proposition 1.3(a) . . . . .	39
5.2	The quadriagonal matrix $P$ : Proof of Proposition 1.3(b) . . . . .	40
<b>6</b>	<b>Total positivity of the production matrices: Multivariate case</b>	<b>41</b>
6.1	The tridiagonal matrix $P^{\circ b}$ : Proof of Proposition 1.6(a) . . . . .	41
6.2	The quadriagonal matrix $P^b$ : Proof of Proposition 1.6(b) . . . . .	41
<b>7</b>	<b>The connection with multiple orthogonal polynomials</b>	<b>48</b>
<b>A</b>	<b>Generalized and modified <math>m</math>-Stieltjes–Rogers polynomials</b>	<b>51</b>
A.1	Classical Stieltjes–Rogers polynomials . . . . .	52
A.2	$m$ -Stieltjes–Rogers polynomials (branched S-fractions) . . . . .	54
A.3	Generalized $m$ -Stieltjes–Rogers polynomials of type $j$ . . . . .	55
A.4	Application to the univariate Laguerre production matrix . . . . .	61
A.4.1	Case $j = 0$ . . . . .	63
A.4.2	Case $j = 1$ . . . . .	65
A.4.3	Case $j = 2$ . . . . .	68
<b>B</b>	<b>When is <math>B_\xi^{-1}PB_\xi</math> <math>(r, 1)</math>-banded?</b>	<b>74</b>
<b>C</b>	<b>Total positivity for a variant class of quadriagonal matrices</b>	<b>76</b>
	<b>Bibliography</b>	<b>80</b>

# 1 Introduction and statement of main results

In a seminal 1980 paper, Flajolet [29] showed that the coefficients in the Taylor expansion of the generic Stieltjes-type (resp. Jacobi-type) continued fraction — which he called the *Stieltjes–Rogers* (resp. *Jacobi–Rogers*) *polynomials* — can be interpreted as the generating polynomials for Dyck (resp. Motzkin) paths with specified height-dependent weights. More recently it was independently discovered by several authors [34, 41, 56, 70] that Thron-type continued fractions also have an interpretation of this kind: namely, their Taylor coefficients — which we call, by analogy, the *Thron–Rogers polynomials* — can be interpreted as the generating polynomials for Schröder paths with specified height-dependent weights.

In a recent paper [59] we presented an infinite sequence of generalizations of the Stieltjes–Rogers and Thron–Rogers polynomials, which are parametrized by an integer  $m \geq 1$  and reduce to the classical Stieltjes–Rogers and Thron–Rogers polynomials when  $m = 1$ ; they are the generating polynomials of  $m$ -Dyck and  $m$ -Schröder paths, respectively, with height-dependent weights, and are also the Taylor coefficients of certain branched continued fractions. We proved that these generalizations all possess the fundamental property of coefficientwise Hankel-total positivity [66, 70], jointly in all the (infinitely many) indeterminates. These facts were known when  $m = 1$  [66, 70] but were new when  $m > 1$ . By specializing the indeterminates we were able to give many examples of Hankel-totally positive sequences whose generating functions do not possess nice classical continued fractions. (The concept of Hankel-total positivity [66, 70] will be explained in more detail later in this Introduction.) Similar considerations apply to the  $m$ -Jacobi–Rogers polynomials, which are the generating polynomials of  $m$ -Łukasiewicz paths; but here the Hankel-total positivity is a more delicate matter, for which a sufficient (but not necessary) condition is the total positivity of a lower-Hessenberg production matrix.

In the present paper we will apply the  $m$ -Jacobi–Rogers theory with  $m = 2$  to prove the coefficientwise Hankel-total positivity for some multivariate generalizations of the Laguerre, rook and Lah polynomials. Our constructions are motivated in part by the work of Coussement and Van Assche [19] on the multiple orthogonal polynomials associated to weights based on modified Bessel functions of the first kind  $I_\alpha$ ; we will explain this unexpected connection in Section 7. Our method — to use production matrices for a binomial row-generating matrix, exploiting the theory of exponential Riordan arrays — was employed previously by ourselves [13, 58, 67] and Zhu [90, 92]; but the present case is more delicate than these previous applications because the total positivity of the quadri-diagonal production matrix (Section 6.2) is decidedly nontrivial.

The starting point of this work was the fact, due to Gantmakher and Krein [36], that a Hankel matrix of real numbers is totally positive if and only if the underlying sequence is a Stieltjes moment sequence (see Section 1.5 below), combined with the classical fact that, for each  $\alpha \geq -1$  and  $x \geq 0$ , the sequence  $(\mathcal{L}_n^{(\alpha)}(x))_{n \geq 0}$  of monic unsigned Laguerre polynomials [defined in (1.3) below] is a Stieltjes moment sequence [cf. (7.1) below]. This led Sylvie Corteel and one of the authors (A.D.S.) to conjecture, a few years ago [18], that the sequence  $(\mathcal{L}_n^{(\alpha)}(x))_{n \geq 0}$  is *coefficientwise* Hankel-totally positive (see Section 1.5 for the definition). The purpose of this paper is to prove this conjecture, along with a vast multivariate generalization. The univariate conjecture was also recently proven by Zhu [90, 92].

## 1.1 Univariate Laguerre polynomials

The Laguerre polynomials  $L_n^{(\alpha)}$  are conventionally defined as [3, 40, 63, 77]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \quad (1.1)$$

and have the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1-t)^{-(1+\alpha)} e^{-xt/(1-t)}. \quad (1.2)$$

However, for our purposes it is more convenient to work with the *monic unsigned Laguerre polynomials*

$$\begin{aligned} \mathcal{L}_n^{(\alpha)}(x) &\stackrel{\text{def}}{=} n! L_n^{(\alpha)}(-x) = \sum_{k=0}^n \frac{n!}{k!} \binom{n+\alpha}{n-k} x^k = \sum_{k=0}^n \binom{n}{k} (n+\alpha)^{n-k} x^k \\ &= (1+\alpha)^{\bar{n}} {}_1F_1\left(\begin{matrix} -n \\ 1+\alpha \end{matrix} \middle| -x\right) \end{aligned} \quad (1.3)$$

[where  $\rho^{\bar{n}} \stackrel{\text{def}}{=} \rho(\rho-1)\cdots(\rho-n+1)$  and  $\rho^{\bar{n}} \stackrel{\text{def}}{=} \rho(\rho+1)\cdots(\rho+n-1)$ ], or the *reversed monic unsigned Laguerre polynomials*

$$\begin{aligned} \bar{\mathcal{L}}_n^{(\alpha)}(x) &\stackrel{\text{def}}{=} n! x^n L_n^{(\alpha)}(-1/x) = \sum_{k=0}^n \binom{n}{k} \binom{n+\alpha}{k} k! x^k = \sum_{k=0}^n \binom{n}{k} (n+\alpha)^{\underline{k}} x^k \\ &= {}_2F_0\left(\begin{matrix} -n, -n-\alpha \\ - \end{matrix} \middle| x\right). \end{aligned} \quad (1.4)$$

Note that  $\mathcal{L}_n^{(\alpha)}(x)$  and  $\bar{\mathcal{L}}_n^{(\alpha)}(x)$  are polynomials jointly in  $x$  and  $\alpha$ , with nonnegative integer coefficients. The first few  $\mathcal{L}_n^{(\alpha)}(x)$  are thus

$$\mathcal{L}_0^{(\alpha)}(x) = 1 \quad (1.5a)$$

$$\mathcal{L}_1^{(\alpha)}(x) = (1+\alpha) + x \quad (1.5b)$$

$$\mathcal{L}_2^{(\alpha)}(x) = (1+\alpha)(2+\alpha) + 2(2+\alpha)x + x^2 \quad (1.5c)$$

$$\mathcal{L}_3^{(\alpha)}(x) = (1+\alpha)(2+\alpha)(3+\alpha) + 3(2+\alpha)(3+\alpha)x + 3(3+\alpha)x^2 + x^3 \quad (1.5d)$$

It follows from (1.2) that these polynomials have the exponential generating functions

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x) \frac{t^n}{n!} = (1-t)^{-(1+\alpha)} e^{xt/(1-t)} \quad (1.6)$$

$$\sum_{n=0}^{\infty} \bar{\mathcal{L}}_n^{(\alpha)}(x) \frac{t^n}{n!} = (1-xt)^{-(1+\alpha)} e^{t/(1-xt)} \quad (1.7)$$

A major role will be played in what follows by the *coefficient matrix* of the monic unsigned Laguerre polynomials: it is the unit-lower-triangular matrix  $\mathbf{L}^{(\alpha)}$  with entries

$$(\mathbf{L}^{(\alpha)})_{n,k} = \frac{n!}{k!} \binom{n+\alpha}{n-k} = \binom{n}{k} (n+\alpha)^{n-k} = \binom{n}{k} (1+\alpha+k)^{\overline{n-k}}. \quad (1.8)$$

This matrix is an exponential Riordan array (see Section 2.5)  $\mathcal{R}[F, G]$  with  $F(t) = (1-t)^{-(1+\alpha)}$  and  $G(t) = t/(1-t)$ : compare (1.6) with (2.33) below. It is also the matrix of generalized Stieltjes–Rogers polynomials of the first kind (see Section A.1) for the Stieltjes-type continued fraction associated to the sequence  $((1+\alpha)^{\overline{n}})_{n \geq 0}$ : see Proposition 3.1.

An important role will also be played by the **binomial row-generating matrix**  $\mathbf{L}^{(\alpha)}B_x$ , where  $B_x$  is the weighted binomial matrix

$$(B_x)_{ij} = \binom{i}{j} x^{i-j} \quad (1.9)$$

(note that it too is unit-lower-triangular). The matrix  $\mathbf{L}^{(\alpha)}B_x$  is an exponential Riordan array  $\mathcal{R}[e^{xG}F, G]$  with  $F$  and  $G$  as above (see Corollary 2.20). The zeroth column of  $\mathbf{L}^{(\alpha)}B_x$  consists of the monic unsigned Laguerre polynomials  $\mathcal{L}_n^{(\alpha)}(x)$ . More generally, it can be shown (see Proposition 3.2) that

$$(\mathbf{L}^{(\alpha)}B_x)_{n,k} = \frac{1}{k!} \frac{d^k}{dx^k} \mathcal{L}_n^{(\alpha)}(x) = \binom{n}{k} \mathcal{L}_{n-k}^{(\alpha+k)}(x). \quad (1.10)$$

Several specializations of the Laguerre polynomials to integer values of  $\alpha$  correspond to well-known combinatorial objects:

1) The **rook polynomial**  $R_B(x)$  of a chessboard  $B$  is the generating polynomial for placements of zero or more nonattacking rooks on  $B$ , with a weight  $x$  for each rook.<sup>1</sup> In particular, for an  $m \times n$  rectangular chessboard, the number of ways of placing  $k$  non-attacking rooks is  $\binom{m}{k} \binom{n}{k} k!$ : we choose  $k$  rows,  $k$  columns, and a permutation connecting them. It follows that the rook polynomial of an  $n \times n$  square chessboard is the reversed monic unsigned Laguerre polynomial  $\overline{\mathcal{L}}_n^{(0)}(x) = \sum_{k=0}^n \binom{n}{k}^2 k! x^k$ ; and more generally, for any integer  $\alpha \geq -n$ , the rook polynomial of an  $n \times (n+\alpha)$  rectangular chessboard is  $\overline{\mathcal{L}}_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+\alpha}{k} k! x^k$ .<sup>2</sup> The first few rook polynomials for square chessboards are

$$\overline{\mathcal{L}}_0^{(0)}(x) = 1 \quad (1.11a)$$

$$\overline{\mathcal{L}}_1^{(0)}(x) = 1 + x \quad (1.11b)$$

$$\overline{\mathcal{L}}_2^{(0)}(x) = 1 + 4x + 2x^2 \quad (1.11c)$$

$$\overline{\mathcal{L}}_3^{(0)}(x) = 1 + 9x + 18x^2 + 6x^3 \quad (1.11d)$$

$$\overline{\mathcal{L}}_4^{(0)}(x) = 1 + 16x + 72x^2 + 96x^3 + 24x^4 \quad (1.11e)$$

2) The **Lah number**  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  is the number of partitions of an  $n$ -element set into  $k$  nonempty linearly ordered blocks (also called *lists*); we set  $\left[ \begin{smallmatrix} 0 \\ k \end{smallmatrix} \right] = \delta_{k0}$ . The **Lah**

<sup>1</sup>See [64, chapters 7 and 8] [72, sections 2.3 and 2.4] for further discussion of rook polynomials.

<sup>2</sup>See [55, A144084] for the triangular array corresponding to the rook polynomial of the  $n \times n$  chessboard, and see [55, A002720] for the row sums.

Note also that a rook configuration of an  $m \times n$  chessboard can equivalently be viewed as a matching of the complete bipartite graph  $K_{m,n}$ ; so the rook polynomial of the  $m \times n$  chessboard is also the matching polynomial [50, p. 334] of  $K_{m,n}$ .

*polynomial* is then defined as  $\text{Lah}_n(x) = \sum_{k=0}^n \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k$ . It is easy to see that the Lah numbers have the explicit expression

$$\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \frac{n!}{k!} \binom{n-1}{n-k} = \begin{cases} \delta_{k0} & \text{if } n = 0 \\ \frac{n!}{k!} \binom{n-1}{k-1} & \text{if } n \geq 1 \end{cases} \quad (1.12)$$

It follows that  $\text{Lah}_n(x)$  is the monic unsigned Laguerre polynomial  $\mathcal{L}_n^{(-1)}(x)$ .<sup>3</sup> The first few Lah polynomials are

$$\mathcal{L}_0^{(-1)}(x) = 1 \quad (1.13a)$$

$$\mathcal{L}_1^{(-1)}(x) = x \quad (1.13b)$$

$$\mathcal{L}_2^{(-1)}(x) = 2x + x^2 \quad (1.13c)$$

$$\mathcal{L}_3^{(-1)}(x) = 6x + 6x^2 + x^3 \quad (1.13d)$$

$$\mathcal{L}_4^{(-1)}(x) = 24x + 36x^2 + 12x^3 + x^4 \quad (1.13e)$$

3) More generally, for any integer  $r \geq 0$ , the *r-Lah number*  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r$  is the number of partitions of an  $(n+r)$ -element set into  $k+r$  nonempty linearly ordered blocks, with the restriction that  $r$  distinguished elements must belong to distinct blocks [54]. The *r-Lah polynomial* is then defined as  $\text{Lah}_n^{[r]}(x) = \sum_{k=0}^n \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r x^k$ . Clearly,  $\text{Lah}_n^{[0]}(x) = \text{Lah}_n(x)$  and  $\text{Lah}_n^{[1]}(x) = \text{Lah}_{n+1}(x)/x$ . It is not difficult to show [54, Theorem 3.7] that the *r-Lah numbers* have the explicit expression

$$\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r = \frac{n!}{k!} \binom{n+2r-1}{n-k} = \begin{cases} \delta_{k0} & \text{if } n = 0 \\ \frac{n!}{k!} \binom{n+2r-1}{k+2r-1} & \text{if } n \geq 1 \end{cases} \quad (1.14)$$

It follows that  $\text{Lah}_n^{[r]}(x)$  is the monic unsigned Laguerre polynomial  $\mathcal{L}_n^{(2r-1)}(x)$ .<sup>4</sup>

In summary, the rook polynomials correspond to  $\alpha = 0$  for square chessboards and  $\alpha = 1, 2, 3, \dots$  for rectangular chessboards, the Lah polynomials correspond to  $\alpha = -1$ , and the *r-Lah polynomials* correspond to  $\alpha = 2r - 1$ .

## 1.2 Combinatorial interpretation in terms of Laguerre digraphs

Four decades ago, Foata and Strehl [31] introduced a beautiful combinatorial interpretation of the Laguerre polynomials, which will form the starting point for our work.<sup>5</sup> Let us define a *Laguerre digraph* to be a digraph in which each vertex has out-degree 0 or 1 and in-degree 0 or 1. It follows that each weakly connected component of a Laguerre digraph is either a directed path of some length  $\ell \geq 0$  (where a path of length 0 is an isolated vertex) or else a directed cycle of some length  $\ell \geq 1$

---

<sup>3</sup>See [55, A105278/A008297/A066667] for further information on the Lah numbers and Lah polynomials.

<sup>4</sup>See [55, A143497/A143498/A143499] for further information on the *r-Lah numbers* and *r-Lah polynomials*.

<sup>5</sup>See also the recent paper of Strehl [76] for further applications of this formalism.

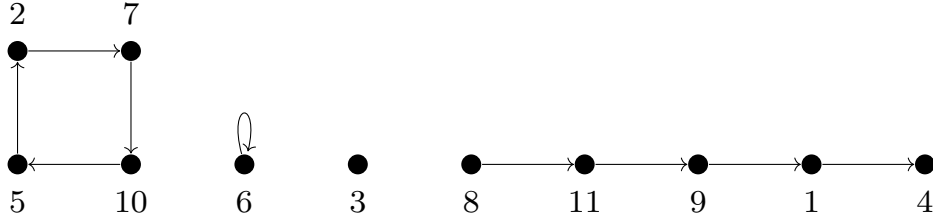


Figure 1: A Laguerre digraph on 11 vertices with 2 cycles (one of which is a loop) and 2 paths (one of which is an isolated vertex).

(where a cycle of length 1 is a loop): see Figure 1. For each integer  $n \geq 0$ , let us write  $\mathbf{LD}_n$  for the set of Laguerre digraphs on the vertex set  $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ ; and for  $n \geq k \geq 0$ , let us write  $\mathbf{LD}_{n,k}$  for the set of Laguerre digraphs on the vertex set  $[n]$  with  $k$  paths. For  $G \in \mathbf{LD}_n$ , we write  $e(G)$  [resp.  $\text{cyc}(G)$ ,  $\text{pa}(G)$ ] for the number of edges (resp. cycles, paths) in  $G$ , and observe that  $e(G) = n - \text{pa}(G)$ . Foata and Strehl [31] then showed that

$$\mathcal{L}_n^{(\alpha)}(x) = \sum_{G \in \mathbf{LD}_n} x^{\text{pa}(G)} (1 + \alpha)^{\text{cyc}(G)} \quad (1.15)$$

or equivalently

$$\overline{\mathcal{L}}_n^{(\alpha)}(x) = \sum_{G \in \mathbf{LD}_n} x^{e(G)} (1 + \alpha)^{\text{cyc}(G)}. \quad (1.16)$$

Indeed, the proof of (1.15)/(1.16) is an easy argument using the exponential formula [73, chapter 5], or equivalently, the theory of species [7]. Let us do (1.15): The number of directed paths on  $n \geq 1$  vertices is  $n!$ , so with a weight  $x$  per path they have exponential generating function  $xt/(1-t)$ . The number of directed cycles on  $n \geq 1$  vertices is  $(n-1)!$ , so with a weight  $1 + \alpha$  per cycle they have exponential generating function  $-(1 + \alpha) \log(1-t)$ . A Laguerre digraph is a disjoint union of paths and cycles, so by the exponential formula it has exponential generating function

$$\exp\left[\frac{xt}{1-t} - (1 + \alpha) \log(1-t)\right] = (1-t)^{-(1+\alpha)} e^{xt/(1-t)}, \quad (1.17)$$

which coincides with (1.6).<sup>6</sup> The principal virtue of the combinatorial model (1.15)/(1.16) is that it treats  $\alpha$  as an indeterminate; it need not be an integer.<sup>7</sup>

The identity (1.15) can equivalently be understood as a combinatorial representation for the coefficient matrix  $\mathbf{L}^{(\alpha)}$ :

$$(\mathbf{L}^{(\alpha)})_{n,k} = \sum_{G \in \mathbf{LD}_{n,k}} (1 + \alpha)^{\text{cyc}(G)}. \quad (1.18)$$

This formula will be the starting point for our multivariate generalizations, in which we will enumerate Laguerre digraphs with additional statistics.

<sup>6</sup>Foata and Strehl [31] also gave a direct combinatorial proof that (1.15)/(1.16) is equivalent to (1.3)/(1.4); this requires a bit more work [31, Lemma 2.1].

<sup>7</sup>The formulae (1.15)/(1.16) also make clear (as Rota [53, p. 206] realized many years ago) that the natural parameter for the Laguerre polynomials is  $\lambda \stackrel{\text{def}}{=} 1 + \alpha$ , not  $\alpha$ . But it would be too confusing to try to change the notation at this late date.



**Remarks.** 1. There is a one-to-one correspondence between Laguerre digraphs and rook configurations on an  $n \times n$  chessboard: for a rook at position  $(i, j)$  we draw an edge  $i \rightarrow j$ . This explains why  $\alpha = 0$  corresponds to the rook polynomial, for which there is no special weighting of cycles. However, we prefer the digraph point of view, where the cycles (and therefore the role of  $\alpha$ ) can be seen more clearly.

2. Note also from (1.15) that the case  $\alpha = -1$  leads to Laguerre digraphs with no cycles — that is, only directed paths — with a weight  $x$  per path. These digraphs are in obvious one-to-one correspondence with partitions of  $[n]$  into linearly ordered blocks; this explains why  $\alpha = -1$  corresponds to the Lah polynomials. ■

### 1.3 A first multivariate generalization

It is now natural to generalize the Laguerre polynomials by introducing further statistics into the digraph model. Here is one way: Let us write  $e_-(G), e_0(G), e_+(G)$  for the number of edges  $i \rightarrow j$  in  $G$  where  $j < i, j = i$  or  $j > i$ , respectively. We then generalize the coefficient matrix (1.18) by introducing separate variables  $v_-, v_0, v_+$  for the three types of edges:

$$\mathbf{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} \stackrel{\text{def}}{=} \sum_{G \in \mathbf{LD}_{n,k}} v_-^{e_-(G)} v_0^{e_0(G)} v_+^{e_+(G)} (1 + \alpha)^{\text{cyc}(G)}. \quad (1.19)$$

This polynomial is homogeneous of degree  $n - k$  in  $v_-, v_0, v_+$ . We call the matrix  $\mathbf{L}^{(\alpha)}(v_-, v_0, v_+)$  the **(first) multivariate Laguerre coefficient matrix**. In Section 4.1 we will compute the bivariate exponential generating function for this matrix: see (4.15). By specializing  $v_- = v_0 = v_+ = v$ , we recover a rescaled version of the univariate Laguerre coefficient matrix

$$\mathbf{L}^{(\alpha)}(v, v, v)_{n,k} = (\mathbf{L}^{(\alpha)})_{n,k} v^{n-k}. \quad (1.20)$$

Let us then introduce the row-generating polynomials of the multivariate Laguerre coefficient matrix:

$$\mathcal{L}_n^{(\alpha)}(x; v_-, v_0, v_+) \stackrel{\text{def}}{=} \sum_{k=0}^n \mathbf{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} x^k \quad (1.21)$$

$$\overline{\mathcal{L}}_n^{(\alpha)}(x; v_-, v_0, v_+) \stackrel{\text{def}}{=} \sum_{k=0}^n \mathbf{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} x^{n-k} \quad (1.22)$$

Of course, by homogeneity (1.22) can also be understood as

$$\overline{\mathcal{L}}_n^{(\alpha)}(x; v_-, v_0, v_+) = \sum_{k=0}^n \mathbf{L}^{(\alpha)}(xv_-, xv_0, xv_+)_{n,k}. \quad (1.23)$$

**Remark.** In the rook model, the three  $v$  variables correspond to giving different weights for rooks below the diagonal, on the diagonal, or above the diagonal. To our knowledge this type of weighting has not previously been considered, even in the rook case  $\alpha = 0$ . However, two special cases generalize previously known results:

1) If we set  $v_+ = 0$ , then the only possible connected components of  $G$  are decreasing paths (of length  $\ell \geq 0$ ) and loops. It follows that

$$\mathbf{L}^{(\alpha)}(v_-, v_0, 0)_{n,k} = \sum_{i=0}^n \binom{n}{i} [(1 + \alpha)v_0]^i \left\{ \begin{matrix} n-i \\ k \end{matrix} \right\} v_-^{n-i-k}, \quad (1.24)$$

where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  denotes the number of partitions of an  $n$ -element set into  $k$  nonempty blocks: we choose  $i$  loops and then partition the remaining vertices into  $k$  nonempty blocks. Further specializing to  $\alpha = 0$  and  $v_0 = 0$  yields the well-known [72, Corollary 2.4.2] formula for the counting of rook configurations on an  $n \times n$  triangular board:

$$\mathbf{L}^{(0)}(1, 0, 0)_{n,k} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}, \quad (1.25)$$

where a configuration with  $k$  paths has  $n - k$  edges and hence  $n - k$  rooks. Similarly, for  $\alpha = 0$  and  $v_0 = 1$  we get

$$\mathbf{L}^{(0)}(1, 1, 0)_{n,k} = \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}, \quad (1.26)$$

which follows from (1.24) by a standard identity [39, eq. (6.15)].

2) At the other extreme, when  $k = 0$ , paths are forbidden, and we obtain the generating polynomial for permutations of  $[n]$  with a weight  $v_+$  for each cycle ascent (= excedance),  $v_-$  for each cycle descent (= anti-excedance),  $v_0$  for each fixed point, and  $1 + \alpha$  for each cycle. For the corresponding exponential generating function, see (4.15) specialized to  $u = 0$ . In particular, when  $v_- = v_0 = v_+ = v$  we obtain

$$\mathbf{L}^{(\alpha)}(v, v, v)_{n,0} = (1 + \alpha)^{\bar{n}} v^n. \quad (1.27)$$

And when  $\alpha = 0$  and  $v_0 = v_-$ , we obtain

$$\mathbf{L}^{(0)}(v_-, v_-, v_+)_{n,0} = \sum_{j=0}^{n-1} \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle v_+^j v_-^{n-j}, \quad (1.28)$$

where the Eulerian number  $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle$  is the number of permutations of  $[n]$  with  $j$  excedances (or  $j$  descents). More generally, these two specializations can be combined: we obtain the generating polynomial for permutations by number of excedances and number of cycles [71, eq. (2.12)] [65, Proposition 3.3]

$$\mathbf{L}^{(\alpha)}(v_-, v_-, v_+)_{n,0} = \sum_{j=0}^n \left\{ \begin{matrix} n \\ j \end{matrix} \right\} (v_+ - v_-)^{n-j} v_-^j (1 + \alpha)^{\bar{j}}. \quad (1.29)$$

And most generally, we can introduce additional fixed points, to obtain

$$\mathbf{L}^{(\alpha)}(v_-, v_0, v_+)_{n,0} = \sum_{i=0}^n \binom{n}{i} [(1 + \alpha)(v_0 - v_-)]^i \mathbf{L}^{(\alpha)}(v_-, v_-, v_+)_{n-i,0}. \quad (1.30)$$

■

## 1.4 A second multivariate generalization

But we can go farther: instead of weighting *edges* according to whether they are increasing, decreasing or fixed points, we can weight *pairs of successive edges* according to whether they are peaks (+−), valleys (−+), double ascents (++), double descents (−−) or fixed points (00). To define these concepts for a Laguerre digraph, we first need to make a convention about *boundary conditions* at the two ends of a path. We will here use 0–0 boundary conditions: that is, we extend the Laguerre digraph  $G$  on the vertex set  $[n]$  to a digraph  $\widehat{G}$  on the vertex set  $[n] \cup \{0\}$  by decreeing that any vertex  $i \in [n]$  that has in-degree (resp. out-degree) 0 in  $G$  will receive an incoming (resp. outgoing) edge from (resp. to) the vertex 0. In this way each vertex  $i \in [n]$  will have a unique predecessor  $p(i) \in [n] \cup \{0\}$  and a unique successor  $s(i) \in [n] \cup \{0\}$ . We then say that a vertex  $i \in [n]$  is a

- *peak* (p) if  $p(i) < i > s(i)$ ;
- *valley* (v) if  $p(i) > i < s(i)$ ;
- *double ascent* (da) if  $p(i) < i < s(i)$ ;
- *double descent* (dd) if  $p(i) > i > s(i)$ ;
- *fixed point* (fp) if  $p(i) = i = s(i)$ .

(Note that “fixed point” is a synonym of “loop”.) When these concepts are applied to the cycles of a Laguerre digraph, we obtain the usual ***cycle classification*** of indices in a permutation as cycle peaks, cycle valleys, cycle double rises, cycle double falls and fixed points [71, 84]. When applied to the paths of a Laguerre digraph, we obtain the usual ***linear classification*** of indices in a permutation (written in word form) as peaks, valleys, double ascents or double descents [72, p. 45]. Note that, because of the 0–0 boundary conditions, an isolated vertex is always a peak, the initial vertex of a path is always a peak or double ascent, and the final vertex of a path is always a peak or double descent; moreover, each path contains at least one peak.

We write  $p(G)$ ,  $v(G)$ ,  $da(G)$ ,  $dd(G)$ ,  $fp(G)$  for the number of vertices  $i \in [n]$  that are, respectively, peaks, valleys, double ascents, double descents or fixed points. We then introduce the ***second multivariate Laguerre coefficient matrix***

$$\widehat{\mathbf{L}}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp})_{n,k} \stackrel{\text{def}}{=} \sum_{G \in \mathbf{LD}_{n,k}} y_p^{p(G)} y_v^{v(G)} y_{da}^{da(G)} y_{dd}^{dd(G)} y_{fp}^{fp(G)} (1 + \alpha)^{\text{cyc}(G)}. \quad (1.31)$$

This polynomial is homogeneous of degree  $n$  in  $y_p, y_v, y_{da}, y_{dd}, y_{fp}$ . In Section 4.1 we will compute the bivariate exponential generating function for this matrix (and indeed for a matrix that further generalizes it).

Because each path contains at least one peak, we can, if we wish, remove a factor  $y_p^k$  and define a unit-lower-triangular matrix by

$$\widehat{\mathbf{L}}^{(\alpha)b}(y_p, y_v, y_{da}, y_{dd}, y_{fp})_{n,k} \stackrel{\text{def}}{=} \widehat{\mathbf{L}}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp})_{n,k} / y_p^k. \quad (1.32)$$

This polynomial is homogeneous of degree  $n - k$  in  $y_p, y_v, y_{da}, y_{dd}, y_{fp}$ .

We can recover the first multivariate Laguerre coefficient matrix by looking only at the second step of each pair: specializing  $y_p = y_{dd} = v_-$ ,  $y_v = y_{da} = v_+$  and

$y_{\text{fp}} = v_0$ , we obtain the previous counting with an extra weight  $v_-$  associated to the final vertex of each path:

$$\widehat{\mathcal{L}}^{(\alpha)}(v_-, v_+, v_+, v_-, v_0)_{n,k} = \mathcal{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} v_-^k \quad (1.33)$$

or equivalently

$$\widehat{\mathcal{L}}^{(\alpha)\flat}(v_-, v_+, v_+, v_-, v_0)_{n,k} = \mathcal{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} . \quad (1.34)$$

Alternatively, we can look only at the first step of each pair: specializing  $y_v = y_{\text{dd}} = v_-$ ,  $y_{\text{p}} = y_{\text{da}} = v_+$  and  $y_{\text{fp}} = v_0$ , we obtain the previous counting with an extra weight  $v_+$  associated to the initial vertex of each path:

$$\widehat{\mathcal{L}}^{(\alpha)}(v_+, v_-, v_+, v_-, v_0)_{n,k} = \mathcal{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} v_+^k \quad (1.35)$$

or equivalently

$$\widehat{\mathcal{L}}^{(\alpha)\flat}(v_+, v_-, v_+, v_-, v_0)_{n,k} = \mathcal{L}^{(\alpha)}(v_-, v_0, v_+)_{n,k} . \quad (1.36)$$

Once again we can introduce the row-generating polynomials:

$$\widehat{\mathcal{L}}_n^{(\alpha)}(x; y_{\text{p}}, y_v, y_{\text{da}}, y_{\text{dd}}, y_{\text{fp}}) \stackrel{\text{def}}{=} \sum_{k=0}^n \mathcal{L}^{(\alpha)}(y_{\text{p}}, y_v, y_{\text{da}}, y_{\text{dd}}, y_{\text{fp}})_{n,k} x^k \quad (1.37)$$

$$\overline{\widehat{\mathcal{L}}}_n^{(\alpha)}(x; y_{\text{p}}, y_v, y_{\text{da}}, y_{\text{dd}}, y_{\text{fp}}) \stackrel{\text{def}}{=} \sum_{k=0}^n \mathcal{L}^{(\alpha)}(y_{\text{p}}, y_v, y_{\text{da}}, y_{\text{dd}}, y_{\text{fp}})_{n,k} x^{n-k} \quad (1.38)$$

## 1.5 Total positivity

The main results of this paper concern the total positivity of various matrices associated to the univariate and multivariate Laguerre polynomials. Recall first that a finite or infinite matrix of real numbers is called *totally positive* (TP) if all its minors are nonnegative, and *totally positive of order  $r$*  (TP $_r$ ) if all its minors of size  $\leq r$  are nonnegative. Background information on totally positive matrices can be found in [28, 35, 42, 60]; they have application to many fields of pure and applied mathematics.<sup>8</sup> In particular, it is known [36, Théorème 9] [60, section 4.6] that an infinite Hankel matrix  $(a_{i+j})_{i,j \geq 0}$  of real numbers is totally positive if and only if the underlying sequence  $(a_n)_{n \geq 0}$  is a Stieltjes moment sequence, i.e. the moments of a positive measure on  $[0, \infty)$ .

But this is only the beginning of the story, because we are here principally concerned, not with sequences and matrices of real numbers, but with sequences and matrices of polynomials (with integer or real coefficients) in one or more indeterminates  $\mathbf{x}$ : they will typically be generating polynomials that enumerate some combinatorial objects with respect to one or more statistics. We equip the polynomial ring  $\mathbb{R}[\mathbf{x}]$  with the coefficientwise partial order: that is, we say that  $P$  is nonnegative (and write  $P \succeq 0$ ) in case  $P$  is a polynomial with nonnegative coefficients. We then say that a matrix with entries in  $\mathbb{R}[\mathbf{x}]$  is *coefficientwise totally positive* if all its minors are polynomials with nonnegative coefficients; and analogously for coefficientwise

---

<sup>8</sup>See [67, footnote 4] for many references.

total positivity of order  $r$ . We say that a sequence  $\mathbf{a} = (a_n)_{n \geq 0}$  with entries in  $\mathbb{R}[\mathbf{x}]$  is **coefficientwise Hankel-totally positive** if its associated infinite Hankel matrix is coefficientwise totally positive; and likewise for the version of order  $r$ . Similar definitions apply to the formal-power-series ring  $\mathbb{R}[[\mathbf{x}]]$ . Most generally, we can consider sequences and matrices with values in an arbitrary partially ordered commutative ring (a precise definition will be given in Section 2.1); total positivity and Hankel-total positivity are then defined in the obvious way. Coefficientwise Hankel-total positivity of a sequence of polynomials  $(P_n(\mathbf{x}))_{n \geq 0}$  *implies* the pointwise Hankel-total positivity (i.e. the Stieltjes moment property) for all  $\mathbf{x} \geq 0$ , but it is vastly stronger.

For instance, it is known (see Section 7) that, for each  $\alpha \geq -1$  and  $x \geq 0$ , the sequence  $(\mathcal{L}_n^{(\alpha)}(x))_{n \geq 0}$  of monic unsigned Laguerre polynomials is a Stieltjes moment sequence. So every minor of the Hankel matrix  $(\mathcal{L}_{i+j}^{(\alpha)}(x))_{i,j \geq 0}$  is a polynomial in  $x$  and  $\alpha$  that is nonnegative whenever  $\alpha \geq -1$  and  $x \geq 0$ . But much more turns out to be true: every minor of the Hankel matrix  $(\mathcal{L}_{i+j}^{(\alpha)}(x))_{i,j \geq 0}$  is in fact a polynomial in  $x$  and  $\alpha$  that is *coefficientwise* nonnegative in  $x$  and  $\lambda \stackrel{\text{def}}{=} 1 + \alpha$ . That is, the sequence  $(\mathcal{L}_n^{(-1+\lambda)}(x))_{n \geq 0}$  is *coefficientwise* Hankel-totally positive in  $x$  and  $\lambda$ : this will be our first main result (Theorem 1.1(c)). We will then extend this result to the multivariate polynomials introduced in Sections 1.3 and 1.4.

Our proofs of total positivity and Hankel-total positivity will be based on the method of *production matrices* [22,23]. We will review this theory in Sections 2.2 and 2.3, so now we state only the bare-bones definitions. Let  $P = (p_{ij})_{i,j \geq 0}$  be an infinite matrix with entries in a commutative ring  $R$ ; we assume that  $P$  is either row-finite (i.e. has only finitely many nonzero entries in each row) or column-finite. Now define an infinite matrix  $A = (a_{nk})_{n,k \geq 0}$  by

$$a_{nk} = (P^n)_{0k} . \quad (1.39)$$

We call  $P$  the **production matrix** and  $A$  the **output matrix**, and we write  $A = \mathcal{O}(P)$ . The two key facts here are the following [70]: if  $R$  is a partially ordered commutative ring,  $P$  is totally positive of order  $r$ , and  $a$  is a nonnegative element of  $R$ , then  $\mathcal{O}(P + aI)$  is totally positive of order  $r$  and the zeroth column of  $\mathcal{O}(P + aI)$  is Hankel-totally positive of order  $r$ . See Section 2.3 for precise statements and proofs.

