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**Large- $N$  Principal Chiral Model in Arbitrary External Fields**Vladimir Kazakov<sup>✉\*</sup>*Laboratoire de physique de l'École normale supérieure, CNRS, Université PSL, Sorbonne Universités,  
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We report the explicit solution for the vacuum state of the two-dimensional  $SU(N)$  principal chiral model at large  $N$  for an arbitrary set of chemical potentials and any interaction strength, a unique result of such kind for an asymptotically free quantum field theory. The solution matches one-loop perturbative calculation at weak coupling, and in the opposite strong-coupling regime exhibits an emergent spatial dimension from the continuum limit of the  $SU(N)$  Dynkin diagram.

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*Introduction.*—The principal chiral model (PCM) is a remarkable representative of a large class of integrable relativistic  $(1 + 1)$ -dimensional QFTs [1,2] possessing key features of QCD: asymptotic freedom, mass-gap generation, and a nontrivial topological expansion in 't Hooft's large- $N$  limit. These features attracted considerable attention to the nonperturbative dynamics of PCM that has been studied via the exact  $S$  matrix [2–4] and Bethe ansatz [5–10] giving access to thermodynamics of the model [2]. While the  $S$  matrix is explicitly known for PCM [5], the linear Bethe equations for the vacuum as a function of chemical potentials or the nonlinear thermodynamic Bethe ansatz equations are difficult to solve at any coupling strength. They can be studied perturbatively [7] or numerically [8–10], but their analytic solution is beyond reach. Fortunately, there is an exception: the large- $N$  limit of the vacuum Bethe equations is exactly solvable [11,12] for a specific choice of external fields (chemical potentials of conserved charges) repeating the mass spectrum profile in their dependence on the  $SU(N)$  Dynkin label [13]. In this Letter we lift this restriction and solve the model at large  $N$

for any set of chemical potentials. It gives us the first exact analytic solution for an asymptotically free theory at general chemical potentials.

The FKW solution [11,12] obeys the semicircle law in rapidity and can be systematically extended to higher orders in  $1/N$ ; moreover, a double-scaling (DS) limit combining large  $N$  with strong coupling resums all orders of the  $1/N$  expansion [15]. In this regime the Dynkin labels give rise to a new emergent dimension in addition to the  $1 + 1$  physical dimensions of space-time, as was pointed out already in [11,12]. The DS limit presumably describes a three-dimensional noncritical string theory, by analogy to the  $c = 1$  matrix quantum mechanics [16] whose DS limit [17–20] is dual to noncritical strings in two dimensions [21–27].

We will report the solution for the vacuum state of large- $N$  PCM with arbitrary chemical potentials. The rapidity dependence happens to be always semicircular with the support  $(-B_a, B_a)$  varying along the Dynkin diagram. We derived an integral equation for the limit shape  $B(\alpha)$  assuming its continuity. We then match the free energy with the known one-loop result [7] at weak coupling. At strong coupling, we observe the locality along the emergent dimension for the small deviations from the first, Perron-Frobenius (PF) mode of the external field. This may have bearing on the currently unknown effective theory of the dual three-dimensional string.

*PCM with general chemical potentials at large  $N$ .*—The PCM is defined by the Lagrangian,

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$$S = \frac{N}{\lambda_0} \int d^2x \text{tr} D_\mu g^\dagger D^\mu g, \quad (1)$$

where the field  $g(x) \in \text{SU}(N)$ . The mass spectrum consists of  $N - 1$  particle species in the bifundamental representations of  $\text{SU}(N) \times \text{SU}(N)$ . The lightest particle acquires its mass by dimensional transmutation while others can be seen as its bound states:

$$m_a = m \frac{\sin \frac{\pi a}{N}}{\sin \frac{\pi}{N}}, \quad a = 1 \dots N - 1. \quad (2)$$

