Strategic Successive Refinement With Interdependent Decoders Cost Functions

Rony Bou Rouphael, Maël Le Treust

Strategic Successive Refinement with Interdependent Decoders Cost Functions

Rony Bou Rouphael

ETIS UMR 8051, CY Cergy-Paris Université

ENSEA, 6 avenue du Ponceau

95014 Cergy-Pontoise CEDEX, FRANCE

Email: rony.bou-rouphael@ensea.fr

Maël Le Treust

IRISA UMR 6074

263 Av. Général Leclerc

35000 Rennes, FRANCE

Email: mael.le-treust@cnrs.fr

Abstract

In decentralized and decision-oriented communication paradigms, autonomous devices strategically implement information compression policies. In this work, we study a strategic communication game between an encoder and two decoders. An i.i.d. information source, observed by the encoder, is transmitted to the decoders via two perfect links, one reaching the first decoder only and the other reaching both decoders, as in the successive refinement setup. All three communicating devices are assumed to be rational, i.e. they want to minimize their respective cost functions, that depend on the source variable and the output symbols of both decoder. The game takes place as follows: the encoder commits to implementing an encoding strategy which induces a Bayesian game among the two decoders. The encoder is the Stackelberg leader and the two decoders are the Stackelberg followers, they select simultaneously the output sequences that minimize their respective long-run costs. We characterize the asymptotic

This work has been conducted during Rony Bou Rouphael's PhD at ETIS UMR 8051. The authors gratefully acknowledges financial support from INS2I CNRS, DIM-RFSI, SRV ENSEA, UFR-ST UCP, The Paris Seine Initiative and IEA Cergy-Pontoise. This research has been conducted as part of the project Labex MME-DII under Grant ANR11-LBX-0023-01.

behavior of the long-run optimal cost of the encoder, when the decoders implement decoding strategies that form a Bayes-Nash equilibrium. We show that this optimal cost converges to a single-letter expression which involves two auxiliary random variables and single-letter incentive constraints of the decoders.

I. Introduction

The lossy source coding setup of Shannon [1] involves an encoder and a decoder that share the common goal of reliably transmitting an information source. What if the communicating agents are self-interested and want to minimize their own cost function? What if the communicating devices have their own incentive constraints? Such situations often arise in adversarial communication settings in which agents have non-aligned and distinct objectives, and the goal of each agent is to optimize its own objective given the information available locally. Such strategic communication scenarios are also frequent in economical interactions, financial transactions, marketing, negotiation, auctions, court, security, and political campaigns among others.

In the seminal paper [2], Kamenica and Gentzkow investigate the Bayesian persuasion game, in which the encoder commits to implementing an encoding strategy. The question of the optimal information disclosure policy arises. In this paper, this problem is referred to as the strategic communication problem. As Machine Learning (ML) is being used in areas such as education, health, employment, and commerce in order to make important decisions about individuals, strategic communication is also being increasingly observed in such human-machine interactions. In adversarial machine learning where data generators and learners have mismatched objectives, classification problems [3] often exhibit opportunities for a strategic agent to get a better classification result by selectively revealing information to the classifier, especially if the transmitter is in possession of information about the classifier like its objective function. Such non-cooperative communication scenarios which frequently arise in real-world circumstances, require new multi-disciplinary approaches and techniques in order to achieve optimal outcomes. The present work studies a non-cooperative successive refinement source coding setup in which communication is subjected to information constraints as well as incentive constraints of the decoders. As information theory is concerned with optimizing the performance of communication systems and game theory analyzes the strategic interactions among rational agents, the problem of strategic information selection and transmission can be naturally studied at the intersection of these two disciplines. In the following, we review the standard cooperative successive refinement source coding setup and the optimal region of achievable rates characterized in [4] and [5].