## 1.6 Statement of main results: Univariate case

Our first main result is the following:

**Theorem 1.1** (Total positivity of the univariate Laguerre polynomials).

- (a) *The Laguerre coefficient matrix  $\mathbf{L}^{(-1+\lambda)}$  is totally positive in the ring  $\mathbb{Z}[\lambda]$  equipped with the coefficientwise order.*
- (b) *The binomial row-generating matrix  $\mathbf{L}^{(-1+\lambda)} B_x$  is totally positive in the ring  $\mathbb{Z}[x, \lambda]$  equipped with the coefficientwise order.*
- (c) *The sequence of monic unsigned Laguerre polynomials  $(\mathcal{L}_n^{(-1+\lambda)}(x))_{n \geq 0}$  is Hankel-totally positive in the ring  $\mathbb{Z}[x, \lambda]$  equipped with the coefficientwise order.*
- (d) *The sequence of reversed monic unsigned Laguerre polynomials  $(\overline{\mathcal{L}}_n^{(-1+\lambda)}(x))_{n \geq 0}$  is Hankel-totally positive in the ring  $\mathbb{Z}[x, \lambda]$  equipped with the coefficientwise order.*

Theorem 1.1(c,d) proves a conjecture made a few years ago by Sylvie Corteel and one of us [18]. Theorem 1.1 was also proven recently by Zhu [90, Proposition 4.14] [92, Proposition 5.11]; his methods are very similar to ours.<sup>9</sup>

The result in part (a) is fairly easy, and we will give two proofs using different methods: one direct proof (Section 3.3), and one proof using production matrices. Part (b) is then an immediate consequence of part (a) together with the coefficientwise total positivity of the binomial matrix  $B_x$  (Lemma 2.2). Also, part (d) is trivially equivalent to part (c) by virtue of the relation  $\overline{\mathcal{L}}_n^{(-1+\lambda)}(x) = x^n \mathcal{L}_n^{(-1+\lambda)}(1/x)$ . So our main effort will be directed to proving part (c): we will do this by constructing the production matrix for the binomial row-generating matrix  $\mathbf{L}^{(-1+\lambda)} B_x$  and then proving its total positivity. More precisely, we will prove the following:

**Proposition 1.2** (Production matrices for the univariate Laguerre polynomials).

(a) *The production matrix of the Laguerre coefficient array  $\mathbf{L}^{(\alpha)}$  is the tridiagonal unit-lower-Hessenberg matrix  $P^\circ = (p_{ij}^\circ)_{i,j \geq 0}$  defined by*

$$p_{n,n+1}^\circ = 1 \tag{1.40a}$$

$$p_{n,n}^\circ = 2n + 1 + \alpha \tag{1.40b}$$

$$p_{n,n-1}^\circ = n(n + \alpha) \tag{1.40c}$$

$$p_{n,k}^\circ = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \tag{1.40d}$$

(b) *The production matrix of the binomial row-generating matrix  $\mathbf{L}^{(\alpha)} B_x$  is the quadridiagonal unit-lower-Hessenberg matrix  $P = (p_{ij})_{i,j \geq 0}$  defined by*

$$p_{n,n+1} = 1 \tag{1.41a}$$

$$p_{n,n} = (2n + 1 + \alpha) + x \tag{1.41b}$$

$$p_{n,n-1} = n(n + \alpha) + 2nx \tag{1.41c}$$

$$p_{n,n-2} = n(n - 1)x \tag{1.41d}$$

$$p_{n,k} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \tag{1.41e}$$

**Remarks.** 1. When  $\alpha \in \{-1, 0, 1\}$ , the production matrix (1.41) arises from a 2-branched S-fraction [59] — more precisely, as the production matrix for the generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$  with  $m = 2$  and  $j \in \{0, 1, 2\}$ . See Appendix A.3 for the theory of these generalized  $m$ -Stieltjes–Rogers polynomials, and Appendix A.4 for the application to (1.41).

2. It will follow easily from our general theory (Lemma 2.7) that the production matrices of Proposition 1.2(a,b) are related by  $P = B_x^{-1} P^\circ B_x$ . What is far from obvious is why  $P$  is quadridiagonal, since it is *not* in general true that  $B_x^{-1} T B_x$  is quadridiagonal whenever  $T$  is tridiagonal. In Appendix B we explain why  $P$  is quadridiagonal in the present case, by answering the more general question: Which lower-Hessenberg matrices  $P$  have the property that  $B_\xi^{-1} P B_\xi$  is  $(r, 1)$ -banded? ■

---

<sup>9</sup>Zhu [90, Proposition 4.14] [92, Proposition 5.11] states a slightly weaker form of Theorem 1.1 in which  $\lambda$  is a nonnegative real number rather than an indeterminate. But his methods actually prove the stronger result claimed here.

**Proposition 1.3** (Total positivity of the univariate production matrices).

- (a) The matrix  $P^\circ = (p_{ij}^\circ)_{i,j \geq 0}$  defined by (1.40), with the change of variable  $\alpha = -1 + \lambda$ , is totally positive in the ring  $\mathbb{Z}[\lambda]$  equipped with the coefficientwise order.
- (b) The matrix  $P = (p_{ij})_{i,j \geq 0}$  defined by (1.41), with the change of variable  $\alpha = -1 + \lambda$ , is totally positive in the ring  $\mathbb{Z}[x, \lambda]$  equipped with the coefficientwise order.

Combining Propositions 1.2 and 1.3 with the general theory of production matrices proves Theorem 1.1.

We will prove Proposition 1.2 in Section 3, and Proposition 1.3 in Section 5.

## 1.7 Statement of main results: Multivariate case

We now state the corresponding results for the first and second multivariate Laguerre polynomials that were introduced in Sections 1.3 and 1.4, respectively. We begin by considering the second multivariate Laguerre coefficient matrix, for which it is convenient to use the form  $\widehat{\mathcal{L}}^{(\alpha)b}(y_p, y_v, y_{da}, y_{dd}, y_{fp})$  that was defined in (1.32). Our main result, generalizing Theorem 1.1, is the following:

**Theorem 1.4** (Total positivity of the second multivariate Laguerre polynomials).

Let  $\lambda, y_p, y_v, y_{da}, y_{dd}, y_{fp}$  be elements of a partially ordered commutative ring  $R$  that satisfy  $\lambda \geq 0$ ,  $\lambda y_{fp} \geq \lambda y_p$ ,  $y_p \geq 0$ ,  $y_v \geq 0$  and  $y_{da} + y_{dd} \geq y_p + y_v$ ; and let  $x$  be an indeterminate. Then:

- (a) The second multivariate Laguerre coefficient array  $\widehat{\mathcal{L}}^{(-1+\lambda)b}(y_p, y_v, y_{da}, y_{dd}, y_{fp})$  is totally positive in the ring  $R$ .
- (b) The binomial row-generating matrix  $\widehat{\mathcal{L}}^{(-1+\lambda)b}(y_p, y_v, y_{da}, y_{dd}, y_{fp}) B_x$  is totally positive in the ring  $R[x]$  equipped with the coefficientwise order.
- (c) The sequence of row-generating polynomials  $\widehat{\mathcal{L}}_n^{(-1+\lambda)}(x; y_p, y_v, y_{da}, y_{dd}, y_{fp})$  defined by (1.37) is Hankel-totally positive in the ring  $R[x]$  equipped with the coefficientwise order.
- (d) The sequence of reversed row-generating polynomials  $\overline{\widehat{\mathcal{L}}}_n^{(-1+\lambda)}(x; y_p, y_v, y_{da}, y_{dd}, y_{fp})$  defined by (1.38) is Hankel-totally positive in the ring  $R[x]$  equipped with the coefficientwise order.

These statements are also true when the roles of  $y_p$  and  $y_v$  are interchanged.

As in the univariate case, part (b) is an immediate consequence of part (a), and part (d) is trivially equivalent to part (c). We therefore concentrate on proving (a) and (c).

As before, we will prove Theorem 1.4 by exhibiting the production matrices and then proving their total positivity.

**Proposition 1.5** (Production matrices for the multivariate Laguerre polynomials).

- (a) The production matrix of the multivariate Laguerre coefficient array  $\widehat{\mathcal{L}}^{(\alpha)b}(y_p, y_v, y_{da}, y_{dd}, y_{fp})$  is the tridiagonal unit-lower-Hessenberg matrix  $P^{\circ b} =$

$(p_{ij}^{\text{ob}})_{i,j \geq 0}$  defined by

$$p_{n,n+1}^{\text{ob}} = 1 \quad (1.42a)$$

$$p_{n,n}^{\text{ob}} = (1 + \alpha)y_{\text{fp}} + n(y_{\text{da}} + y_{\text{dd}}) \quad (1.42b)$$

$$p_{n,n-1}^{\text{ob}} = n(n + \alpha)y_{\text{p}}y_{\text{v}} \quad (1.42c)$$

$$p_{n,k}^{\text{ob}} = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \quad (1.42d)$$

(b) The production matrix of the binomial row-generating matrix  $\widehat{\mathbf{L}}^{(\alpha)\text{b}}(y_{\text{p}}, y_{\text{v}}, y_{\text{da}}, y_{\text{dd}}, y_{\text{fp}})B_x$  is the quadridiagonal unit-lower-Hessenberg matrix  $P^{\text{b}} = (p_{ij}^{\text{b}})_{i,j \geq 0}$  defined by

$$p_{n,n+1}^{\text{b}} = 1 \quad (1.43a)$$

$$p_{n,n}^{\text{b}} = (1 + \alpha)y_{\text{fp}} + n(y_{\text{da}} + y_{\text{dd}}) + x \quad (1.43b)$$

$$p_{n,n-1}^{\text{b}} = n(n + \alpha)y_{\text{p}}y_{\text{v}} + n(y_{\text{da}} + y_{\text{dd}})x \quad (1.43c)$$

$$p_{n,n-2}^{\text{b}} = n(n - 1)y_{\text{p}}y_{\text{v}}x \quad (1.43d)$$

$$p_{n,k}^{\text{b}} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \quad (1.43e)$$

**Proposition 1.6** (Total positivity of the multivariate production matrices).

- (a) The matrix  $P^{\text{ob}} = (p_{ij}^{\text{ob}})_{i,j \geq 0}$  defined by (1.42), with the variables substituted to elements of a partially ordered commutative ring  $R$  that satisfy  $\alpha \geq -1$ ,  $(1 + \alpha)y_{\text{fp}} \geq (1 + \alpha)y_{\text{p}}$ ,  $y_{\text{p}} \geq 0$ ,  $y_{\text{v}} \geq 0$  and  $y_{\text{da}} + y_{\text{dd}} \geq y_{\text{p}} + y_{\text{v}}$ , is totally positive in the ring  $R$ .
- (b) The matrix  $P^{\text{b}} = (p_{ij}^{\text{b}})_{i,j \geq 0}$  defined by (1.43), with the variables substituted to elements of a partially ordered commutative ring  $R$  that satisfy  $\alpha \geq -1$ ,  $(1 + \alpha)y_{\text{fp}} = (1 + \alpha)y_{\text{p}}$ ,  $y_{\text{p}} \geq 0$ ,  $y_{\text{v}} \geq 0$ ,  $y_{\text{da}} + y_{\text{dd}} \geq y_{\text{p}} + y_{\text{v}}$  and  $x \geq 0$ , is totally positive in the ring  $R$ .

These statements are also true when the roles of  $y_{\text{p}}$  and  $y_{\text{v}}$  are interchanged.

We will prove Proposition 1.5 in Section 4.2, and Proposition 1.6 in Section 6. Here the most difficult part is Proposition 1.6(b): namely, proving the total positivity of the quadridiagonal matrix (1.43) under the specified conditions. We will do this by proving the total positivity of a much more general quadridiagonal matrix (Theorem 6.1); and see also Appendix C (Theorem C.1) for a variant of this result.

**Remarks.** 1. The zeroth column of the multivariate Laguerre coefficient array  $\widehat{\mathbf{L}}^{(\alpha)\text{b}}(y_{\text{p}}, y_{\text{v}}, y_{\text{da}}, y_{\text{dd}}, y_{\text{fp}})$  is the sequence of generating polynomials for permutations weighted according to the cycle classification, and the tridiagonal production matrix (1.42) is the production matrix for the corresponding J-fraction, namely, [71, Theorem 2.4] specialized to  $x_1 = u_1 = y_{\text{p}}$ ,  $x_2 = u_2 = y_{\text{dd}}$ ,  $y_1 = v_1 = y_{\text{v}}$ ,  $y_2 = v_2 = y_{\text{da}}$ ,  $w_n = y_{\text{fp}}$ ,  $\lambda = 1 + \alpha$ .

2. Note that in part (a) we require only the *inequality*  $(1 + \alpha)y_{\text{fp}} \geq (1 + \alpha)y_{\text{p}}$ , while in part (b) we require the *equality*  $(1 + \alpha)y_{\text{fp}} = (1 + \alpha)y_{\text{p}}$ . However, this will not preclude us from proving Theorem 1.4 assuming only the inequality, as our general theory (Corollaries 2.12 and 2.16) entitles us to use the production matrix  $P^{\text{b}} + aI$  with  $a = \lambda(y_{\text{fp}} - y_{\text{p}}) \geq 0$ . ■



We can now recover the first multivariate Laguerre coefficient matrix  $\mathbf{L}^{(\alpha)}(v_-, v_0, v_+)$  defined in (1.19), and its corresponding row-generating polynomials  $\mathcal{L}_n^{(\alpha)}(x; v_-, v_0, v_+)$  defined in (1.21), by specializing either

$$y_p = y_{dd} = v_-, \quad y_v = y_{da} = v_+, \quad y_{fp} = v_0 \quad \text{as in (1.33)/(1.34)} \quad (1.44)$$

or

$$y_v = y_{dd} = v_-, \quad y_p = y_{da} = v_+, \quad y_{fp} = v_0 \quad \text{as in (1.35)/(1.36)}. \quad (1.45)$$

By applying these specializations to Theorem 1.4, we obtain:

**Corollary 1.7** (Total positivity of the first multivariate Laguerre polynomials).

- (a) *The first multivariate Laguerre coefficient matrix  $\mathbf{L}^{(-1+\lambda)}(v_-, v_0, v_+)$ , defined in (1.19), is totally positive when the variables are substituted to elements of a partially ordered commutative ring  $R$  that satisfy  $\lambda \geq 0$ ,  $v_- \geq 0$ ,  $v_+ \geq 0$ , and either  $v_0 \geq v_-$  or  $v_0 \geq v_+$ .*

*Equivalently, the matrices  $\mathbf{L}^{(-1+\lambda)}(v_-, v_- + w, v_+)$  and  $\mathbf{L}^{(-1+\lambda)}(v_-, v_+ + w, v_+)$  are totally positive in the ring  $\mathbb{Z}[v_-, v_+, w, \lambda]$  equipped with the coefficientwise order.*

- (b) *The sequences  $(\mathcal{L}_n^{(\alpha)}(x; v_-, v_-, v_+))_{n \geq 0}$  and  $(\mathcal{L}_n^{(\alpha)}(x; v_-, v_+, v_+))_{n \geq 0}$  of row-generating polynomials, defined in (1.21), are Hankel-totally positive, in the ring  $R[x]$  equipped with the coefficientwise order, when the variables are substituted to elements of a partially ordered commutative ring  $R$  that satisfy  $\lambda \geq 0$ ,  $v_- \geq 0$ ,  $v_+ \geq 0$ , and either  $v_0 \geq v_-$  or  $v_0 \geq v_+$ .*

*Equivalently, the sequences  $(\mathcal{L}_n^{(\alpha)}(x; v_-, v_- + w, v_+))_{n \geq 0}$  and  $(\mathcal{L}_n^{(\alpha)}(x; v_-, v_+ + w, v_+))_{n \geq 0}$  are Hankel-totally positive in the ring  $\mathbb{Z}[x, v_-, v_+, w, \lambda]$  equipped with the coefficientwise order.*

The multivariate case (Theorem 1.4 and Propositions 1.5 and 1.6) obviously subsumes the univariate case (Theorem 1.1 and Propositions 1.2 and 1.3), so it is in principle redundant to consider the latter. But since the proofs in the univariate case are quite a bit simpler, and since this case exhibits some special features that do not carry over to the fully multivariate case, we think it useful to present the univariate case first.

## 1.8 Plan of this paper

Although the present paper is a follow-up to our papers [58, 59], we have endeavored, for the convenience of the reader, to make it as self-contained as possible. We have therefore begun, in Section 2, with a brief review of the key definitions and results from [58, 59] (plus a few other things) that will be needed in the sequel. We then proceed as follows: In Section 3 we determine the production matrices for the univariate Laguerre coefficient matrix and binomial row-generating matrix. In Section 4 we compute the exponential generating function for the multivariate Laguerre coefficient matrix (1.31) and employ this to determine the corresponding production matrices, using the theory of exponential Riordan arrays. In Sections 5 and 6 we prove the total positivity of the univariate and multivariate production matrices, respectively. In

particular, in Section 6.2 (Theorem 6.1) we prove the coefficientwise total positivity of a rather general class of quadridiagonal matrices; we think that this result, and its method of proof, may be of some independent interest. In Section 7 we explain the unexpected connection with multiple orthogonal polynomials; in particular, we explain how we were led to guess the production matrices (1.41) and (1.43).

In Appendix A we review the theory of classical and branched S-fractions, and then introduce the generalized  $m$ -Stieltjes–Rogers polynomials and modified  $m$ -Stieltjes–Rogers polynomials of type  $j \geq 0$ ; this expands and supersedes some of the discussion in [59, sections 5, 7 and 9]. We also apply this theory to the univariate Laguerre production matrix (1.41). In Appendix B we answer the question: Which lower-Hessenberg matrices  $P$  have the property that  $B_\xi^{-1}PB_\xi$  is  $(r, 1)$ -banded? In Appendix C we prove the total positivity of a class of quadridiagonal matrices, giving a variant of what was done in Section 6.2.

The first proof of the production matrices in this paper was found in 2019 by one of us (M.P.) using a bijection from Laguerre digraphs to labeled 2-Łukasiewicz paths. We hope to revisit this powerful method in a subsequent paper.

## 2 Preliminaries

In this section we review some definitions and results from [13, 58, 59, 67, 70] that will be needed in what follows. After a brief introduction to total positivity in a partially ordered commutative ring (Section 2.1), we provide a précis of the theory of production matrices (Section 2.2) and its application to total positivity (Section 2.3); these latter results form the theoretical foundation for our work. Then we introduce the concept of binomial row-generating matrices (Section 2.4). Finally, we review the theory of exponential Riordan arrays (Section 2.5); this theory will be our main technical tool.

For the benefit of readers who are familiar with the papers [13, 58, 59, 67], let us mention two things that are new here (and crucial for the present paper): the tridiagonal comparison theorem (Proposition 2.6), and adding a multiple of the identity to the production matrix (Corollaries 2.12 and 2.16).

### 2.1 Partially ordered commutative rings and total positivity

In this paper all rings will be assumed to have an identity element 1 and to be nontrivial ( $1 \neq 0$ ).

A **partially ordered commutative ring** is a pair  $(R, \mathcal{P})$  where  $R$  is a commutative ring and  $\mathcal{P}$  is a subset of  $R$  satisfying

- (a)  $0, 1 \in \mathcal{P}$ .
- (b) If  $a, b \in \mathcal{P}$ , then  $a + b \in \mathcal{P}$  and  $ab \in \mathcal{P}$ .
- (c)  $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$ .

We call  $\mathcal{P}$  the *nonnegative elements* of  $R$ , and we define a partial order on  $R$  (compatible with the ring structure) by writing  $a \leq b$  as a synonym for  $b - a \in \mathcal{P}$ . Please note that, unlike the practice in real algebraic geometry [9, 46, 51, 61], we do *not* assume here that squares are nonnegative; indeed, this property fails completely for our

prototypical example, the ring of polynomials with the coefficientwise order, since  $(1-x)^2 = 1 - 2x + x^2$  is not coefficientwise nonnegative.

Now let  $(R, \mathcal{P})$  be a partially ordered commutative ring and let  $\mathbf{x} = \{x_i\}_{i \in I}$  be a collection of indeterminates. In the polynomial ring  $R[\mathbf{x}]$  and the formal-power-series ring  $R[[\mathbf{x}]]$ , let  $\mathcal{P}[\mathbf{x}]$  and  $\mathcal{P}[[\mathbf{x}]]$  be the subsets consisting of polynomials (resp. series) with nonnegative coefficients. Then  $(R[\mathbf{x}], \mathcal{P}[\mathbf{x}])$  and  $(R[[\mathbf{x}]], \mathcal{P}[[\mathbf{x}]])$  are partially ordered commutative rings; we refer to this as the *coefficientwise order* on  $R[\mathbf{x}]$  and  $R[[\mathbf{x}]]$ .

A finite or infinite matrix with entries in a partially ordered commutative ring is called *totally positive* (TP) if all its minors are nonnegative; it is called *totally positive of order  $r$*  ( $\text{TP}_r$ ) if all its minors of size  $\leq r$  are nonnegative. It follows immediately from the Cauchy–Binet formula that the product of two TP (resp.  $\text{TP}_r$ ) matrices is TP (resp.  $\text{TP}_r$ ).<sup>10</sup> This fact is so fundamental to the theory of total positivity that we shall henceforth use it without comment.

We say that a sequence  $\mathbf{a} = (a_n)_{n \geq 0}$  with entries in a partially ordered commutative ring is *Hankel-totally positive* (resp. *Hankel-totally positive of order  $r$* ) if its associated infinite Hankel matrix  $H_\infty(\mathbf{a}) = (a_{i+j})_{i,j \geq 0}$  is TP (resp.  $\text{TP}_r$ ). We say that  $\mathbf{a}$  is *Toeplitz-totally positive* (resp. *Toeplitz-totally positive of order  $r$* ) if its associated infinite Toeplitz matrix  $T_\infty(\mathbf{a}) = (a_{i-j})_{i,j \geq 0}$  (where  $a_n \stackrel{\text{def}}{=} 0$  for  $n < 0$ ) is TP (resp.  $\text{TP}_r$ ).<sup>11</sup>

We will need a few easy facts about the total positivity of special matrices:

**Lemma 2.1** (Bidiagonal matrices). *Let  $A$  be a matrix with entries in a partially ordered commutative ring, with the property that all its nonzero entries belong to two consecutive diagonals. Then  $A$  is totally positive if and only if all its entries are nonnegative.*

PROOF. The nonnegativity of the entries (i.e.  $\text{TP}_1$ ) is obviously a necessary condition for TP. Conversely, for a matrix of this type it is easy to see that every nonzero minor is simply a product of some entries.  $\square$

**Lemma 2.2** (Binomial matrix). *In the ring  $\mathbb{Z}$ , the binomial matrix  $B = \left(\binom{n}{k}\right)_{n,k \geq 0}$  is totally positive. More generally, the weighted binomial matrix*

$$B_{x,y} = \left(x^{n-k}y^k \binom{n}{k}\right)_{n,k \geq 0} \tag{2.1}$$

*is totally positive in the ring  $\mathbb{Z}[x, y]$  equipped with the coefficientwise order.*

We also write  $B_x$  as a shorthand for  $B_{x,1}$ .

PROOF OF LEMMA 2.2. It is well known that the binomial matrix  $B$  is totally positive, and this can be proven by a variety of methods: e.g. using production

---

<sup>10</sup>For infinite matrices, we need some condition to ensure that the product is well-defined. For instance, the product  $AB$  is well-defined whenever  $A$  is row-finite (i.e. has only finitely many nonzero entries in each row) or  $B$  is column-finite.

<sup>11</sup>When  $R = \mathbb{R}$ , Toeplitz-totally positive sequences are traditionally called *Pólya frequency sequences* (PF), and Toeplitz-totally positive sequences of order  $r$  are called *Pólya frequency sequences of order  $r$*  ( $\text{PF}_r$ ). See [42, chapter 8] for a detailed treatment.

matrices [42, pp. 136–137, Example 6.1] [60, pp. 108–109], by diagonal similarity to a totally positive Toeplitz matrix [60, p. 109], by exponentiation of a nonnegative lower-subdiagonal matrix [28, p. 63], or by an application of the Lindström–Gessel–Viennot lemma [33, p. 24].

Then  $B_{x,y} = DBD'$  where  $D = \text{diag}((x^n)_{n \geq 0})$  and  $D' = \text{diag}((x^{-k}y^k)_{k \geq 0})$ . By Cauchy–Binet,  $B_{x,y}$  is totally positive in the ring  $\mathbb{Z}[x, x^{-1}, y]$  equipped with the coefficientwise order. But because  $B$  is lower-triangular, the elements of  $B_{x,y}$  actually lie in the subring  $\mathbb{Z}[x, y]$ .  $\square$

See also Example 2.10 below for an *ab initio* proof of Lemma 2.2 using production matrices.

**Lemma 2.3** (Introducing indeterminates). *Let  $A = (a_{ij})_{i,j \geq 0}$  be a lower-triangular matrix with entries in a partially ordered commutative ring  $R$ , and let  $\mathbf{x} = (x_i)_{i \geq 1}$ . Define the lower-triangular matrix  $B = (b_{ij})_{i,j \geq 0}$  by*

$$b_{ij} = x_{j+1}x_{j+2} \cdots x_i a_{ij} . \quad (2.2)$$

*Then:*

- (a) *If  $A$  is  $TP_r$  and  $\mathbf{x}$  are indeterminates, then  $B$  is  $TP_r$  in the ring  $R[\mathbf{x}]$  equipped with the coefficientwise order.*
- (b) *If  $A$  is  $TP_r$  and  $\mathbf{x}$  are nonnegative elements of  $R$ , then  $B$  is  $TP_r$  in the ring  $R$ .*

PROOF. (a) Let  $\mathbf{x} = (x_i)_{i \geq 1}$  be commuting indeterminates, and let us work in the ring  $R[\mathbf{x}, \mathbf{x}^{-1}]$  equipped with the coefficientwise order. Let  $D = \text{diag}(1, x_1, x_1x_2, \dots)$ . Then  $D$  is invertible, and  $D^{-1} = \text{diag}(1, x_1^{-1}, x_1^{-1}x_2^{-1}, \dots)$  has nonnegative elements. It follows that  $B = DAD^{-1}$  is  $TP_r$  in the ring  $R[\mathbf{x}, \mathbf{x}^{-1}]$  equipped with the coefficientwise order. But the matrix elements  $b_{ij}$  actually belong to the subring  $R[\mathbf{x}] \subseteq R[\mathbf{x}, \mathbf{x}^{-1}]$ . So  $B$  is  $TP_r$  in the ring  $R[\mathbf{x}]$  equipped with the coefficientwise order.

(b) follows from (a) by specializing indeterminates.  $\square$

Finally, we need some special facts about the total positivity of tridiagonal matrices. We recall that a *contiguous principal minor* of a matrix  $A$  is a minor  $\det A_{II}$  for  $I = \{r, r+1, r+2, \dots, s\}$  for some  $r \leq s$ .

**Lemma 2.4.** *Every nonzero  $k \times k$  minor of a tridiagonal matrix is a product of off-diagonal elements and contiguous principal minors, with sizes adding up to  $k$ .*

The standard proof for real matrices [60, p. 98] is valid in any commutative ring.

**Corollary 2.5** (Total positivity of tridiagonal matrices). *Let  $A$  be a tridiagonal matrix with entries in a partially ordered commutative ring. Then  $A$  is totally positive of order  $r$  if and only if all its off-diagonal elements and all its contiguous principal minors of size  $\leq r$  are nonnegative.*

In other words, the well-known [60, Theorem 4.3] criterion for total positivity of tridiagonal matrices with real entries extends without change to matrices with entries in an arbitrary partially ordered commutative ring.

This has the following important consequence:

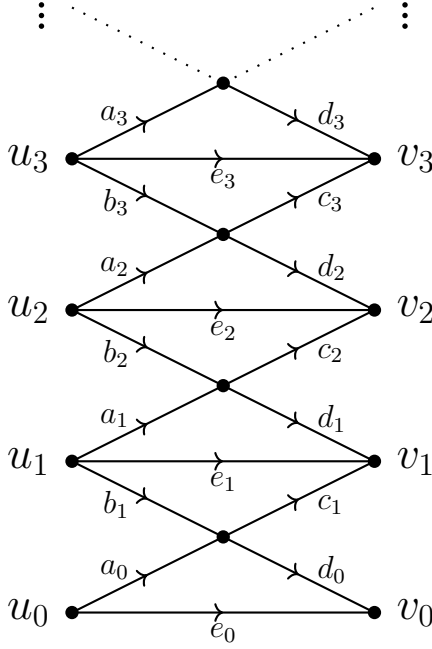


Figure 2: Digraph representation of the matrix  $LU+D$  where  $L$  is the lower-bidiagonal matrix with  $a_0, a_1, \dots$  on the diagonal and  $b_1, b_2, \dots$  on the subdiagonal,  $U$  is the upper-bidiagonal matrix with  $c_1, c_2, \dots$  on the superdiagonal and  $d_0, d_1, \dots$  on the diagonal, and  $D$  is a diagonal matrix with entries  $e_0, e_1, \dots$ .

**Proposition 2.6** (Tridiagonal comparison theorem, weak form). *Let  $A$  and  $D$  be matrices with entries in a partially ordered commutative ring, with  $A$  being tridiagonal and  $D$  being diagonal. If  $A$  is totally positive of order  $r$  and  $D$  is nonnegative, then  $A + D$  is totally positive of order  $r$ .*

PROOF. We apply Corollary 2.5. The off-diagonal elements of  $A + D$  are the same as those of  $A$ . The principal minors of  $A + D$  are

$$\det (A + D)_{II} = \sum_{J \subseteq I} (\det A_{JJ}) \prod_{i \in I \setminus J} d_i, \quad (2.3)$$

where  $d_i$  are the diagonal elements of  $D$ .  $\square$

**Remarks.** 1. There is a stronger form of the tridiagonal comparison theorem [70] [91, Proposition 3.1] in which we can increase the diagonal elements *and* decrease the off-diagonal elements (while keeping them nonnegative). But we will not need this stronger result here.

2. In the special case where  $A = LU$  with  $L$  nonnegative lower-bidiagonal and  $U$  nonnegative upper-bidiagonal, Proposition 2.6 can be proven by a Lindström–Gessel–Viennot argument [2, Chapter 32] using the digraph shown in Figure 2.  $\blacksquare$

## 2.2 Production matrices

The method of production matrices [22, 23] has become in recent years an important tool in enumerative combinatorics. In the special case of a tridiagonal production matrix, this construction goes back to Stieltjes' [74, 75] work on continued fractions: the production matrix of a classical S-fraction or J-fraction is tridiagonal. In the present paper, by contrast, we shall need production matrices that are lower-Hessenberg (i.e. vanish above the first superdiagonal) but are not in general tridiagonal. We therefore begin by reviewing briefly the basic theory of production matrices. The important connection of production matrices with total positivity will be treated in the next subsection.

Let  $P = (p_{ij})_{i,j \geq 0}$  be an infinite matrix with entries in a commutative ring  $R$ . In order that powers of  $P$  be well-defined, we shall assume that  $P$  is either row-finite (i.e. has only finitely many nonzero entries in each row) or column-finite.

Let us now define an infinite matrix  $A = (a_{nk})_{n,k \geq 0}$  by

$$a_{nk} = (P^n)_{0k} \quad (2.4)$$

(in particular,  $a_{0k} = \delta_{0k}$ ). Writing out the matrix multiplications explicitly, we have

$$a_{nk} = \sum_{i_1, \dots, i_{n-1}} p_{0i_1} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-2} i_{n-1}} p_{i_{n-1} k}, \quad (2.5)$$

so that  $a_{nk}$  is the total weight for all  $n$ -step walks in  $\mathbb{N}$  from  $i_0 = 0$  to  $i_n = k$ , in which the weight of a walk is the product of the weights of its steps, and a step from  $i$  to  $j$  gets a weight  $p_{ij}$ . Yet another equivalent formulation is to define the entries  $a_{nk}$  by the recurrence

$$a_{nk} = \sum_{i=0}^{\infty} a_{n-1,i} p_{ik} \quad \text{for } n \geq 1 \quad (2.6)$$

with the initial condition  $a_{0k} = \delta_{0k}$ .

We call  $P$  the **production matrix** and  $A$  the **output matrix**, and we write  $A = \mathcal{O}(P)$ . Note that if  $P$  is row-finite, then so is  $\mathcal{O}(P)$ ; if  $P$  is lower-Hessenberg, then  $\mathcal{O}(P)$  is lower-triangular; if  $P$  is lower-Hessenberg with invertible superdiagonal entries, then  $\mathcal{O}(P)$  is lower-triangular with invertible diagonal entries; and if  $P$  is unit-lower-Hessenberg (i.e. lower-Hessenberg with entries 1 on the superdiagonal), then  $\mathcal{O}(P)$  is unit-lower-triangular. In all the applications in this paper,  $P$  will be lower-Hessenberg, most often unit-lower-Hessenberg.

The matrix  $P$  can also be interpreted as the adjacency matrix for a weighted directed graph on the vertex set  $\mathbb{N}$  (where the edge  $i \rightarrow j$  is omitted whenever  $p_{ij} = 0$ ). Then  $P$  is row-finite (resp. column-finite) if and only if every vertex has finite out-degree (resp. finite in-degree); and  $P$  is lower-Hessenberg if and only if all edges  $i \rightarrow j$  satisfy  $j \leq i + 1$ .

This iteration process can be given a compact matrix formulation. Let us define the **augmented production matrix**

$$\tilde{P} \stackrel{\text{def}}{=} \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 & \cdots \\ & P & & & \end{array} \right], \quad (2.7)$$

or in other words

$$(\tilde{P})_{n,k} = \begin{cases} \delta_{0k} & \text{if } n = 0 \\ p_{n-1,k} & \text{if } n \geq 1 \end{cases} \quad (2.8)$$

Then the recurrence (2.6) together with the initial condition  $a_{0k} = \delta_{0k}$  can be written as

$$A = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & \cdots \\ \hline & & & & \end{array} \right]_{AP} = \left[ \begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & A \end{array} \right] \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & \cdots \\ \hline & & & & \end{array} \right]_P = \left[ \begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & A \end{array} \right] \tilde{P}. \quad (2.9)$$

This identity can be iterated to give the factorization

$$A = \cdots \left[ \begin{array}{c|c} I_3 & \mathbf{0} \\ \hline \mathbf{0} & \tilde{P} \end{array} \right] \left[ \begin{array}{c|c} I_2 & \mathbf{0} \\ \hline \mathbf{0} & \tilde{P} \end{array} \right] \left[ \begin{array}{c|c} I_1 & \mathbf{0} \\ \hline \mathbf{0} & \tilde{P} \end{array} \right] \tilde{P} \quad (2.10)$$

where  $I_k$  is the  $k \times k$  identity matrix; and conversely, (2.10) implies (2.9).

Now let  $\Delta = (\delta_{i+1,j})_{i,j \geq 0}$  be the matrix with 1 on the superdiagonal and 0 elsewhere. Then for any matrix  $M$  with rows indexed by  $\mathbb{N}$ , the product  $\Delta M$  is simply  $M$  with its zeroth row removed and all other rows shifted upwards. (Some authors use the notation  $\overline{M} \stackrel{\text{def}}{=} \Delta M$ .) The recurrence (2.6) can then be written as

$$\Delta \mathcal{O}(P) = \mathcal{O}(P) P. \quad (2.11)$$

It follows that if  $A$  is a row-finite matrix that has a row-finite inverse  $A^{-1}$  and has first row  $a_{0k} = \delta_{0k}$ , then  $P = A^{-1} \Delta A$  is the unique matrix such that  $A = \mathcal{O}(P)$ . This holds, in particular, if  $A$  is lower-triangular with invertible diagonal entries and  $a_{00} = 1$ ; then  $A^{-1}$  is lower-triangular and  $P = A^{-1} \Delta A$  is lower-Hessenberg. And if  $A$  is unit-lower-triangular, then  $P = A^{-1} \Delta A$  is unit-lower-Hessenberg.

Later we shall need the following easy but fundamental fact, which shows how the production matrix transforms when the output matrix  $A$  is *right*-multiplied by another matrix  $B$ :

**Lemma 2.7** (Right-multiplication lemma). *Let  $P = (p_{ij})_{i,j \geq 0}$  be a row-finite matrix (with entries in a commutative ring  $R$ ), with output matrix  $A = \mathcal{O}(P)$ ; and let  $B = (b_{ij})_{i,j \geq 0}$  be a lower-triangular matrix with invertible (in  $R$ ) diagonal entries. Then*

$$AB = b_{00} \mathcal{O}(B^{-1}PB). \quad (2.12)$$

*That is, up to a factor  $b_{00}$ , the matrix  $AB$  has production matrix  $B^{-1}PB$ .*

**PROOF.** Since  $P$  is row-finite, so is  $A = \mathcal{O}(P)$ ; then the matrix products  $AB$  and  $B^{-1}PB$  arising in the lemma are well-defined. Now

$$a_{nk} = \sum_{i_1, \dots, i_{n-1}} p_{0i_1} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-2} i_{n-1}} p_{i_{n-1} k}, \quad (2.13)$$

while

$$\mathcal{O}(B^{-1}PB)_{nk} = \sum_{j, i_1, \dots, i_{n-1}, i_n} (B^{-1})_{0j} p_{j i_1} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-2} i_{n-1}} p_{i_{n-1} i_n} b_{i_n k}. \quad (2.14)$$

But  $B$  is lower-triangular with invertible diagonal entries, so  $B$  is invertible and  $B^{-1}$  is lower-triangular, with  $(B^{-1})_{0j} = b_{00}^{-1} \delta_{j0}$ . It follows that  $AB = b_{00} \mathcal{O}(B^{-1}PB)$ .  $\square$

**Remark.** If  $b_{00} = 1$ , then  $(AB)_{00} = 1$  and hence  $AB$  can be the output matrix of some production matrix; and in this case (2.12) is an immediate consequence of (2.11). Indeed, right-multiplying (2.11) by  $B$  yields

$$\Delta(\mathcal{O}(P)B) = \mathcal{O}(P)PB = (\mathcal{O}(P)B)B^{-1}PB. \quad (2.15)$$

■

We will frequently apply this lemma with  $B$  taken to be the binomial matrix  $B_x$ . Unfortunately we know very little about what happens to the production matrix when the output matrix is *left*-multiplied by another matrix  $B$ . But we do know one special case: see Lemma 2.11 below.

## 2.3 Production matrices and total positivity

Let  $P = (p_{ij})_{i,j \geq 0}$  be a matrix with entries in a partially ordered commutative ring  $R$ . We will use  $P$  as a production matrix; let  $A = \mathcal{O}(P)$  be the corresponding output matrix. As before, we assume that  $P$  is either row-finite or column-finite.