A finite density of particles is introduced by gauging the  $\text{SU}(N) \times \text{SU}(N)$  global symmetry by constant chemical potentials:  $D_0 = \partial_0 g - (i/2)(Hg + gH)$ ,  $D_1 = \partial_1$ , where  $H = \text{diag}(q_1, \dots, q_N)$  is a traceless diagonal matrix with the eigenvalues organized in the descending order:  $q_1 \geq q_2 \geq \dots \geq q_N$ . The chemical potential of the  $a$ th species, transforming in the rank- $a$  antisymmetric representation, is then  $h_a = \sum_{b=1}^a q_b$  (ordering of  $q_a$ 's is important for this conclusion). At large  $N$  we introduce a continuous coordinate  $\alpha = \pi a/N \in (0, \pi)$  and define the limit shape functions  $q(\alpha) = q_a$  and  $h(\alpha) = \pi h_a/N$ , smooth at  $N \rightarrow \infty$  and related by  $h'(\alpha) = q(\alpha)$ . Trace and ordering conditions on  $q_a$  translate into the boundary conditions on  $h(\alpha)$ :

$$h(0) = 0 = h(\pi), \quad h''(\alpha) < 0. \quad (3)$$

The Bethe equations for the ground state of PCM take the form of integral equations for the pseudo-energies [11]:

$$\varepsilon_a(\theta) + \sum_{b=1}^{N-1} \int_{-B_b}^{B_b} d\theta' R_{ab}(\theta - \theta') \varepsilon_b(\theta') = h_a - m_a \cosh \theta, \quad (4)$$

which hold on the intervals  $(-B_a, B_a)$  where the pseudo-energies are positive. The condition that they vanish at the end points,  $\varepsilon_a(\pm B_a) = 0$ , closes the system. Once the pseudo-energies are known, the vacuum energy is obtained by simple integration:

$$E = - \sum_{a=1}^{N-1} m_a \int_{-B_a}^{B_a} \frac{d\theta}{2\pi} \varepsilon_a(\theta) \cosh \theta. \quad (5)$$

The kernels of the integral equation originate from scattering between particles in the Fermi sea and can be extracted from the exact  $S$  matrix of PCM (their Fourier transform is explicitly given in [11]):

$$R_{ab}(\theta) = G\left(\frac{\pi(a+b)}{N} + i\theta\right) - G\left(\frac{\pi|a-b|}{N} + i\theta\right) + \text{c.c.}, \quad (6)$$

where

$$4\pi^2 G(x) = \psi\left(\frac{x}{2\pi}\right) + \psi\left(-\frac{x}{2\pi}\right) - \frac{2\pi}{x}, \quad (7)$$

and  $\psi(x)$  is the logarithmic derivative of the gamma function.

Taking the large- $N$  limit is now straightforward: the Dynkin labels are promoted to continuous variables  $\alpha$  and  $\beta$ , while summation is replaced by integration. Quite remarkably, the Dynkin coordinate and rapidity combine into complex variables:

$$z = \alpha + i\theta, \quad w = \beta + i\theta', \quad (8)$$

such that the kernel (6) becomes almost holomorphic:

$$R(z, \bar{z}; w, \bar{w}) = G(z + \bar{w}) - \frac{1}{2}G(z - w) - \frac{1}{2}G(w - z) + \frac{\text{sign}(\alpha - \beta)}{2\pi(z - w)} + \text{c.c.} \quad (9)$$

We remark here that for symmetric  $\varepsilon(\beta, \theta') = \varepsilon(\beta, -\theta')$  the first term  $G(z + \bar{w})$  is equivalent to  $G(z + w)$  and then the first line becomes holomorphic in both variables, but this is immaterial for our purposes: holomorphy in  $z$  alone will suffice. More importantly, the second line breaks holomorphy explicitly, but just locally on the line  $\alpha = \beta$ . The sign function originates from  $|a - b|$  in the kernel (6), before the large- $N$  limit  $|a - \beta|$  comes in multiples of  $\pi/N$ , and the sign function should be smeared on the interval  $(-\varepsilon, \varepsilon)$ , where  $\varepsilon$  is a proxy for  $\pi/N$ . At the end  $\varepsilon$  will just define the contour deformation prescription.