A. Preliminary: Successive Refinement Coding

Successive refinement is a special case of multiple description code for a discrete memoryless source U drawn according to an i.i.d. distribution $\mathcal{P}_U$ and observed by an encoder $\mathcal{E}$ and communicated via two perfect links to two decoders $\mathcal{D}_1$ and $\mathcal{D}_2$. In the standard cooperative communication scheme, formulated as a rate-distortion problem, a source stream is encoded into two descriptions M_0 at rate R_0 and M_1 at rate R_1 in a progressive order, which allows the later description to be used in order to refine the early one. This improves the quality of the reconstructions progressively. $\mathcal{D}_1$ receives both descriptions M_0 and M_1, while $\mathcal{D}_2$ receives only one description M_0. Both decoders are endowed with distortion measures, also called cost functions, $c_1(U, V_1)$ and $c_2(U, V_2)$ where V_1 and V_2 are their respective reconstructions of the source. The early works [4] and [5], focused on the fundamental problem of characterizing the optimal rate-distortion region, in which such a progressive coding does not result in any performance loss, compared to single stage coding systems, see also [6, Chapter 13.5]. For a given conditional distribution $\mathcal{Q}_{V_1V_2|U}$ and a pair of costs (C_1, C_2) such that $\mathbb{E}[c_1(U, V_1)] \leq C_1$ and $\mathbb{E}[c_2(U, V_2)] \leq C_2$, the optimal successive refinement rate distortion region is the set of pairs (R_0, R_1) such that

$$R_0 \geq I(U; V_1), \tag{1}$$

$$R_0 + R_1 \geq I(U; V_1, V_2). \tag{2}$$

In this work, the goal is not to find the optimal rate region for fixed cost values. We consider an equivalent analysis which consists of minimizing the expected cost function for fixed rates, subject to a conditional distribution that satisfies the information constraints (1) and (2).

We investigate a strategic communication problem through a decentralized, successive refinement network with restricted communication, between one informed encoder and two decoders making the ultimate decisions that affect the encoder, as depicted in Fig. 2. The encoder and the decoders are endowed with non-aligned objectives captured by distinct and arbitrary cost functions. The goal of the encoder consists of designing an information disclosure policy specifying what bits of information should be revealed in order to attain a specific objective subject to the challenges imposed by the successive refinement coding setup. We consider interdependent decoders cost functions, i.e. the cost function of each decoder depends on the state and on the choice of action of the other decoder. The goal of each player is to minimize its respective cost function. We assume that the players are aware of the objectives of the other players, i.e cost functions are known by all players and thus encoding and decoding functions are selected accordingly.

The game starts when the encoder commits to and reveals its encoding function to both decoders. Each encoding function induces a finite Bayesian game among the decoders, as in [7]. This Bayesian game admits Bayes-Nash equilibria at which the pair of decoding functions will be played. Once the source sequence of independent and identically distributed random variables is drawn, the encoder observes it and transmits a public signal to both decoders and a private signal to the first decoder only. Then, each decoder observes its received messages, and draws a sequence of actions according to its respective decoding function. This setting, also referred to as the Bayesian persuasion game, enables the use of entropy constraints in order to derive the fundamental information-theoretic limits of compression and communication. We are interested in characterizing the coding scheme that minimizes the encoder's long-run cost function subject to the constraints imposed by the channel.

B. Motivation

Consider a navigation app $\mathcal{E}$ that is also concerned with traffic regulation, sending signal recommendations to two different users, drivers $\mathcal{D}_1$ and $\mathcal{D}_2$, about road traffic: busy (b) or light (l). Assume that one of the users, driver $\mathcal{D}_1$, has a premium account which gives her access to an additional private information about the state of the road. Each driver, based on the symbol(s) received from the app, will choose between two actions: Take the road v_{i0} or not v_{i1} for $i \in \{1, 2\}$. We assume no driver has incentive in taking the road when the traffic is busy. Initially, $\mathcal{D}_1$ is better off if both drivers choose the same action, i.e either they both take the road, or they both don't. However; driver $\mathcal{D}_2$ is better off if each takes a different action. The app however, is better off if only one of the drivers takes the road. Assume that the signaling strategy, i.e. the conditional distribution over the symbol spaces $\{l, b\} \times \{l, b\}$ given the state of the traffic {light, busy}, of the app is known by both drivers. The app and each of the drivers are endowed with cost functions that depend on the state of the road and on the actions taken by both drivers. These cost functions are given in the following tables when the traffic is busy and when the traffic is light.

Busy traffic	v_{20}	v_{21}
v_{10}	$2, 3, 1$	$1, 2, 2$
v_{11}	$1, 2, 2$	$2, 3, 1$

Light traffic	v_{20}	v_{21}
v_{10}	$1, 2, 0$	$0, 1, 1$
v_{11}	$0, 1, 1$	$1, 2, 0$

Fig. 1: Cost functions of $\mathcal{E}$, $\mathcal{D}_1$, $\mathcal{D}_2$ with respect to the state of traffic and drivers' action pairs. If the traffic is busy, $\mathcal{D}_1$ plays v_{10} and $\mathcal{D}_2$ plays v_{20}, the cost of $\mathcal{E}$ is 2, the cost of $\mathcal{D}_1$ is 3, and the cost of $\mathcal{D}_2$ is 1.