When  $P$  is totally positive, it turns out [70] that the output matrix  $\mathcal{O}(P)$  has *two* total-positivity properties: firstly, it is totally positive; and secondly, its zeroth column is Hankel-totally positive. More generally, the same properties hold whenever  $P = P_0 + aI$ , where  $P_0$  is totally positive and  $a$  is nonnegative. Since [70] is not yet publicly available, we shall present briefly here (with proof) the main results that will be needed in the sequel.

The fundamental fact that drives the whole theory is the following:

**Proposition 2.8** (Minors of the output matrix). *Every  $k \times k$  minor of the output matrix  $A = \mathcal{O}(P)$  can be written as a sum of products of minors of size  $\leq k$  of the production matrix  $P$ .*

In this proposition the matrix elements  $\mathbf{p} = \{p_{ij}\}_{i,j \geq 0}$  should be interpreted in the first instance as indeterminates: for instance, we can fix a row-finite or column-finite set  $S \subseteq \mathbb{N} \times \mathbb{N}$  and define the matrix  $P^S = (p_{ij}^S)_{i,j \in \mathbb{N}}$  with entries

$$p_{ij}^S = \begin{cases} p_{ij} & \text{if } (i, j) \in S \\ 0 & \text{if } (i, j) \notin S \end{cases} \quad (2.16)$$

Then the entries (and hence also the minors) of both  $P$  and  $A$  belong to the polynomial ring  $\mathbb{Z}[\mathbf{p}]$ , and the assertion of Proposition 2.8 makes sense. Of course, we can subsequently specialize the indeterminates  $\mathbf{p}$  to values in any commutative ring  $R$ .

**PROOF OF PROPOSITION 2.8.** For any infinite matrix  $X = (x_{ij})_{i,j \geq 0}$ , let us write  $X_N = (x_{ij})_{0 \leq i \leq N-1, j \geq 0}$  for the submatrix consisting of the first  $N$  rows (and *all* the columns) of  $X$ . Every  $k \times k$  minor of  $A$  is of course a  $k \times k$  minor of  $A_N$  for some  $N$ , so it suffices to prove that the claim about minors holds for all the  $A_N$ . But this is easy: the fundamental identity (2.9) implies

$$A_N = \left[ \begin{array}{c|c} 1 & \mathbf{0} \\ \mathbf{0} & A_{N-1} \end{array} \right] \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & \cdots \\ \hline & & & & P \end{array} \right]. \quad (2.17)$$



So the result follows by induction on  $N$ , using the Cauchy–Binet formula.  $\square$

If we now specialize the indeterminates  $\mathbf{p}$  to values in some partially ordered commutative ring  $R$ , we can immediately conclude:

**Theorem 2.9** (Total positivity of the output matrix). *Let  $P$  be an infinite matrix that is either row-finite or column-finite, with entries in a partially ordered commutative ring  $R$ . If  $P$  is totally positive of order  $r$ , then so is  $A = \mathcal{O}(P)$ .*

**Remarks.** 1. In the case  $R = \mathbb{R}$ , Theorem 2.9 is due to Karlin [42, pp. 132–134]; see also [60, Theorem 1.11]. Karlin’s proof is different from ours.

2. Our quick inductive proof of Proposition 2.8 follows an idea of Zhu [86, proof of Theorem 2.1], which was in turn inspired in part by Aigner [1, pp. 45–46]. The same idea recurs in recent work of several authors [87, Theorem 2.1] [11, Theorem 2.1(i)] [12, Theorem 2.3(i)] [47, Theorem 2.1] [14, Theorems 2.1 and 2.3] [37]. However, all of these results concerned only special cases: [1, 12, 47, 86] treated the case in which the production matrix  $P$  is tridiagonal; [87] treated a (special) case in which  $P$  is upper-bidiagonal; [11] treated the case in which  $P$  is the production matrix of a Riordan array; [14, 37] treated (implicitly) the case in which  $P$  is upper-triangular and Toeplitz. But the argument is in fact completely general, as we have just seen; there is no need to assume any special form for the matrix  $P$ .

3. A slightly different version of this proof was presented in [58, 59]. The simplified reformulation given here, using the augmented production matrix, is due to Mu and Wang [52].  $\blacksquare$

**Example 2.10** (Binomial matrix). Let  $P$  be the upper-bidiagonal Toeplitz matrix  $xI + y\Delta$ , where  $x$  and  $y$  are indeterminates. By Lemma 2.1,  $P$  is TP in the ring  $\mathbb{Z}[x, y]$  equipped with the coefficientwise order. An easy computation shows that  $\mathcal{O}(xI + y\Delta) = B_{x,y}$ , the weighted binomial matrix with entries  $(B_{x,y})_{nk} = x^{n-k}y^k \binom{n}{k}$ . So Theorem 2.9 implies that  $B_{x,y}$  is TP in the ring  $\mathbb{Z}[x, y]$  equipped with the coefficientwise order. This gives an *ab initio* proof of Lemma 2.2.  $\blacksquare$

More generally, we have:

**Lemma 2.11.**  $\mathcal{O}(aI + bP) = B_{a,b} \mathcal{O}(P)$ .

Note that Example 2.10 is the special case  $P = \Delta$ .

PROOF OF LEMMA 2.11. Since

$$(aI + bP)^n = \sum_{j=0}^n a^{n-j} b^j \binom{n}{j} P^j, \quad (2.18)$$

we have

$$\mathcal{O}(aI + bP)_{nk} \stackrel{\text{def}}{=} [(aI + bP)^n]_{0k} = \sum_{j=0}^n a^{n-j} b^j \binom{n}{j} (P^j)_{0k} \quad (2.19a)$$

$$= \sum_{j=0}^n (B_{a,b})_{nj} \mathcal{O}(P)_{jk} \quad (2.19b)$$

$$= [B_{a,b} \mathcal{O}(P)]_{nk}. \quad (2.19c)$$

□

Since the binomial matrix  $B_{a,b}$  is totally positive by Lemma 2.2, we conclude:

**Corollary 2.12** (Total positivity of output matrix, improved). *Let  $P$  be an infinite matrix that is either row-finite or column-finite, with entries in a partially ordered commutative ring  $R$ . Then  $P$  is  $TP_r \implies \mathcal{O}(P)$  is  $TP_r \iff \mathcal{O}(aI + bP)$  is  $TP_r$  for all  $a, b \geq 0$ .*

Here  $a$  and  $b$  can be nonnegative elements of  $R$ ; or  $a$  and  $b$  can be indeterminates and we work in the ring  $R[a, b]$  equipped with the coefficientwise order. Corollary 2.12 allows us to prove the total positivity of the output matrix  $\mathcal{O}(aI + bP)$  whenever  $P$  is TP, *even if  $aI + bP$  is not TP*. This will play a crucial role in our proof of Theorem 1.4, by allowing us to take  $\lambda_{y_{fp}} \geq \lambda_{y_p}$  rather than just  $\lambda_{y_{fp}} = \lambda_{y_p}$ .

**Remark.** Lemma 2.11 and Corollary 2.12 are a special case of an idea of Zhu [88]; see [70] for a general version. ■

Now define  $\mathcal{o}_0(P)$  to be the zeroth-column sequence of  $\mathcal{O}(P)$ , i.e.

$$\mathcal{o}_0(P)_n \stackrel{\text{def}}{=} \mathcal{O}(P)_{n0} \stackrel{\text{def}}{=} (P^n)_{00}. \quad (2.20)$$

Then the Hankel matrix of  $\mathcal{o}_0(P)$  has matrix elements

$$\begin{aligned} H_\infty(\mathcal{o}_0(P))_{nn'} &= \mathcal{o}_0(P)_{n+n'} = (P^{n+n'})_{00} = \sum_{k=0}^{\infty} (P^n)_{0k} (P^{n'})_{k0} = \\ &= \sum_{k=0}^{\infty} (P^n)_{0k} ((P^T)^{n'})_{0k} = \sum_{k=0}^{\infty} \mathcal{O}(P)_{nk} \mathcal{O}(P^T)_{n'k} = [\mathcal{O}(P) \mathcal{O}(P^T)^T]_{nn'}. \end{aligned} \quad (2.21)$$

(Note that the sum over  $k$  has only finitely many nonzero terms: if  $P$  is row-finite, then there are finitely many nonzero  $(P^n)_{0k}$ , while if  $P$  is column-finite, there are finitely many nonzero  $(P^{n'})_{k0}$ .<sup>12</sup>) We have therefore proven:

**Lemma 2.13** (Identity for Hankel matrix of the zeroth column). *Let  $P$  be a row-finite or column-finite matrix with entries in a commutative ring  $R$ . Then*

$$H_\infty(\mathcal{o}_0(P)) = \mathcal{O}(P) \mathcal{O}(P^T)^T. \quad (2.22)$$

Combining Proposition 2.8 with Lemma 2.13 and the Cauchy–Binet formula, we obtain:

**Corollary 2.14** (Hankel minors of the zeroth column). *Every  $k \times k$  minor of the infinite Hankel matrix  $H_\infty(\mathcal{o}_0(P)) = ((P^{n+n'})_{00})_{n,n' \geq 0}$  can be written as a sum of products of the minors of size  $\leq k$  of the production matrix  $P$ .*

---

<sup>12</sup>Or to put it another way: If  $P$  is row-finite, then  $\mathcal{O}(P)$  is row-finite;  $\mathcal{O}(P^T)$  need not be row- or column-finite, but the product  $\mathcal{O}(P) \mathcal{O}(P^T)^T$  is anyway well-defined. Likewise, if  $P$  is column-finite, then  $\mathcal{O}(P^T)^T$  is column-finite;  $\mathcal{O}(P)$  need not be row- or column-finite, but the product  $\mathcal{O}(P) \mathcal{O}(P^T)^T$  is anyway well-defined.

And specializing the indeterminates  $\mathbf{p}$  to nonnegative elements in a partially ordered commutative ring, in such a way that  $P$  is row-finite or column-finite, we deduce:

**Theorem 2.15** (Hankel-total positivity of the zeroth column). *Let  $P = (p_{ij})_{i,j \geq 0}$  be an infinite row-finite or column-finite matrix with entries in a partially ordered commutative ring  $R$ , and define the infinite Hankel matrix  $H_\infty(\mathcal{O}_0(P)) = ((P^{n+n'})_{00})_{n,n' \geq 0}$ . If  $P$  is totally positive of order  $r$ , then so is  $H_\infty(\mathcal{O}_0(P))$ .*

Once again we can we can improve these results to replace  $P$  by  $aI + bP$ . It suffices to note that

$$H_\infty(\mathcal{O}_0(aI + bP)) = \mathcal{O}(aI + bP) \mathcal{O}(aI + bP^\top)^\top \quad (2.23a)$$

$$= B_{a,b} \mathcal{O}(P) \mathcal{O}(P^\top)^\top (B_{a,b})^\top \quad (2.23b)$$

$$= B_{a,b} H_\infty(\mathcal{O}_0(P)) (B_{a,b})^\top \quad (2.23c)$$

by Lemmas 2.11 and 2.13. Since the binomial matrix  $B_{a,b}$  is totally positive by Lemma 2.2, we conclude:

**Corollary 2.16** (Hankel-total positivity of zeroth column, improved). *Let  $P$  be an infinite matrix that is either row-finite or column-finite, with entries in a partially ordered commutative ring  $R$ . Then  $P$  is  $TP_r \implies \mathcal{O}(P)$  and  $\mathcal{O}(P^\top) \implies TP_r \implies \mathcal{O}_0(P)$  is Hankel- $TP_r \iff \mathcal{O}_0(aI + bP)$  is Hankel- $TP_r$  for all  $a, b \geq 0$ .*

Once again,  $a$  and  $b$  can be nonnegative elements of  $R$ , or  $a$  and  $b$  can be indeterminates and we work in the ring  $R[a, b]$  equipped with the coefficientwise order. Corollary 2.16 allows us to prove the Hankel-total positivity of the output sequence  $\mathcal{O}_0(aI + bP)$  whenever  $P$  is TP, even if  $aI + bP$  is not TP. It will play an important role in our proof of Theorem 1.4.

**Remark.** We see from (2.19b) specialized to  $k = 0$  that  $\mathcal{O}_0(aI + bP)$  is a binomial transform of  $\mathcal{O}_0(P)$ :

$$\mathcal{O}_0(aI + bP)_n = \sum_{j=0}^n a^{n-j} b^j \binom{n}{j} \mathcal{O}_0(P)_j. \quad (2.24)$$

Now, it is known [70, 89] that the binomial transform preserves Hankel- $TP_r$  of arbitrary sequences in a partially ordered commutative ring. So Corollary 2.16 is just a special case of this general result combined with Theorem 2.15. ■

## 2.4 Binomial row-generating matrices

Let  $A = (a_{nk})_{n,k \geq 0}$  be a row-finite matrix with entries in a commutative ring  $R$ . (In most applications, including all those in the present paper, the matrix  $A$  will be lower-triangular.) We define its **row-generating polynomials** in the usual way:

$$A_n(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a_{nk} x^k, \quad (2.25)$$

where the sum is actually finite because  $A$  is row-finite. More generally, let us define its **binomial partial row-generating polynomials**

$$A_{n,k}(x) \stackrel{\text{def}}{=} \sum_{\ell=k}^{\infty} a_{n\ell} \binom{\ell}{k} x^{\ell-k} \quad (2.26a)$$

$$= \frac{1}{k!} \frac{d^k}{dx^k} A_n(x). \quad (2.26b)$$

(Note that the operator  $(1/k!)d^k/dx^k$  has a well-defined action on the polynomial ring  $R[x]$  even if  $R$  does not contain the rationals, since  $(1/k!)(d^k/dx^k)x^n = \binom{n}{k}x^{n-k}$ .) The polynomials  $A_{n,k}(x)$  are the matrix elements of the **binomial row-generating matrix**  $AB_x$ :

$$(AB_x)_{nk} = A_{n,k}(x), \quad (2.27)$$

where  $B_x = B_{x,1}$  is the weighted binomial matrix defined in (1.9). The zeroth column of the matrix  $AB_x$  consists of the row-generating polynomials  $A_n(x) = A_{n,0}(x)$ .

In this paper the matrix  $A$  will be either the Laguerre coefficient matrix  $L^{(\alpha)}$  or one of its multivariate generalizations.

We can now explain the method that we will use to prove Theorems 1.1 and 1.4:

**Proposition 2.17.** *Let  $P$  be a row-finite matrix with entries in a partially ordered commutative ring  $R$ , and let  $A = \mathcal{O}(P)$ .*

- (a) *If  $P$  is totally positive of order  $r$ , then so is  $A$ .*
- (b) *If the matrix  $B_x^{-1}PB_x$  is totally positive of order  $r$  in the ring  $R[x]$  equipped with the coefficientwise order, then the sequence  $(A_n(x))_{n \geq 0}$  of row-generating polynomials is Hankel-totally positive of order  $r$  in the ring  $R[x]$  equipped with the coefficientwise order.*

Indeed, (a) is just a restatement of Theorem 2.9; and (b) is an immediate consequence of Lemma 2.7 and Theorem 2.15 together with the fact that the zeroth column of the matrix  $AB_x$  consists of the row-generating polynomials  $A_n(x)$ .

**Remark.** The binomial row-generating matrix  $AB_x$  can also be considered as a modified Wronskian matrix. To see this, let  $N \in \mathbb{N} \cup \{\infty\}$ , and fix formal power series  $f_1(x), \dots, f_N(x) \in R[[x]]$ . We define the  $N \times \infty$  **Wronskian matrix**

$$W(f_1, \dots, f_N)(x) = (f_n^{(k)}(x))_{1 \leq n \leq N, k \geq 0} \quad (2.28)$$

(where  $(^{(k)})$  denotes the  $k$ th derivative) and the  $N \times \infty$  **modified Wronskian matrix**

$$\widetilde{W}(f_1, \dots, f_N)(x) = (f_n^{(k)}(x)/k!)_{1 \leq n \leq N, k \geq 0}; \quad (2.29)$$

these are matrices with entries in  $R[[x]]$ . Of course, we have

$$W(f_1, \dots, f_N)(x) = \widetilde{W}(f_1, \dots, f_N)(x) D \quad (2.30)$$

where  $D = \text{diag}((k!)_{k \geq 0})$ . The key fact is that

$$\widetilde{W}(f_1, \dots, f_N)(x) = \widetilde{W}(f_1, \dots, f_N)(0) B_x. \quad (2.31)$$

In particular, if the ring  $R$  carries a partial order, the total positivity of order  $r$  of the matrix  $\widetilde{W}(f_1, \dots, f_N)(0)$  in the ring  $R$  implies the total positivity of order  $r$  of the matrix  $\widetilde{W}(f_1, \dots, f_N)(x)$  in the ring  $R[[x]]$  equipped with the coefficientwise order. ■

## 2.5 Exponential Riordan arrays

Let  $R$  be a commutative ring containing the rationals, and let  $F(t) = \sum_{n=0}^{\infty} f_n t^n / n!$  and  $G(t) = \sum_{n=1}^{\infty} g_n t^n / n!$  be formal power series with coefficients in  $R$ ; we set  $g_0 = 0$ . Then the **exponential Riordan array** [5, 23, 24] associated to the pair  $(F, G)$  — or equivalently to the pair of sequences  $\mathbf{f} = (f_n)_{n \geq 0}$  and  $\mathbf{g} = (g_n)_{n \geq 1}$  — is the infinite lower-triangular matrix  $\mathcal{R}[F, G] = (\mathcal{R}[F, G]_{nk})_{n, k \geq 0}$  defined by

$$\mathcal{R}[F, G]_{nk} = \frac{n!}{k!} [t^n] F(t) G(t)^k. \quad (2.32)$$

That is, the  $k$ th column of  $\mathcal{R}[F, G]$  has exponential generating function  $F(t)G(t)^k/k!$ . It follows that the bivariate exponential generating function of  $\mathcal{R}[F, G]$  is

$$\sum_{n, k=0}^{\infty} \mathcal{R}[F, G]_{nk} \frac{t^n}{n!} u^k = F(t) e^{uG(t)}. \quad (2.33)$$

Please note that the diagonal elements of  $\mathcal{R}[F, G]$  are  $\mathcal{R}[F, G]_{nn} = f_0 g_1^n$ , so the matrix  $\mathcal{R}[F, G]$  is invertible in the ring  $R_{\text{lt}}^{\mathbb{N} \times \mathbb{N}}$  of lower-triangular matrices if and only if  $f_0$  and  $g_1$  are invertible in  $R$ .

We shall use an easy but important result that is sometimes called the *fundamental theorem of exponential Riordan arrays* (FTERA):

**Lemma 2.18** (Fundamental theorem of exponential Riordan arrays). *Let  $\mathbf{b} = (b_n)_{n \geq 0}$  be a sequence with exponential generating function  $B(t) = \sum_{n=0}^{\infty} b_n t^n / n!$ . Considering  $\mathbf{b}$  as a column vector and letting  $\mathcal{R}[F, G]$  act on it by matrix multiplication, we obtain a sequence  $\mathcal{R}[F, G]\mathbf{b}$  whose exponential generating function is  $F(t) B(G(t))$ .*

PROOF. We compute

$$\sum_{k=0}^n \mathcal{R}[F, G]_{nk} b_k = \sum_{k=0}^{\infty} \frac{n!}{k!} [t^n] F(t) G(t)^k b_k \quad (2.34a)$$

$$= n! [t^n] F(t) \sum_{k=0}^{\infty} b_k \frac{G(t)^k}{k!} \quad (2.34b)$$

$$= n! [t^n] F(t) B(G(t)). \quad (2.34c)$$

□

Let us now consider the product of two exponential Riordan arrays  $\mathcal{R}[F_1, G_1]$  and  $\mathcal{R}[F_2, G_2]$ . Applying the FTERA to the  $k$ th column of  $\mathcal{R}[F_2, G_2]$ , whose exponential generating function is  $F_2(t)G_2(t)^k/k!$ , we readily obtain:

**Lemma 2.19** (Product of two exponential Riordan arrays). *We have*

$$\mathcal{R}[F_1, G_1] \mathcal{R}[F_2, G_2] = \mathcal{R}[(F_2 \circ G_1)F_1, G_2 \circ G_1] \quad (2.35)$$

where  $\circ$  denotes composition of formal power series.

In particular, if we let  $\mathcal{R}[F_2, G_2]$  be the weighted binomial matrix  $B_{\xi} = \mathcal{R}[e^{\xi t}, t]$ , we obtain:

**Corollary 2.20** (Binomial row-generating matrix of an exponential Riordan array).  
We have

$$\mathcal{R}[F, G] B_\xi = \mathcal{R}[e^{\xi G} F, G]. \quad (2.36)$$

We can now determine the production matrix of an exponential Riordan array  $\mathcal{R}[F, G]$ . Let  $\mathbf{a} = (a_n)_{n \geq 0}$  and  $\mathbf{z} = (z_n)_{n \geq 0}$  be sequences in a commutative ring  $R$ , with ordinary generating functions  $A(s) = \sum_{n=0}^{\infty} a_n s^n$  and  $Z(s) = \sum_{n=0}^{\infty} z_n s^n$ . We then define the **exponential AZ matrix** associated to the sequences  $\mathbf{a}$  and  $\mathbf{z}$  to be the lower-Hessenberg matrix with entries

$$\text{EAZ}(\mathbf{a}, \mathbf{z})_{nk} = \frac{n!}{k!} (z_{n-k} + k a_{n-k+1}) \quad (2.37)$$

(where  $z_{-1} \stackrel{\text{def}}{=} 0$ ), or equivalently (if  $R$  contains the rationals)

$$\text{EAZ}(\mathbf{a}, \mathbf{z}) = D T_\infty(\mathbf{z}) D^{-1} + D T_\infty(\mathbf{a}) D^{-1} \Delta \quad (2.38)$$

where  $D = \text{diag}((n!)_{n \geq 0})$ . We also write  $\text{EAZ}(A, Z)$  as a synonym for  $\text{EAZ}(\mathbf{a}, \mathbf{z})$ .

**Theorem 2.21** (Production matrices of exponential Riordan arrays). *Let  $L$  be a lower-triangular matrix (with entries in a commutative ring  $R$  containing the rationals) with invertible diagonal entries and  $L_{00} = 1$ , and let  $P = L^{-1} \Delta L$  be its production matrix. Then  $L$  is an exponential Riordan array if and only if  $P$  is an exponential AZ matrix.*

*More precisely,  $L = \mathcal{R}[F, G]$  if and only if  $P = \text{EAZ}(A, Z)$ , where the generating functions  $(F(t), G(t))$  and  $(A(s), Z(s))$  are connected by*

$$G'(t) = A(G(t)), \quad \frac{F'(t)}{F(t)} = Z(G(t)) \quad (2.39)$$

or equivalently

$$A(s) = G'(\bar{G}(s)), \quad Z(s) = \frac{F'(\bar{G}(s))}{F(\bar{G}(s))} \quad (2.40)$$

where  $\bar{G}(s)$  is the compositional inverse of  $G(t)$ .

PROOF (mostly contained in [5, pp. 217–218]). Suppose that  $L = \mathcal{R}[F, G]$ . The hypotheses on  $L$  imply that  $f_0 = 1$  and that  $g_1$  is invertible in  $R$ ; so  $G(t)$  has a compositional inverse. Now let  $P = (p_{nk})_{n, k \geq 0}$  be a matrix; its column exponential generating functions are, by definition,  $P_k(t) = \sum_{n=0}^{\infty} p_{nk} t^n / n!$ . Applying the FTERA to each column of  $P$ , we see that  $\mathcal{R}[F, G]P$  is a matrix whose column exponential generating functions are  $(F(t) P_k(G(t)))_{k \geq 0}$ . On the other hand,  $\Delta \mathcal{R}[F, G]$  is the matrix  $\mathcal{R}[F, G]$  with its zeroth row removed and all other rows shifted upwards, so it has column exponential generating functions

$$\frac{d}{dt} (F(t) G(t)^k / k!) = \frac{1}{k!} \left[ F'(t) G(t)^k + k F(t) G(t)^{k-1} G'(t) \right]. \quad (2.41)$$

Comparing these two results, we see that  $\Delta \mathcal{R}[F, G] = \mathcal{R}[F, G]P$  if and only if

$$P_k(G(t)) = \frac{1}{k!} \frac{F'(t) G(t)^k + k F(t) G(t)^{k-1} G'(t)}{F(t)}, \quad (2.42)$$

or in other words

$$P_k(t) = \frac{1}{k!} \left[ \frac{F'(\bar{G}(t))}{F(\bar{G}(t))} t^k + k t^{k-1} G'(\bar{G}(t)) \right]. \quad (2.43)$$

Therefore

$$p_{nk} = \frac{n!}{k!} [t^n] \left[ \frac{F'(\bar{G}(t))}{F(\bar{G}(t))} t^k + k t^{k-1} G'(\bar{G}(t)) \right] \quad (2.44a)$$

$$= \frac{n!}{k!} \left[ [t^{n-k}] \frac{F'(\bar{G}(t))}{F(\bar{G}(t))} + k [t^{n-k+1}] G'(\bar{G}(t)) \right] \quad (2.44b)$$

$$= \frac{n!}{k!} (z_{n-k} + k a_{n-k+1}) \quad (2.44c)$$

where  $\mathbf{a} = (a_n)_{n \geq 0}$  and  $\mathbf{z} = (z_n)_{n \geq 0}$  are given by (2.40).

Conversely, suppose that  $P = \text{EAZ}(A, Z)$ . Define  $F(t)$  and  $G(t)$  as the unique solutions (in the formal-power-series ring  $R[[t]]$ ) of the differential equations (2.39) with initial conditions  $F(0) = 1$  and  $G(0) = 0$ . Then running the foregoing computation backwards shows that  $\Delta \mathcal{R}[F, G] = \mathcal{R}[F, G] P$ .  $\square$

A central role will be played later in this paper by a simple but remarkable identity for  $B_x^{-1} \text{EAZ}(\mathbf{a}, \mathbf{z}) B_x$ , where  $B_x$  is the  $x$ -binomial matrix defined in (1.9):

**Lemma 2.22** (Identity for  $B_x^{-1} \text{EAZ}(\mathbf{a}, \mathbf{z}) B_x$ ). *Let  $\mathbf{a} = (a_n)_{n \geq 0}$ ,  $\mathbf{z} = (z_n)_{n \geq 0}$  and  $x$  be indeterminates. Then*

$$B_x^{-1} \text{EAZ}(\mathbf{a}, \mathbf{z}) B_x = \text{EAZ}(\mathbf{a}, \mathbf{z} + x\mathbf{a}) \quad (2.45)$$

as an identity in  $\mathbb{Z}[\mathbf{a}, \mathbf{z}]$ .

The special case  $\mathbf{z} = \mathbf{0}$  of this lemma was proven in [58, Lemma 3.6]; a simpler proof was given in [67, Lemma 2.16]. Here we give the easy generalization to include  $\mathbf{z}$ , taken from [13]:

PROOF. We work temporarily in the ring  $\mathbb{Q}[\mathbf{a}, \mathbf{z}]$  and use the matrix definition (2.38):

$$\text{EAZ}(\mathbf{a}, \mathbf{z}) = D T_\infty(\mathbf{z}) D^{-1} + D T_\infty(\mathbf{a}) D^{-1} \Delta \quad (2.46)$$

where  $D = \text{diag}((n!)_{n \geq 0})$ . Since  $\text{EAZ}(\mathbf{a}, \mathbf{z}) = \text{EAZ}(\mathbf{a}, \mathbf{0}) + \text{EAZ}(\mathbf{0}, \mathbf{z})$ , it suffices to consider separately the two contributions.

The key observation is that  $B_x = D T_\infty((x^n/n!)_{n \geq 0}) D^{-1}$ . Now two Toeplitz matrices always commute:  $T_\infty(\mathbf{a}) T_\infty(\mathbf{b}) = T_\infty(\mathbf{a} \star \mathbf{b}) = T_\infty(\mathbf{b}) T_\infty(\mathbf{a})$ . It follows that  $D T_\infty(\mathbf{z}) D^{-1}$  and  $D T_\infty(\mathbf{a}) D^{-1}$  commute with  $B_x$ . Therefore

$$B_x^{-1} \text{EAZ}(\mathbf{0}, \mathbf{z}) B_x = \text{EAZ}(\mathbf{0}, \mathbf{z}). \quad (2.47)$$

On the other hand, the classic recurrence for binomial coefficients implies

$$\Delta B_x = B_x (xI + \Delta) \quad (2.48)$$

(cf. Example 2.10). Therefore

$$B_x^{-1} \text{EAZ}(\mathbf{a}, \mathbf{0}) B_x = B_x^{-1} DT_\infty(\mathbf{a}) D^{-1} \Delta B_x \quad (2.49a)$$

$$= B_x^{-1} DT_\infty(\mathbf{a}) D^{-1} B_x (xI + \Delta) \quad (2.49b)$$

$$= DT_\infty(\mathbf{a}) D^{-1} (xI + \Delta) \quad (2.49c)$$

$$= \text{EAZ}(\mathbf{a}, x\mathbf{a}) . \quad (2.49d)$$

Adding (2.47) and (2.49) yields (2.45).  $\square$

The identity (2.45) can alternatively be proven by combining Lemma 2.7 and Corollary 2.20 with Theorem 2.21: see [13] for details.

### 3 Production matrices: Univariate case

In this section we will prove Proposition 1.2, which gives the production matrices for the Laguerre coefficient matrix  $L^{(\alpha)}$  [defined in (1.8)] and for the binomial row-generating matrix  $L^{(\alpha)} B_x$ . The proofs are in fact quite easy. We also give a simple direct proof of the coefficientwise total positivity of the Laguerre coefficient matrix  $L^{(-1+\lambda)}$ ,

#### 3.1 Coefficient matrix $L^{(\alpha)}$ : Proof of Proposition 1.2(a)

We begin by proving Proposition 1.2(a), which asserts that the production matrix for the Laguerre coefficient matrix  $L^{(\alpha)}$  is the tridiagonal unit-lower-Hessenberg matrix  $P^\circ = (p_{ij}^\circ)_{i,j \geq 0}$  defined by

$$p_{n,n+1}^\circ = 1 \quad (3.1a)$$

$$p_{n,n}^\circ = 2n + 1 + \alpha \quad (3.1b)$$

$$p_{n,n-1}^\circ = n(n + \alpha) \quad (3.1c)$$

$$p_{n,k}^\circ = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \quad (3.1d)$$

We give two proofs: one by direct computation, and one using the theory of exponential Riordan arrays.

FIRST PROOF. Recall that the Laguerre coefficient matrix  $L^{(\alpha)}$  has entries

$$\ell_{n,k}^{(\alpha)} \stackrel{\text{def}}{=} \binom{n}{k} (\alpha + 1 + k)^{\overline{n-k}} . \quad (3.2)$$

We need to verify that  $\sum_j \ell_{n,j}^{(\alpha)} p_{j,k}^\circ = \ell_{n+1,k}^{(\alpha)}$ , or in other words that

$$\ell_{n,k-1}^{(\alpha)} + (2k + 1 + \alpha) \ell_{n,k}^{(\alpha)} + (k + 1)(k + 1 + \alpha) \ell_{n,k+1}^{(\alpha)} = \ell_{n+1,k}^{(\alpha)} . \quad (3.3)$$

This is a straightforward computation.  $\square$

SECOND PROOF. It follows from (1.6) and (2.33) that the matrix  $L^{(\alpha)}$  is an exponential Riordan array  $\mathcal{R}[F, G]$  with  $F(t) = (1 - t)^{-(1+\alpha)}$  and  $G(t) = t/(1 - t)$ . Since

$$G'(t) = [1 + G(t)]^2 , \quad (3.4)$$



from (2.39) we have

$$A(s) = 1 + 2s + s^2. \quad (3.5)$$

Then

$$\frac{F'(t)}{F(t)} = \frac{1 + \alpha}{1 - t} = (1 + \alpha)[1 + G(t)], \quad (3.6)$$

so from (2.39) we have

$$Z(s) = (1 + \alpha)(1 + s). \quad (3.7)$$

By Theorem 2.21, the production matrix for  $\mathbf{L}^{(\alpha)}$  is the exponential AZ matrix  $\text{EAZ}(A, Z)$ . Inserting (3.5)/(3.7) into (2.37) yields (3.1).  $\square$

We now wish to make an observation about the connection of the Laguerre coefficient matrix  $\mathbf{L}^{(\alpha)}$  with continued fractions. The zeroth column of  $\mathbf{L}^{(\alpha)}$  is the sequence of rising powers (= Stirling cycle polynomials)  $(\lambda^{\bar{n}})_{n \geq 0}$ , where  $\lambda \stackrel{\text{def}}{=} \alpha + 1$ . And this sequence has a well-known classical S-fraction for its ordinary generating function, which was found more than two-and-a-half centuries ago by Euler [26, section 26] [27]<sup>13</sup>:

$$\sum_{n=0}^{\infty} \lambda^{\bar{n}} t^n = \frac{1}{1 - \frac{\lambda t}{1 - \frac{t}{1 - \frac{(\lambda + 1)t}{1 - \frac{2t}{1 - \dots}}}}} \quad (3.8)$$

with coefficients  $\alpha_{2k-1} = \lambda + k - 1$  and  $\alpha_{2k} = k$ . Now, any S-fraction has an associated production matrix

$$P = \begin{bmatrix} \alpha_1 & 1 & & & \\ \alpha_1 \alpha_2 & \alpha_2 + \alpha_3 & 1 & & \\ & \alpha_3 \alpha_4 & \alpha_4 + \alpha_5 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \quad (3.9)$$

(see eq. (A.7) in Appendix A.1), which has as its output matrix the triangular array of generalized Stieltjes–Rogers polynomials of the first kind. And the production matrix corresponding to the coefficients  $\alpha_{2k-1} = \lambda + k - 1$  and  $\alpha_{2k} = k$  is precisely (3.1). In view of Proposition 1.2(a), this shows:

**Proposition 3.1.** *The Laguerre coefficient matrix  $\mathbf{L}^{(\alpha)}$  is the matrix  $\mathbf{S} = (S_{n,\ell}(\boldsymbol{\alpha}))_{n,\ell \geq 0}$  of generalized Stieltjes–Rogers polynomials of the first kind corresponding to the coefficients  $\alpha_{2k-1} = k + \alpha$  and  $\alpha_{2k} = k$ .*

**Remark.** Proposition 3.1 looks at first sight a bit bizarre, because the coefficient matrix for the monic orthogonal polynomials associated to a Stieltjes moment sequence  $\mathbf{a} = (a_n)_{n \geq 0}$  is in general the *inverse matrix* of the corresponding matrix

---

<sup>13</sup>The paper [26], which is E247 in Eneström’s [25] catalogue, was probably written circa 1746; it was presented to the St. Petersburg Academy in 1753, and published in 1760. The paper [27], which is E616 in Eneström’s [25] catalogue, was apparently presented to the St. Petersburg Academy in 1776, and published posthumously in 1788.

$S$  of generalized Stieltjes–Rogers polynomials<sup>14</sup>; therefore, the coefficient matrix for the monic unsigned Laguerre polynomials should be the *unsigned inverse matrix* of  $S$ , i.e.

$$S^\# \stackrel{\text{def}}{=} Q S^{-1} Q \quad (3.10)$$

where  $Q = ((-1)^i \delta_{ij})_{i,j \geq 0}$  is the diagonal matrix of alternating 1's and  $-1$ 's. This apparent paradox is explained by the curious fact that the matrix  $L^{(\alpha)}$  equals its own unsigned inverse: that is,

$$L^{(\alpha)} = Q (L^{(\alpha)})^{-1} Q. \quad (3.11)$$

PROOF. From (1.8) we see that  $L^{(\alpha)} = D B^{(\alpha)} D^{-1}$  where  $B^{(\alpha)}$  is the generalized binomial matrix

$$(B^{(\alpha)})_{nk} = \binom{n+\alpha}{n-k} \quad (3.12)$$

and  $D = \text{diag}((n!)_{n \geq 0})$ . So it suffices to prove that  $B^{(\alpha)}$  equals its own unsigned inverse:

$$\sum_j \binom{n+\alpha}{n-j} \binom{j+\alpha}{j-k} (-1)^{j-k} = \binom{n+\alpha}{n-k} \sum_j \binom{n-k}{j-k} (-1)^{j-k} \quad (3.13a)$$

$$= \delta_{nk} \quad (3.13b)$$

(cf. [39, eq. (5.21)]). ■

### 3.2 Binomial row-generating matrix $L^{(\alpha)} B_x$ : Proof of Proposition 1.2(b)

We now prove Proposition 1.2(b), which asserts that the production matrix for the binomial row-generating matrix  $L^{(\alpha)} B_x$  is the quadridiagonal unit-lower-Hessenberg matrix  $P = (p_{ij})_{i,j \geq 0}$  defined by

$$p_{n,n+1} = 1 \quad (3.14a)$$

$$p_{n,n} = (2n+1+\alpha) + x \quad (3.14b)$$

$$p_{n,n-1} = n(n+\alpha) + 2nx \quad (3.14c)$$

$$p_{n,n-2} = n(n-1)x \quad (3.14d)$$

$$p_{n,k} = 0 \quad \text{if } k < n-2 \text{ or } k > n+1 \quad (3.14e)$$

This is an easy consequence of the theory of exponential Riordan arrays. In the preceding subsection we saw that the production matrix  $P^\circ$  for the Laguerre coefficient matrix  $L^{(\alpha)}$  is the exponential AZ matrix  $\text{EAZ}(A, Z)$  where  $A$  and  $Z$  are given by (3.5)/(3.7). By Lemma 2.7 the production matrix for  $L^{(\alpha)} B_x$  is  $P = B_x^{-1} P^\circ B_x$ ; and by Lemma 2.22 we have  $B_x^{-1} P^\circ B_x = \text{EAZ}(A, Z + xA)$ . Using this together with (3.5)/(3.7) and (2.37) yields (3.14).