Finally we get the Bethe equations in the two-dimensional form:

$$\int_{\mathcal{D}} d^2w R(z, \bar{z}; w, \bar{w}) \varepsilon(w, \bar{w}) = h(\alpha) - \frac{m}{2}(\sin z + \sin \bar{z}), \quad (10)$$

where integration is over a complex domain,

$$\mathcal{D} = \{\alpha + i\theta | 0 < \alpha < \pi, -B(\alpha) < \theta < B(\alpha)\}, \quad (11)$$

and the equation holds for  $z \in \mathcal{D}$ .

*Solution of Bethe equation.*—Solution of the large- $N$  integral equation crucially relies on holomorphy of the kernel. Every holomorphic function is harmonic, and thus, applying the Laplacian  $\partial^2/\partial z \partial \bar{z}$  to both sides of (10) picks only the anomalous term which moreover becomes local in the Dynkin variable. At the end we get a singular integral equation for each  $\alpha$ :

$$\frac{2}{\pi} \int_{-B(\alpha)}^{B(\alpha)} d\theta' \frac{\varepsilon(\alpha, \theta')}{(\theta - \theta')^2} = h''(\alpha), \quad (12)$$

the solution of which is the famous semicircle law:

$$\varepsilon(\alpha, \theta) = -\frac{h''(\alpha)}{2} \sqrt{B(\alpha)^2 - \theta^2}. \quad (13)$$

Defining a function,

$$F(z) = \int_{\mathcal{D}} d^2w \left[ 2G(z + \bar{w}) - G(z - w) - G(w - z) + \frac{\text{sign}(\alpha - \beta)}{\pi(z - w)} \right] \varepsilon(w, \bar{w}) - ih'(\alpha)\theta - h(\alpha), \quad (14)$$

we see that it will be holomorphic in  $\mathcal{D}$  as soon as  $\varepsilon(w, \bar{w})$  satisfies the integral equation (12). This can be checked by computing  $\partial F / \partial \bar{z}$ . On the other hand, by definition,

$$\int_{\mathcal{D}} d^2w R(z, \bar{z}; w, \bar{w}) \varepsilon(w, \bar{w}) = \frac{F(z)}{2} + \frac{F(\bar{z})}{2} + h(\alpha). \quad (15)$$

In virtue of this equation, (10) factorizes into holomorphic and antiholomorphic parts and boils down to

$$F(z) = -m \sin z. \quad (16)$$

This functional condition can be used for fixing the profile of the Fermi surface  $B(\alpha)$ . We thus call it the *shape equation*.

A useful form of the shape equation is obtained by turning the branch cuts of the square root inside out, in other words, replacing the short cut  $[-B(\alpha), B(\alpha)]$  passing through zero by a long cut passing through infinity. The resulting expression is manifestly analytic in  $\mathcal{D}$ :

$$F(z) = \int_0^\pi \frac{d\beta}{2\pi i} h''(\beta) \sin \beta \int_{B(\beta)}^\infty \frac{d\xi \sqrt{\xi^2 - B(\beta)^2}}{\cosh(\xi - iz) + \cos \beta} - (z \rightarrow -z). \quad (17)$$

We mention in passing that transformation from short to long cuts is well familiar in the theory of integrable systems; for example, it is a crucial step in deriving the quantum spectral curve for AdS/CFT [28,29]. Apart from making holomorphy manifest, this representation reveals the symmetries of the problem: written that way,  $F(z)$  is manifestly  $2\pi$  periodic and antisymmetric. A convenient basis in the space of such functions is  $\{\sin pz | p \in \mathbb{N}\}$ . Expanding (16) and (17) in this basis and matching the coefficients gives:

$$\int_0^\pi d\alpha h''(\alpha) B(\alpha) K_1(pB(\alpha)) \sin p\alpha = -\frac{\pi m}{2} \delta_{p1}, \quad (18)$$

where  $K_1$  is the modified Bessel function. For the energy, we get accordingly:

$$E = \frac{N^2 m}{4\pi^2} \int_0^\pi d\alpha h''(\alpha) B(\alpha) I_1(B(\alpha)) \sin \alpha. \quad (19)$$

These two equations completely characterize the ground state of PCM at arbitrary chemical potentials.