In this example, for every signaling strategy of the app, the decoders will play a *matching pennies game* which admits an equilibrium. What is the optimal signaling strategy of the navigation app in order to minimize its cost? In other words, how can the app persuade one of the drivers to take the road and the other not to by sending a public signal to both drivers and a private signal to driver $\mathcal{D}_1$?

In this paper, we follow the approach of [8], [9], and extend our previous work in [10] which considers strategic communication via a successive refinement network where the action of one decoder does not affect the cost function of the other decoder. In the following, we provide a brief review of related literature before proceeding with our system model and the formulation of the problem and the solution.

C. Related Literature

Originally referred to as the sender-receiver game, the problem of strategic information transmission was formulated in the game theory literature with no communication constraints. The Nash equilibrium solution of the cheap talk game was investigated by Crawford and Sobel in their seminal paper [11], in which the encoder and the decoder are endowed with distinct objectives and choose their coding strategies simultaneously. In [2], Kamenica and Gentzkow formulate the Stackelberg version of the strategic communication game, in which the encoder is the Stackelberg leader and the decoder is the Stackelberg follower, choosing its strategies as a best-response to the encoder's strategy. This setting, referred to as the Bayesian persuasion game, is the one under study in this paper by considering the successive refinement coding setup with two decoders.

Information design with multiple senders interacting with a set of receivers is studied in [7]. The authors establish the existence of equilibria, and show that the senders do not need to randomize if the public message is selected from the continuum. In [12], [13], the Nash equilibrium solution is investigated for multi-dimensional sources and quadratic cost functions. In [14], both Nash and Stackelberg equilibria are studied for dynamic multi-stage signaling games under quadratic criteria.

Strategic compression in the context of strategic information transmission was addressed in [15]. The problem is modeled as a Stackelberg game with mismatched objectives, and asymptotic limits of equilibrium strategies are characterized along with their associated costs for three different problem settings. The case where the decoder privately observes a signal correlated to the state, also referred to as the Wyner-Ziv setting [16], is studied in [15], [17] and [9]. Vora and Kulkarni investigate the achievable rates for the strategic communication problem in [18], [19] where the decoder is the Stackelberg leader. In [20], Deori and Kulkarni characterize the minimum number of distinct source symbols that can be

correctly recovered by a receiver in any Stackelberg equilibrium of the sender-receiver game where the encoder is the Stackelberg leader.

The computational aspects of the persuasion game are considered in [21]. The strategic communication problem through a noisy channel has been addressed in [8]. The persuader and the decision maker are endowed with mismatched utility functions. Their solution is formulated as a function of the channel's capacity and characterized using the concavification method of [2]. The problem considered in [8] is analogous to the one studied in this paper which consists of persuading two interdependent decoders instead of one decoder endowed with a utility measure.

In our communication scenario, the Bayesian persuasion game played between the encoder and both decoders is a Stackelberg game in which players do not choose their strategies simultaneously, but the encoder, who is the Stackelberg leader, chooses its strategy first, then the decoders, who are two Stackelberg followers, play a subsequent Bayesian game and choose their respective strategies simultaneously after observing the encoder's strategy.

D. Contributions

We establish a strategic model for the successive refinement lossy source coding setup in which the encoder is endowed with a cost function that depends on the actions of both decoders as well as the state. We study the information-theoretic limits of the strategic communication problem by considering several independent and identical copies of the Bayesian persuasion game of [2]. We combine the incentive constraints of the decoders with the information constraints imposed by the successive refinement coding setup. We assume that the encoder $\mathcal{E}$ commits to and reveals an encoding strategy before observing the source. We use the successive refinement coding technique to characterize the set of target distributions which will be used in the solution formulation. In [10], the cost function of decoder $\mathcal{D}_i$ depends only on its own action V_i, for $i \in \{1, 2\}$. In this work, we consider cost functions that depend on the state and actions of both decoders. Thus, each commitment of the encoder induces a finite Bayesian game among the decoders. Since the set of information policies is compact, such a Bayesian game admits an essential Bayes-Nash equilibrium [22]. In order to get a robust solution concept, we assume that decoders will select the pair of output sequences that minimizes their respective costs and maximizes the encoder's cost. We fully characterize the sets of target distributions of the encoder and the decoders as well as the optimal single-letter expected cost of the encoder and the single-letter Bayesian game played among the decoders. Our main result describes the asymptotic behavior of the encoder's long-run cost function and