---

<sup>14</sup>This is a well-known fact: see, for instance [79, p. III-2, Théorème 1]. And for a generalization to unit-lower-Hessenberg production matrices that are not necessarily tridiagonal, see [68, Proposition 3.2 and Remark 1 following it].

In Appendix B we will explain why  $P = B_x^{-1}P^\circ B_x$  is quadridiagonal, by answering the more general question: Which lower-Hessenberg matrices  $P$  have the property that  $B_\xi^{-1}PB_\xi$  is  $(r, 1)$ -banded?

We also have an explicit formula for the binomial row-generating matrix:

**Proposition 3.2** (Binomial row-generating matrix of the Laguerre coefficient array). *The matrix elements of the binomial row-generating matrix  $\mathbf{L}^{(\alpha)}B_x$  are*

$$(\mathbf{L}^{(\alpha)}B_x)_{n,k} = \frac{1}{k!} \frac{d^k}{dx^k} \mathcal{L}_n^{(\alpha)}(x) = \binom{n}{k} \mathcal{L}_{n-k}^{(\alpha+k)}(x). \quad (3.15)$$

PROOF. The first equality is a special case of the general property (2.26) of binomial row-generating polynomials. The second equality follows by repeated use of the differentiation formula

$$\frac{d}{dx} \mathcal{L}_n^{(\alpha)}(x) = n \mathcal{L}_{n-1}^{(\alpha+1)}(x). \quad (3.16)$$

□

It can also be shown, by direct computation using classical identities for the Laguerre polynomials, that the matrix (3.15) is indeed the output matrix for the production matrix (3.14). But since this proof is rather lengthy, we refrain from showing it here.

### 3.3 Total positivity of the coefficient matrix $\mathbf{L}^{(-1+\lambda)}$

Finally, let us give a simple direct proof of the coefficientwise total positivity of the Laguerre coefficient matrix  $\mathbf{L}^{(-1+\lambda)}$ . Start from the binomial matrix  $a_{nk} = \binom{n}{k}$ , which is totally positive by Lemma 2.2. Now use Lemma 2.3(b) with  $x_i = \lambda + i - 1$  for  $i \geq 1$ , which are nonnegative elements of  $\mathbb{Z}[\lambda]$ . Then

$$b_{nk} = x_{k+1}x_{k+2} \cdots x_n a_{nk} \quad (3.17a)$$

$$= (\lambda + k)^{\overline{n-k}} \binom{n}{k}, \quad (3.17b)$$

which is (1.8) with  $\alpha = -1 + \lambda$ .

## 4 Multivariate Laguerre polynomials: Exponential generating functions and production matrices

In this section we compute the column and bivariate exponential generating functions for the multivariate Laguerre coefficient matrix (1.31); we then use these exponential generating functions, together with the theory of exponential Riordan arrays, to deduce the corresponding production matrices.

## 4.1 Exponential generating functions

In this subsection we shall compute the column and bivariate exponential generating functions for the coefficient matrix  $\widehat{\mathbf{L}}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp})$  that was defined in (1.31), which enumerates Laguerre digraphs according to the status of vertices as peaks, valleys, double ascents, double descents or fixed points. Indeed, we shall go farther, by assigning different weights for the vertices belonging to a cycle or to a path. For a Laguerre digraph  $G$ , let us write  $\text{pcyc}(G)$ ,  $\text{vcyc}(G)$ ,  $\text{dacyc}(G)$ ,  $\text{ddcyc}(G)$ ,  $\text{fp}(G)$  for the number of peaks, valleys, double ascents, double descents and fixed points that belong to a cycle of  $G$ , and  $\text{ppa}(G)$ ,  $\text{vpa}(G)$ ,  $\text{dapa}(G)$ ,  $\text{ddpa}(G)$  for the number of peaks, valleys, double ascents and double descents that belong to a path of  $G$  (of course fixed points can only belong to a cycle). We then assign weights  $y_p, y_v, y_{da}, y_{dd}, y_{fp}$  to the vertices belonging to a cycle, and weights  $z_p, z_v, z_{da}, z_{dd}$  to the vertices belonging to a path. We therefore define the **generalized second multivariate Laguerre coefficient matrix**

$$\begin{aligned} & \widetilde{\mathbf{L}}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd})_{n,k} \\ & \stackrel{\text{def}}{=} \sum_{G \in \mathbf{LD}_{n,k}} y_p^{\text{pcyc}(G)} y_v^{\text{vcyc}(G)} y_{da}^{\text{dacyc}(G)} y_{dd}^{\text{ddcyc}(G)} y_{fp}^{\text{fp}(G)} z_p^{\text{ppa}(G)} z_v^{\text{vpa}(G)} z_{da}^{\text{dapa}(G)} z_{dd}^{\text{ddpa}(G)} \times \\ & \quad (1 + \alpha)^{\text{cyc}(G)}. \end{aligned} \quad (4.1)$$

This polynomial is homogeneous of degree  $n$  in  $y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd}$ .

Because of the 0–0 boundary conditions, each path contains at least one peak; so we can, if we wish, remove a factor  $z_p^k$  and define a unit-lower-triangular matrix by

$$\widetilde{\mathbf{L}}^{(\alpha)b}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd})_{n,k} \stackrel{\text{def}}{=} \widetilde{\mathbf{L}}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd})_{n,k} / z_p^k. \quad (4.2)$$

This polynomial is homogeneous of degree  $n - k$  in  $y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd}$ .

We now proceed to compute the exponential generating functions for the matrices (4.1) and (4.2). We do this by combining the known exponential generating functions for permutations with cyclic statistics [84, Théorème 1] and permutations with linear statistics [84, Proposition 4]. These formulae are as follows:

- 1) Permutations with cyclic statistics are enumerated by the polynomials

$$P_n^{\text{cyc}}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, \lambda) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} y_p^{\text{pcyc}(\sigma)} y_v^{\text{vcyc}(\sigma)} y_{da}^{\text{dacyc}(\sigma)} y_{dd}^{\text{ddcyc}(\sigma)} y_{fp}^{\text{fp}(\sigma)} \lambda^{\text{cyc}(\sigma)} \quad (4.3)$$

where  $\text{pcyc}(\sigma)$ ,  $\text{vcyc}(\sigma)$ ,  $\text{dacyc}(\sigma)$ ,  $\text{ddcyc}(\sigma)$ ,  $\text{fp}(\sigma)$  denote the number of cycle peaks, cycle valleys, cycle double rises, cycle double falls and fixed points in  $\sigma$ , and  $\text{cyc}(\sigma)$  denotes the number of cycles in  $\sigma$ . By convention we set  $P_0^{\text{cyc}} = 1$ . The polynomial  $P_n^{\text{cyc}}$  is homogeneous of degree  $n$  in  $y_p, y_v, y_{da}, y_{dd}, y_{fp}$ . We write

$$F(t; y_p, y_v, y_{da}, y_{dd}, y_{fp}, \lambda) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} P_n^{\text{cyc}}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, \lambda) \frac{t^n}{n!} \quad (4.4)$$

for the corresponding exponential generating function.

**Lemma 4.1.** [84, Théorème 1] *We have*

$$F(t; y_p, y_v, y_{da}, y_{dd}, y_{fp}, \lambda) = e^{\lambda y_{fp} t} \left( \frac{r_1 - r_2}{r_1 e^{r_2 t} - r_2 e^{r_1 t}} \right)^\lambda \quad (4.5a)$$

$$= F(t; y_p, y_v, y_{da}, y_{dd}, y_{fp}, 1)^\lambda \quad (4.5b)$$

where  $r_1 r_2 = y_p y_v$  and  $r_1 + r_2 = y_{da} + y_{dd}$ . Otherwise put,  $r_1$  and  $r_2$  are the roots (in either order) of the quadratic equation  $\rho^2 - (y_{da} + y_{dd})\rho + y_p y_v = 0$ . Concretely,

$$r_{1,2} = \frac{y_{da} + y_{dd} \pm \sqrt{(y_{da} + y_{dd})^2 - 4y_p y_v}}{2}. \quad (4.6)$$

2) Permutations with linear statistics and 0–0 boundary conditions are enumerated by the polynomials

$$P_n^{\text{lin}(00)}(z_p, z_v, z_{da}, z_{dd}) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} z_p^{\text{ppa}(\sigma)} z_v^{\text{vpa}(\sigma)} z_{da}^{\text{dapa}(\sigma)} z_{dd}^{\text{ddpa}(\sigma)} \quad (4.7)$$

where  $\text{ppa}(\sigma)$ ,  $\text{vpa}(\sigma)$ ,  $\text{dapa}(\sigma)$ ,  $\text{ddpa}(\sigma)$  denote the number of peaks, valleys, double ascents and double descents in the permutation  $\sigma$  written as a word  $\sigma_1 \cdots \sigma_n$ , where we impose the boundary conditions  $\sigma_0 = \sigma_{n+1} = 0$ . By convention we restrict attention to  $n \geq 1$ . The polynomial  $P_n^{\text{lin}(00)}$  is homogeneous of degree  $n$  in  $z_p, z_v, z_{da}, z_{dd}$ . We write

$$G(t; z_p, z_v, z_{da}, z_{dd}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} P_n^{\text{lin}(00)}(z_p, z_v, z_{da}, z_{dd}) \frac{t^n}{n!} \quad (4.8)$$

for the corresponding exponential generating function (note that the sum starts at  $n = 1$ ).

**Lemma 4.2.** [84, Proposition 4] *We have*

$$G(t; z_p, z_v, z_{da}, z_{dd}) = z_p \left( \frac{e^{r_1 t} - e^{r_2 t}}{r_1 e^{r_2 t} - r_2 e^{r_1 t}} \right) \quad (4.9)$$

where  $r_1 r_2 = z_p z_v$  and  $r_1 + r_2 = z_{da} + z_{dd}$  analogously to Lemma 4.1. This function satisfies the differential equation

$$G'(t) = z_p + (z_{da} + z_{dd})G(t) + z_v G(t)^2. \quad (4.10)$$

We can now put these ingredients together to determine the exponential generating functions for the matrix (4.1). A Laguerre digraph  $G \in \mathbf{LD}_{n,k}$  consists of a permutation (that is, a collection of disjoint cycles) on some subset  $S \subseteq [n]$  together with  $k$  disjoint paths on  $[n] \setminus S$ . Each of these paths can be considered as a permutation written in word form. By the exponential formula, the exponential generating function for the  $k$ th column of the matrix (4.1) is then

$$\begin{aligned} & \sum_{n=0}^{\infty} \tilde{\mathbf{L}}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd})_{n,k} \frac{t^n}{n!} \\ &= F(t; y_p, y_v, y_{da}, y_{dd}, y_{fp}, 1)^{1+\alpha} \frac{G(t; z_p, z_v, z_{da}, z_{dd})^k}{k!}, \end{aligned} \quad (4.11)$$

where the  $1/k!$  comes because the paths are indistinguishable. The bivariate exponential generating function is therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n \tilde{L}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd})_{n,k} \frac{t^n}{n!} u^k \\ &= F(t; y_p, y_v, y_{da}, y_{dd}, y_{fp}, 1)^{1+\alpha} \exp[u G(t; z_p, z_v, z_{da}, z_{dd})] . \end{aligned} \quad (4.12)$$

Comparing this with (2.32)/(2.33) shows:

**Proposition 4.3** (Generalized second multivariate Laguerre coefficient matrix as exponential Riordan array). *The matrix  $\tilde{L}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}, z_p, z_v, z_{da}, z_{dd})$ , which was defined in (4.1), is an exponential Riordan array  $\mathcal{R}[F, G]$  where  $F$  is given by (4.5) with  $\lambda = 1 + \alpha$  and  $G$  is given by (4.9).*

For the matrix  $\tilde{L}^{(\alpha)b}$  defined in (4.2), the formulae are the same except that  $G$  is replaced by

$$G^b(t; z_p, z_v, z_{da}, z_{dd}) \stackrel{\text{def}}{=} G(t; z_p, z_v, z_{da}, z_{dd})/z_p = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 e^{r_2 t} - r_2 e^{r_1 t}} , \quad (4.13)$$

which satisfies the differential equation

$$(G^b)'(t) = 1 + (z_{da} + z_{dd})G^b(t) + z_p z_v G^b(t)^2 . \quad (4.14)$$

**Remark.** For the first multivariate coefficient matrix (1.19), the formulae simplify: taking  $y_p = y_{dd} = z_p = z_{dd} = v_-$ ,  $y_v = y_{da} = z_v = z_{da} = v_-$  and  $y_{fp} = v_0$ , we get  $r_1 = v_+$ ,  $r_2 = v_-$  and hence, using (1.33),

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n L^{(\alpha)}(v_-, v_0, v_+)_{n,k} \frac{t^n}{n!} u^k \\ &= e^{(1+\alpha)v_0 t} \left( \frac{v_+ - v_-}{v_+ e^{v_- t} - v_- e^{v_+ t}} \right)^{1+\alpha} \exp \left[ u \left( \frac{e^{v_+ t} - e^{v_- t}}{v_+ e^{v_- t} - v_- e^{v_+ t}} \right) \right] . \end{aligned} \quad (4.15)$$

## 4.2 Production matrices: Proof of Proposition 1.5(a,b)

We now restrict attention to the case in which the vertices belonging to paths and cycles are given the same weights, i.e.  $(z_p, z_v, z_{da}, z_{dd}) = (y_p, y_v, y_{da}, y_{dd})$ . In this situation it is easy to find the series  $A(s)$  and  $Z(s)$  that satisfy the relations (2.39) leading to the production matrix of an exponential Riordan array. Namely, from (4.10) we have immediately

$$A(s) = y_p + (y_{da} + y_{dd})s + y_v s^2 . \quad (4.16)$$

And a straightforward computation shows that the logarithmic derivative of  $F$  equals  $\lambda y_{fp} + \lambda y_v G$ , so we have

$$Z(s) = \lambda y_{fp} + \lambda y_v s . \quad (4.17)$$

Theorem 2.21 then yields:

**Proposition 4.4** (Production matrix of the second multivariate Laguerre coefficient matrix). *The second multivariate Laguerre coefficient matrix  $\widehat{L}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp})$ , which was defined in (1.31), has production matrix  $P^\circ = \text{EAZ}(A, Z)$  where  $A$  and  $Z$  are given by (4.16)/(4.17) with  $\lambda = 1 + \alpha$ . In detail,  $P^\circ$  is the tridiagonal matrix*

$$p_{n,n+1}^\circ = y_p \quad (4.18a)$$

$$p_{n,n}^\circ = (1 + \alpha)y_{fp} + n(y_{da} + y_{dd}) \quad (4.18b)$$

$$p_{n,n-1}^\circ = n(n + \alpha)y_v \quad (4.18c)$$

$$p_{n,k}^\circ = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \quad (4.18d)$$

From this it is straightforward to deduce the production matrix for the binomial row-generating matrix  $\widehat{L}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}) B_x$ . Namely, by Lemma 2.7 this production matrix is  $P = B_x^{-1}P^\circ B_x$ ; and by Lemma 2.22 we have  $B_x^{-1}P^\circ B_x = \text{EAZ}(A, Z + xA)$ . We have therefore proven:

**Proposition 4.5** (Production matrix of the second multivariate Laguerre binomial row-generating matrix). *The binomial row-generating matrix  $\widehat{L}^{(\alpha)}(y_p, y_v, y_{da}, y_{dd}, y_{fp}) B_x$  has production matrix  $P = \text{EAZ}(A, Z + xA)$  where  $A$  and  $Z$  are given by (4.16)/(4.17) with  $\lambda = 1 + \alpha$ . In detail,  $P$  is the quadridiagonal lower-Hessenberg matrix*

$$p_{n,n+1} = y_p \quad (4.19a)$$

$$p_{n,n} = (1 + \alpha)y_{fp} + n(y_{da} + y_{dd}) + y_p x \quad (4.19b)$$

$$p_{n,n-1} = n(n + \alpha)y_v + n(y_{da} + y_{dd})x \quad (4.19c)$$

$$p_{n,n-2} = n(n - 1)y_v x \quad (4.19d)$$

$$p_{n,k} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \quad (4.19e)$$

For the matrix  $\widetilde{L}^{(\alpha)b}$ , the formulae are the same except that  $A$  and  $Z$  are replaced by

$$A^b(s) = 1 + (y_{da} + y_{dd})s + y_p y_v s^2 \quad (4.20)$$

$$Z^b(s) = \lambda y_{fp} + \lambda y_p y_v s \quad (4.21)$$

This leads to the production matrices

$$p_{n,n+1}^{\text{ob}} = 1 \quad (4.22a)$$

$$p_{n,n}^{\text{ob}} = (1 + \alpha)y_{fp} + n(y_{da} + y_{dd}) \quad (4.22b)$$

$$p_{n,n-1}^{\text{ob}} = n(n + \alpha)y_p y_v \quad (4.22c)$$

$$p_{n,k}^{\text{ob}} = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \quad (4.22d)$$

and

$$p_{n,n+1}^b = 1 \quad (4.23a)$$

$$p_{n,n}^b = (1 + \alpha)y_{fp} + n(y_{da} + y_{dd}) + x \quad (4.23b)$$

$$p_{n,n-1}^b = n(n + \alpha)y_p y_v + n(y_{da} + y_{dd})x \quad (4.23c)$$

$$p_{n,n-2}^b = n(n - 1)y_p y_v x \quad (4.23d)$$

$$p_{n,k}^b = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \quad (4.23e)$$

This completes the proof of Proposition 1.5.

## 5 Total positivity of the production matrices: Univariate case

In this section we will prove the total positivity of the production matrices for the univariate Laguerre polynomials: namely, the tridiagonal production matrix  $P^\circ$  [defined in (1.40)] for the Laguerre coefficient matrix  $L^{(\alpha)}$ , and the quadri-diagonal production matrix  $P$  [defined in (1.41)] for the binomial row-generating matrix  $L^{(\alpha)} B_x$ . Throughout, we use the parameter  $\lambda = 1 + \alpha$ .

The proofs are easy factorizations; in the quadri-diagonal case we will also need to invoke the tridiagonal comparison theorem (Proposition 2.6).

### 5.1 The tridiagonal matrix $P^\circ$ : Proof of Proposition 1.3(a)

The tridiagonal matrix  $P^\circ$  is given by

$$p_{n,n+1}^\circ = 1 \tag{5.1a}$$

$$p_{n,n}^\circ = 2n + \lambda \tag{5.1b}$$

$$p_{n,n-1}^\circ = n(n - 1 + \lambda) \tag{5.1c}$$

$$p_{n,k}^\circ = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \tag{5.1d}$$

As observed preceding Proposition 3.1, this is the production matrix (A.7) corresponding to the classical S-fraction with coefficients  $\alpha_{2k-1} = k - 1 + \lambda$  and  $\alpha_{2k} = k$  (see Appendix A.1). And as observed in (A.8), the production matrix (A.7) can be factorized in the form  $LU$  where  $L$  is the lower-bidiagonal matrix with 1 on the diagonal and  $\alpha_2, \alpha_4, \dots$  on the subdiagonal, and  $U$  is the upper-bidiagonal matrix with 1 on the superdiagonal and  $\alpha_1, \alpha_3, \dots$  on the diagonal. And finally, as also observed there, this factorization shows, by Lemma 2.1, that the matrix (A.7) is coefficientwise totally positive in the indeterminates  $\alpha$ . Putting this all together, we conclude:

**Proposition 5.1** (Factorization and total positivity of  $P^\circ$ ).

(a) *The tridiagonal matrix  $P^\circ$  defined by (5.1) has the factorization  $P^\circ = LU$  where*

- *$L$  is the lower-bidiagonal matrix with 1 on the diagonal and  $1, 2, 3, \dots$  on the subdiagonal, and*
- *$U$  is the upper-bidiagonal matrix with 1 on the superdiagonal and  $\lambda, \lambda + 1, \lambda + 2, \dots$  on the diagonal.*

(b)  *$P^\circ$  is totally positive in the ring  $\mathbb{Z}[\lambda]$  equipped with the coefficientwise order.*

In particular, this proves Proposition 1.3(a).

Note that we can also write  $U = L\Delta + \lambda I$ , since  $L\Delta$  is the upper-bidiagonal matrix with 1 on the superdiagonal and  $0, 1, 2, \dots$  on the diagonal.



## 5.2 The quadridiagonal matrix $P$ : Proof of Proposition 1.3(b)

The quadridiagonal matrix  $P$  is given by

$$p_{n,n+1} = 1 \quad (5.2a)$$

$$p_{n,n} = (2n + \lambda) + x \quad (5.2b)$$

$$p_{n,n-1} = n(n - 1 + \lambda) + 2nx \quad (5.2c)$$

$$p_{n,n-2} = n(n - 1)x \quad (5.2d)$$

$$p_{n,k} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \quad (5.2e)$$

**Proposition 5.2** (Factorization and total positivity of  $P$ ).

(a) *The quadridiagonal matrix  $P$  defined by (5.2) has the factorization  $P = L(LU_x + \lambda I)$  where*

- *$L$  is the lower-bidiagonal matrix with 1 on the diagonal and  $1, 2, 3, \dots$  on the subdiagonal, and*
- *$U_x = \Delta + xI$  is the upper-bidiagonal matrix with 1 on the superdiagonal and  $x$  on the diagonal.*

(b)  *$P$  is totally positive in the ring  $\mathbb{Z}[x, \lambda]$  equipped with the coefficientwise order.*

PROOF. (a) It is easy to see that  $P = P^\circ + xLL$ . Since  $P^\circ = L(L\Delta + \lambda I)$  by Proposition 5.1(a), we have  $P = L[L(\Delta + xI) + \lambda I]$ , which proves part (a).

(b) By Lemma 2.1, the bidiagonal matrices  $L$  and  $U_x$  are totally positive in the ring  $\mathbb{Z}[x]$  equipped with the coefficientwise order. It follows that the tridiagonal matrix  $LU_x$  is also totally positive in the ring  $\mathbb{Z}[x]$  equipped with the coefficientwise order. Then the tridiagonal comparison theorem (Proposition 2.6) shows that the tridiagonal matrix  $LU_x + \lambda I$  is totally positive in the ring  $\mathbb{Z}[x, \lambda]$  equipped with the coefficientwise order. Therefore the same holds for  $P = L(LU_x + \lambda I)$ .  $\square$

This completes the proof of Proposition 1.3(b).

**Remarks.** 1. For  $\lambda = 0$  ( $\alpha = -1$ ), we have  $P = LLU_x$ , which is the production matrix for the known 2-S-fraction for the Lah polynomials [58, Theorem 1.5 with  $r = 2$ ], which has coefficients

$$\alpha_{3n-1} = x \quad (5.3a)$$

$$\alpha_{3n} = n \quad (5.3b)$$

$$\alpha_{3n+1} = n \quad (5.3c)$$

(For the production matrix associated to a 2-S-fraction, see [59, Propositions 7.2 and 8.2 and eqn. (7.8)] or the case  $j = 0$  of Proposition A.3 below.) However, it turns out that this same production matrix has a one-parameter family of distinct factorizations  $P = L_1L_2U_x$ , arising from the 2-S-fractions with

$$\alpha_{3n-1} = x \quad (5.4a)$$

$$\alpha_{3n} = c_n n \quad (5.4b)$$

$$\alpha_{3n+1} = (2 - c_n) n \quad (5.4c)$$

where

$$c_n \stackrel{\text{def}}{=} \frac{(n-1) - (n-2)\kappa}{n - (n-1)\kappa} \quad \text{for } n \geq 1 \quad (5.5)$$

and  $\kappa \in [0, 1]$ : see Proposition A.8 below. The case  $\kappa = 1$  is the known 2-S-fraction; all the others appear to be new.

2. It is easy to see that  $L\Delta + I = \Delta L$  (this is a special case of (A.23) below) and hence  $LU_x + I = U_x L$ . This implies that for  $\lambda = 1$  ( $\alpha = 0$ ), we have  $P = L(LU_x + I) = LU_x L$ . This is the production matrix for the generalized 2-Stieltjes–Rogers polynomials of type  $j = 1$  with the coefficients (5.3): see Proposition A.3 for the general theory, and Proposition A.12 for this application. ■

## 6 Total positivity of the production matrices: Multivariate case

In this section we will prove Proposition 1.6 on the total positivity of the production matrices for the multivariate Laguerre polynomials. The proof of Proposition 1.6(a), dealing with the tridiagonal matrix  $P^{\text{ob}}$ , is an easy factorization combined with an appeal to the tridiagonal comparison theorem. By contrast, the proof of Proposition 1.6(b), dealing with the quadri-diagonal matrix  $P^{\text{b}}$ , is decidedly nontrivial.

### 6.1 The tridiagonal matrix $P^{\text{ob}}$ : Proof of Proposition 1.6(a)

PROOF OF PROPOSITION 1.6(a). Let  $Q$  the matrix (A.8) with  $\alpha_{2k-1} = (k+\alpha)y_p$  and  $\alpha_{2k} = ky_v$  for  $k \geq 1$ ; note that these coefficients are nonnegative in the ring  $R$  because of the hypotheses  $\alpha \geq -1$ ,  $y_p \geq 0$  and  $y_v \geq 0$ . It then follows from Lemma 2.1 that  $Q$  is totally positive in the ring  $R$ . The matrix  $Q$  has elements

$$q_{n,n+1} = 1 \quad (6.1a)$$

$$q_{n,n} = (1 + \alpha)y_p + n(y_p + y_v) \quad (6.1b)$$

$$q_{n,n-1} = n(n + \alpha)y_p y_v \quad (6.1c)$$

$$q_{n,k} = 0 \quad \text{if } k < n - 1 \text{ or } k > n + 1 \quad (6.1d)$$

Then  $P^{\text{ob}} = Q + D$ , where  $D$  is the diagonal matrix with entries  $d_n = (1 + \alpha)(y_{\text{fp}} - y_p) + n(y_{\text{da}} + y_{\text{dd}} - y_p - y_v)$ . By hypothesis these entries are nonnegative in the ring  $R$ , so the tridiagonal comparison theorem (Proposition 2.6) implies that  $P^{\text{ob}}$  is totally positive in the ring  $R$ . □

### 6.2 The quadri-diagonal matrix $P^{\text{b}}$ : Proof of Proposition 1.6(b)

We will now prove Proposition 1.6(b), which asserts the total positivity of the production matrix (1.43) when the variables are substituted to elements of a partially ordered commutative ring  $R$  that satisfy  $\lambda \geq 0$ ,  $\lambda y_{\text{fp}} = \lambda y_p$ ,  $y_p \geq 0$ ,  $y_v \geq 0$ ,  $y_{\text{da}} + y_{\text{dd}} \geq$

$y_p + y_v$  and  $x \geq 0$ . We will do this by proving the *coefficientwise* total positivity of a *much more general* quadridiagonal lower-Hessenberg matrix  $P$ , defined by

$$P \stackrel{\text{def}}{=} L_1 U L_2 + L_1 D_1 + D_2 L_2 \quad (6.2a)$$

$$= L_1 (U L_2 + D_1) + D_2 L_2 \quad (6.2b)$$

$$= (L_1 U + D_2) L_2 + L_1 D_1 \quad (6.2c)$$

where

- $L_1$  is the lower-bidiagonal matrix with the sequence  $a_0, a_1, \dots$  on the diagonal, the sequence  $b_1, b_2, \dots$  on the subdiagonal, and zeroes elsewhere;
- $U$  is the upper-bidiagonal matrix with the sequence  $c_1, c_2, \dots$  on the superdiagonal, the sequence  $d_0, d_1, \dots$  on the diagonal, and zeroes elsewhere;
- $L_2$  is the lower-bidiagonal matrix with the sequence  $e_0, e_1, \dots$  on the diagonal, the sequence  $f_1, f_2, \dots$  on the subdiagonal, and zeroes elsewhere;
- $D_1$  is the diagonal matrix with entries  $g_0, g_1, \dots$ ;
- $D_2$  is the diagonal matrix with entries  $h_0, h_1, \dots$ ;

and  $\mathbf{a} = (a_n)_{n \geq 0}$ ,  $\mathbf{b} = (b_n)_{n \geq 1}$ ,  $\mathbf{c} = (c_n)_{n \geq 1}$ ,  $\mathbf{d} = (d_n)_{n \geq 0}$ ,  $\mathbf{e} = (e_n)_{n \geq 0}$ ,  $\mathbf{f} = (f_n)_{n \geq 1}$ ,  $\mathbf{g} = (g_n)_{n \geq 0}$ , and  $\mathbf{h} = (h_n)_{n \geq 0}$  are all indeterminates. The entries in the  $n$ th row of  $P = (p_{n,k})_{k \geq 0}$  are given by

$$p_{n,n+1} = a_n c_{n+1} e_{n+1} \quad (6.3a)$$

$$p_{n,n} = a_n d_n e_n + b_n c_n e_n + a_n c_{n+1} f_{n+1} + a_n g_n + h_n e_n \quad (6.3b)$$

$$p_{n,n-1} = a_n d_n f_n + b_n c_n f_n + b_n d_{n-1} e_{n-1} + b_n g_{n-1} + h_n f_n \quad (6.3c)$$

$$p_{n,n-2} = b_n d_{n-1} f_{n-1} \quad (6.3d)$$

$$p_{n,k} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \quad (6.3e)$$

where by definition  $b_0 = c_0 = f_0 = 0$  and  $a_n = b_n = c_n = d_n = e_n = f_n = g_n = h_n = 0$  whenever  $n < 0$ . Our main result is:

**Theorem 6.1** (Total positivity of the generalized production matrix). *The matrix  $P$  defined by (6.2)/(6.3) is totally positive, coefficientwise in the indeterminates  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}$ .*

PROOF OF PROPOSITION 1.6(B) ASSUMING THEOREM 6.1. Specialize the matrix (6.3) by setting

$$a_n = c_n = e_n = 1 \quad (6.4a)$$

$$b_n = n y_v \quad (6.4b)$$

$$d_n = n y_p \quad (6.4c)$$

$$f_n = x \quad (6.4d)$$

$$g_n = \lambda y_p (= \lambda y_{fp}) \quad (6.4e)$$

$$h_n = n(y_{da} + y_{dd} - y_p - y_v) \quad (6.4f)$$

□

Theorem 6.1 will be proven as follows. Define the matrix  $Q = (q_{n,k})_{n,k \geq 0}$  by

$$Q \stackrel{\text{def}}{=} P|_{\mathbf{h}=\mathbf{0}} = L_1(UL_2 + D_1), \quad (6.5)$$

or in other words

$$q_{n,n+1} = a_n c_{n+1} e_{n+1} \quad (6.6a)$$

$$q_{n,n} = a_n d_n e_n + b_n c_n e_n + a_n c_{n+1} f_{n+1} + a_n g_n \quad (6.6b)$$

$$q_{n,n-1} = a_n d_n f_n + b_n c_n f_n + b_n d_{n-1} e_{n-1} + b_n g_{n-1} \quad (6.6c)$$

$$q_{n,n-2} = b_n d_{n-1} f_{n-1} \quad (6.6d)$$

$$q_{n,k} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \quad (6.6e)$$

Then

$$P = Q + D_2 L_2. \quad (6.7)$$

We will begin by proving (Lemma 6.6) that  $Q$  is coefficientwise totally positive; this proof uses the factorization  $Q = L_1(UL_2 + D_1)$  together with the tridiagonal comparison theorem, in close analogy with Propositions 1.6(a) and 5.2. It follows that for every integer  $m \geq 0$ , the matrix consisting of first  $m$  rows of  $Q$  is also coefficientwise totally positive.

The rest of the proof shows how to restore the terms in  $P$  involving  $\mathbf{h}$ . In terms of the row vectors  $(\mathbf{p}_n)_{n \geq 0}$ ,  $(\mathbf{q}_n)_{n \geq 0}$ ,  $(\boldsymbol{\ell}_n)_{n \geq 0}$  associated to the matrices  $P, Q, L_2$ , equation (6.7) can be rewritten as

$$\mathbf{p}_n = \mathbf{q}_n + h_n \boldsymbol{\ell}_n, \quad (6.8)$$

where

$$\boldsymbol{\ell}_0 = [e_0 \quad \mathbf{0}_{1 \times \infty}] \quad (6.9a)$$

$$\boldsymbol{\ell}_n = [\mathbf{0}_{1 \times n-1} \quad f_n \quad e_n \quad \mathbf{0}_{1 \times \infty}] \quad \text{for } n \geq 1 \quad (6.9b)$$

**Remark.** Here we have chosen to write  $Q \stackrel{\text{def}}{=} P|_{\mathbf{h}=\mathbf{0}} = L_1(UL_2 + D_1)$  and  $P = Q + D_2 L_2$  and to argue by rows (because  $D_2$  acts on the left). We could equally well have started from  $Q' \stackrel{\text{def}}{=} P|_{\mathbf{g}=\mathbf{0}} = (L_1 U + D_2) L_2$  and  $P = Q' + L_1 D_1$  and to argue by columns (because  $D_1$  acts on the right). The reader is invited to work out this variant proof. ■

Our proof is based on considering the matrix 
$$\begin{bmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_{n-1} \\ \mathbf{p}_n \\ \vdots \\ \mathbf{p}_m \end{bmatrix},$$
 which we shall sometimes

write henceforth for typographical simplicity as  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \mathbf{p}_n, \dots, \mathbf{p}_m]]$  (and likewise for other matrices written by rows). We will show (Lemma 6.10) that for every pair of integers  $0 \leq n \leq m + 1$ , this matrix is totally positive; and we will do this, for

each fixed  $m \geq 0$ , by induction on  $n = m+1, m, m-1, \dots, 0$ . The base case  $n = m+1$  of this induction is thus Lemma 6.6, and the final case  $n = 0$  is Theorem 6.1. The proof of Lemma 6.10 will involve the following steps:

**Lemma 6.7:** The matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \ell_n]]$  is totally positive.

**Lemma 6.8:** If the matrix  $[[\mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$  is totally positive, then so is  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \ell_n, \mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$ .

**The induction step (Lemma 6.9):** If the matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_n, \mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$  is totally positive, then so is  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \mathbf{p}_n, \dots, \mathbf{p}_m]]$ .

Putting this all together will prove Lemma 6.10 and hence Theorem 6.1.

As preparation, we need some simple lemmas concerning operations that preserve total positivity. All of these hold in an arbitrary partially ordered commutative ring, and apply to rectangular (i.e. not necessarily square) matrices of suitably conformable dimensions.

**Lemma 6.2** (Increasing the upper-left entry). *Let  $A$  be a (finite or infinite) matrix with entries in a partially ordered commutative ring  $R$ ; let  $c$  be a nonnegative element of  $R$ ; and let  $A'$  be the matrix obtained from  $A$  by adding  $c$  to the upper-left entry and leaving all other entries unchanged. If  $A$  is totally positive of order  $r$ , then so is  $A'$ .*

PROOF. Consider a minor on rows  $I$  and columns  $J$ . If  $I$  and  $J$  do not both contain 1, then obviously  $\det(A')_{IJ} = \det A_{IJ}$ . If  $I$  and  $J$  both contain 1, then  $\det(A')_{IJ} = \det A_{IJ} + c \det A_{I \setminus 1, J \setminus 1}$ .  $\square$

**Lemma 6.3** (Matrices that coincide except in one row).

Let  $A = \begin{bmatrix} M \\ \boldsymbol{\alpha} \\ N \end{bmatrix}$  and  $B = \begin{bmatrix} M \\ \boldsymbol{\beta} \\ N \end{bmatrix}$  be matrices that coincide except in one row; let  $a, b \geq 0$ ; and let  $C = \begin{bmatrix} M \\ a\boldsymbol{\alpha} + b\boldsymbol{\beta} \\ N \end{bmatrix}$ . If  $A$  and  $B$  are totally positive of order  $r$ , then so is  $C$ .

PROOF. This is an immediate consequence of the row linearity of determinants.  $\square$

Of course, an analogous result holds for matrices that coincide except in one column.

**Lemma 6.4** (Block-diagonal matrices). *If the matrices  $A$  and  $B$  are totally positive of order  $r$ , then so is the matrix  $C = \left[ \begin{array}{c|c} A & \\ \hline & B \end{array} \right]$ .*

PROOF. This is trivial: every nonzero minor of  $C$  is a product of minors of  $A$  and  $B$ .  $\square$

A slightly less trivial result [42, p. 398] concerns matrices that are not quite block-diagonal, being shifted so as to overlap in one row:

**Lemma 6.5** (Almost-block-diagonal matrices). *Let  $A = \begin{bmatrix} A' \\ \boldsymbol{\alpha} \end{bmatrix}$  and  $B = \begin{bmatrix} \boldsymbol{\beta} \\ B' \end{bmatrix}$  be two matrices, where the row vector  $\boldsymbol{\alpha}$  is the last row of  $A$  and the row vector  $\boldsymbol{\beta}$  is the first row of  $B$ . Now form the matrix*

$$M = \left[ \begin{array}{c|c} A' & \mathbf{0} \\ \hline \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \hline \mathbf{0} & B' \end{array} \right], \quad (6.10)$$

in which the blocks  $A$  and  $B$  overlap in one row. If  $A$  and  $B$  are totally positive of order  $r$ , then so is  $M$ .

PROOF. By Lemma 6.4, the block-diagonal matrices

$$M_1 \stackrel{\text{def}}{=} \left[ \begin{array}{c|c} A & \\ \hline & B' \end{array} \right] = \left[ \begin{array}{c|c} A' & \\ \hline \boldsymbol{\alpha} & \mathbf{0} \\ \hline & B' \end{array} \right] \quad \text{and} \quad M_2 \stackrel{\text{def}}{=} \left[ \begin{array}{c|c} A' & \\ \hline & B \end{array} \right] = \left[ \begin{array}{c|c} A' & \\ \hline \mathbf{0} & \boldsymbol{\beta} \\ \hline & B' \end{array} \right] \quad (6.11)$$

are totally positive of order  $r$ . But then, by Lemma 6.3, so is  $M$ .  $\square$

Of course, an analogous result holds for matrices that overlap in one column.

We now begin the proof of Theorem 6.1.