If the chemical potentials repeat the mass spectrum, with only the lowest PF mode present:  $h(\alpha) = \mathfrak{h} \sin \alpha$ , the solution is constant:  $B(\alpha) = \mathbb{B}$ . Indeed, the shape equations (18) with  $p \geq 2$  then follow from orthogonality of  $\sin k\alpha$  with different  $k$ , and the shape equation for  $p = 1$  determines  $\mathbb{B}$ :

$$\mathbb{B} K_1(\mathbb{B}) = \frac{m}{\mathfrak{h}}, \quad E = -\frac{N^2 m \mathfrak{h}}{8\pi} \mathbb{B} I_1(\mathbb{B}), \quad (20)$$

recovering the results of [11,12]. At large chemical potential the theory is weakly coupled, and the Fermi rapidity grows logarithmically:

$$\mathbb{B} \stackrel{\mathfrak{h} \gg m}{\simeq} \ln \frac{\mathfrak{h}}{m} + \frac{1}{2} \ln \ln \frac{\mathfrak{h}}{m} + \frac{1}{2} \ln \frac{\pi}{2} + \dots \quad (21)$$

It can be shown to satisfy the two-loop renormalization group equation and can thus be identified with the inverse of the running coupling in a particular renormalization scheme [11]:

$$\frac{4\pi}{\lambda(\mathfrak{h})} = \mathbb{B}. \quad (22)$$

The Fermi rapidity thus directly controls the interaction strength. In the opposite, strong-coupling regime (small  $\mathbb{B}$ ), we have [11]

$$\mathbb{B} \stackrel{\mathfrak{h} \rightarrow m}{\simeq} \sqrt{\frac{4\Delta}{|\ln \Delta|}}, \quad \Delta = \frac{\mathfrak{h}}{m} - 1. \quad (23)$$

In general, the shape equations are an infinite set of nonlinear integral conditions for a real function  $B(\alpha)$ . These equations can be solved numerically, as illustrated in Fig. 1, however, deriving an analytic solution for a general profile proves challenging. Therefore, our analysis is confined to small perturbations around the constant solution.

*Small fluctuations around PF mode.*—Consider the chemical potential of the form

$$h(\alpha) = \mathfrak{h} \sin \alpha + \sum_{p=2}^{\infty} h_p \sin p\alpha, \quad (24)$$

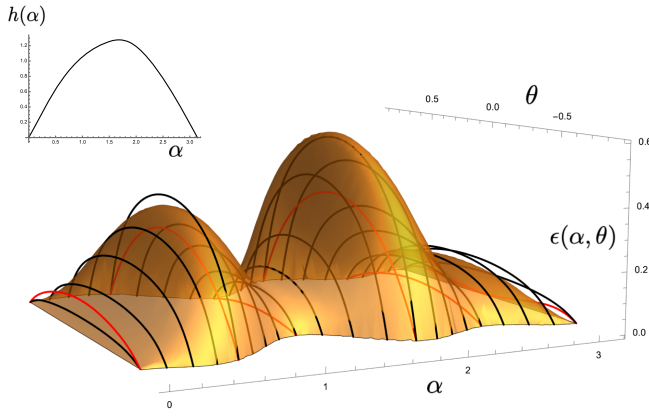


FIG. 1. The semitransparent surface represents the large- $N$  pseudo-energy (13) with the profile  $B(\alpha)$  computed from the shape equation (18). For comparison we plot the numerical solution of the exact integral equations (4) for  $N = 7$  (red lines) and  $N = 22$  (black lines). The inset shows the chemical potential  $h(\alpha)$ .

with small  $h_p \ll \mathfrak{h}$ . The Fermi surface will deviate little from a constant:  $B(\alpha) = \mathbb{B} + b(\alpha)$ , and we may expand the shape equations in  $b$ . To streamline this expansion we introduce convenient notations.