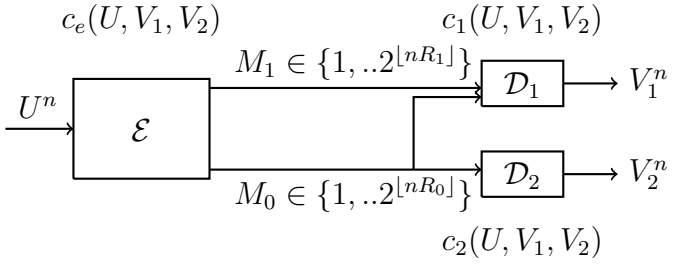

Fig. 2: Strategic successive refinement coding setup with cost-interdependent decoders

its lower bound that could be achieved with strategic compression and transmission. In order to prove achievability, we propose an alternative formulation of the solution by considering a dense subset of target distributions which induce essential equilibria only. We control the Bayesian beliefs of each decoder about the state and about the type of the other decoder. In our converse proof we identify the auxiliary random variables and show for each pair of single-letter distributions, there exists at least one pair of decoding strategies that forms a Bayes-Nash equilibrium. We also show that the single-letter optimal cost of the encoder is a lower bound to its long-run optimal cost.

This paper is organized as follows. In Sec. II, we introduce the system model, and we provide the main result in Sec. III. The converse and the achievability proofs are stated in Sec. IV and Sec. V.

E. Notations

Let U^n and V_i^n denote the sequences of random variables of source information $u^n = (u_1, ..., u_n) \in \mathcal{U}^n$, and decoder $\mathcal{D}_i$'s actions $v_i^n \in \mathcal{V}_i^n$ respectively for $i \in \{1, 2\}$. Calligraphic fonts $\mathcal{U}$ and $\mathcal{V}_i$ denote the finite alphabets of the state space and the action set of decoder $\mathcal{D}_i$ respectively. Lowercase letters u and v_i denote the symbol realizations. For a discrete random variable X, we denote by $\Delta(\mathcal{X})$ the probability simplex, i.e. the set of probability distributions over $\mathcal{X}$, and by $\mathcal{P}_X(x)$ the probability mass function $\mathbb{P}\{X = x\}$. We use the notation of $\text{supp}\mathcal{P}_X$ to refer to the support of $\mathcal{P}_X$. We denote by $\mathcal{T}_\delta^n(P_X)$ the set of all $\delta-$typical sequences corresponding to P_X. This definition can be extended to $k-$tuples of sequences $(x_1^n, x_2^n, ...x_k^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times ... \times \mathcal{X}_k^n$, that are jointly $\delta-$typical with respect to the joint probability $P_{X_1...X_k}$. The set of all such $k-$tuples is denoted by $\mathcal{T}_\delta^n(\mathcal{P}_{X_1...X_k})$. The *Kullback-Leiber* (KL) divergence between two distributions $\mathcal{P}$ and $\mathcal{Q}$ is denoted by $D(\mathcal{P}\big|\big|\mathcal{Q})$. Notation $X \multimap Y \multimap Z$ stands for the Markov chain property $\mathcal{P}_{Z|XY} = \mathcal{P}_{Z|Y}$. The information source U follows the independent and identically distributed (i.i.d) probability distribution $\mathcal{P}_U \in \Delta(\mathcal{U})$.

In the following, we introduce the model and we define the strategic communication game.

II. System Model

In this section, we define the coding functions, as well as single-letter and long-run cost functions. We specify the consisting elements of the Bayesian game induced by each encoding and we describe the corresponding set of Bayes-Nash equilibria. Finally we state the strategic encoding problem to be solved by the encoder.