**Lemma 6.6** (Total positivity of  $Q$ ). *The matrix  $Q$  defined by (6.5)/(6.6) is totally positive, coefficientwise in the indeterminates  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}$ .*

*In particular, for every integer  $m \geq 0$ , the matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_m]]$  is coefficientwise totally positive.*

PROOF. From (6.2)/(6.5) we have

$$Q = L_1(UL_2 + D_1). \quad (6.12)$$

By Lemma 2.1, the matrices  $L_1$ ,  $L_2$ ,  $U$  and  $D_1$  are all totally positive. Therefore,  $UL_2$  is totally positive and tridiagonal, while  $D_1$  is totally positive and diagonal. The tridiagonal comparison theorem (Proposition 2.6) then implies that  $UL_2 + D_1$  is totally positive. The result then follows.  $\square$

**Lemma 6.7.** *For each integer  $n \geq 0$ , the matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \boldsymbol{\ell}_n]]$  is coefficientwise totally positive.*

PROOF. The case when  $n = 0$  is trivial. So we assume that  $n \geq 1$ .

Let  $\tilde{\mathbf{q}}_{n-1} = (\tilde{q}_{n-1,k})_{k \geq 0}$  be the row vector with entries

$$\tilde{q}_{n-1,n} = a_{n-1}c_n e_n \quad (6.13a)$$

$$\tilde{q}_{n-1,n-1} = a_{n-1}d_{n-1}e_{n-1} + b_{n-1}c_{n-1}e_{n-1} + a_{n-1}g_{n-1} \quad (6.13b)$$

$$\tilde{q}_{n-1,n-2} = a_{n-1}d_{n-1}f_{n-1} + b_{n-1}c_{n-1}f_{n-1} + b_{n-1}d_{n-2}e_{n-2} + b_{n-1}g_{n-2} \quad (6.13c)$$

$$\tilde{q}_{n-1,n-3} = b_{n-1}d_{n-2}f_{n-2} \quad (6.13d)$$

$$\tilde{q}_{n-1,k} = 0 \quad \text{if } k < n-3 \text{ or } k > n \quad (6.13e)$$

Thus,  $\tilde{\mathbf{q}}_{n-1}$  is identical to  $\mathbf{q}_{n-1}$  except that the entry  $\tilde{q}_{n-1,n-1}$  does not contain the summand  $a_{n-1}c_n f_n$ . Also, let  $\tilde{\boldsymbol{\ell}}_n = (\tilde{l}_{n,k})_{k \geq 0}$  be the row vector

$$\tilde{\boldsymbol{\ell}}_n = [\mathbf{0}_{1 \times n} \quad e_n \quad \mathbf{0}_{1 \times \infty}] . \quad (6.14)$$

Then

$$q_{n-1,n-1} = \tilde{q}_{n-1,n-1} + \frac{f_n}{e_n} \tilde{q}_{n-1,n} \quad (6.15a)$$

$$l_{n,n-1} = \tilde{l}_{n,n-1} + \frac{f_n}{e_n} \tilde{l}_{n,n} \quad (6.15b)$$

(actually  $\tilde{l}_{n,n-1} = 0$  but it is convenient to write it anyway).

By Lemma 6.6, the matrix

$$M_1 = [[\mathbf{q}_0, \dots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1}]] \quad (6.16)$$

is coefficientwise totally positive. Therefore the matrix

$$M_2 = [[\mathbf{q}_0, \dots, \mathbf{q}_{n-2}, \tilde{\mathbf{q}}_{n-1}]] , \quad (6.17)$$

which is obtained from  $M_1$  by setting  $f_n = 0$ , is also coefficientwise totally positive. (Setting  $f_n = 0$  also affects rows  $n$  and  $n + 1$  of  $Q$ , but these are not contained in  $M_1$ .) Then

$$M_3 = [[\mathbf{q}_0, \dots, \mathbf{q}_{n-2}, \tilde{\mathbf{q}}_{n-1}, \tilde{\boldsymbol{\ell}}_n]] \quad (6.18)$$

is obtained from  $M_2$  by adjoining a row that has  $e_n$  in the  $n$ th column and 0 elsewhere; this is totally positive because  $M_2$  is zero beyond the  $n$ th column (this reasoning is a very special case of Lemma 6.5). Finally, the matrix

$$M = [[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \boldsymbol{\ell}_n]] \quad (6.19)$$

can be obtained by right-multiplying  $M_3$  by the lower-bidiagonal matrix that has 1 on the diagonal,  $f_n/e_n$  in position  $(n, n - 1)$ , and 0 elsewhere (this adds  $f_n/e_n$  times the  $n$ th column to the  $(n - 1)$ st column, which is the content of (6.15)). This completes the proof of Lemma 6.7.  $\square$

**Remark.** To justify the use of the quantity  $f_n/e_n$ , it suffices to work in a ring of Laurent polynomials with indeterminates  $\mathbf{e} = (e_n)_{n \geq 0}$  and  $\mathbf{e}^{-1} = (e_n^{-1})_{n \geq 0}$ , and then to consider coefficientwise total positivity in this ring. At the end, all quantities will belong to the subring consisting of ordinary polynomials with nonnegative exponents.  $\blacksquare$

**Lemma 6.8.** *Fix integers  $0 \leq n \leq m$ . If the matrix  $[[\mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$  is coefficientwise totally positive, then so is the matrix  $M_0 \stackrel{\text{def}}{=} [[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \boldsymbol{\ell}_n, \mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$ .*

PROOF. The case  $n = m$  is Lemma 6.7; so assume that  $n < m$ .

Let  $\mathbf{t}_{n+1}$  be obtained from  $\mathbf{p}_{n+1}$  by specializing  $d_n$  to zero; then

$$\mathbf{p}_{n+1} = \mathbf{t}_{n+1} + b_{n+1} d_n \boldsymbol{\ell}_n . \quad (6.20)$$

Note that this specialization does not affect  $\mathbf{p}_\ell$  for  $\ell > n + 1$ . By hypothesis

$$M_1 \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{p}_{n+1} \\ \vdots \\ \mathbf{p}_m \end{bmatrix} \quad (6.21)$$

is coefficientwise totally positive; this implies, by specialization, that

$$M_2 \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{t}_{n+1} \\ \mathbf{p}_{n+2} \\ \vdots \\ \mathbf{p}_m \end{bmatrix} \quad (6.22)$$

is coefficientwise totally positive. Also let

$$M_3 \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_{n-1} \\ \ell_n \end{bmatrix}. \quad (6.23)$$

(Here for clarity we have avoided the notation  $[[ \cdots ]]$ .)

Now observe that the matrix  $S \stackrel{\text{def}}{=} \begin{bmatrix} M_3 \\ M_2 \end{bmatrix}$  consists of two blocks overlapping in a single column (namely, column  $n$  when the columns are numbered starting at 0):

$$S = \begin{bmatrix} M_3 \\ M_2 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_0 \\ \vdots \\ \mathbf{q}_{n-1} \\ \ell_n \\ \mathbf{t}_{n+1} \\ \mathbf{p}_{n+2} \\ \vdots \\ \mathbf{p}_m \end{bmatrix} = \left[ \begin{array}{c|c|c|c} * & * & * & \mathbf{0}_{n \times \infty} \\ \mathbf{0}_{1 \times n-1} & f_n & e_n & \mathbf{0}_{1 \times \infty} \\ 0 & 0 & * & * \end{array} \right] \quad (6.24)$$

where the asterisks stand for blocks of unspecified entries (which may be zero or nonzero). By Lemma 6.7, the matrix  $M_3$  is totally positive; and we have just shown that the matrix  $M_2$  is totally positive. So the transpose of Lemma 6.5 implies that the matrix  $S$  is totally positive.

On the other hand, the matrix  $M_0$  can be obtained from  $S$  by left-multiplying it by the lower-bidiagonal matrix that has 1 on the diagonal,  $b_{n+1}d_n$  in position  $(n+1, n)$  and zeroes elsewhere (this adds  $b_{n+1}d_n$  times row  $n$  to row  $n+1$ , which is the content of (6.20)). This proves that the desired matrix  $M_0$  is totally positive.  $\square$

**Lemma 6.9** (Induction step).

*Fix integers  $0 \leq n \leq m$ . If the matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_n, \mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$  is coefficientwise totally positive, then so is  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \mathbf{p}_n, \dots, \mathbf{p}_m]]$ .*



PROOF. Notice that  $\mathbf{p}_n = \mathbf{q}_n + h_n \boldsymbol{\ell}_n$ . The matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \mathbf{q}_n, \mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$  is totally positive by hypothesis; and the matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \boldsymbol{\ell}_n, \mathbf{p}_{n+1}, \dots, \mathbf{p}_m]]$  is totally positive by Lemma 6.8. So the conclusion follows by Lemma 6.3 with  $a = 1$  and  $b = h_n$ .  $\square$

**Lemma 6.10.**

*For every pair of integers  $0 \leq n \leq m + 1$ , the matrix  $[[\mathbf{q}_0, \dots, \mathbf{q}_{n-1}, \mathbf{p}_n, \dots, \mathbf{p}_m]]$  is coefficientwise totally positive.*

PROOF. Start from Lemma 6.6, and apply Lemma 6.9 for  $n = m, m - 1, \dots, 0$ .  $\square$

This completes the proof of Theorem 6.1.

We conclude by posing the following open problem:

**Problem 6.11.** *Find a combinatorial interpretation for the output matrix  $A = \mathcal{O}(P)$  generated by the production matrix (6.2)/(6.3), or by interesting specializations thereof.*

## 7 The connection with multiple orthogonal polynomials

In this section we would like to explain how we were led to guess the quadridiagonal production matrix (1.41), by virtue of an unexpected connection with multiple orthogonal polynomials.

Let us first show that, for each  $\alpha \geq -1$  and  $x \geq 0$ , the sequence  $(\mathcal{L}_n^{(\alpha)}(x))_{n \geq 0}$  is a Stieltjes moment sequence, by explicitly exhibiting its Stieltjes moment representation. Indeed, we have the integral representation (compare [77, p. 103, eq. (5.4.1)])

$$\mathcal{L}_n^{(\alpha)}(x) = e^{-x} x^{-\alpha/2} \int_0^\infty u^{n+\alpha/2} e^{-u} I_\alpha(2\sqrt{xu}) du \quad (7.1)$$

valid for  $\alpha > -1$ , where  $I_\alpha$  is a modified Bessel function of the first kind [83, p. 77]

$$I_\alpha(z) = \sum_{k=0}^\infty \frac{(z/2)^{\alpha+2k}}{k! \Gamma(\alpha + k + 1)}. \quad (7.2)$$

For  $n = 0$ , the formula (7.1) is easily verified by inserting (7.2) and integrating term-by-term. The general formula then follows by computing the exponential generating function of the right-hand side of (7.1), i.e. multiplying (7.1) by  $t^n/n!$  and summing over  $n \geq 0$ ; the result is  $(1-t)^{-(\alpha+1)} e^{xt/(1-t)}$ , which is indeed [by (1.2)] the exponential generating function of  $\mathcal{L}_n^{(\alpha)}(x) = n! L_n^{(\alpha)}(x)$ . And since  $I_\alpha$  is nonnegative on  $[0, \infty)$ , the integral representation (7.1) is a Stieltjes moment representation, with representing measure

$$d\mu_{\alpha,x}(u) = e^{-x} x^{-\alpha/2} u^{\alpha/2} e^{-u} I_\alpha(2\sqrt{xu}) du \quad \text{on } [0, \infty) \quad (7.3)$$

for  $\alpha > -1$  (with the corresponding limiting measure when  $\alpha = -1$ ).<sup>15</sup>

We now invoke a recently discovered [68] connection between multiple orthogonal polynomials and branched continued fractions. Recall that multiple orthogonal polynomials [40, Chapter 23] are a generalization of conventional orthogonal polynomials [15, 40, 77] in which the polynomials satisfy orthogonality relations with respect to several measures  $\mu_1, \dots, \mu_r$  rather than just one. In particular, the multiple orthogonal polynomials of type II, denoted  $P_{\mathbf{n}}(x)$  and indexed by a multi-index  $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ , are monic polynomials of degree  $|\mathbf{n}| = \sum_{i=1}^r n_i$  that satisfy the orthogonality relations

$$\int x^k P_{\mathbf{n}}(x) d\mu_i(x) = 0 \quad \text{for } 1 \leq i \leq r \text{ and } 0 \leq k < n_i. \quad (7.4)$$

(Here we restrict attention to the case in which the system of measures  $\mu_1, \dots, \mu_r$  is *perfect*: namely, there exists, for each multi-index  $\mathbf{n}$ , a *unique* monic polynomial  $P_{\mathbf{n}}(x)$  of degree  $|\mathbf{n}|$  satisfying (7.4). Several general sufficient conditions for a system to be perfect are known (Angelesco systems, AT systems, Nikishin systems, ...): see [40, Chapter 23] [78].)

Now recall [81, Theorems 50.1 and 51.1] [79, Chapitre V] [17, Theorem 2.3 and Corollary 2.5] [85, Section 5.2.1] that the conventional orthogonal polynomials associated to a measure  $\mu$  (which we here normalize to total mass 1) satisfy a three-term recurrence relation of the form

$$P_{n+1}(x) = (x - \gamma_n) P_n(x) - \beta_n P_{n-1}(x), \quad (7.5)$$

where the coefficients  $\gamma_n$  and  $\beta_n$  occurring in this recurrence relation are identical to those occurring in the classical J-fraction for the moments of  $\mu$ , i.e.

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{1 - \gamma_0 t - \frac{\beta_1 t^2}{1 - \gamma_1 t - \frac{\beta_2 t^2}{1 - \gamma_2 t - \frac{\beta_3 t^2}{1 - \gamma_3 t - \dots}}}} \quad (7.6)$$

where  $a_n = \int x^n d\mu(x)$ . Now, Flajolet [29] showed that the Taylor coefficient  $a_n$  in (7.6) is the generating polynomial for Motzkin paths from  $(0, 0)$  to  $(n, 0)$  in which each rise gets weight 1, each level step at height  $i$  gets weight  $\gamma_i$ , and each fall from height  $i$  gets weight  $\beta_i$ . In other words,  $a_n$  is the  $(n, 0)$  matrix element of the output matrix  $A = \mathcal{O}(\Pi)$ , where  $\Pi$  is the tridiagonal production matrix

$$\Pi = \begin{bmatrix} \gamma_0 & 1 & & & \\ \beta_1 & \gamma_1 & 1 & & \\ & \beta_2 & \gamma_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \quad (7.7)$$

It turns out [68] that a generalization of this connection holds for multiple orthogonal polynomials. Fix measures  $\mu_1, \dots, \mu_r$ , and let  $(P_{\mathbf{n}}(x))_{\mathbf{n} \in \mathbb{N}^r}$  be the corresponding multiple orthogonal polynomials of type II. Among these, let us consider

---

<sup>15</sup>These facts were observed many years ago by Karlin [42, p. 440] [43, p. 62].

the multi-indices  $\mathbf{n} = (n_1, \dots, n_r)$  lying on the so-called **stepline**: this is the near-diagonal sequence starting at  $(0, 0, \dots, 0)$  and following the path  $(n, n, \dots, n) \rightarrow (n+1, n, \dots, n) \rightarrow (n+1, n+1, \dots, n) \rightarrow \dots \rightarrow (n+1, n+1, \dots, n+1) \rightarrow \dots$ . That is, we define a singly-indexed sequence  $(\tilde{P}_n(x))_{n \geq 0}$  by

$$\tilde{P}_n(x) = P_{(n_1, \dots, n_r)}(x) \quad \text{where } n_i = \left\lfloor \frac{n+r-i}{r} \right\rfloor \text{ for } 1 \leq i \leq r. \quad (7.8)$$

It is well known [40, Theorem 23.1.7] that the stepline sequence  $(\tilde{P}_n(x))_{n \geq 0}$  satisfies an  $(r+2)$ -term recurrence of the form

$$\tilde{P}_{n+1}(x) = (x - \pi_{nn}) \tilde{P}_n(x) - \sum_{i=1}^r \pi_{n, n-i} \tilde{P}_{n-i}(x), \quad (7.9)$$

or equivalently

$$x \tilde{P}_n(x) = \sum_{k=n-r}^{n+1} \pi_{nk} \tilde{P}_k(x) \quad (7.10)$$

where  $\pi_{n, n+1} = 1$ . Here the production matrix  $\Pi = (\pi_{nk})_{n, k \geq 0}$  is an  $(r, 1)$ -banded unit-lower-Hessenberg matrix, i.e.  $\pi_{nk} = 0$  for  $k < n-r$  or  $k > n+1$ . Furthermore, it turns out [68] — and this is the key fact — that the zeroth column of the output matrix  $A = \mathcal{O}(\Pi)$  is precisely the moment sequence of the measure  $\mu_1$ . Therefore, if one knows the recurrence relation (7.9)/(7.10) for the multiple orthogonal polynomials along the stepline, one also knows a production matrix for the moment sequence of the measure  $\mu_1$ .

**Remark.** The subsequent columns of the output matrix  $A = \mathcal{O}(\Pi)$  also have interpretations [68] as linear combinations of the moment sequences of  $\mu_1, \dots, \mu_r$ . For instance, for  $0 \leq i \leq r-1$ , the  $i$ th column of the output matrix is that particular linear combination of the moment sequences of  $\mu_1, \mu_2, \dots, \mu_{i+1}$  that annihilates the first  $i$  entries and makes the next entry equal to 1. But we will not need this fact here. ■

Return now to our Laguerre polynomials and their Stieltjes moment representation with the measure  $\mu_{\alpha, x}$  defined in (7.3). We then benefit from exceedingly good fortune: some years ago, Coussement and Van Assche [19] studied the multiple orthogonal polynomials with  $r = 2$  associated to the pair of measures  $(\mu_1, \mu_2) = (\mu_{\alpha, x}, \mu_{\alpha+1, x})$ . (They actually used a slightly different normalization, so that their moments are a rescaled version of the reversed monic unsigned Laguerre polynomials  $\bar{\mathcal{L}}_n^{(\alpha)}(x)$ : compare [19, Lemma 1] to our (1.4).) And they computed explicitly, among other things, the four-term recurrence relation for the multiple orthogonal polynomials of type II along the stepline [19, Theorem 9]. After translating their normalization to ours, this four-term recurrence corresponds precisely to the quadridiagonal production matrix (1.41) for the monic unsigned Laguerre polynomials  $\mathcal{L}_n^{(\alpha)}(x)$ . This is how we first discovered these production matrices.

From here it was a small step to discover the production matrix (1.43) for our multivariate Laguerre polynomials. Indeed, once we had exploited the Foata–Strehl [31] combinatorial model to define the multivariate Laguerre polynomials (1.37), it was not difficult to guess — helped by a bit of computer experimentation — how the

production matrix (1.41), with its single variable  $x$ , should be refined to introduce the further variables  $y_p, y_v, y_{da}, y_{dd}, y_{fp}$ .

Let us also observe that there is another way that we could have discovered the production matrix (1.41) for the univariate Laguerre polynomials. As observed preceding Proposition 3.1, we already knew a tridiagonal production matrix for the Laguerre polynomials at  $x = 0$ , which are  $\mathcal{L}_n^{(\alpha)}(0) = (1 + \alpha)^{\bar{n}}$ . It would then have been natural to try perturbing this tridiagonal matrix by terms linear in  $x$ ; and if one also realized that one ought to consider introducing a second subdiagonal, one could try the Ansatz

$$p_{n,n+1} = 1 \tag{7.11a}$$

$$p_{n,n} = (2n + 1 + \alpha) + A_n x \tag{7.11b}$$

$$p_{n,n-1} = n(n + \alpha) + B_n x \tag{7.11c}$$

$$p_{n,n-2} = C_n x \tag{7.11d}$$

$$p_{n,k} = 0 \quad \text{if } k < n - 2 \text{ or } k > n + 1 \tag{7.11e}$$

A little computer work — setting the zeroth column of the output matrix equal to  $\mathcal{L}_n^{(\alpha)}(x)$  — would then lead quickly to the values of  $A_n, B_n, C_n$  for small  $n$ , from which one could easily guess that  $A_n = 1, B_n = 2n$  and  $C_n = n(n - 1)$ . This is not, in fact, the way we first found the production matrix (1.41), but it easily could have been.

## Acknowledgments

One of us (A.D.S.) wishes to thank the organizers of the 15th International Symposium on Orthogonal Polynomials, Special Functions and Applications (Hagenberg, Austria, 22–26 July 2019) for inviting him to give a talk there; this allowed him to meet Walter Van Assche and to discover the unexpected connection [68] between branched continued fractions and multiple orthogonal polynomials, which was the starting point for this work.

We wish to thank Xi Chen, Sylvie Corteel and Walter Van Assche for helpful discussions.

This research was supported in part by the U.K. Engineering and Physical Sciences Research Council grant EP/N025636/1 and by the research fellowship DY 133/1-1 from the Deutsche Forschungsgemeinschaft. The first author is currently supported by DIMERS project ANR-18-CE40-0033 funded by Agence Nationale de la Recherche (ANR, France).

## A Generalized and modified $m$ -Stieltjes–Rogers polynomials

In this Appendix we introduce, for each integer  $m \geq 1$ , an infinite sequence of triangular arrays, indexed by an integer  $j \geq 0$ , whose matrix elements we call the *generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$* . We examine in particular the entries in these matrices' zeroth columns, which we call the *modified  $m$ -Stieltjes–Rogers*

*polynomials of type  $j$* . When  $j = 0$  these polynomials reduce to the (generalized)  $m$ -Stieltjes–Rogers polynomials introduced in [59, sections 5, 7 and 9]. When  $m = 1$  they reduce to the generalized classical Stieltjes–Rogers polynomials of the first ( $j = 0$ ) and second ( $j = 1$ ) kinds. The discussion here expands and supersedes the treatment in [59].

We begin (Section A.1) by reviewing the theory of classical Stieltjes–Rogers polynomials, and then (Section A.2) the basic ideas from [59] concerning the  $m$ -Stieltjes–Rogers polynomials (of type 0). Then (Section A.3) we introduce the generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$ . Finally (Section A.4), we apply this theory to the univariate Laguerre production matrix (1.41).

## A.1 Classical Stieltjes–Rogers polynomials

A **Dyck path** is a path in the upper half-plane  $\mathbb{Z} \times \mathbb{N}$ , starting and ending on the horizontal axis, using steps  $(1, 1)$  [“rise” or “up step”] and  $(1, -1)$  [“fall” or “down step”]. More generally, a **Dyck path at level  $k$**  is a path in  $\mathbb{Z} \times \mathbb{N}_{\geq k}$ , starting and ending at height  $k$ , using steps  $(1, 1)$  and  $(1, -1)$ . Clearly a Dyck path must be of even length; we denote by  $\mathcal{D}_{2n}$  the set of Dyck paths from  $(0, 0)$  to  $(2n, 0)$ .

Now let  $\boldsymbol{\alpha} = (\alpha_i)_{i \geq 1}$  be an infinite set of indeterminates, and let  $S_n(\boldsymbol{\alpha})$  be the generating polynomial for Dyck paths from  $(0, 0)$  to  $(2n, 0)$  in which each rise gets weight 1 and each fall from height  $i$  gets weight  $\alpha_i$ . Clearly  $S_n(\boldsymbol{\alpha})$  is a homogeneous polynomial of degree  $n$  with nonnegative integer coefficients; following Flajolet [29], we call it the **Stieltjes–Rogers polynomial** of order  $n$ .

Let  $f_0(t) = \sum_{n=0}^{\infty} S_n(\boldsymbol{\alpha}) t^n$  be the ordinary generating function for Dyck paths with these weights (considered as a formal power series in  $t$ ); and more generally, let  $f_k(t)$  be the ordinary generating function for Dyck paths at level  $k$  with these same weights. (Obviously  $f_k$  is just  $f_0$  with each  $\alpha_i$  replaced by  $\alpha_{i+k}$ ; but we shall not explicitly use this fact.) Then a straightforward “renewal” argument [29] gives the functional equation

$$f_k(t) = 1 + \alpha_{k+1} t f_k(t) f_{k+1}(t) \tag{A.1}$$

or equivalently

$$f_k(t) = \frac{1}{1 - \alpha_{k+1} t f_{k+1}(t)}. \tag{A.2}$$

Iterating (A.2), we see immediately that  $f_k$  is given by the continued fraction

$$f_k(t) = \frac{1}{1 - \frac{\alpha_{k+1} t}{1 - \frac{\alpha_{k+2} t}{1 - \frac{\alpha_{k+3} t}{1 - \dots}}}} \tag{A.3}$$

and in particular that  $f_0$  is given by

$$f_0(t) = \frac{1}{1 - \frac{\alpha_1 t}{1 - \frac{\alpha_2 t}{1 - \frac{\alpha_3 t}{1 - \dots}}}} \tag{A.4}$$

The right-hand sides of (A.3)/(A.4) are called (classical) *Stieltjes-type continued fractions*, or (classical) *S-fractions* for short. This combinatorial interpretation of S-fractions in terms of weighted Dyck paths is due to Flajolet [29].

We now generalize the Stieltjes–Rogers polynomials to a triangular array, as follows: We use the term *partial Dyck path* to denote a path in the upper half-plane  $\mathbb{Z} \times \mathbb{N}$ , using steps  $(1, 1)$  and  $(1, -1)$ , that starts on the horizontal axis but is allowed to end anywhere in the upper half-plane. For  $n, \ell \geq 0$ , we define the *generalized Stieltjes–Rogers polynomial of the first kind*  $S_{n,\ell}(\boldsymbol{\alpha})$  to be the generating polynomial for partial Dyck paths starting at  $(0, 0)$  and ending at  $(2n, 2\ell)$ , in which each rise gets weight 1 and each fall from height  $i$  gets weight  $\alpha_i$ . Obviously  $S_{n,\ell}$  is nonvanishing only for  $0 \leq \ell \leq n$ , so we have an infinite lower-triangular array  $\mathbf{S} = (S_{n,\ell}(\boldsymbol{\alpha}))_{n,\ell \geq 0}$  in which the zeroth column displays the ordinary Stieltjes–Rogers polynomials  $S_{n,0} = S_n$ . In particular we have  $S_{n,n} = 1$  and  $S_{n,n-1} = \sum_{i=1}^{2n-1} \alpha_i$ .

Analogously, we define the *generalized Stieltjes–Rogers polynomial of the second kind*  $S'_{n,\ell}(\boldsymbol{\alpha})$  to be the generating polynomial for partial Dyck paths starting at  $(0, 0)$  and ending at  $(2n + 1, 2\ell + 1)$ , in which again each rise gets weight 1 and each fall from height  $i$  gets weight  $\alpha_i$ . Since  $S'_{n,\ell}$  is nonvanishing only for  $0 \leq \ell \leq n$ , we obtain a second infinite lower-triangular array  $\mathbf{S}' = (S'_{n,\ell}(\boldsymbol{\alpha}))_{n,\ell \geq 0}$ . In particular we have  $S'_{n,n} = 1$  and  $S'_{n,n-1} = \sum_{i=1}^{2n} \alpha_i$ .

The polynomials  $S_{n,\ell}(\boldsymbol{\alpha})$  and  $S'_{n,\ell}(\boldsymbol{\alpha})$  manifestly satisfy the joint recurrence

$$S'_{n,\ell} = S_{n,\ell} + \alpha_{2\ell+2} S_{n,\ell+1} \quad (\text{A.5a})$$

$$S_{n+1,\ell} = S'_{n,\ell-1} + \alpha_{2\ell+1} S'_{n,\ell} \quad (\text{A.5b})$$

for  $n, \ell \geq 0$ , with the initial condition  $S_{0,\ell} = \delta_{\ell 0}$  (where of course we also set  $S'_{n,-1} = 0$ ). It follows that the  $S_{n,\ell}$  satisfy the recurrence

$$S_{n+1,\ell} = S_{n,\ell-1} + (\alpha_{2\ell} + \alpha_{2\ell+1}) S_{n,\ell} + \alpha_{2\ell+1} \alpha_{2\ell+2} S_{n,\ell+1} \quad (\text{A.6})$$

with the initial condition  $S_{0,\ell} = \delta_{\ell 0}$  (where we set  $S_{n,-1} = 0$  and  $\alpha_0 = 0$ ). In other words, the unit-lower-triangular array  $\mathbf{S}$  has the tridiagonal production matrix

$$P = \begin{bmatrix} \alpha_1 & 1 & & & \\ \alpha_1 \alpha_2 & \alpha_2 + \alpha_3 & 1 & & \\ & \alpha_3 \alpha_4 & \alpha_4 + \alpha_5 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \quad (\text{A.7})$$

Please observe now that the production matrix (A.7) can be factorized as a product of two bidiagonal matrices:

$$\begin{bmatrix} \alpha_1 & 1 & & & \\ \alpha_1 \alpha_2 & \alpha_2 + \alpha_3 & 1 & & \\ & \alpha_3 \alpha_4 & \alpha_4 + \alpha_5 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \alpha_2 & 1 & & & \\ & \alpha_4 & 1 & & \\ & & \ddots & \ddots & \end{bmatrix} \begin{bmatrix} \alpha_1 & 1 & & & \\ & \alpha_3 & 1 & & \\ & & \alpha_5 & 1 & \\ & & & \ddots & \ddots \end{bmatrix} \quad (\text{A.8})$$

By Lemma 2.1 this shows [82, Lemma 3.3] that the production matrix (A.7) is totally positive in the ring  $Z[\boldsymbol{\alpha}]$  equipped with the coefficientwise order. It then follows [70] from Theorems 2.9 and 2.15 that the unit-lower-triangular array  $\mathbf{S}$  of generalized

Stieltjes–Rogers polynomials is coefficientwise totally positive, and that the sequence  $(S_n(\boldsymbol{\alpha}))_{n \geq 0}$  of Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive.

Similar considerations apply to the matrix  $S'$  of generalized Stieltjes–Rogers polynomials of the second kind [59, eq. (7.12)]; we leave the details as an exercise for the reader.

## A.2 $m$ -Stieltjes–Rogers polynomials (branched S-fractions)

In what follows we fix an integer  $m \geq 1$ . We recall [4, 10, 59, 62] that an  **$m$ -Dyck path** is a path in the upper half-plane  $\mathbb{Z} \times \mathbb{N}$ , starting and ending on the horizontal axis, using steps  $(1, 1)$  [“rise” or “up step”] and  $(1, -m)$  [“ $m$ -fall” or “down step”]. More generally, an  **$m$ -Dyck path at level  $k$**  is a path in  $\mathbb{Z} \times \mathbb{N}_{\geq k}$ , starting and ending at height  $k$ , using steps  $(1, 1)$  and  $(1, -m)$ . Since the number of up steps must equal  $m$  times the number of down steps, the length of an  $m$ -Dyck path must be a multiple of  $m + 1$ .

Now let  $\boldsymbol{\alpha} = (\alpha_i)_{i \geq m}$  be an infinite set of indeterminates. Then [59] the  **$m$ -Stieltjes–Rogers polynomial** of order  $n$ , denoted  $S_n^{(m)}(\boldsymbol{\alpha})$ , is the generating polynomial for  $m$ -Dyck paths from  $(0, 0)$  to  $((m + 1)n, 0)$  in which each rise gets weight 1 and each  $m$ -fall from height  $i$  gets weight  $\alpha_i$ . Clearly  $S_n^{(m)}(\boldsymbol{\alpha})$  is a homogeneous polynomial of degree  $n$  with nonnegative integer coefficients.

Let  $f_0(t) = \sum_{n=0}^{\infty} S_n^{(m)}(\boldsymbol{\alpha}) t^n$  be the ordinary generating function for  $m$ -Dyck paths with these weights; and more generally, let  $f_k(t)$  be the ordinary generating function for  $m$ -Dyck paths at level  $k$  with these same weights. (Obviously  $f_k$  is just  $f_0$  with each  $\alpha_i$  replaced by  $\alpha_{i+k}$ ; but we shall not explicitly use this fact.) Then straightforward combinatorial arguments [59] lead to the functional equation

$$f_k(t) = 1 + \alpha_{k+m} t f_k(t) f_{k+1}(t) \cdots f_{k+m}(t) \quad (\text{A.9})$$

or equivalently

$$f_k(t) = \frac{1}{1 - \alpha_{k+m} t f_{k+1}(t) \cdots f_{k+m}(t)}. \quad (\text{A.10})$$

Iterating (A.10), we see immediately that  $f_k$  is given by the branched continued fraction

$$f_k(t) = \frac{1}{1 - \alpha_{k+m} t \prod_{i_1=1}^m \frac{1}{1 - \alpha_{k+m+i_1} t \prod_{i_2=1}^m \frac{1}{1 - \alpha_{k+m+i_1+i_2} t \prod_{i_3=1}^m \frac{1}{1 - \cdots}}}} \quad (\text{A.11a})$$

$$= \frac{1}{1 - \frac{\alpha_{k+m} t}{\left(1 - \frac{\alpha_{k+m+1} t}{\left(1 - \frac{\alpha_{k+m+2} t}{\cdots}\right) \cdots \left(1 - \frac{\alpha_{k+2m+1} t}{\cdots}\right)}\right)} \cdots \left(1 - \frac{\alpha_{k+2m} t}{\left(1 - \frac{\alpha_{k+2m+1} t}{\cdots}\right) \cdots \left(1 - \frac{\alpha_{k+3m} t}{\cdots}\right)}\right)} \quad (\text{A.11b})$$

and in particular that  $f_0$  is given by the specialization of (A.11) to  $k = 0$ . We shall call the right-hand side of (A.11) an  **$m$ -branched Stieltjes-type continued fraction**, or  **$m$ -S-fraction** for short.

**Remark.** In truth, we hardly ever use the branched continued fraction (A.11); instead, we work directly with the  $m$ -Dyck paths and/or with the recurrence (A.9)/(A.10) that their generating functions satisfy. ■

### A.3 Generalized $m$ -Stieltjes–Rogers polynomials of type $j$

Fix again an integer  $m \geq 1$ . A *partial  $m$ -Dyck path* is a path in the upper half-plane  $\mathbb{Z} \times \mathbb{N}$ , starting on the horizontal axis but ending anywhere in the upper half-plane, using steps  $(1, 1)$  [“rise”] and  $(1, -m)$  [“ $m$ -fall”]. A partial  $m$ -Dyck path starting at  $(0, 0)$  must stay always within the set  $V_m = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : x = y \bmod (m + 1)\}$ . An  *$m$ -Dyck path* is simply a partial  $m$ -Dyck path that ends on the horizontal axis.

Now let  $\alpha = (\alpha_i)_{i \geq m}$  be an infinite set of indeterminates. We recall (Section A.2) that the  $m$ -Stieltjes–Rogers polynomial  $S_n^{(m)}(\alpha)$  is defined to be the generating polynomial for  $m$ -Dyck paths from  $(0, 0)$  to  $((m + 1)n, 0)$ , in which each rise gets weight 1 and each  $m$ -fall from height  $i$  gets weight  $\alpha_i$ . Now, more generally, for any integer  $j \geq 0$ , we define the *modified  $m$ -Stieltjes–Rogers polynomial of type  $j$* , denoted  $S_n^{(m;j)}(\alpha)$ , to be the generating polynomial for partial  $m$ -Dyck paths from  $(0, 0)$  to  $((m + 1)n + j, j)$ , in which each rise gets weight 1 and each  $m$ -fall from height  $i$  gets weight  $\alpha_i$ .<sup>16</sup>

For  $j = 0$ , the modified  $m$ -Stieltjes–Rogers polynomials are of course the usual  $m$ -Stieltjes–Rogers polynomials  $S_n^{(m;0)}(\alpha) = S_n^{(m)}(\alpha)$ , with ordinary generating function  $f_0(t)$ . For general  $j \geq 0$ , the ordinary generating function of the modified  $m$ -Stieltjes–Rogers polynomials of type  $j$  was found in [59, section 2.3] (though this terminology was not used there). Recall that  $f_k(t)$  is the ordinary generating function for  $m$ -Dyck paths at level  $k$  with the weights  $\alpha$ . We then have:

**Proposition A.1** (Generating function of modified  $m$ -Stieltjes–Rogers polynomials). *For each integer  $j \geq 0$ , we have*

$$\sum_{n=0}^{\infty} S_n^{(m;j)}(\alpha) t^n = f_0(t) f_1(t) \cdots f_j(t). \quad (\text{A.12})$$

The proof is a simple combinatorial argument based on splitting the partial  $m$ -Dyck path at its last return to level 0, then its last return to level 1, etc. [59, sections 2.1 and 2.3].

For  $0 \leq j \leq m$ , the modified  $m$ -Stieltjes–Rogers polynomials  $S_n^{(m;j)}(\alpha)$  also have interpretations in terms of the ordinary  $m$ -Stieltjes–Rogers polynomials  $S_n^{(m)}(\alpha)$ . Besides the trivial case  $j = 0$ , the simplest case is  $j = m$ : since an  $m$ -Dyck path of nonzero length must always end with an  $m$ -fall from height  $m$ , we have

$$S_n^{(m;m)}(\alpha) = \frac{S_{n+1}^{(m)}(\alpha)}{\alpha_m}. \quad (\text{A.13})$$

---

<sup>16</sup>In [59] we used the notation  $S_{n|j}^{(m)}(\alpha)$  for what we are now calling  $S_n^{(m;j)}(\alpha)$ .



Observe also that, multiplying both sides of (A.13) by  $t^n$  and summing over  $n \geq 0$  and then using (A.12), we obtain

$$f_0(t) f_1(t) \cdots f_m(t) = \frac{f_0(t) - 1}{\alpha_m t}, \quad (\text{A.14})$$

which is precisely the fundamental functional equation (A.9) at  $k = 0$ .