For two functions on the interval  $(0, \pi)$  we define a pairing:

$$\langle f|g \rangle = -\frac{2}{\pi} \int_0^\pi d\alpha h''(\alpha) f(\alpha) g(\alpha). \quad (25)$$

We also introduce two sets of functions:

$$\langle p| = \sin p\alpha, \quad |p \rangle = U_{p-1}(\cos \alpha), \quad (26)$$

where  $U_n$  are Chebyshev polynomials. Those are orthogonal for the pure PF profile, while for a more general profile (24)

$$\langle p|p' \rangle = \mathfrak{h} \delta_{pp'} + \mathcal{O}(h_k). \quad (27)$$

The  $\mathcal{O}(h_k)$  terms can be derived explicitly, but we will only need these three:

$$\langle 1|p \rangle = \langle p|1 \rangle = p^2 h_p, \quad \langle 1|1 \rangle = \mathfrak{h}. \quad (28)$$

The shape equations and the energy are concisely written as

$$\langle p|\mathcal{K}(pB) \rangle = m\delta_{p1}, \quad E = -\frac{N^2 m}{8\pi} \langle 1|\mathcal{I}(B) \rangle, \quad (29)$$

where  $\mathcal{K}(x) = xK_1(x)$  and  $\mathcal{I}(x) = xI_1(x)$ .

It is now straightforward to expand the shape equations around the constant solution:

$$\mathcal{K}\langle p|1 \rangle + p\mathcal{K}'\langle p|b \rangle + \frac{1}{2}p^2\mathcal{K}''\langle p|b^2 \rangle + \dots = m\delta_{p1}. \quad (30)$$

If  $p \geq 2$ , we may keep only the linear order:

$$\langle p|b \rangle = -\frac{p\mathcal{K}}{\mathcal{K}'} h_p, \quad (31)$$

from which

$$b(\alpha) = \sum_{p=2}^{\infty} \frac{pK_1(p\mathbb{B})}{\mathfrak{h}K_0(p\mathbb{B})} h_p U_{p-1}(\cos \alpha). \quad (32)$$

The constant ( $p = 1$ ) mode in  $b(\alpha)$  appears at the quadratic order and can be found from the  $p = 1$  equation because the linear term there cancels identically and so it is valid at  $\mathcal{O}(b^2)$ :

$$\langle 1|b \rangle = -\frac{\mathcal{K}''}{2\mathcal{K}'} \langle 1|b^2 \rangle. \quad (33)$$

For  $b$  appearing on the right-hand side, we can use the linear approximation above.

Corrections to the energy are also quadratic:

$$E = -\frac{N^2 m}{8\pi} \left( \mathcal{I}\langle 1|1 \rangle + \mathcal{I}'\langle 1|b \rangle + \frac{1}{2}\mathcal{I}''\langle 1|b^2 \rangle + \dots \right). \quad (34)$$

Using (33) and plugging in the solution for  $b$ , we get

$$E = -\frac{N^2 m}{8\pi \mathfrak{h}} \left( \mathbb{B}I_1(\mathbb{B})\mathfrak{h}^2 + \sum_{p=2}^{\infty} \frac{p^2 K_1(p\mathbb{B})^2}{2K_0(\mathbb{B})K_0(p\mathbb{B})^2} h_p^2 \right). \quad (35)$$

This result is valid both at weak and strong coupling; the only assumption used in its derivation is smallness of deviations from the PF mode in  $h(\alpha)$ .