Definition 1. *Let $R_0, R_1 \in \mathbb{R}_+^2 = [0, +\infty[^2$, and $n \in \mathbb{N}^\star = \mathbb{N}\backslash\{0\}$. The encoding function σ and the decoding functions τ_i of the encoder $\mathcal{E}$ and decoders $\mathcal{D}_i$ for $i \in \{1,2\}$ respectively, are given by*

$$\begin{aligned}
&\sigma : \mathcal{U}^n \mapsto \Delta\big(\{1,..2^{\lfloor nR_0 \rfloor}\} \times \{1,..2^{\lfloor nR_1 \rfloor}\}\big),\\
&\tau_1 : \{1,2,..2^{\lfloor nR_0 \rfloor}\} \times \{1,2,..2^{\lfloor nR_1 \rfloor}\} \mapsto \Delta(\mathcal{V}_1^n),\\
&\tau_2 : \{1,2,..2^{\lfloor nR_0 \rfloor}\} \mapsto \Delta(\mathcal{V}_2^n).
\end{aligned} \tag{3}$$

The coding triplet (σ, τ_1, τ_2) induces a joint probability distribution

$$\mathcal{P}^{\sigma,\tau_1,\tau_2}_{U^n M_0 M_1 V_1^n V_2^n} = \bigg(\prod_{t=1}^{n} \mathcal{P}_{U_t}\bigg) \mathcal{P}^{\sigma}_{M_0 M_1|U^n} \mathcal{P}^{\tau_1}_{V_1^n|M_0 M_1} \mathcal{P}^{\tau_2}_{V_2^n|M_0}. \tag{4}$$

Definition 2. *We consider the single-letter cost functions $c_e : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \mapsto \mathbb{R}$ of the encoder and $c_i : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \mapsto \mathbb{R}$ of the decoder $\mathcal{D}_i$, for $i \in \{1,2\}$. A coding triplet (σ, τ_1, τ_2) induces a long-run cost function*

$$c_e^n(\sigma, \tau_1, \tau_2) = \sum_{u^n, v_1^n, v_2^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}_{U^n V_1^n V_2^n}(u^n, v_1^n, v_2^n) \cdot \left[\frac{1}{n} \sum_{t=1}^{n} c_e(u_t, v_{1,t}, v_{2,t})\right],$$

where $\mathcal{P}^{\sigma,\tau_1,\tau_2}_{U^n V_1^n V_2^n}$ denotes the marginal distributions of $\mathcal{P}^{\sigma,\tau_1,\tau_2}_{U^n M_0 M_1 V_1^n V_2^n}$ over the n-sequences (U^n, V_1^n, V_2^n).

We consider the strategic communication game in which the decoders aim to minimize their respective cost functions. In our model, the decoders cost functions are interdependent. As a consequence, each encoding function σ induces a Bayesian game G^σ among the decoders. The consisting elements of the Bayesian game G^σ are listed in the following definition.

Definition 3. *For each encoding σ, the finite Bayesian game G^σ consists of:*

- *The decoders $\mathcal{D}_1$ and $\mathcal{D}_2$ are the players of the game,*
- *The sets of actions $\mathcal{V}_1^n$, $\mathcal{V}_2^n$ are the set of decoder output sequences.*

- *The pair of messages (M_0, M_1) corresponds to the type of decoder $\mathcal{D}_1$, the message M_0 is the type of decoder $\mathcal{D}_2$.*
- *The decoding functions τ_1, τ_2 are the behavior[1] strategies of the decoders.*
- *The decoder $\mathcal{D}_1$ knows the type of the decoder $\mathcal{D}_2$. The decoder $\mathcal{D}_2$ forms a belief $\mathcal{P}^{\sigma}_{M_1|M_0}$ regarding the type of the decoder $\mathcal{D}_1$.*
- *The cost function $C_i^{\sigma} : \{1, 2, ..2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, ..2^{\lfloor nR_1 \rfloor}\} \times \mathcal{V}_1^n \times \mathcal{V}_2^n \mapsto \mathbb{R}$ of decoder $\mathcal{D}_i$, $i \in \{1, 2\}$ is defined by*

$$C_i^{\sigma}(m_0, m_1, v_1^n, v_2^n) = \sum_{u^n} \mathcal{P}^{\sigma}_{U^n|M_0M_1}(u^n|m_0, m_1) \left[\frac{1}{n} \sum_{t=1}^{n} c_i(u_t, v_{1,t}, v_{2,t}) \right], \ \forall v_1^n, v_2^n, m_0, m_1. \tag{5}$$