Now consider (A.13) specialized to  $\alpha_m = 0$ : it generates partial  $m$ -Dyck paths from  $(0, 0)$  to  $((m+1)n + m, m)$  with the constraint that they must stay always at height  $\geq 1$  (except of course for the starting point). Removing the first step, which is necessarily a rise, we have paths from  $(1, 1)$  to  $((m+1)n + m, m)$  with the constraint that must stay always at height  $\geq 1$ ; or translating everything by  $(-1, -1)$ , we have partial  $m$ -Dyck paths from  $(0, 0)$  to  $((m+1)n + m - 1, m - 1)$ . It follows that

$$S_n^{(m; m-1)}(\boldsymbol{\alpha}) = \frac{S_{n+1}^{(m)}(\boldsymbol{\alpha})}{\alpha_m} \Big|_{\alpha_m=0, \alpha_i \rightarrow \alpha_{i-1}}. \quad (\text{A.15})$$

Continuing in the same way, we have for  $0 \leq \ell \leq m$ ,

$$S_n^{(m; m-\ell)}(\boldsymbol{\alpha}) = \frac{S_{n+1}^{(m)}(\boldsymbol{\alpha})}{\alpha_m} \Big|_{\alpha_m=\alpha_{m+1}=\dots=\alpha_{m+\ell-1}=0, \alpha_i \rightarrow \alpha_{i-\ell}}. \quad (\text{A.16})$$

So all the modified  $m$ -Stieltjes–Rogers polynomials  $S_n^{(m; j)}(\boldsymbol{\alpha})$  for  $0 \leq j \leq m$  can be interpreted as  $S_{n+1}^{(m)}(\boldsymbol{\alpha})/\alpha_m$  with some initial coefficients  $\alpha_i$  set to zero (and the variables renamed). Multiplying both sides of (A.16) by  $t^n$  and summing over  $n \geq 0$ , we obtain an alternate form for the generating function of Proposition A.1 when  $j \leq m$ :

**Proposition A.2** (Alternate generating function of modified  $m$ -Stieltjes–Rogers polynomials). *For each integer  $j$  satisfying  $0 \leq j \leq m$ , we have*

$$\sum_{n=0}^{\infty} S_n^{(m; j)}(\boldsymbol{\alpha}) t^n = \frac{f_0(t) - 1}{\alpha_m t} \Big|_{\alpha_m=\alpha_{m+1}=\dots=\alpha_{2m-j-1}=0, \alpha_i \rightarrow \alpha_{i+j-m}}. \quad (\text{A.17})$$

We now generalize this construction by introducing, for each integer  $j \geq 0$ , a triangular array. For each triplet of integers  $j, n, k \geq 0$ , let  $S_{n,k}^{(m; j)}(\boldsymbol{\alpha})$  be the generating polynomial for partial  $m$ -Dyck paths from  $(0, 0)$  to  $((m+1)n + j, (m+1)k + j)$ , in which each rise gets weight 1 and each  $m$ -fall from height  $i$  gets weight  $\alpha_i$ . We call the  $S_{n,k}^{(m; j)}$  the **generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$** . Obviously  $S_{n,k}^{(m; j)}$  is nonvanishing only for  $0 \leq k \leq n$ , and  $S_{n,n}^{(m; j)} = 1$ . We therefore have, for each integer  $j \geq 0$ , an infinite unit-lower-triangular array  $\mathbf{S}^{(m; j)} = (S_{n,k}^{(m; j)}(\boldsymbol{\alpha}))_{n,k \geq 0}$ . The zeroth column of this array ( $k = 0$ ) consists of the modified  $m$ -Stieltjes–Rogers polynomials of type  $j$  defined earlier:  $S_{n,0}^{(m; j)}(\boldsymbol{\alpha}) = S_n^{(m; j)}(\boldsymbol{\alpha})$ ,

Let us now take a closer look at the triangular arrays  $\mathbf{S}^{(m; j)}$ . Note first that the arrays for  $0 \leq j \leq m$  are the most fundamental, in the sense that any array  $\mathbf{S}^{(m; j)}$  for  $j \geq m+1$  is a submatrix of the array  $\mathbf{S}^{(m; j')}$  with  $j' \stackrel{\text{def}}{=} j \bmod (m+1)$ : namely,

if  $j = j' + (m + 1)\ell$ , then  $S_{n,k}^{(m;j)} = S_{n+\ell,k+\ell}^{(m;j')}$ . In terms of the matrix  $\Delta$  with 1 on the superdiagonal and 0 elsewhere, we can write

$$\mathfrak{S}^{(m;j'+(m+1)\ell)} = \Delta^\ell \mathfrak{S}^{(m;j')} (\Delta^\top)^\ell. \quad (\text{A.18})$$

So we shall concentrate, wherever necessary, on the cases  $0 \leq j \leq m$ .

The polynomials  $S_{n,k}^{(m;j)}$  satisfy a simple pair of recurrences, based on examining the two possibilities (rise or  $m$ -fall) for the final step of a partial  $m$ -Dyck path:

$$S_{n,k}^{(m;j+1)} = S_{n,k}^{(m;j)} + \alpha_{(m+1)(k+1)+j} S_{n,k+1}^{(m;j)} \quad (\text{A.19})$$

$$S_{n+1,k}^{(m;j)} = S_{n,k-1}^{(m;j+m)} + \alpha_{(m+1)k+j+m} S_{n,k}^{(m;j+m)} \quad (\text{A.20})$$

In each case, the first (resp. second) term on the right-hand side corresponds to the final step being a rise (resp. an  $m$ -fall). For the classical case  $m = 1$  and  $j = 0$ , these recurrences are (A.5).

The production matrix for the triangle  $\mathfrak{S}^{(m;0)}$  was found in [59, sections 7.1 and 8.2]; and the production matrix for the triangle  $\mathfrak{S}^{(m;m)}$  was in essence found in [59, section 7.2] (though this terminology was not used there). Here we will generalize this analysis, and find the production matrices for all the triangles  $\mathfrak{S}^{(m;j)}$  with  $0 \leq j \leq m$ .

We begin by defining some special matrices  $M = (m_{ij})_{i,j \geq 0}$ :

- $L(s_1, s_2, \dots)$  is the lower-bidiagonal matrix with 1 on the diagonal and  $s_1, s_2, \dots$  on the subdiagonal:

$$L(s_1, s_2, \dots) = \begin{bmatrix} 1 & & & & & \\ s_1 & 1 & & & & \\ & s_2 & 1 & & & \\ & & s_3 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \end{bmatrix}. \quad (\text{A.21})$$

- $U^*(s_1, s_2, \dots)$  is the upper-bidiagonal matrix with 1 on the superdiagonal and  $s_1, s_2, \dots$  on the diagonal:

$$U^*(s_1, s_2, \dots) = \begin{bmatrix} s_1 & 1 & & & & \\ & s_2 & 1 & & & \\ & & s_3 & 1 & & \\ & & & s_4 & 1 & \\ & & & & \ddots & \ddots \end{bmatrix}. \quad (\text{A.22})$$

Note that

$$\Delta L(s_1, s_2, \dots) = U^*(s_1, s_2, \dots) \quad (\text{A.23a})$$

$$L(s_1, s_2, \dots) \Delta = U^*(0, s_1, s_2, \dots) \quad (\text{A.23b})$$

Let us now define, for each  $r \geq 0$ , the matrices

$$L_r \stackrel{\text{def}}{=} L(\alpha_{m+r}, \alpha_{2m+r+1}, \alpha_{3m+r+2}, \dots) \quad (\text{A.24})$$

and

$$U_r \stackrel{\text{def}}{=} \Delta L_r = U^*(\alpha_{m+r}, \alpha_{2m+r+1}, \alpha_{3m+r+2}, \dots). \quad (\text{A.25})$$

We then have:

**Proposition A.3** (Production matrices for the generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$ ). For  $0 \leq j \leq m$ , the generalized  $m$ -Stieltjes–Rogers matrix of type  $j$ , namely  $\mathbf{S}^{(m;j)} = (S_{n,k}^{(m;j)}(\boldsymbol{\alpha}))_{n,k \geq 0}$ , has production matrix  $P^{(m;j)} \stackrel{\text{def}}{=} (\mathbf{S}^{(m;j)})^{-1} \Delta \mathbf{S}^{(m;j)}$  given by

$$P^{(m;j)} = L_{j+1} L_{j+2} \cdots L_m U_0 L_1 L_2 \cdots L_j. \quad (\text{A.26})$$

(Here  $L_{j+1} L_{j+2} \cdots L_m$  is the identity matrix when  $j = m$ , and  $L_1 L_2 \cdots L_j$  is the identity matrix when  $j = 0$ .)

This result was also found independently, and almost simultaneously, by Helder Lima [49].

**Remark.** Any production matrix  $P^{(m;j)}$  as in (A.26) can be rewritten as  $P^{(m;j')}$  for any  $j' \in [j, m]$  with different parameters  $\boldsymbol{\alpha}$ , by using (A.23) to move the matrix  $\Delta$  to the left:  $L(s_1, s_2, \dots) \Delta = \Delta L(0, s_1, s_2, \dots)$ . A little thought then shows that this result can be stated as

$$P^{(m;j)}(\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \dots) = P^{(m;j+1)}(0, \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \dots) \quad \text{for } 0 \leq j \leq m-1. \quad (\text{A.27})$$

There is also a simple combinatorial argument for this identity: as explained below,  $(P^{(m;j)})_{k,k'}$  is the sum over  $(m+1)$ -step walks in  $\mathbb{N}$  going from height  $(m+1)k + j$  to height  $(m+1)k' + j$ , using steps  $(1, 1)$  and  $(1, -m)$ , with weight  $\alpha_i$  for each  $m$ -fall from height  $i$ ; and these quantities manifestly satisfy (A.27).

However, the converse is not true: it is not in general possible to *reduce*  $j$ . In particular, the modified  $m$ -Stieltjes–Rogers polynomials of type  $j > 0$  are a genuine generalization of those of type 0. ■

The proof of Proposition A.3 is in fact a very minor modification of the proof of [59, Proposition 7.2], but for clarity we include a few more details:

**PROOF.** Let us first clarify what the matrices  $L_r$  and  $U_0$  do. The  $i$ th row of the matrix  $L_r$  corresponds to steps starting at height  $(m+1)i + r - 1$ : more precisely, the diagonal ( $i \rightarrow i$ ) entries in the matrix  $L_r$  give the weights for rises from height  $(m+1)i + r - 1$  to height  $(m+1)i + r$ , while the subdiagonal ( $i \rightarrow i - 1$ ) entries in the matrix  $L_r$  give the weights for  $m$ -falls from height  $(m+1)i + r - 1$  to height  $(m+1)(i-1) + r$ . Similarly, the  $i$ th row of the matrix  $U_0$  corresponds to steps starting at height  $(m+1)i + m$ : more precisely, the superdiagonal ( $i \rightarrow i + 1$ ) entries in the matrix  $U_0$  give the weights for rises from height  $(m+1)i + m$  to height  $(m+1)(i+1)$ , while the diagonal ( $i \rightarrow i$ ) entries in the matrix  $U_0$  give the weights for  $m$ -falls from height  $(m+1)i + m$  to height  $(m+1)i$ .

The production matrix of  $\mathbf{S}^{(m;j)}$  tells us how to get from the  $n$ th row of  $\mathbf{S}^{(m;j)}$  to the  $(n+1)$ st row. That is, to get the  $(k, k')$  matrix element of  $P^{(m;j)}$ , we need to enumerate the  $(m+1)$ -step walks in  $\mathbb{N}$  going from height  $(m+1)k + j$  to height  $(m+1)k' + j$ , using steps  $(1, 1)$  and  $(1, -m)$ , with weights  $\boldsymbol{\alpha}$ ; we will show that the right-hand side of (A.26) does the job. The first step starts at height  $(m+1)k + j$  and uses the matrix  $L_{j+1}$ : it is either a rise, using the diagonal ( $i \rightarrow i$ ) with weight 1, or an  $m$ -fall, using the subdiagonal ( $i \rightarrow i - 1$ ) with weight  $\alpha_{(m+1)k+j}$ . We are now at height  $(m+1)k_1 + j + 1$ , where  $k_1$  is either  $k$  (if we made a rise) or  $k - 1$  (if we made an  $m$ -fall). The next step uses the matrix  $L_{j+2}$  in an analogous way, and we end up at height  $(m+1)k_2 + j + 2$ , where  $k_2$  is either  $k_1$  (if we made a rise) or  $k_1 - 1$  (if we

made an  $m$ -fall). And so forth, through the matrix  $L_m$ . After using the matrix  $L_m$ , we are at height  $(m+1)k_{m-j} + m$ , and we use the matrix  $U_0$ : this step is either a rise, using the superdiagonal ( $i \rightarrow i+1$ ) with weight 1, or else an  $m$ -fall, using the diagonal ( $i \rightarrow i$ ) with weight  $\alpha_{(m+1)k_{m-j}+m}$ . We are now at height  $(m+1)\widehat{k}$ , where  $\widehat{k}$  equals  $k+1$  minus the number of  $m$ -falls that have occurred so far. Now we use the matrix  $L_1$ : it is either a rise, using the diagonal ( $i \rightarrow i$ ) with weight 1, or an  $m$ -fall, using the subdiagonal ( $i \rightarrow i-1$ ) with weight  $\alpha_{(m+1)\widehat{k}}$ . We are now at height  $(m+1)\widehat{k}_1 + 1$ , where  $\widehat{k}_1$  is either  $\widehat{k}$  (if we made a rise) or  $\widehat{k}-1$  (if we made an  $m$ -fall). The next step uses the matrix  $L_2$  in an analogous way, and so forth through  $L_j$ . We end at height  $(m+1)\widehat{k}_j + j$ , where  $\widehat{k}_j$  equals  $k+1$  minus the number of  $m$ -falls that have occurred throughout the whole process. But  $k+1$  minus the number of  $m$ -falls is exactly the index we have arrived at in the matrix product  $L_{j+1}L_{j+2} \cdots L_m U_0 L_1 L_2 \cdots L_j$ , because each rise (resp.  $m$ -fall) corresponds to a step  $i \rightarrow i$  (resp.  $i \rightarrow i-1$ ), except for the case of  $U_0$ , for which a rise (resp.  $m$ -fall) corresponds to a step  $i \rightarrow i+1$  (resp.  $i \rightarrow i$ ). That is,  $\widehat{k}_j = k'$ , and the proof is complete.  $\square$

It will be useful to write out explicitly the production matrices for the case  $m=2$ , which is the one arising in this paper. For  $j=0$  we have [68, eq. (7.11)]

$$P_{n,n} = \alpha_{3n} + \alpha_{3n+1} + \alpha_{3n+2} \quad (\text{A.28a})$$

$$P_{n,n-1} = \alpha_{3n-2}\alpha_{3n} + \alpha_{3n-1}\alpha_{3n} + \alpha_{3n-1}\alpha_{3n+1} \quad (\text{A.28b})$$

$$P_{n,n-2} = \alpha_{3n-4}\alpha_{3n-2}\alpha_{3n} \quad (\text{A.28c})$$

provided that we make the convention  $\alpha_0 = \alpha_1 = 0$ . (If  $\alpha_{3n}$  and  $\alpha_{3n+1}$  are given by polynomial expressions in  $n$  that do *not* vanish when  $n=0$ , then  $P_{n,n}$  and  $P_{n,n-1}$  are given by the corresponding polynomial expressions *plus* correction terms proportional to  $\delta_{n,0}$  and  $\delta_{n,1}$ .) For  $j=1$  we have

$$P_{n,n} = \alpha_{3n+1} + \alpha_{3n+2} + \alpha_{3n+3} \quad (\text{A.29a})$$

$$P_{n,n-1} = \alpha_{3n-1}\alpha_{3n+1} + \alpha_{3n}\alpha_{3n+1} + \alpha_{3n}\alpha_{3n+2} \quad (\text{A.29b})$$

$$P_{n,n-2} = \alpha_{3n-3}\alpha_{3n-1}\alpha_{3n+1} \quad (\text{A.29c})$$

provided that we make the convention  $\alpha_1 = 0$ . For  $j=2$  we have

$$P_{n,n} = \alpha_{3n+2} + \alpha_{3n+3} + \alpha_{3n+4} \quad (\text{A.30a})$$

$$P_{n,n-1} = \alpha_{3n}\alpha_{3n+2} + \alpha_{3n+1}\alpha_{3n+2} + \alpha_{3n+1}\alpha_{3n+3} \quad (\text{A.30b})$$

$$P_{n,n-2} = \alpha_{3n-2}\alpha_{3n}\alpha_{3n+2} \quad (\text{A.30c})$$

Note that these expressions arise from (A.28) simply by incrementing all the indices by  $j$ ; this fact is equivalent to (A.27).

Let us now prove some theorems on total positivity. We begin with the total positivity of the production matrices:

**Proposition A.4** (Total positivity of the production matrices). *For  $0 \leq j \leq m$ , the production matrix  $P^{(m;j)} \stackrel{\text{def}}{=} (\mathbf{S}^{(m;j)})^{-1} \Delta \mathbf{S}^{(m;j)}$  is totally positive in the ring  $\mathbb{Z}[\boldsymbol{\alpha}]$  equipped with the coefficientwise order.*

PROOF. This is an immediate consequence of the factorization (A.26) combined with Lemma 2.1.  $\square$

Next we prove the total positivity of the triangular arrays  $\mathbf{S}^{(m;j)}$ . As in [59], we will give two proofs of each result: a graphical proof, based on the Lindström–Gessel–Viennot lemma; and an algebraic proof, based on production matrices.

**Theorem A.5** (Total positivity for generalized  $m$ -Stieltjes–Rogers polynomials). *For each integer  $j \geq 0$ , the lower-triangular matrix  $\mathbf{S}^{(m;j)}$  is totally positive in the ring  $\mathbb{Z}[\boldsymbol{\alpha}]$  equipped with the coefficientwise order.*

GRAPHICAL PROOF. We apply the Lindström–Gessel–Viennot lemma (see [59, section 9.4] for a summary) to the directed graph  $G_m = (V_m, E_m)$  with vertex set

$$V_m = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : x = y \bmod (m + 1)\} \quad (\text{A.31})$$

and edge set

$$E_m = \{((x_1, y_1), (x_2, y_2)) \in V_m \times V_m : x_2 - x_1 = 1 \text{ and } y_2 - y_1 \in \{1, -m\}\}. \quad (\text{A.32})$$

It is easy to see that  $G_m$  is planar and acyclic.

Now consider an  $\ell \times \ell$  minor of  $\mathbf{S}^{(m;j)}$ : that is, we choose sets of integers  $I = \{i_1, i_2, \dots, i_\ell\}$  with  $0 \leq i_1 < i_2 < \dots < i_\ell$  and  $J = \{j_1, j_2, \dots, j_\ell\}$  with  $0 \leq j_1 < j_2 < \dots < j_\ell$ , and we consider the  $\ell \times \ell$  submatrix

$$S_{IJ}^{(m;j)} = (S_{i_r, j_s}^{(m;j)}(\boldsymbol{\alpha}))_{1 \leq r, s \leq \ell} \quad (\text{A.33})$$

and the corresponding minor

$$\Delta_{IJ}(\mathbf{S}^{(m;j)}) = \det S_{IJ}^{(m;j)}. \quad (\text{A.34})$$

We can write the elements of the submatrix  $\mathbf{S}_{IJ}^{(m;j)}$  as sums over walks in the directed graph  $G_m$ : it is easy to see that  $S_{i_r, j_s}^{(m;j)}(\boldsymbol{\alpha})$  is the sum over walks from  $(-(m + 1)i_r, 0)$  to  $(j, (m + 1)j_s + j)$ , with a weight 1 on each rising directed edge and a weight  $\alpha_i$  on each  $m$ -falling directed edge starting at height  $i$ . Note that the sets of vertices  $I^* = \{(0, 0) = i_0^* < (-(m + 1), 0) = i_1^* < (-2(m + 1), 0) = i_2^* < \dots\}$  and  $J^* = \{(j, j) = j_0^* < (j, m + 1 + j) = j_1^* < (j, 2(m + 1) + j) = j_2^* < \dots\}$  lie on the boundary of  $G_m \upharpoonright ((-\infty, j] \times \mathbb{N})$  in the order “first  $I^*$  in reverse order, then  $J^*$  in order”; therefore, by [59, Lemma 9.18] they form a fully nonpermutable pair. Applying the Lindström–Gessel–Viennot lemma, we conclude that the minor  $\Delta_{IJ}(\mathbf{S}^{(m;j)})$  is the generating polynomial for families of vertex-disjoint paths  $P_1, \dots, P_\ell$  in the digraph  $G_m$ , where path  $P_r$  starts at  $(-(m + 1)i_r, 0)$  and ends at  $(j, (m + 1)j_r + j)$ , in which each rise gets weight 1 each  $m$ -fall from height  $i$  gets weight  $\alpha_i$ . Since every such polynomial manifestly has nonnegative integer coefficients, Theorem A.5 is proven.  $\square$

ALGEBRAIC PROOF. Since the array  $\mathbf{S}^{(m;j)}$  for  $j \geq m + 1$  is a submatrix of the array  $\mathbf{S}^{(m;j')}$  with  $j' \stackrel{\text{def}}{=} j \bmod (m + 1)$ , it suffices to prove the theorem for  $0 \leq j \leq m$ . And the latter is an immediate consequence of the total positivity of the production matrix  $P^{(m;j)}$  (Proposition A.4) combined with Theorem 2.9.  $\square$

**Remark.** Theorem A.5 was proven for the special case  $j = 0$  in [59, Theorem 9.10], with both graphical and algebraic proofs. The graphical proof given here for general  $j \geq 0$  is the obvious generalization of the graphical proof given there. The algebraic proof given here reduces to the one given there when  $j = 0$ , but for  $1 \leq j \leq m$  it requires the new production matrices (A.26), and for  $j \geq m + 1$  it requires the obvious submatrix argument. ■

The second result concerns the Hankel-total positivity of the modified  $m$ -Stieltjes–Rogers polynomials  $S_n^{(m;j)}(\boldsymbol{\alpha})$ . It was already proven in [59, Theorem 9.12], so we simply state the result here:

**Theorem A.6** (Hankel-total positivity for modified  $m$ -Stieltjes–Rogers polynomials). *For  $0 \leq j \leq m$ , the sequence  $\mathbf{S}^{(m;j)} = (S_n^{(m;j)}(\boldsymbol{\alpha}))_{n \geq 0}$  of modified  $m$ -Stieltjes–Rogers polynomials of type  $j$  is a Hankel-totally positive sequence in the polynomial ring  $\mathbb{Z}[\boldsymbol{\alpha}]$  equipped with the coefficientwise partial order.*

The proof of Theorem A.6 using the Lindström–Gessel–Viennot lemma is similar to that of Theorem A.5, but using sink vertices  $J^{**} = \{(j, j) = j_0^{**} < (m + 1 + j, j) = j_1^{**} < (2(m + 1) + j, j) = j_2^{**} < \dots\}$  in place of  $J^*$ . For  $j \leq m$  (but *only* then), the source and sink vertices lie on the boundary of the graph  $G_m$ , and the pair  $(I^*, J^{**})$  is fully nonpermutable. The algebraic proof of Theorem A.6 uses the production matrix (A.26) together with Theorem 2.15; once again, it works only for  $j \leq m$ , since the “submatrix argument” does not apply to the zeroth column. Indeed, the restriction of these proofs to  $j \leq m$  is no accident: as explained in [59, end of section 9.4], the conclusion of Theorem A.6 is *false* for  $j > m$ ; in fact, it fails even for the  $2 \times 2$  minors.

**Final remark.** For each fixed  $m \geq 1$ , the quantities  $S_{n,k}^{(m;j)}$  depend on the three parameters  $n, k, j$ . We have chosen here to fix  $j$  and then assemble these quantities into a lower-triangular matrix  $\mathbf{S}^{(m;j)} = (S_{n,k}^{(m;j)}(\boldsymbol{\alpha}))_{n,k \geq 0}$ . Lima [48] has pointed out that one can also fix  $k = 0$  and assemble them into a (full) matrix  $\widehat{\mathbf{S}}^{(m)} = (S_{n,0}^{(m;j)}(\boldsymbol{\alpha}))_{n,j \geq 0}$ . He then shows [48, Proposition 3.1] the wonderful  $LU$  factorization

$$\widehat{\mathbf{S}}^{(m)} = \mathbf{S}^{(m;0)} \boldsymbol{\Lambda}^{(m)} \tag{A.35}$$

for an explicit upper-triangular matrix  $\boldsymbol{\Lambda}^{(m)}$ . Furthermore, he shows [48, Proposition 3.2] that  $\boldsymbol{\Lambda}^{(m)}$  is coefficientwise totally positive. Since  $\mathbf{S}^{(m;0)}$  is also coefficientwise totally positive by [59, Theorem 9.10] (i.e., the case  $j = 0$  of Theorem A.5), this shows that  $\widehat{\mathbf{S}}^{(m)}$  is coefficientwise totally positive [48, Theorem 3.3]. ■

## A.4 Application to the univariate Laguerre production matrix

Let us now apply this theory to determine the conditions under which the univariate Laguerre production matrix (1.41), namely

$$p_{n,n+1} = 1 \tag{A.36a}$$

$$p_{n,n} = (2n + 1 + \alpha) + x \tag{A.36b}$$

$$p_{n,n-1} = n(n + \alpha) + 2nx \tag{A.36c}$$

$$p_{n,n-2} = n(n - 1)x \tag{A.36d}$$

	$j = 0$	$j = 1$	$j = 2$
$\alpha = -1$	Proposition A.8 $\alpha_{3n-1} = x$ $\alpha_{3n} = c_n n$ $\alpha_{3n+1} = (2 - c_n) n$	Proposition A.11 $\alpha_2 = 0$ $\alpha_{3n} = x$ $\alpha_{3n+1} = c_n n$ $\alpha_{3n+2} = (2 - c_n) n$	Proposition A.14 $\alpha_2 = \alpha_3 = 0$ $\alpha_{3n+1} = x$ $\alpha_{3n+2} = c_n n$ $\alpha_{3n+3} = (2 - c_n) n$
$\alpha = 0$		Proposition A.12 $\alpha_{3n-1} = x$ $\alpha_{3n} = n$ $\alpha_{3n+1} = n$	Proposition A.15 $\alpha_2 = 0$ $\alpha_{3n} = x$ $\alpha_{3n+1} = n$ $\alpha_{3n+2} = n$
$\alpha = 1$			Proposition A.16 $\alpha_{3n-1} = x$ $\alpha_{3n} = c_n n$ $\alpha_{3n+1} = (2 - c_n) n$

Table 1: Cases where the univariate Laguerre production matrix (1.41)/(A.36) can be written as the production matrix  $P^{(m;j)}$  for the generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$  with  $m = 2$  and  $j = 0, 1$  or  $2$ . Here  $c_n$  is given by (A.39), with  $\kappa \in [0, 1]$ . Note that each passage to the right (at fixed  $\alpha$ ) in this table is given by the transformation (A.27).

can be written as the production matrix  $P^{(m;j)}$  [cf. (A.26)] for the generalized  $m$ -Stieltjes–Rogers polynomials of type  $j$  with  $m = 2$  and  $j = 0, 1$  or  $2$  and some  $\alpha$ . To do this, we will use the formulae (A.28)–(A.30) for  $m = 2$  and  $j = 0, 1, 2$ , remembering that  $\alpha_0 = \alpha_1 \stackrel{\text{def}}{=} 0$ . In this analysis, we will interpret the Laguerre parameter  $\alpha$  as a fixed real number  $\geq -1$ . We will require that the Stieltjes parameters  $\alpha$  be polynomials with real coefficients in the indeterminate  $x$ , and we will most often require that these be polynomials with *nonnegative* real coefficients.

Since some of the computations are fairly lengthy, it may be helpful to the reader to give now a summary of our results. We will find that

- $j = 0$  works for  $\alpha = -1$ , and only that value;
- $j = 1$  works for  $\alpha = -1, 0$ , and only those values;
- $j = 2$  works for  $\alpha = -1, 0, 1$ , and only those values.

The allowed parameter sets  $\alpha$  for each pair  $(j, \alpha)$  are summarized in Table 1. We stress that the different columns in each row correspond to different factorizations of the *same* production matrix.

#### A.4.1 Case $j = 0$

**Proposition A.7** ( $j = 0$  implies  $\alpha = -1$ ). *The univariate Laguerre production matrix (1.41)/(A.36) can be written as the production matrix  $P^{(2;0)}$ , where  $\alpha_2, \alpha_4, \alpha_6$  are polynomials in  $x$ , only if  $\alpha = -1$ .*

PROOF. From (A.36) and (A.28) we have

$$p_{0,0} = 1 + \alpha + x = \alpha_2 \quad (\text{A.37a})$$

$$p_{2,0} = 2x = \alpha_2 \alpha_4 \alpha_6 \quad (\text{A.37b})$$

Since  $\alpha_4$  and  $\alpha_6$  are polynomials in  $x$ , the polynomial  $p_{0,0} = 1 + \alpha + x$  must divide  $p_{2,0} = 2x$ , which occurs only if  $\alpha = -1$ .  $\square$

**Proposition A.8** (Solutions for  $j = 0$  and  $\alpha = -1$ ). *The univariate Laguerre production matrix (1.41)/(A.36) with  $\alpha = -1$  can be written as the production matrix  $P^{(2;0)}$ , where the  $\alpha$  are polynomials in  $x$  with nonnegative coefficients, in the following ways (and only the following ways):*

Let  $\kappa$  be any real number in the interval  $[0, 1]$ , and set

$$\alpha_{3n-1} = x \quad (\text{A.38a})$$

$$\alpha_{3n} = c_n n \quad (\text{A.38b})$$

$$\alpha_{3n+1} = (2 - c_n) n \quad (\text{A.38c})$$

where

$$c_n \stackrel{\text{def}}{=} \frac{(n-1) - (n-2)\kappa}{n - (n-1)\kappa} \quad \text{for } n \geq 1. \quad (\text{A.39})$$

Please note that  $0 \leq \kappa = c_1 \leq c_2 \leq c_3 \leq \dots \leq 1$  and  $\lim_{n \rightarrow \infty} c_n = 1$ .

PROOF. We first prove the necessity, then the sufficiency.

**Necessity.** For  $j = 0$  and  $\alpha = -1$  we have

$$p_{0,0} = x = \alpha_2 \quad (\text{A.40a})$$

$$p_{n,n} = 2n + x = \alpha_{3n} + \alpha_{3n+1} + \alpha_{3n+2} \quad \text{for } n \geq 1 \quad (\text{A.40b})$$

$$p_{1,0} = 2x = \alpha_2(\alpha_3 + \alpha_4) \quad (\text{A.40c})$$

$$p_{n,n-1} = n(n-1) + 2nx = \alpha_{3n-2}\alpha_{3n} + \alpha_{3n-1}(\alpha_{3n} + \alpha_{3n+1}) \quad \text{for } n \geq 2 \quad (\text{A.40d})$$

$$p_{n,n-2} = n(n-1)x = \alpha_{3n-4}\alpha_{3n-2}\alpha_{3n} \quad (\text{A.40e})$$

From  $p_{0,0}$  we get  $\alpha_2 = x$ ; from  $p_{1,0}$  we then get  $\alpha_3 + \alpha_4 = 2$ ; and from  $p_{2,0}$  we then get  $\alpha_4\alpha_6 = 2$ . We will now prove by induction that for all  $n \geq 1$ , we have

$$\alpha_{3n-1} = x \quad (\text{A.41a})$$

$$\alpha_{3n} + \alpha_{3n+1} = 2n \quad (\text{A.41b})$$

$$\alpha_{3n+1}\alpha_{3n+3} = n(n+1) \quad (\text{A.41c})$$



The base case  $n = 1$  has just been proven. Using the inductive hypothesis  $\alpha_{3n} + \alpha_{3n+1} = 2n$  and the formula for  $p_{n,n}$ , we deduce  $\alpha_{3n+2} = x$ . Then using this together with the inductive hypothesis  $\alpha_{3n+1}\alpha_{3n+3} = n(n+1)$  and the formula for  $p_{n+1,n}$ , we deduce  $\alpha_{3n+3} + \alpha_{3n+4} = 2n + 2$ . And finally, from  $\alpha_{3n+2} = x$  and the formula for  $p_{n+2,n}$ , we deduce  $\alpha_{3n+4}\alpha_{3n+6} = (n+1)(n+2)$ . This completes the induction.

Writing  $\alpha_{3n} = c_n n$ , (A.41b,c) say that  $\alpha_{3n+1} = (2 - c_n)n$  and  $c_{n+1} = 1/(2 - c_n)$ . It is then easily proven by induction that the solution to this recurrence, with initial condition  $c_1 = \kappa$ , is given by (A.39).

By hypothesis we must have  $c_1 \geq 0$ , i.e.  $\kappa \geq 0$ . On the other hand, if  $\kappa > 1$ , then there exists a positive integer  $n$  satisfying

$$\frac{n-1}{n-2} < \kappa \leq \frac{n}{n-1}, \quad (\text{A.42})$$

which makes either  $c_n < 0$  or  $c_n = \infty$ , both of which are forbidden. So we must have  $\kappa \in [0, 1]$ .

**Sufficiency.** It is easy to verify that, for any  $\kappa \in [0, 1]$ , the parameters  $\alpha$  defined by (A.38)/(A.39) satisfy (A.41) and thence (A.40).  $\square$

**Remarks.** 1. The reasoning in this proof can be abstracted as a *general* method for writing a quadridiagonal unit-lower-Hessenberg matrix  $P$  as the production matrix  $P^{(2;0)}$  of a 2-S-fraction of type  $j = 0$  [cf. (A.28)] whenever this is possible in the chosen ring, as follows: For  $n \geq 1$ , define

$$a_n = \alpha_{3n-1} \quad (\text{A.43a})$$

$$b_n = \alpha_{3n} + \alpha_{3n+1} \quad (\text{A.43b})$$

$$c_n = \alpha_{3n+1}\alpha_{3n+3} \quad (\text{A.43c})$$

Then the equations for  $p_{0,0}$ ,  $p_{1,0}$  and  $p_{2,0}$  give

$$a_1 = p_{0,0} \quad (\text{A.44a})$$

$$b_1 = \frac{p_{1,0}}{p_{0,0}} \quad (\text{A.44b})$$

$$c_1 = \frac{p_{2,0}}{p_{0,0}} \quad (\text{A.44c})$$

if these divisions make sense in the chosen ring. Then the equations for  $p_{n,n}$ ,  $p_{n+1,n}$  and  $p_{n+2,n}$  give successively

$$a_{n+1} = p_{n,n} - b_n \quad (\text{A.45a})$$

$$b_{n+1} = \frac{p_{n+1,n} - c_n}{a_{n+1}} = \frac{p_{n+1,n} - c_n}{p_{n,n} - b_n} \quad (\text{A.45b})$$

$$c_{n+1} = \frac{p_{n+2,n}}{a_{n+1}} = \frac{p_{n+2,n}}{p_{n,n} - b_n} \quad (\text{A.45c})$$

if these divisions make sense. Now define, for  $n \geq 1$ ,

$$d_n = \alpha_{3n} \quad (\text{A.46a})$$

$$e_n = \alpha_{3n+1} \quad (\text{A.46b})$$

Then  $d_1 = \alpha_3$  can be a freely chosen ring element, call it  $\kappa$ ; and we have the recurrences

$$e_n = b_n - d_n \quad (\text{A.47a})$$

$$d_{n+1} = \frac{c_n}{e_n} = \frac{c_n}{b_n - d_n} \quad (\text{A.47b})$$

if these divisions make sense.

2. There are two extreme cases in Proposition A.8. On the one hand,  $\kappa = 1$  implies  $c_n = 1$  and hence

$$\alpha = (\alpha_i)_{i \geq 2} = x, 1, 1, x, 2, 2, x, 3, 3, x, 4, 4, \dots \quad (\text{A.48})$$

This is the known 2-S-fraction for the Lah polynomials [58, Theorem 1.5 with  $r = 2$ ]. On the other hand,  $\kappa = 0$  implies  $c_n = (n - 1)/n$  and hence

$$\alpha = (\alpha_i)_{i \geq 2} = x, 0, 2, x, 1, 3, x, 2, 4, x, 3, 5, \dots \quad (\text{A.49})$$

which is apparently new. But there is in fact a one-parameter family, indexed by  $\kappa \in [0, 1]$ , that interpolates between these two extreme cases. ■

#### A.4.2 Case $j = 1$

**Proposition A.9** ( $j = 1$  implies  $\alpha = -1$  or  $0$ ). *The univariate Laguerre production matrix (1.41)/(A.36) can be written as the production matrix  $P^{(2;1)}$ , where  $\alpha_2, \alpha_3, \dots, \alpha_{13}$  are polynomials in  $x$  with nonnegative real coefficients, only if  $\alpha = -1$  or  $0$ .*

PROOF. From (A.36) and (A.29) we have

$$p_{0,0} = 1 + \alpha + x = \alpha_2 + \alpha_3 \quad (\text{A.50a})$$

$$p_{1,1} = 3 + \alpha + x = \alpha_4 + \alpha_5 + \alpha_6 \quad (\text{A.50b})$$

$$p_{2,2} = 5 + \alpha + x = \alpha_7 + \alpha_8 + \alpha_9 \quad (\text{A.50c})$$

$$p_{1,0} = 1 + \alpha + 2x = (\alpha_2 + \alpha_3)\alpha_4 + \alpha_3\alpha_5 \quad (\text{A.50d})$$

$$p_{2,1} = 4 + 2\alpha + 4x = (\alpha_5 + \alpha_6)\alpha_7 + \alpha_6\alpha_8 \quad (\text{A.50e})$$

$$p_{2,0} = 2x = \alpha_3\alpha_5\alpha_7 \quad (\text{A.50f})$$

$$p_{3,1} = 6x = \alpha_6\alpha_8\alpha_{10} \quad (\text{A.50g})$$

$$p_{4,2} = 12x = \alpha_9\alpha_{11}\alpha_{13} \quad (\text{A.50h})$$

Since  $\alpha_2$  and  $\alpha_3$  have nonnegative coefficients, from  $p_{0,0}$  we see that  $\alpha \geq -1$ . If  $\alpha = -1$  we are done; so we henceforth assume that  $\alpha > -1$ .