*Weak coupling.*—At weak coupling (large  $\mathbb{B}$ ), we have

$$E \stackrel{\mathbb{B} \gg 1}{\simeq} -\frac{N^2}{16\pi} \left[ \mathbb{B}\mathfrak{h}^2 + \mathbb{B} \sum_p h_p^2 + \frac{1}{2} \sum_p p(p+2)h_p^2 \right]. \quad (36)$$

This can be compared with the perturbative expression known to the one-loop order [7] for generic chemical potentials [30]:

$$E^{1\text{-loop}} = -\frac{N^2}{2\pi\lambda(\mathfrak{h})} \int d\alpha q(\alpha)^2 - \frac{N^2}{16\pi^3} \int d\alpha d\beta [q(\alpha) - q(\beta)]^2 \ln \frac{2|q(\alpha) - q(\beta)|}{e\mathfrak{h}}, \quad (37)$$

where  $q(\alpha) = h'(\alpha)$  and the running coupling is defined by (22). Plugging in  $h(\alpha)$  from (24) and expanding in  $h_p$  to

the quadratic order, we get the complete agreement with our integrability-based result.

*Strong coupling.*—In the strong-coupling regime  $\mathfrak{h}$  approaches  $m$  and  $\mathbb{B}$  goes to zero, then

$$E \stackrel{\mathbb{B} \ll 1}{\simeq} -\frac{N^2}{16\pi} \left( \mathbb{B}^2 \mathfrak{h}^2 + \frac{1}{\mathbb{B}^2 |\ln \mathbb{B}|^3} \sum_p \mathfrak{h}_p^2 \right). \quad (38)$$

Quite remarkably, the energy becomes local along the Dynkin dimension. Using (23), and denoting  $\varphi(\alpha) = (h(\alpha) - \mathfrak{h} \sin \alpha)/(4\Delta)$ , we get

$$E \stackrel{\Delta \rightarrow 0}{\simeq} -\frac{N^2 \Delta}{4\pi |\ln \Delta|} \left( m^2 + \frac{1}{|\ln \Delta|} \int d\alpha \varphi(\alpha)^2 \right). \quad (39)$$

Deviations from the PF mode are naturally of order  $\Delta$ , thus  $\varphi$  can be viewed as an order-one variable, albeit we had linearized in fluctuations and thus neglected contributions of higher than quadratic order in  $\varphi$ ; moreover, one can show that the very leading term will be the same for any small profile  $B(\alpha)$ . The fluctuations beyond the PF mode appear to be logarithmically suppressed at strong coupling. Perhaps the PF contribution should be regarded as a subtraction constant and the remainder could possibly be interpreted in terms of an effective theory for fluctuations, of which we calculated just the quadratic term.

*Discussion.*—We extended the exact solution of large- $N$  PCM [11,12] to the case of arbitrary chemical potentials. The pseudo-energies of physical excitations still follow the semicircular law but now the Fermi surface becomes a new functional degree of freedom that has a nontrivial profile along the Dynkin diagram. This plays the role of a new continuous dimension emerging in the large- $N$  limit. With such new functional degrees of freedom at hand one can model various weak-coupling regimes which may reveal new types of nonperturbative trans-series expansions [31]. At strong coupling, intriguing signs of locality arise, perhaps pointing toward string theory interpretation. It would be interesting to split the Dynkin diagram into two intervals and calculate entanglement entropy as a diagnostic for locality [32].

Similarly to our previous work [15], one can identify the double-scaling limit in the case of general chemical potentials; we will return to this point in the forthcoming publication [33].

Another interesting direction would be to analyze the large- $N$  limit of PCM compactified on the cylinder with twisted boundary conditions; the corresponding analysis at finite  $N$  was already initiated in [10]. This corresponds to the putative dual string at finite temperature. Also, it would be interesting to go beyond the vacuum state and analyze the excited states.

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