- *Given the decoding functions (τ_1, τ_2) and the types (m_0, m_1), we define the expected decoder costs by*

$$\Psi_1^{\sigma}(\tau_1, \tau_2, m_0, m_1) = \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}_{V_1^n|M_0M_1}(v_1^n|m_0, m_1) \mathcal{P}^{\tau_2}_{V_2^n|M_0}(v_2^n|m_0) C_1^{\sigma}(m_0, m_1, v_1^n, v_2^n), \tag{6}$$

$$\Psi_2^{\sigma}(\tau_1, \tau_2, m_0) = \sum_{m_1} \mathcal{P}^{\sigma}_{M_1|M_0}(m_1|m_0) \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}_{V_1^n|M_0M_1}(v_1^n|m_0, m_1) \mathcal{P}^{\tau_2}_{V_2^n|M_0}(v_2^n|m_0) C_2^{\sigma}(m_0, m_1, v_1^n, v_2^n). \tag{7}$$

Remark 1. *Unlike the case in the Bayesian persuasion game of Kamenica-Gentzkow [2] in which the decoder plays a pure strategy, our model allows randomization at the decoders which play behavioral strategies as defined in Definition 1. This assumption is required as it guarantees the existence of Bayes-Nash equilibria for the Bayesian game G^{σ}.*

The Bayesian game G^{σ} as defined in Definition 3, is finite and the players of this game which are the decoders use behavioral strategies, therefore Nash Theorem [23] ensures the existence of at least one Bayes-Nash equilibrium for G^{σ}. At equilibria, no player has incentive in deviating from its equilibrium strategy. In the following, we define the set of Bayes-Nash equilibria of the Bayesian game G^{σ}, and we formulate the encoding problem accordingly.

Definition 4. *Given σ, for each behavior strategy τ_2 (resp. τ_1), decoder $\mathcal{D}_1$ (resp. $\mathcal{D}_2$), computes the set*

[1]A behavior strategy τ_i of player i in a Bayesian game is a mapping $\mathcal{T}_i \mapsto \Delta(\mathcal{V}_i)$, where $\mathcal{T}_i$ is the set of types of player i and V_i is the set of actions.

$BR_1(\sigma,\tau_2)$ *(resp.* $BR_2(\sigma,\tau_1)$*) of best-response strategies as follows*

$$BR_1(\sigma,\tau_2) = \big\{\tau_1, \Psi_1^\sigma(\tau_1,\tau_2,m_0,m_1) \leq \Psi_1^\sigma(\tilde{\tau_1},\tau_2,m_0,m_1), \forall\ \tilde{\tau_1}, m_0, m_1\big\}.$$
$$BR_2(\sigma,\tau_1) = \{\tau_2, \Psi_2^\sigma(\tau_1,\tau_2,m_0) \leq \Psi_2^\sigma(\tilde{\tau_2},\tau_2,m_0)\ \forall\ \tilde{\tau_2}, m_0\}.$$

Definition 5. *For each encoding strategy* σ*, we define the set* $BNE(\sigma)$ *of Bayes-Nash equilibria* (τ_1,τ_2) *of* G^σ *as follows*

$$BNE(\sigma) = \{(\tau_1,\tau_2),\ \tau_1 \in BR_1(\sigma,\tau_2) \text{ and } \tau_2 \in BR_2(\sigma,\tau_1)\}.$$

Remark 2. *The assumption that state and action sets are finite can be generalized to compact sets as in [2]. In fact, there exist Bayes-Nash equilibria for Bayesian games with compact convex spaces and quasi-concave functions. More details on existence of Bayes-Nash equilibria can be found in [24, Chap. 6].*

The communication game takes place following the timeline given below:

- The encoder $\mathcal{E}$ chooses, announces the encoding σ.
- The sequence U^n is drawn i.i.d with distribution $\mathcal{P}_U$.
- The messages (M_0, M_1) are encoded according to $\mathcal{P}^\sigma_{M_0M_1|U^n}$, and the game between the decoders G^σ begins.
- Knowing σ, the decoders select $(\tau_1,\tau_2) \in BNE(\sigma)$. In case of multiple Bayes-Nash equilibria (τ_1,τ_2), we assume the decoders select the worst one for the encoder perspective. A sequence V_1^n is drawn according to τ_1 and a sequence V_2^n is drawn according to the strategy τ_2.
- The encoder cost is given by $c_e^n(\sigma,\tau_1,\tau_2)$.