Comparing the equations for  $p_{0,0}$  and  $p_{1,0}$  gives

$$1 + \alpha + 2x = (1 + \alpha + x)\alpha_4 + \alpha_3\alpha_5. \quad (\text{A.51})$$

Since  $\alpha_3$  and  $\alpha_5$  have nonnegative coefficients, we can avoid an  $x^2$  (or higher-order) term on the right-hand side only if  $\alpha_4$  is some real constant  $c$ ; moreover, by comparing the constant terms in (A.51) and using  $\alpha > -1$ , we see that  $0 \leq c \leq 1$ .

Next we observe that  $\alpha_3$  must divide  $p_{2,0} = 2x$ , so either  $\alpha_3 = d$  or  $\alpha_3 = dx$ , for some real constant  $d > 0$ .

**Case  $\alpha_3 = d$ .** From  $p_{0,0}$  with  $\alpha_3 = d$  we get  $d \leq 1 + \alpha$ . From (A.51) with  $\alpha_4 = c$  and  $\alpha_3 = d$  we get

$$\alpha_5 = \frac{(1 + \alpha)(1 - c)}{d} + \frac{2 - c}{d}x. \quad (\text{A.52})$$

Since  $\alpha_5$  divides  $p_{2,0} = 2x$ , we must either have  $(1 + \alpha)(1 - c) = 0$  or  $c = 2$ . Since  $\alpha > -1$  and  $0 \leq c \leq 1$ , we conclude that  $c = 1$ . Then  $\alpha_5 = x/d$ , and from  $p_{1,1}$  we get

$$\alpha_6 = (2 + \alpha) + \left(1 - \frac{1}{d}\right)x. \quad (\text{A.53})$$

Since  $\alpha_6$  must divide  $p_{3,1} = 6x$ , we must either have  $\alpha = -2$  (which is not allowed) or  $d = 1$ ; so  $d = 1$  and  $\alpha_6 = 2 + \alpha$ . From  $p_{2,1}$  we then get

$$4 + 2\alpha + 4x = (2 + \alpha + x)\alpha_7 + (2 + \alpha)\alpha_8. \quad (\text{A.54})$$

From  $p_{3,1}$  we see that  $\alpha_8$  is either a constant or a constant times  $x$ : the first possibility gives  $\alpha_7 = 4$  and  $\alpha_8 = -2$ , which is forbidden; the second possibility gives  $\alpha_7 = 2$  and

$$\alpha_8 = \frac{2}{2 + \alpha}x. \quad (\text{A.55})$$

Then using  $p_{2,2}$  we get

$$\alpha_9 = (3 + \alpha) + \frac{\alpha}{2 + \alpha}x. \quad (\text{A.56})$$

But  $\alpha_9$  must divide  $p_{4,2} = 12x$ , which is possible only if  $\alpha = -3$  (which is not allowed) or  $\alpha = 0$ . This completes the proof in the case  $\alpha_3 = d$ .

**Case  $\alpha_3 = dx$ .** From (A.51) with  $\alpha_4 = c$  and  $\alpha_3 = dx$ , comparing constant terms and using  $\alpha \neq -1$  we conclude that  $c = 1$ ; then  $\alpha_5 = 1/d$ . From  $p_{1,1}$  we then get

$$\alpha_6 = \left(2 + \alpha - \frac{1}{d}\right) + x. \quad (\text{A.57})$$

Since  $\alpha_6$  must divide  $p_{3,1} = 6x$ , we must have  $d = 1/(2 + \alpha)$  and  $\alpha_6 = x$ . From  $p_{2,0}$  we get  $\alpha_7 = 2$ , and then  $p_{2,1}$  gives  $\alpha_8 = 2$ . Then using  $p_{2,2}$  we get  $\alpha_9 = 1 + \alpha + x$ . But  $\alpha_9$  must divide  $p_{4,2} = 12x$ , so  $\alpha = -1$ , contrary to hypothesis. We conclude that the case  $\alpha_3 = dx$  cannot occur when  $\alpha \neq -1$ .  $\square$

**Question A.10.** Does this result hold if we omit the condition that the coefficients of the polynomials  $\alpha$  are nonnegative?

**Proposition A.11** (Solutions for  $j = 1$  and  $\alpha = -1$ ). *The univariate Laguerre production matrix (1.41)/(A.36) with  $\alpha = -1$  can be written as the production matrix  $P^{(2;1)}$  where the  $\alpha$  are polynomials in  $x$  with nonnegative coefficients, in the following ways (and only the following ways):*

$$\alpha_2 = 0 \quad (\text{A.58a})$$

$$\alpha_{3n} = x \quad (\text{A.58b})$$

$$\alpha_{3n+1} = c_n n \quad (\text{A.58c})$$

$$\alpha_{3n+2} = (2 - c_n)n \quad \text{for } n \geq 1 \quad (\text{A.58d})$$

where  $c_n$  is given by (A.39), with  $\kappa \in [0, 1]$ .

Note that these coefficients  $\alpha$  are precisely those that are obtained from Proposition A.8 by the transformation (A.27). So the sufficiency of these coefficients is an immediate consequence of the sufficiency half of Proposition A.8. The interesting part of the present result is therefore the necessity: namely, these are the *only* solutions for  $j = 1$  and  $\alpha = -1$ .

**PROOF. Necessity.** In the proof of Proposition A.9 we saw that either  $\alpha_3 = d$  or  $\alpha_3 = dx$ , for some real constant  $d > 0$ ; and that if  $\alpha_3 = d$ , then  $d \leq 1 + \alpha$ . But for  $\alpha = -1$  this contradicts  $d > 0$ . Therefore, for  $\alpha = -1$  we must have  $\alpha_3 = dx$ .

For  $j = 1$  and  $\alpha = -1$  we have

$$p_{0,0} = x = \alpha_2 + \alpha_3 \quad (\text{A.59a})$$

$$p_{n,n} = 2n + x = \alpha_{3n+1} + \alpha_{3n+2} + \alpha_{3n+3} \quad \text{for } n \geq 1 \quad (\text{A.59b})$$

$$p_{n,n-1} = n(n-1) + 2nx = \alpha_{3n-1}\alpha_{3n+1} + \alpha_{3n}\alpha_{3n+1} + \alpha_{3n}\alpha_{3n+2} \quad (\text{A.59c})$$

$$p_{n,n-2} = n(n-1)x = \alpha_{3n-3}\alpha_{3n-1}\alpha_{3n+1} \quad (\text{A.59d})$$

From  $\alpha_3 = dx$  and  $p_{0,0}$  we have  $\alpha_2 = (1-d)x$ . Hence  $0 < d \leq 1$ . From  $p_{1,0}$  we deduce that  $\alpha_4 + \alpha_5d = 2$ ; so  $\alpha_4 = c$  for some real constant  $c$ , and  $\alpha_5 = (2-c)/d$ . Then from  $p_{1,1}$  we get

$$\alpha_6 = \left[ 2 - \left( c + \frac{2-c}{d} \right) \right] + x, \quad (\text{A.60})$$

Since  $\alpha_6$  must divide  $p_{3,1} = 6x$ , we conclude that  $c + (2-c)/d = 2$  and hence either  $c = 2$  or  $d = 1$ . Then either way we have  $\alpha_6 = x$ , and from  $p_{2,1}$  we get

$$2 + 4x = \left( \frac{2-c}{d} + x \right) \alpha_7 + \alpha_8 x \quad (\text{A.61a})$$

$$= \frac{2-c}{d} \alpha_7 + (\alpha_7 + \alpha_8)x. \quad (\text{A.61b})$$

It follows that  $\alpha_7 = 2d/(2-c)$  and  $\alpha_7 + \alpha_8 = 1$ ; therefore,  $c = 2$  is impossible, and we have  $d = 1$  and hence  $\alpha_2 = 0$ .

From here on the proof is identical to that of Proposition A.8, with each  $\alpha_i$  replaced by  $\alpha_{i+1}$ .

**Sufficiency.** It is easily verified that (A.58), inserted into (A.29), yields (A.36) with  $\alpha = -1$ .  $\square$

**Proposition A.12** (Solutions for  $j = 1$  and  $\alpha = 0$ ). *The univariate Laguerre production matrix (1.41)/(A.36) with  $\alpha = 0$  can be written as the production matrix  $P^{(2;1)}$  where the  $\alpha$  are polynomials in  $x$  with nonnegative coefficients, in the following way (and only the following way):*

$$\alpha_{3n-1} = x \quad (\text{A.62a})$$

$$\alpha_{3n} = n \quad (\text{A.62b})$$

$$\alpha_{3n+1} = n \quad (\text{A.62c})$$

**PROOF. Necessity.** In the proof of Proposition A.9 we saw that either  $\alpha_3 = d$  or  $\alpha_3 = dx$ , for some real constant  $d > 0$ ; and we saw that  $\alpha_3 = dx$  leads to a

contradiction whenever  $\alpha \neq -1$ . So for  $\alpha = 0$  we must have  $\alpha_3 = d$ ; and in that case we saw that  $d = 1$  (hence  $\alpha_3 = 1$ ) and  $\alpha_4 = 1$ ,  $\alpha_5 = x$ ,  $\alpha_6 = 2$ ,  $\alpha_7 = 2$ ,  $\alpha_8 = x$ ,  $\alpha_9 = 3$ . And from  $p_{0,0}$  and  $\alpha_3 = 1$  we get  $\alpha_2 = x$ .

For  $j = 1$  and  $\alpha = 0$  we have

$$p_{0,0} = 1 + x = \alpha_2 + \alpha_3 \quad (\text{A.63a})$$

$$p_{n,n} = 2n + 1 + x = \alpha_{3n+1} + \alpha_{3n+2} + \alpha_{3n+3} \quad \text{for } n \geq 1 \quad (\text{A.63b})$$

$$p_{n,n-1} = n^2 + 2nx = \alpha_{3n-1}\alpha_{3n+1} + \alpha_{3n}\alpha_{3n+1} + \alpha_{3n}\alpha_{3n+2} \quad (\text{A.63c})$$

$$p_{n,n-2} = n(n-1)x = \alpha_{3n-3}\alpha_{3n-1}\alpha_{3n+1} \quad (\text{A.63d})$$

We now prove by induction that (A.62) holds for all  $n \geq 1$ . The base case  $n = 1$  has already been proven (as has  $n = 2$ ). Using the inductive hypothesis, from  $p_{n,n-1}$  we get  $\alpha_{3n+2} = x$ . Then from  $p_{n,n}$  we get  $\alpha_{3n+3} = n + 1$ . And finally from  $p_{n+1,n-1}$  we get  $\alpha_{3n+4} = n + 1$ .

**Sufficiency.** It is easily verified that (A.62), inserted into (A.29), yields (A.36) with  $\alpha = 0$ .  $\square$

Proposition A.12 shows that the monic unsigned Laguerre polynomials with  $\alpha = 0$  are given by the *modified* 2-Stieltjes–Rogers polynomials of type  $j = 1$  with the coefficients

$$\boldsymbol{\alpha} = (\alpha_i)_{i \geq 2} = x, 1, 1, x, 2, 2, x, 3, 3, x, 4, 4, \dots \quad (\text{A.64})$$

Equivalently, the rook polynomials of an  $n \times n$  chessboard, which are the reversed monic unsigned Laguerre polynomials with  $\alpha = 0$ , are given by the modified 2-Stieltjes–Rogers polynomials of type  $j = 1$  with the coefficients

$$\boldsymbol{\alpha} = (\alpha_i)_{i \geq 2} = 1, x, x, 1, 2x, 2x, 1, 3x, 3x, 1, 4x, 4x, \dots \quad (\text{A.65})$$

It was through this particular example that we first discovered the importance of the modified  $m$ -Stieltjes–Rogers polynomials.

Please note that the coefficients (A.64) are precisely (A.48), i.e. the  $\kappa = 1$  case of the coefficients in Proposition A.8. Thus, these coefficients with  $j = 0$  give  $\alpha = -1$ , while the same coefficients with  $j = 1$  give  $\alpha = 0$ ; and we will see in Proposition A.16 that the same coefficients with  $j = 2$  give  $\alpha = 1$ . But it is curious that for  $j = 1$ , only  $\kappa = 1$  is allowed, while for  $j = 0$  and  $j = 2$ , we can take any  $\kappa \in [0, 1]$ .

### A.4.3 Case $j = 2$

**Proposition A.13** ( $j = 2$  implies  $\alpha = -1$  or  $0$  or  $1$ ). *The univariate Laguerre production matrix (1.41)/(A.36) can be written as the production matrix  $P^{(2;2)}$ , where  $\alpha_2, \alpha_3, \dots$  are polynomials in  $x$  with nonnegative coefficients, only if  $\alpha = -1$  or  $0$  or  $1$ .*

PROOF. From (A.36) and (A.30) we have

$$p_{0,0} = 1 + \alpha + x = \alpha_2 + \alpha_3 + \alpha_4 \quad (\text{A.66a})$$

$$p_{1,1} = 3 + \alpha + x = \alpha_5 + \alpha_6 + \alpha_7 \quad (\text{A.66b})$$

$$p_{2,2} = 5 + \alpha + x = \alpha_8 + \alpha_9 + \alpha_{10} \quad (\text{A.66c})$$

$$p_{1,0} = 1 + \alpha + 2x = (\alpha_3 + \alpha_4)\alpha_5 + \alpha_4\alpha_6 \quad (\text{A.66d})$$

$$p_{2,1} = 4 + 2\alpha + 4x = (\alpha_6 + \alpha_7)\alpha_8 + \alpha_7\alpha_9 \quad (\text{A.66e})$$

$$p_{3,2} = 9 + 3\alpha + 6x = (\alpha_9 + \alpha_{10})\alpha_{11} + \alpha_{10}\alpha_{12} \quad (\text{A.66f})$$

$$p_{2,0} = 2x = \alpha_4\alpha_6\alpha_8 \quad (\text{A.66g})$$

$$p_{3,1} = 6x = \alpha_7\alpha_9\alpha_{11} \quad (\text{A.66h})$$

$$p_{4,2} = 12x = \alpha_{10}\alpha_{12}\alpha_{14} \quad (\text{A.66i})$$

Since  $\alpha_2, \alpha_3, \alpha_4$  have nonnegative coefficients, from  $p_{0,0}$  we see that  $\alpha \geq -1$ . If  $\alpha = -1$  we are done; so we henceforth assume that  $\alpha > -1$ .

We also conclude from  $p_{0,0}$  that

$$\alpha_3 + \alpha_4 = c_1 + c_2x \quad (\text{A.67})$$

with  $0 \leq c_1 \leq 1 + \alpha$  and  $0 \leq c_2 \leq 1$ . Furthermore, from  $p_{2,0}$  we see that either  $\alpha_4\alpha_6 = d$  or  $\alpha_4\alpha_6 = dx$ , for some  $d > 0$ . Finally, we observe that  $\alpha_3 + \alpha_4$  cannot be identically zero, because  $p_{1,0} = 1 + \alpha + 2x$  has both constant and linear terms nonzero (since  $\alpha > -1$ ) but  $\alpha_4\alpha_6$  has only a constant term *or* a linear term. So we must have either  $c_1 > 0$  or  $c_2 > 0$ .

**Case  $\alpha_4\alpha_6 = d$ .** From  $p_{2,0}$  we get  $\alpha_8 = (2/d)x$ . Furthermore, from  $p_{1,0}$  we get

- If  $c_2 = 0$ , then  $c_1 > 0$  and  $\alpha_5 = \frac{(1 + \alpha - d) + 2x}{c_1}$ .

- If  $c_2 > 0$ , then  $\alpha_5 = \frac{2}{c_2}$  and  $d = (1 + \alpha) - c_1\alpha_5$ .

**When  $c_2 = 0$ ,** we substitute  $\alpha_5$  into  $p_{1,1}$  to get

$$\alpha_6 + \alpha_7 = \left(3 + \alpha - \frac{1 + \alpha - d}{c_1}\right) + \left(1 - \frac{2}{c_1}\right)x. \quad (\text{A.68})$$

From  $p_{2,1}$  we have

$$4 + 2\alpha + 4x = (\alpha_6 + \alpha_7)\frac{2}{d}x + \alpha_7\alpha_9. \quad (\text{A.69})$$

It follows that  $\alpha_6 + \alpha_7$  cannot have a term  $x$ , hence  $c_1 = 2$ . This implies  $\alpha_6 + \alpha_7 = (5 + \alpha + d)/2$ . Inserting this into (A.69) we conclude that

$$\alpha_7\alpha_9 = (4 + 2\alpha) + \frac{3d - 5 - \alpha}{d}x. \quad (\text{A.70})$$

Now  $\alpha_7\alpha_9$  must divide  $p_{3,1} = 6x$ ; but since  $4 + 2\alpha > 0$ , we must have  $d = (5 + \alpha)/3$ . Then  $\alpha_7\alpha_9 = 4 + 2\alpha$  and  $\alpha_{11} = \frac{3}{2 + \alpha}x$ . Also, since  $\alpha_8 = (2/d)x$ , we have  $\alpha_8 =$

$\frac{6}{5+\alpha}x$ . Next, from  $p_{2,2}$  we get

$$\alpha_9 + \alpha_{10} = (5 + \alpha) + \left(1 - \frac{6}{5 + \alpha}\right)x. \quad (\text{A.71})$$

From  $p_{3,2}$  we have

$$9 + 3\alpha + 6x = (\alpha_9 + \alpha_{10})\frac{3}{2 + \alpha}x + \alpha_{10}\alpha_{12}. \quad (\text{A.72})$$

Combining (A.71) and (A.72), to avoid an  $x^2$  term we must have  $6/(5 + \alpha) = 1$ , i.e.  $\alpha = 1$ .

**When  $c_2 > 0$ ,** we substitute  $\alpha_5$  into  $p_{1,1}$  to get

$$\alpha_6 + \alpha_7 = \left(3 + \alpha - \frac{2}{c_2}\right) + x. \quad (\text{A.73})$$

But then  $\alpha_8 = (2/d)x$  produces an  $x^2$  term on the right-hand side of  $p_{2,1}$ , which is a contradiction. So  $c_2 > 0$  is impossible.

This completes the case  $\alpha_4\alpha_6 = d$ .

**Case  $\alpha_4\alpha_6 = dx$ .** From  $p_{2,0}$  we get  $\alpha_8 = 2/d$ . Furthermore, from  $p_{1,0}$  we get

- If  $c_2 = 0$ , then  $c_1 > 0$  and  $\alpha_5 = \frac{1 + \alpha + (2 - d)x}{c_1}$ .

- If  $c_2 > 0$ , then  $c_1 > 0$  and  $\alpha_5 = \frac{1 + \alpha}{c_1} = \frac{2 - d}{c_2}$ .

**When  $c_2 = 0$ ,** we substitute  $\alpha_5$  into  $p_{1,1}$  to get

$$\alpha_6 + \alpha_7 = \left(3 + \alpha - \frac{1 + \alpha}{c_1}\right) + \left(1 - \frac{2 - d}{c_1}\right)x. \quad (\text{A.74})$$

From  $p_{2,1}$  we have

$$4 + 2\alpha + 4x = (\alpha_6 + \alpha_7)\frac{2}{d} + \alpha_7\alpha_9, \quad (\text{A.75})$$

so

$$\alpha_7\alpha_9 = \left[4 + 2\alpha - \left(3 + \alpha - \frac{1 + \alpha}{c_1}\right)\frac{2}{d}\right] + \left[4 - \left(1 - \frac{2 - d}{c_1}\right)\frac{2}{d}\right]x \stackrel{\text{def}}{=} A + Bx. \quad (\text{A.76})$$

Now  $\alpha_7\alpha_9$  must divide  $p_{3,1} = 6x$ , so either  $A = 0$  or  $B = 0$ . From  $p_{2,2}$  we get

$$\alpha_9 + \alpha_{10} = \left(5 + \alpha - \frac{2}{d}\right) + x, \quad (\text{A.77})$$

and from  $p_{3,2}$  we have

$$9 + 3\alpha + 6x = (\alpha_9 + \alpha_{10})\alpha_{11} + \alpha_{10}\alpha_{12}. \quad (\text{A.78})$$

Combining (A.77) and (A.78), to avoid an  $x^2$  term,  $\alpha_{11}$  must be a constant; therefore, from  $p_{3,1}$  we conclude that  $A = 0$ , hence  $\alpha_7\alpha_9 = Bx$  and  $\alpha_{11} = 6/B$ . It follows that

$$\alpha_{10}\alpha_{12} = \left[9 + 3\alpha - \left(5 + \alpha - \frac{2}{d}\right)\frac{6}{B}\right] + \left(6 - \frac{6}{B}\right)x. \quad (\text{A.79})$$

Now  $\alpha_{10}\alpha_{12}$  must divide  $p_{4,2} = 12x$ , so either  $B = \frac{10 + 2\alpha - \frac{4}{d}}{3 + \alpha}$  or  $B = 1$ . Combining this with  $\alpha > -1$ , it can be shown that there are no solutions.<sup>17</sup>

**When  $c_2 > 0$ ,** from  $p_{1,1}$  we get

$$\alpha_6 + \alpha_7 = (3 + \alpha - \alpha_5) + x. \quad (\text{A.80})$$

From  $p_{2,1}$  we then have

$$4 + 2\alpha + 4x = \left[ (3 + \alpha - \alpha_5) + x \right] \frac{2}{d} + \alpha_7\alpha_9, \quad (\text{A.81})$$

so

$$\alpha_7\alpha_9 = \left[ 4 + 2\alpha - (3 + \alpha - \alpha_5) \frac{2}{d} \right] + \left( 4 - \frac{2}{d} \right) x \stackrel{\text{def}}{=} A' + B'x. \quad (\text{A.82})$$

Now  $\alpha_7\alpha_9$  must divide  $p_{3,1} = 6x$ , so either  $A' = 0$  or  $B' = 0$ . From  $p_{2,2}$  we get as before

$$\alpha_9 + \alpha_{10} = \left( 5 + \alpha - \frac{2}{d} \right) + x, \quad (\text{A.83})$$

and from  $p_{3,2}$  we get as before

$$9 + 3\alpha + 6x = (\alpha_9 + \alpha_{10})\alpha_{11} + \alpha_{10}\alpha_{12}; \quad (\text{A.84})$$

so to avoid an  $x^2$  term,  $\alpha_{11}$  must again be a constant. Therefore, from  $p_{3,1}$  we conclude that  $A' = 0$ , hence  $\alpha_7\alpha_9 = B'x$  and  $\alpha_{11} = 6/B'$ . As before we get

$$\alpha_{10}\alpha_{12} = \left[ 9 + 3\alpha - \left( 5 + \alpha - \frac{2}{d} \right) \frac{6}{B'} \right] + \left( 6 - \frac{6}{B'} \right) x; \quad (\text{A.85})$$

and since  $\alpha_{10}\alpha_{12}$  must divide  $p_{4,2} = 12x$ , we have either  $B' = 1$  or  $B' = \frac{10 + 2\alpha - \frac{4}{d}}{3 + \alpha}$ . In the first case we must have  $d = 2/3$  to satisfy (A.82), and hence  $\alpha_{10}\alpha_{12} = -3(1 + \alpha) < 0$ , contrary to hypothesis. In the second case we must have  $d = 1$  and hence  $B' = 2$ ,  $\alpha_4\alpha_6 = x$ ,  $\alpha_8 = 2$ ,  $\alpha_7\alpha_9 = 2x$ ,  $\alpha_{11} = 3$  and  $\alpha_{10}\alpha_{12} = 3x$ , and then (from  $A' = 0$ )  $\alpha_5 = 1$ ,  $c_2 = 1$  and  $c_1 = 1 + \alpha$ ; here  $\alpha > -1$  is arbitrary. But we now argue as follows: Since  $\alpha_6 + \alpha_7 = 2 + \alpha + x$  and  $\alpha_6$  divides  $p_{2,0} = 2x$  and  $\alpha_7$  divides  $p_{3,1} = 6x$ , we must have either

- $\alpha_6 = 2 + \alpha$  and  $\alpha_7 = x$ , which implies  $\alpha_9 = 2$  and  $\alpha_{10} = 1 + \alpha + x$ ; but  $\alpha_{10}$  must divide  $p_{4,2} = 12x$ , which is a contradiction since  $\alpha \neq -1$ .

---

<sup>17</sup> **If  $B = 1$ ,** then  $d = 2(c_1 - 2)/(3c_1 - 2)$ ; substituting this into  $A = 0$  gives

$$(2 + 2\alpha) - (1 + \alpha)c_1 + (5 + \alpha)c_1^2 = 0.$$

But the discriminant of this quadratic is  $-(1 + \alpha)(39 + 7\alpha)$ , which is  $< 0$  whenever  $\alpha > -1$ .

**If  $B = \frac{10 + 2\alpha - \frac{4}{d}}{3 + \alpha}$ ,** then  $d = [6 + 2\alpha - (1 + \alpha)c_1]/[3 + \alpha - (1 + \alpha)c_1]$ ; substituting this into  $A = 0$  gives (assuming  $\alpha \neq -1$ )

$$(3 + \alpha) + 2c_1 + c_1^2 = 0.$$

Now the discriminant is  $-4(2 + \alpha)$ , which is  $< 0$  whenever  $\alpha > -2$ .



- $\alpha_6 = x$  and  $\alpha_7 = 2 + \alpha$ , which implies  $\alpha_9 = 2x/(2 + \alpha)$  and  $\alpha_{10} = 3 + \alpha + [1 - 2/(2 + \alpha)]x$ ; but  $\alpha_{10}$  must divide  $p_{4,2} = 12x$ , which implies  $\alpha = 0$ .

□

**Proposition A.14** (Solutions for  $j = 2$  and  $\alpha = -1$ ). *The univariate Laguerre production matrix (1.41)/(A.36) with  $\alpha = -1$  can be written as the production matrix  $P^{(2;2)}$  where the  $\alpha$  are polynomials in  $x$  with nonnegative coefficients, in the following ways (and only the following ways):*

$$\alpha_2 = \alpha_3 = 0 \tag{A.86a}$$

$$\alpha_{3n+1} = x \tag{A.86b}$$

$$\alpha_{3n+2} = c_n n \quad \text{for } n \geq 1 \tag{A.86c}$$

$$\alpha_{3n+3} = (2 - c_n) n \quad \text{for } n \geq 1 \tag{A.86d}$$

where  $c_n$  is given by (A.39), with  $\kappa \in [0, 1]$ .

Note that these coefficients  $\alpha$  are precisely those that are obtained from Proposition A.8 by applying twice the transformation (A.27). So the sufficiency of these coefficients is an immediate consequence of the sufficiency half of Proposition A.8. The interesting part of the present result is therefore the necessity: namely, these are the *only* solutions for  $j = 2$  and  $\alpha = -1$ .

**PROOF. Necessity.** For  $j = 2$  and  $\alpha = -1$  we have

$$p_{n,n} = 2n + x = \alpha_{3n+2} + \alpha_{3n+3} + \alpha_{3n+4} \tag{A.87a}$$

$$p_{n,n-1} = n(n-1) + 2nx = (\alpha_{3n} + \alpha_{3n+1})\alpha_{3n+2} + \alpha_{3n+1}\alpha_{3n+3} \tag{A.87b}$$

$$p_{n,n-2} = n(n-1)x = \alpha_{3n-2}\alpha_{3n}\alpha_{3n+2} \tag{A.87c}$$

From  $p_{0,0}$  we have  $\alpha_2 + \alpha_3 + \alpha_4 = x$ , hence  $\alpha_3 + \alpha_4 = c_2x$  and  $\alpha_4 = \widehat{c}x$  with  $0 \leq \widehat{c} \leq c_2 \leq 1$ . From  $p_{2,0}$  we have  $\alpha_4\alpha_6\alpha_8 = 2x$ , hence  $\widehat{c} > 0$  and  $\alpha_4\alpha_6 = dx$  for some  $d > 0$ ; this gives  $\alpha_6 = d/\widehat{c}$  and  $\alpha_8 = 2/d$ .

Next from  $p_{1,0}$  we have  $2x = c_2x\alpha_5 + dx$ , hence  $\alpha_5 = (2 - d)/c_2$ . From  $p_{1,1}$  we then get

$$\alpha_7 = \left(2 - \frac{2-d}{c_2} - \frac{d}{\widehat{c}}\right) + x \stackrel{\text{def}}{=} A + x. \tag{A.88}$$

Since  $\alpha_7$  must divide  $p_{3,1} = 6x$ , we conclude that  $A = 0$ , hence

$$0 = 2 - \frac{2-d}{c_2} - \frac{d}{\widehat{c}} \leq 2 - \frac{2-d}{c_2} - \frac{d}{c_2} = 2 - \frac{2}{c_2}. \tag{A.89}$$

Since  $c_2 \leq 1$ , this implies  $c_2 = 1$ , and then equality here implies also  $\widehat{c} = c_2 = 1$ . This implies  $\alpha_2 = \alpha_3 = 0$ .

From here on the proof is identical to that of Proposition A.8, with each  $\alpha_i$  replaced by  $\alpha_{i+2}$ . □

**Proposition A.15** (Solutions for  $j = 2$  and  $\alpha = 0$ ). *The univariate Laguerre production matrix (1.41)/(A.36) with  $\alpha = 0$  can be written as the production matrix  $P^{(2;2)}$  where the  $\alpha$  are polynomials in  $x$  with nonnegative coefficients, in the following way (and only the following way):*

$$\alpha_2 = 0 \tag{A.90a}$$

$$\alpha_{3n} = x \tag{A.90b}$$

$$\alpha_{3n+1} = n \tag{A.90c}$$

$$\alpha_{3n+2} = n \tag{A.90d}$$

**PROOF. Necessity.** In the proof of Proposition A.13 we saw that either  $\alpha_4\alpha_6 = d$  or  $\alpha_4\alpha_6 = dx$ , for some real constant  $d > 0$ ; and we saw that  $\alpha_4\alpha_6 = d$  leads to  $\alpha = 1$  whenever  $\alpha \neq -1$ . So, for  $\alpha = 0$  we must have  $\alpha_4\alpha_6 = dx$ . Furthermore, we saw that  $\alpha_3 + \alpha_4 = c_1 + c_2x$ , and whenever  $\alpha \neq -1$  and  $\alpha_4\alpha_6 = dx$ , to obtain  $\alpha = 0$ , we need  $c_2 > 0$ . In this case, we saw that we must have  $d = 1$  with  $\alpha_6 = x$  and thus,  $\alpha_4 = 1$ . We also obtained  $c_2 = 1$  and  $c_1 = 1$ , and thus,  $\alpha_3 = x$  and from  $p_{0,0}$ ,  $\alpha_2 = 0$ .

From here on the proof is identical to that of Proposition A.12, with each  $\alpha_i$  replaced by  $\alpha_{i+1}$ .

**Sufficiency.** This is easily checked.  $\square$

**Proposition A.16** (Solutions for  $j = 2$  and  $\alpha = 1$ ). *The univariate Laguerre production matrix (1.41)/(A.36) with  $\alpha = 1$  can be written as the production matrix  $P^{(2;2)}$  where the  $\alpha$  are polynomials in  $x$  with nonnegative coefficients, in the following ways (and only the following ways):*

$$\alpha_{3n-1} = x \tag{A.91a}$$

$$\alpha_{3n} = c_n n \tag{A.91b}$$

$$\alpha_{3n+1} = (2 - c_n)n \tag{A.91c}$$

where  $c_n$  is given by (A.39), with  $\kappa \in [0, 1]$ .

**PROOF. Necessity.** In the proof of Proposition A.13 we saw that either  $\alpha_4\alpha_6 = d$  or  $\alpha_4\alpha_6 = dx$ , for some real constant  $d > 0$ ; and we saw that  $\alpha_4\alpha_6 = dx$  leads to  $\alpha = 0$  whenever  $\alpha \neq -1$ . So, for  $\alpha = 1$  we must have  $\alpha_4\alpha_6 = d$ . Furthermore, in this case we found that  $c_2 = 0$  and  $c_1 = 2$ , hence  $\alpha_3 + \alpha_4 = 2$ . We also found that  $d = (5 + \alpha)/3 = 2$ , hence  $\alpha_4\alpha_6 = 2$ .

For  $j = 2$  and  $\alpha = 1$  we have

$$p_{n,n} = 2n + 2 + x = \alpha_{3n+2} + \alpha_{3n+3} + \alpha_{3n+4} \tag{A.92a}$$

$$p_{n,n-1} = n(n+1) + 2nx = (\alpha_{3n} + \alpha_{3n+1})\alpha_{3n+2} + \alpha_{3n+1}\alpha_{3n+3} \tag{A.92b}$$

$$p_{n,n-2} = n(n-1)x = \alpha_{3n-2}\alpha_{3n}\alpha_{3n+2} \tag{A.92c}$$

From  $p_{0,0}$  and  $\alpha_3 + \alpha_4 = 2$ , we see that  $\alpha_2 = x$ . We will now prove by induction that for all  $n \geq 1$ , we have

$$\alpha_{3n-1} = x \tag{A.93a}$$

$$\alpha_{3n} + \alpha_{3n+1} = 2n \tag{A.93b}$$

$$\alpha_{3n+1}\alpha_{3n+3} = n(n+1) \tag{A.93c}$$

The base case  $n = 1$  has just been proven. Using the inductive hypotheses  $\alpha_{3n} + \alpha_{3n+1} = 2n$  and  $\alpha_{3n+1}\alpha_{3n+3} = n(n+1)$  and the formula for  $p_{n,n-1}$ , we deduce  $\alpha_{3n+2} = x$ . Then using this together with the formula for  $p_{n,n}$ , we deduce  $\alpha_{3n+3} + \alpha_{3n+4} = 2n+2$ . Using  $\alpha_{3n+1}\alpha_{3n+3} = n(n+1)$  and the formula for  $p_{n+1,n-1}$ , we deduce that  $\alpha_{3n+5} = x$ . And using this together with  $\alpha_{3n+3} + \alpha_{3n+4} = 2n+2$  and the formula for  $p_{n+1,n}$ , we deduce that  $\alpha_{3n+4}\alpha_{3n+6} = (n+1)(n+2)$ . This completes the induction.

From here on the proof is identical to that of Proposition A.8.

**Sufficiency.** It is easy to verify that, for any  $\kappa \in [0, 1]$ , the parameters  $\alpha$  defined by (A.91)/(A.39) satisfy (A.93) and thence (A.92).  $\square$

## B When is $B_\xi^{-1}PB_\xi$ $(r, 1)$ -banded?

We say that a matrix  $A = (a_{ij})_{i,j \geq 0}$  is  $(r, s)$ -*banded* if  $a_{ij} = 0$  whenever  $j < i - r$  or  $j > i + s$ . Otherwise put, the nonzero elements  $a_{ij}$  can occur only on the diagonal, in the first  $r$  bands below the diagonal, and in the first  $s$  bands above the diagonal. We say that a matrix is *lower-Hessenberg* if it is  $(\infty, 1)$ -banded.

Let  $P = (p_{ij})_{i,j \geq 0}$  be a lower-Hessenberg matrix with entries in a commutative ring  $R$ . We are interested in the matrix  $B_\xi^{-1}PB_\xi$ , where  $B_\xi$  is the  $\xi$ -binomial matrix (1.9) and  $\xi$  is an indeterminate, understood as a matrix with entries in the polynomial ring  $R[\xi]$ . Obviously  $B_\xi^{-1}PB_\xi$  is lower-Hessenberg; we want to know under what conditions it is  $(r, 1)$ -banded for some integer  $r \geq 0$ . An obvious necessary condition is that  $P$  must itself be  $(r, 1)$ -banded, since  $P$  can be recovered from  $B_\xi^{-1}PB_\xi$  by specializing  $\xi$  to zero. The full necessary and sufficient condition can be written in terms of the superdiagonal sequence  $(p_{n,n+1})_{n \geq 0}$  and the  $m$ th subdiagonal sequences  $(p_{n,n-m})_{n \geq m}$  for  $0 \leq m \leq r$ , and goes as follows:

**Proposition B.1.** *Fix an integer  $r \geq 0$ , and let  $P = (p_{ij})_{i,j \geq 0}$  be an  $(r, 1)$ -banded matrix with entries in a commutative ring  $R$  containing the rationals. Then the following are equivalent:*

- (a)  $B_\xi^{-1}PB_\xi$  is  $(r, 1)$ -banded. [That is, the  $m$ th subdiagonal of  $B_\xi^{-1}PB_\xi$  vanishes for all  $m \geq r + 1$ .]
- (b) The  $(r + 1)$ st subdiagonal of  $B_\xi^{-1}PB_\xi$  vanishes.
- (c) The superdiagonal satisfies

$$p_{n,n+1} = f_{-1}(n) \tag{B.1}$$

where  $f_{-1}(\cdot)$  is a polynomial of degree at most  $r$  with coefficients in  $R$ ; and the  $m$ th subdiagonals for  $0 \leq m \leq r$  satisfy

$$p_{n,n-m} = n(n-1) \cdots (n-m+1) f_m(n), \tag{B.2}$$

where in each case  $f_m(\cdot)$  is a polynomial of degree at most  $r - m$  with coefficients in  $R$ .

PROOF. (a)  $\implies$  (b) is trivial; we will prove (b)  $\implies$  (c) and (c)  $\implies$  (a).

We begin by computing the matrix element  $(B_\xi^{-1}PB_\xi)_{k+t,k}$  for  $k, t \geq 0$ :

$$(B_\xi^{-1}PB_\xi)_{k+t,k} = (B_{-\xi}PB_\xi)_{k+t,k} = \sum_{m=-1}^r \sum_{j=k}^{k+t-m} \binom{k+t}{j+m} (-\xi)^{(k+t)-(j+m)} p_{j+m,j} \binom{j}{k} \xi^{j-k} \quad (\text{B.3a})$$

$$= \sum_{m=-1}^r \xi^{t-m} \sum_{j=k}^{k+t-m} (-1)^{(t-m)-(j-k)} \binom{k+t}{j+m} \binom{j}{k} p_{j+m,j}. \quad (\text{B.3b})$$

(Here we let the binomial coefficient  $\binom{N}{-1} = 0$  and  $p_{-1,0} = 0$ . This happens in (B.3) when  $j = k = 0$  and  $m = -1$ .)

With  $l = j - k$ , we can rewrite equation (B.3) as

$$[\xi^{t-m}] (B_\xi^{-1}PB_\xi)_{k+t,k} = \sum_{l=0}^{t-m} (-1)^{(t-m)-l} \binom{k+t}{k+l+m} \binom{k+l}{k} p_{k+l+m,k+l} \quad (\text{B.4a})$$

$$= \frac{(k+t)!}{k!(t-m)!} \sum_{l=0}^{t-m} (-1)^{(t-m)-l} \binom{t-m}{l} \frac{p_{k+l+m,k+l}}{(k+l+m)!/(k+l)!}, \quad (\text{B.4b})$$

valid when  $k, t \geq 0$  and  $-1 \leq m \leq t$  (remembering that  $p_{-1,0} = 0$ ).

So, for  $m \geq -1$ , let  $g_m$  be the function defined on the domain  $\mathbb{N}$  (with values in  $R$ ) by

$$g_m(k) \stackrel{\text{def}}{=} \frac{p_{k+m,k}}{(k+m)!/k!} \quad (\text{B.5})$$

with the convention that  $g_{-1}(0) = 0$ . Then we have

$$[\xi^{t-m}] (B_\xi^{-1}PB_\xi)_{k+t,k} = \frac{(k+t)!}{k!(t-m)!} \sum_{l=0}^{t-m} (-1)^{(t-m)-l} \binom{t-m}{l} g_m(k+l) \quad (\text{B.6a})$$

$$= \frac{(k+t)!}{k!(t-m)!} (\Delta^{t-m} g_m)(k) \quad (\text{B.6b})$$

where  $\Delta$  denotes the forward difference operator  $(\Delta f)(n) = f(n+1) - f(n)$ ; this is valid when  $k, t \geq 0$  and  $-1 \leq m \leq t$ . It follows that  $[\xi^{t-m}] (B_\xi^{-1}PB_\xi)_{k+t,k}$  vanishes for all  $k \geq 0$  if and only if  $g_m$  is a polynomial of degree  $< t - m$ .

Applying this now under hypothesis (b) for  $t = r + 1$  and  $m \in [-1, r]$ , we conclude that  $g_m$  is a polynomial of degree  $\leq r - m$ . When  $m \geq 0$ , we set  $n = k + m$ , which gives us (B.2). When  $m = -1$ , equation (B.5) gives us

$$p_{k-1,k} = \frac{g_{-1}(k)}{k} \quad \text{for } k \geq 1 \quad (\text{B.7})$$

where  $g_{-1}$  is a polynomial of degree  $\leq r + 1$  that satisfies  $g_{-1}(0) = 0$ . Setting  $n = k - 1$  gives us (B.1). We have therefore proven that (b) implies (c).

On the other hand, if  $g_m$  is a polynomial of degree  $\leq r - m$  for all  $m \in [0, r]$ , and  $p_{k-1,k} = f_{-1}(k)$  where  $f_{-1}$  is a polynomial of degree at most  $r$ , it follows from (B.6b) that  $(B_\xi^{-1}PB_\xi)_{k+t,k} = 0$  for all  $t \geq r + 1$ . So (c) implies (a).  $\square$

## C Total positivity for a variant class of quadridiagonal matrices

In this appendix we prove total positivity for a class of quadridiagonal matrices. The result is very similar to Theorem 6.1, but for a slightly different class of matrices; and the proof is also very similar, though slightly more difficult. We present this variant result because we think that it may be useful in future work. Indeed, in one current project of ours [20, 21], the total positivity of the production matrix can be proven using a special case of Theorem C.1, but not, as far as we can tell, using Theorem 6.1.

Consider the quadridiagonal lower-Hessenberg matrix  $P$ , defined by

$$P \stackrel{\text{def}}{=} L_1 L_2 U + L_1 D_1 + L_2 D_2 \quad (\text{C.1a})$$

$$= L_1(L_2 U + D_1) + L_2 D_2 \quad (\text{C.1b})$$

$$= L_2(L_1 U + D_2) + L_1 D_1 \quad (\text{C.1c})$$

where

$$L_1 = \alpha I + xL \quad (\text{C.2a})$$

$$L_2 = \beta I + yL \quad (\text{C.2b})$$

and

- $L$  is the lower-bidiagonal matrix with the sequence  $a_0, a_1, \dots$  on the diagonal, the sequence  $b_1, b_2, \dots$  on the subdiagonal, and zeroes elsewhere;
- $U$  is the upper-bidiagonal matrix with the sequence  $c_1, c_2, \dots$  on the superdiagonal, the sequence  $d_0, d_1, \dots$  on the diagonal, and zeroes elsewhere;
- $D_1$  is the diagonal matrix with entries  $e_0, e_1, \dots$ ;
- $D_2$  is the diagonal matrix with entries  $f_0, f_1, \dots$ ;

and  $\alpha, \beta, x, y, \mathbf{a} = (a_n)_{n \geq 0}, \mathbf{b} = (b_n)_{n \geq 1}, \mathbf{c} = (c_n)_{n \geq 1}, \mathbf{d} = (d_n)_{n \geq 0}, \mathbf{e} = (e_n)_{n \geq 0}, \mathbf{f} = (f_n)_{n \geq 0}$  are all indeterminates. Note that (C.1b) = (C.1c) because by construction  $L_1$  and  $L_2$  commute:

$$L_1 L_2 = L_2 L_1. \quad (\text{C.3})$$

This commutation is what makes the present situation more restrictive than that of Theorem 6.1; in compensation, the diagonal matrices  $D_1$  and  $D_2$  can both act on the same side (here the right).

The entries in the  $k$ th column of  $P = (p_{n,k})_{n \geq 0}$  are given by

$$p_{k-1,k} = (\alpha + xa_{k-1})(\beta + ya_{k-1})c_k \quad (\text{C.4a})$$

$$p_{k,k} = (\alpha + xa_k)(\beta + ya_k)d_k + (\alpha + xa_k)yb_k c_k + xb_k(\beta + ya_{k-1})c_k + (\alpha + xa_k)e_k + (\beta + ya_k)f_k \quad (\text{C.4b})$$

$$p_{k+1,k} = (\alpha + xa_{k+1})yb_{k+1}d_k + xb_{k+1}(\beta + ya_k)d_k + xb_{k+1}yb_k c_k + xb_{k+1}e_k + yb_{k+1}f_k \quad (\text{C.4c})$$

$$p_{k+2,k} = xyb_{k+2}b_{k+1}d_k \quad (\text{C.4d})$$

$$p_{n,k} = 0 \quad \text{if } n < k - 1 \text{ or } n > k + 2 \quad (\text{C.4e})$$

where by definition  $b_0 = c_0 = 0$  and  $a_n = b_n = c_n = d_n = e_n = f_n = 0$  whenever  $n < 0$ . Our main result is:

**Theorem C.1** (Total positivity of the generalized production matrix). *The matrix  $P$  defined by (C.1)/(C.2)/(C.4) is totally positive, coefficientwise in the indeterminates  $\alpha, \beta, x, y, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ .*

The proof of Theorem C.1 follows the same pattern as that of Theorem 6.1, but works with columns rather than rows (because the matrices  $D_1$  and  $D_2$  act on the right). More precisely, Theorem C.1 will be proven as follows: Define the matrix  $Q = (q_{n,k})_{n,k \geq 0} = (\mathbf{q}_0, \mathbf{q}_1, \dots)$  by

$$Q \stackrel{\text{def}}{=} P|_{\mathbf{f}=0} = L_1(L_2U + D_1), \quad (\text{C.5})$$

so that

$$q_{k-1,k} = (\alpha + xa_{k-1})(\beta + ya_{k-1})c_k \quad (\text{C.6a})$$

$$q_{k,k} = (\alpha + xa_k)(\beta + ya_k)d_k + (\alpha + xa_k)yb_kc_k + xb_k(\beta + ya_{k-1})c_k + (\alpha + xa_k)e_k \quad (\text{C.6b})$$

$$q_{k+1,k} = (\alpha + xa_{k+1})yb_{k+1}d_k + xb_{k+1}(\beta + ya_k)d_k + xb_{k+1}yb_kc_k + xb_{k+1}e_k \quad (\text{C.6c})$$

$$q_{k+2,k} = xyb_{k+2}b_{k+1}d_k \quad (\text{C.6d})$$

$$q_{n,k} = 0 \quad \text{if } n < k - 1 \text{ or } n > k + 2 \quad (\text{C.6e})$$

Then

$$P = Q + L_2D_2. \quad (\text{C.7})$$

We will begin by proving (Lemma C.2) that  $Q$  is coefficientwise totally positive; this proof uses the factorization  $Q = L_1(L_2U + D_1)$  together with the tridiagonal comparison theorem. We omit the proof as it is almost identical to that of Lemma 6.6. It follows that for every integer  $m \geq 0$ , the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_m)$  is totally positive.

The rest of the proof shows how to restore the terms in  $P$  involving  $\mathbf{f}$ . In terms of the column vectors  $(\mathbf{p}_k)_{k \geq 0}, (\mathbf{q}_k)_{k \geq 0}, (\boldsymbol{\ell}_k)_{k \geq 0}$  associated to the matrices  $P, Q, L_2$ , equation (C.7) can be rewritten as

$$\mathbf{p}_k = \mathbf{q}_k + f_k \boldsymbol{\ell}_k, \quad (\text{C.8})$$

where

$$\boldsymbol{\ell}_k = \begin{bmatrix} \mathbf{0}_{k \times 1} \\ \beta + ya_k \\ yb_{k+1} \\ \mathbf{0}_{\infty \times 1} \end{bmatrix} \quad \text{for } k \geq 0. \quad (\text{C.9})$$

We will show (Lemma C.6) that for every pair of integers  $0 \leq k \leq m + 1$ , the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \mathbf{p}_k, \dots, \mathbf{p}_m)$  is totally positive; and we will do this, for each fixed  $m \geq 0$ , by induction on  $k = m + 1, m, m - 1, \dots, 0$ . The base case  $k = m + 1$  of this induction is thus Lemma C.2, and the final case  $k = 0$  is Theorem C.1. Similar to the proof of Lemma 6.10, the proof of Lemma C.6 will involve the following steps:

**Lemma C.3:** The matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k)$  is totally positive.

**Lemma C.4:** If the matrix  $(\mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  is totally positive, then so is  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$ .

**The induction step (Lemma C.5):** If the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  is totally positive, then so is  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \mathbf{p}_k, \dots, \mathbf{p}_m)$ .

Putting this all together will prove Lemma C.6 and hence Theorem C.1. Here Lemmas C.3 and C.4 are closely analogous to Lemmas 6.7 and 6.8, but their proofs are somewhat more difficult.

We now begin the proof of Theorem C.1.

**Lemma C.2** (Total positivity of  $Q$ ). *The matrix  $Q$  defined by (C.5)/(C.6) is totally positive, coefficientwise in the indeterminates  $\alpha, \beta, x, y, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ .*

*In particular, for every integer  $m \geq 0$ , the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_m)$  is coefficientwise totally positive.*

Lemma C.2 follows from the factorization  $Q = L_1(L_2U + D_1)$  by the same argument as in Lemma 6.6.

**Lemma C.3.** *For each integer  $k \geq 0$ , the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k)$  is totally positive.*

PROOF. We prove this by induction on  $k$ . The base case  $k = 0$  is trivial. So we need to show the inductive step: if  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \boldsymbol{\ell}_{k-1})$  is totally positive, then so is  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k)$ .

Define  $\tilde{\mathbf{q}}_{k-1} \stackrel{\text{def}}{=} \mathbf{q}_{k-1}|_{d_{k-1}=0}$ , so that  $\tilde{\mathbf{q}}_{k-1} = (\tilde{q}_{n,k-1})_{n \geq 0}^\top$  where

$$\tilde{q}_{k-2,k-1} = (\alpha + xa_{k-2})(\beta + ya_{k-2})c_{k-1} \quad (\text{C.10a})$$

$$\begin{aligned} \tilde{q}_{k-1,k-1} &= (\alpha + xa_{k-1})yb_{k-1}c_{k-1} + xb_{k-1}(\beta + ya_{k-2})c_{k-1} \\ &\quad + (\alpha + xa_{k-1})e_{k-1} \end{aligned} \quad (\text{C.10b})$$

$$\tilde{q}_{k,k-1} = xb_k y b_{k-1} c_{k-1} + x b_k e_{k-1} \quad (\text{C.10c})$$

$$\tilde{q}_{n,k-1} = 0 \quad \text{if } n < k-2 \text{ or } n > k. \quad (\text{C.10d})$$

Then

$$\mathbf{q}_{k-1} = \tilde{\mathbf{q}}_{k-1} + d_{k-1}(\alpha + xa_{k-1})\boldsymbol{\ell}_{k-1} + d_{k-1}xb_k\boldsymbol{\ell}_k. \quad (\text{C.11})$$

We will successively handle the second and third terms on the right-hand side of this formula.

By Lemma C.2 the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1})$  is totally positive; and specializing this matrix to  $d_{k-1} = 0$  yields  $M_1 \stackrel{\text{def}}{=} (\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \tilde{\mathbf{q}}_{k-1})$ , which is therefore also totally positive. And by the induction hypothesis, the matrix  $M_2 \stackrel{\text{def}}{=} (\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \boldsymbol{\ell}_{k-1})$  is totally positive. Applying Lemma 6.3 (or rather its transpose) to the matrices  $M_1$  and  $M_2$ , we conclude that the matrix  $M_3 \stackrel{\text{def}}{=} (\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \tilde{\mathbf{q}}_{k-1} + d_{k-1}(\alpha + xa_{k-1})\boldsymbol{\ell}_{k-1})$  is totally positive. Note that  $M_3$  has  $k+1$  nonzero rows (i.e.  $0 \leq n \leq k$ ); all

subsequent rows are zero. On the other hand, the column vector  $\ell_k$  begins with  $k$  zeroes. Therefore, Lemma 6.5 implies that

$$M_4 \stackrel{\text{def}}{=} (M_3 | \ell_k) = (\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \tilde{\mathbf{q}}_{k-1} + d_{k-1}(\alpha + xa_{k-1})\ell_{k-1}, \ell_k) \quad (\text{C.12})$$

is totally positive. Right-multiplying  $M_4$  by the lower-bidiagonal matrix that has 1 on the diagonal,  $d_{k-1}xb_k$  in position  $(k, k-1)$  and zeroes elsewhere — in other words, adding  $d_{k-1}xb_k$  times the last column of  $M_4$  to its next-to-last column — we obtain the matrix

$$(\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \tilde{\mathbf{q}}_{k-1} + d_{k-1}(\alpha + xa_{k-1})\ell_{k-1} + d_{k-1}xb_k\ell_k, \ell_k) = (\mathbf{q}_0, \dots, \mathbf{q}_{k-2}, \mathbf{q}_{k-1}, \ell_k) \quad (\text{C.13})$$

and prove its total positivity, completing the inductive step.  $\square$

**Lemma C.4.** *Fix integers  $0 \leq k \leq m$ . If the matrix  $(\mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  is totally positive, then so is the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \ell_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$ .*

PROOF. The case  $k = m$  is Lemma C.3; so assume that  $k < m$ . Let  $\mathbf{t}_{k+1}$  be obtained from  $\mathbf{p}_{k+1}$  by first specializing  $b_{k+1} = 0$ , and followed by the substitution  $f_{k+1} \rightarrow f_{k+1} + xb_{k+1}c_{k+1}$ : that is,

$$\mathbf{t}_{k+1} = \left( \mathbf{p}_{k+1} \Big|_{b_{k+1}=0} \right) \Big|_{f_{k+1} \rightarrow f_{k+1} + xb_{k+1}c_{k+1}}. \quad (\text{C.14})$$

The entries in the column vector  $\mathbf{t}_{k+1}$  are  $\mathbf{t}_{k+1} = (t_{n,k+1})_{n \geq 0}^\top$  where

$$t_{k,k+1} = (\alpha + xa_k)(\beta + ya_k)c_{k+1} \quad (\text{C.15a})$$

$$\begin{aligned} t_{k+1,k+1} &= (\alpha + xa_{k+1})(\beta + ya_{k+1})d_{k+1} + xy a_{k+1} b_{k+1} c_{k+1} + \beta x b_{k+1} c_{k+1} \\ &\quad + (\alpha + xa_{k+1})e_{k+1} + (\beta + ya_{k+1})f_{k+1} \end{aligned} \quad (\text{C.15b})$$

$$\begin{aligned} t_{k+2,k+1} &= (\alpha + xa_{k+2})y b_{k+2} d_{k+1} + x b_{k+2} (\beta + ya_{k+1}) d_{k+1} + x b_{k+2} y b_{k+1} c_{k+1} \\ &\quad + x b_{k+2} e_{k+1} + y b_{k+2} f_{k+1} \end{aligned} \quad (\text{C.15c})$$

$$t_{k+3,k+1} = xy b_{k+3} b_{k+2} d_{k+1} \quad (\text{C.15d})$$

$$t_{n,k+1} = 0 \quad \text{if } n < k \text{ or } n > k + 3 \quad (\text{C.15e})$$

Note that these substitutions would not affect  $\mathbf{p}_\ell$  for  $\ell > k + 1$ . Next, let  $\tilde{\mathbf{p}}_{k+1}$  be identical to  $\mathbf{t}_{k+1}$  except that the entry  $t_{k,k+1}$  is now made equal to 0, i.e.  $\tilde{\mathbf{p}}_{k+1} = (\tilde{p}_{n,k+1})_{n \geq 0}^\top$  where

$$\begin{aligned} \tilde{p}_{k+1,k+1} &= (\alpha + xa_{k+1})(\beta + ya_{k+1})d_{k+1} + xy a_{k+1} b_{k+1} c_{k+1} + \beta x b_{k+1} c_{k+1} \\ &\quad + (\alpha + xa_{k+1})e_{k+1} + (\beta + ya_{k+1})f_{k+1} \end{aligned} \quad (\text{C.16a})$$

$$\begin{aligned} \tilde{p}_{k+2,k+1} &= (\alpha + xa_{k+2})y b_{k+2} d_{k+1} + x b_{k+2} (\beta + ya_{k+1}) d_{k+1} + x b_{k+2} y b_{k+1} c_{k+1} \\ &\quad + x b_{k+2} e_{k+1} + y b_{k+2} f_{k+1} \end{aligned} \quad (\text{C.16b})$$

$$\tilde{p}_{k+3,k+1} = xy b_{k+3} b_{k+2} d_{k+1} \quad (\text{C.16c})$$

$$\tilde{p}_{n,k+1} = 0 \quad \text{if } n < k + 1 \text{ or } n > k + 3 \quad (\text{C.16d})$$

Notice that

$$\mathbf{p}_{k+1} = \tilde{\mathbf{p}}_{k+1} + (\alpha + xa_k)c_{k+1}\ell_k. \quad (\text{C.17})$$



By hypothesis  $M_1 \stackrel{\text{def}}{=} (\mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  is totally positive; this implies, by substitution, that  $M_2 \stackrel{\text{def}}{=} (\mathbf{t}_{k+1}, \mathbf{p}_{k+2}, \dots, \mathbf{p}_m)$  is totally positive; and finally, this implies that  $M_3 \stackrel{\text{def}}{=} (\tilde{\mathbf{p}}_{k+1}, \mathbf{p}_{k+2}, \dots, \mathbf{p}_m)$  is totally positive, because the nonzero rows of  $M_3$  form a submatrix of  $M_2$ .

Now observe that the matrix  $S \stackrel{\text{def}}{=} (\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k, \tilde{\mathbf{p}}_{k+1}, \mathbf{p}_{k+2}, \dots, \mathbf{p}_m)$  consists of two blocks overlapping in a single row:

$$S = (\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k \mid \tilde{\mathbf{p}}_{k+1}, \mathbf{p}_{k+2}, \dots, \mathbf{p}_m) = \left[ \begin{array}{cc|c} * & \mathbf{0}_{k \times 1} & \mathbf{0}_{k \times (m-k)} \\ * & \beta + ya_k & \mathbf{0}_{1 \times (m-k)} \\ \hline * & yb_{k+1} & * \\ \hline \mathbf{0}_{\infty \times k} & \mathbf{0}_{\infty \times 1} & * \end{array} \right] \quad (\text{C.18})$$

where the asterisks stand for blocks of unspecified entries (which may be zero or nonzero). By Lemma C.3, the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k)$  is totally positive; and we have just shown that the matrix  $(\tilde{\mathbf{p}}_{k+1}, \mathbf{p}_{k+2}, \dots, \mathbf{p}_m)$  is totally positive. So Lemma 6.5 implies that the matrix  $S$  is totally positive.

On the other hand, using equation (C.17) we can conclude that the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  can be obtained from  $S$  by right-multiplying it by the upper-bidiagonal matrix that has 1 on the diagonal,  $(\alpha + xa_k)c_{k+1}$  in position  $(k, k+1)$  and zeroes elsewhere. This proves that the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \boldsymbol{\ell}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  is totally positive.  $\square$

The next two lemmas are the following:

**Lemma C.5.** *Fix integers  $0 \leq k \leq m$ . If the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_k, \mathbf{p}_{k+1}, \dots, \mathbf{p}_m)$  is totally positive, then so is  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \mathbf{p}_k, \dots, \mathbf{p}_m)$ .*

**Lemma C.6.** *For every pair of integers  $0 \leq k \leq m+1$ , the matrix  $(\mathbf{q}_0, \dots, \mathbf{q}_{k-1}, \mathbf{p}_k, \dots, \mathbf{p}_m)$  is totally positive.*

We omit the proofs of Lemmas C.5 and C.6 as they are analogous to the proofs of Lemmas 6.9 and 6.10, respectively, but using columns instead of rows.

This completes the proof of Theorem C.1.

We conclude by posing the following open problem:

**Problem C.7.** *Find a combinatorial interpretation for the output matrix  $A = \mathcal{O}(P)$  generated by the production matrix (C.1)/(C.4), or by interesting specializations thereof.*

## References

- [1] M. Aigner, Catalan-like numbers and determinants, *J. Combin. Theory A* **87**, 33–51 (1999).
- [2] M. Aigner and G.M. Ziegler, *Proofs from THE BOOK*, 6th ed. (Springer-Verlag, Berlin–Heidelberg, 2018).
- [3] G.E. Andrews, R. Askey and R. Roy, *Special Functions* (Cambridge University Press, Cambridge, 1999).

- [4] J.-C. Aval, Multivariate Fuss-Catalan numbers, *Discrete Math.* **308**, 4660–4669 (2008).
- [5] P. Barry, *Riordan Arrays: A Primer* (Logic Press, County Kildare, Ireland, 2016).
- [6] F. Bergeron, Combinatoire des polynômes orthogonaux classiques: une approche unifiée, *European J. Combin.* **11**, 393–401 (1990).
- [7] F. Bergeron, G. Labelle and P. Leroux, *Combinatorial Species and Tree-Like Structures* (Cambridge University Press, Cambridge–New York, 1998).
- [8] P. Biane, Permutations suivant le type d’excédance et le nombre d’inversions et interprétation combinatoire d’une fraction continue de Heine, *European J. Combin.* **14**, 277–284 (1993).
- [9] G.W. Brumfiel, *Partially Ordered Rings and Semi-Algebraic Geometry*, London Mathematical Society Lecture Note Series #37 (Cambridge University Press, Cambridge–New York, 1979).
- [10] N.T. Cameron and J.E. McLeod, Returns and hills on generalized Dyck paths, *J. Integer Seq.* **19**, article 16.6.1 (2016), 28 pp.
- [11] X. Chen, H. Liang and Y. Wang, Total positivity of Riordan arrays, *European J. Combin.* **46**, 68–74 (2015).
- [12] X. Chen, H. Liang and Y. Wang, Total positivity of recursive matrices, *Lin. Alg. Appl.* **471**, 383–393 (2015).
- [13] X. Chen and A.D. Sokal, Total positivity of some polynomial matrices that enumerate labeled trees and forests, II. Rooted labeled trees and partial functional digraphs, preprint (February 2023), arXiv:2302.03999 [math.CO].
- [14] X. Chen and Y. Wang, Notes on the total positivity of Riordan arrays, *Lin. Alg. Appl.* **569**, 156–161 (2019).
- [15] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York–London–Paris, 1978). Reprinted by Dover, Mineola NY, 2011.
- [16] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions* (Reidel, Dordrecht–Boston, 1974). [French original: *Analyse Combinatoire*, tomes I et II, Presses Universitaires de France, Paris, 1970.]
- [17] S. Corteel, J.S. Kim and D. Stanton, Moments of orthogonal polynomials and combinatorics, in *Recent Trends in Combinatorics*, edited by A. Beveridge, J.R. Griggs, L. Hogben, G. Musiker and P. Tetali, IMA Volumes in Mathematics and its Applications #159 (Springer, Cham, 2016), pp. 545–578.
- [18] S. Corteel and A.D. Sokal, unpublished, June 2017.
- [19] E. Coussement and W. Van Assche, Multiple orthogonal polynomials associated with the modified Bessel functions of the first kind, *Constr. Approx.* **19**, 237–263 (2003).

- [20] B. Deb and A.D. Sokal, Coefficientwise Hankel-total positivity of the Schett polynomials, preprint (November 2023), arXiv:2311.????? [math.CO].
- [21] B. Deb and A.D. Sokal, Jacobian elliptic and hyperelliptic functions, Schett polynomials and Schett symmetric functions, and coefficientwise Hankel-total positivity, in preparation.
- [22] E. Deutsch, L. Ferrari and S. Rinaldi, Production matrices, *Adv. Appl. Math.* **34**, 101–122 (2005).
- [23] E. Deutsch, L. Ferrari and S. Rinaldi, Production matrices and Riordan arrays, *Ann. Comb.* **13**, 65–85 (2009).
- [24] E. Deutsch and L. Shapiro, Exponential Riordan arrays, handwritten lecture notes, Nankai University, 26 February 2004, available on-line at <http://www.combinatorics.net/ppt2004/Louis%20W.%20Shapiro/shapiro.pdf>
- [25] G. Eneström, *Die Schriften Eulers chronologisch nach den Jahren geordnet, in denen sie verfaßt worden sind*, Jahresbericht der Deutschen Mathematiker-Vereinigung (Teubner, Leipzig, 1913).
- [26] L. Euler, De seriebus divergentibus, *Novi Commentarii Academiae Scientiarum Petropolitanae* **5**, 205–237 (1760); reprinted in *Opera Omnia*, ser. 1, vol. 14, pp. 585–617. [Latin original and English and German translations available at <http://eulerarchive.maa.org/pages/E247.html>]
- [27] L. Euler, De transformatione seriei divergentis  $1 - mx + m(m+n)x^2 - m(m+n)(m+2n)x^3 + \text{etc.}$  in fractionem continuam, *Nova Acta Academiae Scientiarum Imperialis Petropolitanae* **2**, 36–45 (1788); reprinted in *Opera Omnia*, ser. 1, vol. 16, pp. 34–46. [Latin original and English and German translations available at <http://eulerarchive.maa.org/pages/E616.html>]
- [28] S.M. Fallat and C.R. Johnson, *Totally Nonnegative Matrices* (Princeton University Press, Princeton NJ, 2011).
- [29] P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Math.* **32**, 125–161 (1980).
- [30] D. Foata and J. Labelle, Modèles combinatoires pour les polynômes de Meixner, *European J. Combin.* **4**, 305–311 (1983).
- [31] D. Foata and V. Strehl, Combinatorics of Laguerre polynomials, in *Enumeration and Design*, edited by D.M. Jackson and S.A. Vanstone (Academic Press, Toronto, 1984), pp. 123–140.
- [32] D. Foata and D. Zeilberger, Denert’s permutation statistic is indeed Euler-Mahonian, *Stud. Appl. Math.* **83**, 31–59 (1990).
- [33] S. Fomin and A. Zelevinsky, Total positivity: tests and parametrizations, *Math. Intelligencer* **22**, no. 1, 23–33 (2000).

- [34] E. Fusy and E. Guitter, Comparing two statistical ensembles of quadrangulations: A continued fraction approach, *Ann. Inst. Henri Poincaré D* **4**, 125–176 (2017).
- [35] F.R. Gantmacher and M.G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems* (AMS Chelsea Publishing, Providence RI, 2002). Based on the second Russian edition, 1950.
- [36] F. Gantmakher and M. Krein, Sur les matrices complètement non négatives et oscillatoires, *Compositio Math.* **4**, 445–476 (1937).
- [37] A.L.L. Gao, M. Pétréolle, A.D. Sokal, A.L.B. Yang and B.-X. Zhu, Total positivity of a class of Riordan-like matrices, implying a class of non-triangular linear transforms that preserve Hankel-total positivity, in preparation.
- [38] I.M. Gessel, Generalized rook polynomials and orthogonal polynomials, in *q-Series and Partitions*, edited by D. Stanton, IMA Volumes in Mathematics and its Applications #18 (Springer-Verlag, New York, 1989), pp. 159–176.
- [39] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed. (Addison-Wesley, Reading, Mass., 1994).
- [40] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, with two chapters by W. Van Assche and a foreword by R.A. Askey (Cambridge University Press, Cambridge, 2005).
- [41] M. Josuat-Vergès, A  $q$ -analog of Schläfli and Gould identities on Stirling numbers, *Ramanujan J.* **46**, 483–507 (2018).
- [42] S. Karlin, *Total Positivity* (Stanford University Press, Stanford CA, 1968).
- [43] S. Karlin, Sign regularity properties of classical orthogonal polynomials, in: *Orthogonal Expansions and their Continuous Analogues*, edited by Deborah Tepper Haimo (Southern Illinois Univ. Press, Carbondale IL, 1968), pp. 55–74.
- [44] J. Labelle and Y.N. Yeh, The combinatorics of Laguerre, Charlier, and Hermite polynomials, *Stud. Appl. Math.* **80**, 25–36 (1989).
- [45] J. Labelle and Y.N. Yeh, Combinatorial proofs of some limit formulas involving orthogonal polynomials, *Discrete Math.* **79**, 77–93 (1989/90).
- [46] T.Y. Lam, An introduction to real algebra, *Rocky Mountain J. Math.* **14**, 767–814 (1984).
- [47] H. Liang, L. Mu and Y. Wang, Catalan-like numbers and Stieltjes moment sequences, *Discrete Math.* **339**, 484–488 (2016).
- [48] H. Lima, Multiple orthogonal polynomials associated with branched continued fractions for ratios of hypergeometric series, *Adv. Appl. Math.* **147**, 102505 (2023).

- [49] H. Lima, Bidiagonal matrix factorisations associated with symmetric multiple orthogonal polynomials and lattice paths, preprint (August 2023), arXiv:2308.03561 [math.CA].
- [50] L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland Mathematics Studies #121 / Annals of Discrete Mathematics #29 (North-Holland, Amsterdam-New York / Akadémiai Kiadó, Budapest, 1986). [Reprinted with corrections by the American Mathematical Society, Providence RI, 2009.]
- [51] M. Marshall, *Positive Polynomials and Sums of Squares*, Mathematical Surveys and Monographs #146 (American Mathematical Society, Providence RI, 2008).
- [52] L. Mu and Y. Wang, private communication (2020).
- [53] R. Mullin and G.-C. Rota, On the foundations of combinatorial theory. III. Theory of binomial enumeration, in *Graph Theory and Its Applications*, edited by B. Harris (Academic Press, New York–London, 1970), pp. 167–213.
- [54] G. Nyul and G. Rácz, The  $r$ -Lah numbers, *Discrete Math.* **338**, 1660–1666 (2015).
- [55] The On-Line Encyclopedia of Integer Sequences, published electronically at <http://oeis.org>
- [56] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, *Electron. J. Combin.* **22**, no. 2, #P2.8 (2015).
- [57] T.K. Petersen, *Eulerian Numbers* (Birkhäuser–Springer, New York–Heidelberg, 2015).
- [58] M. Pétréolle and A.D. Sokal, Lattice paths and branched continued fractions, II: Multivariate Lah polynomials and Lah symmetric functions, *European J. Combin.* **92**, 103235 (2021).
- [59] M. Pétréolle, A.D. Sokal and B.-X. Zhu, Lattice paths and branched continued fractions: An infinite sequence of generalizations of the Stieltjes–Rogers and Thron–Rogers polynomials, with coefficientwise Hankel-total positivity, arXiv:1807.03271v2 [math.CO] at arXiv.org, to appear in the *Memoirs of the American Mathematical Society*.
- [60] A. Pinkus, *Totally Positive Matrices* (Cambridge University Press, Cambridge, 2010).
- [61] A. Prestel and C.N. Delzell, *Positive Polynomials: From Hilbert’s 17th Problem to Real Algebra* (Springer-Verlag, Berlin, 2001).
- [62] H. Prodinger, Returns, hills, and  $t$ -ary trees, *J. Integer Seq.* **19**, article 16.7.2 (2016), 8 pp.
- [63] E.D. Rainville, *Special Functions* (Macmillan, New York, 1960).
- [64] J. Riordan, *An Introduction to Combinatorial Analysis* (Wiley, New York, 1958). [Reprinted by Dover, Mineola NY, 2002.]

- [65] J. Salas and A.D. Sokal, The Graham–Knuth–Patashnik recurrence: Symmetries and continued fractions, *Electron. J. Combin.* **28**, paper #P2.18 (2021).
- [66] A.D. Sokal, Coefficientwise total positivity (via continued fractions) for some Hankel matrices of combinatorial polynomials, talk at the Séminaire de Combinatoire Philippe Flajolet, Institut Henri Poincaré, Paris, 5 June 2014; transparencies available at <http://semflajolet.math.cnrs.fr/index.php/Main/2013-2014>
- [67] A.D. Sokal, Total positivity of some polynomial matrices that enumerate labeled trees and forests: I. Forests of rooted labeled trees, *Monatsh. Math.* **200**, 389–452 (2023).
- [68] A.D. Sokal, Multiple orthogonal polynomials,  $d$ -orthogonal polynomials, production matrices, and branched continued fractions, preprint (April 2022), arXiv:2204.11528 [math.CA] at arXiv.org, to appear in *Trans. Amer. Math. Soc.*
- [69] A.D. Sokal, A simple algorithm for expanding a power series as a continued fraction, *Expositiones Mathematicae* **41**, 245–287 (2023).
- [70] A.D. Sokal, Coefficientwise Hankel-total positivity, monograph in preparation.
- [71] A.D. Sokal and J. Zeng, Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions, *Adv. Appl. Math.* **138**, 102341 (2022).
- [72] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, 2nd ed. (Cambridge University Press, Cambridge–New York, 2012).
- [73] R.P. Stanley, *Enumerative Combinatorics*, vol. 2 (Cambridge University Press, Cambridge–New York, 1999).
- [74] T.J. Stieltjes, Sur la réduction en fraction continue d’une série procédant selon les puissances descendantes d’une variable, *Ann. Fac. Sci. Toulouse* **3**, H1–H17 (1889).
- [75] T.J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sci. Toulouse* **8**, J1–J122 (1894) and **9**, A1–A47 (1895). [Reprinted, together with an English translation, in T.J. Stieltjes, *Œuvres Complètes/Collected Papers* (Springer-Verlag, Berlin, 1993), vol. II, pp. 401–566 and 609–745.]
- [76] V. Strehl, Lacunary Laguerre series from a combinatorial perspective, *Séminaire Lotharingien de Combinatoire* **76**, article B76c (2016).
- [77] G. Szegő, *Orthogonal Polynomials*, 4th ed. (American Mathematical Society, Providence RI, 1975). [First edition 1939; second edition 1959; third edition 1967.]
- [78] W. Van Assche, Multiple orthogonal polynomials, in *Encyclopedia of Special Functions: The Askey–Bateman Project*, volume III, edited by M.E.H. Ismail and W. Van Assche (Cambridge University Press, Cambridge, to appear).

- [79] G. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, Notes de conférences données à l'Université du Québec à Montréal, septembre-octobre 1983. Available on-line at [http://www.xavierviennot.org/xavier/polynomes\\_orthogonaux.html](http://www.xavierviennot.org/xavier/polynomes_orthogonaux.html)
- [80] G. Viennot, A combinatorial theory for general orthogonal polynomials with extensions and applications, in *Polynômes Orthogonaux et Applications* (Lecture Notes in Mathematics #1171), edited by C. Brezinski *et al.* (Springer-Verlag, Berlin, 1985), pp. 139–157.
- [81] H.S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, New York, 1948). Reprinted by the American Mathematical Society, Providence RI, 2000.
- [82] Y. Wang and B.-X. Zhu, Log-convex and Stieltjes moment sequences, *Adv. Appl. Math.* **81**, 115–127 (2016).
- [83] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge University Press, Cambridge, 1944, reprinted 1995).
- [84] J. Zeng, Énumérations de permutations et J-fractions continues, *European J. Combin.* **14**, 373–382 (1993).
- [85] J. Zeng, Combinatorics of orthogonal polynomials and their moments, in *Lectures on Orthogonal Polynomials and Special Functions*, edited by H.S. Cohl and M.E.H. Ismail, London Mathematical Society Lecture Note Series #464 (Cambridge University Press, Cambridge, 2021), pp. 280–334.
- [86] B.-X. Zhu, Log-convexity and strong  $q$ -log-convexity for some triangular arrays, *Adv. Appl. Math.* **50**, 595–606 (2013).
- [87] B.-X. Zhu, Some positivities in certain triangular arrays, *Proc. Amer. Math. Soc.* **142**, 2943–2952 (2014).
- [88] B.-X. Zhu, Total positivity, continued fractions and Stieltjes moment sequences, unpublished manuscript (2018).
- [89] B.-X. Zhu, Positivity and continued fractions from the binomial transformation, *Proc. Roy. Soc. Edinburgh A* **149**, 831–847 (2019).
- [90] B.-X. Zhu, Total positivity from the exponential Riordan arrays, *SIAM J. Discrete Math.* **35**, 2971–3003 (2021).
- [91] B.-X. Zhu, Stieltjes moment properties and continued fractions from combinatorial triangles, *Adv. Appl. Math.* **130**, 102232 (2021).
- [92] B.-X. Zhu, Coefficientwise Hankel-total positivity of row-generating polynomials for the  $m$ -Jacobi-Rogers triangle, preprint (March 2022), arXiv:2202.03793 [math.CO].