For $(R_0,R_1) \in \mathbb{R}_+^2$ and $n \in \mathbb{N}^\star$, the coding problem under study is given by

$$\Gamma_e^n(R_0,R_1) = \inf_{\sigma} \max_{(\tau_1,\tau_2)\in BNE(\sigma),} c_e^n(\sigma,\tau_1,\tau_2). \tag{8}$$

In [2] and [10], the decoders are assumed to select the best-response strategy that is worst for the encoder's cost. In this model, we also consider the Bayes-Nash equilibrium which is worst for the encoder's cost in order to get a robust solution concept. In the following, we present our main result which consists of two statements describing the asymptotic behaviour of the encoder's long-run cost given in (8).

III. Main Result

A. Characterization

We consider two auxiliary random variables $W_0 \in \mathcal{W}_0$ and $W_1 \in \mathcal{W}_1$ with $|\mathcal{W}_0| = |\mathcal{V}_1| \times |\mathcal{V}_2|$, and $|\mathcal{W}_1| = |\mathcal{V}_1|$.

Definition 6. *For $(R_0, R_1) \in \mathbb{R}_+^2$, we define the set of target distributions as follows*

$$\mathbb{Q}_0(R_0,R_1) = \Big\{ \mathcal{Q}_{W_0W_1|U},\ R_0 \geq I(U;W_0),\ R_0 + R_1 \geq I(U;W_1,W_0) \Big\}, \tag{9}$$

Definition 7. *For each distribution $\mathcal{Q}_{W_0W_1|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1)^{|\mathcal{U}|}$, we define a single-letter Bayesian game as follows:*

- *The type of the decoder $\mathcal{D}_1$ is (w_0, w_1), whereas the type of the decoder $\mathcal{D}_2$ is w_0.*
- *The decoder $\mathcal{D}_1$ knows the type of the decoder $\mathcal{D}_2$, the decoder $\mathcal{D}_2$ forms a Bayesian belief $\mathcal{Q}_{W_1|W_0}$ regarding the type of the decoder $\mathcal{D}_1$.*
- *The single-letter cost functions $C_i^\star : \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W}_0 \times \mathcal{W}_1 \mapsto \mathbb{R}$ of decoder $\mathcal{D}_i$ are defined for all (v_1, v_2, w_0, w_1) by*

$$C_i^\star(v_1, v_2, w_0, w_1) = \sum_u \mathcal{Q}(u|w_0, w_1) c_i(u, v_1, v_2),$$

- *For each pair $(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0})$ and types (w_0, w_1), the single-letter expected costs are defined by*

$$\Psi_1^\star(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0, w_1) = \sum_{v_1,v_2} \mathcal{Q}(v_1|w_0, w_1)\mathcal{Q}(v_2|w_0) C_1^\star(v_1, v_2, w_0, w_1). \tag{10}$$

$$\Psi_2^\star(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0) = \sum_{w_1} \mathcal{Q}(w_1|w_0) \sum_{v_1,v_2} \mathcal{Q}(v_1|w_0, w_1)\mathcal{Q}(v_2|w_0) C_2^\star(v_1, v_2, w_0, w_1). \tag{11}$$

Definition 8. *For each distribution $\mathcal{Q}_{W_0W_1|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1)^{|\mathcal{U}|}$, the auxiliary set of Bayes-Nash equilibria is given by*

$$\mathbb{BNE}(\mathcal{Q}_{W_0W_1|U}) = \Big\{ (\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}), \quad \Psi_1^\star(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0, w_1) \leq \Psi_1^\star(\tilde{\mathcal{Q}}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0, w_1)$$
$$\forall \tilde{\mathcal{Q}}_{V_1|W_0W_1}, w_0, w_1, \Psi_2^\star(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0) \leq \Psi_2^\star(\mathcal{Q}_{V_1|W_0W_1}, \tilde{\mathcal{Q}}_{V_2|W_0}, w_0)\ \forall \tilde{\mathcal{Q}}_{V_2|W_0}, w_0 \Big\}.$$

Definition 9. *The encoder's optimal single-letter cost is defined by*

$$\Gamma_e^\star(R_0,R_1)=\inf_{\substack{\mathcal{Q}_{W_0W_1|U}\\ \in\mathbb{Q}_0(R_0,R_1)}}\ \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1},\mathcal{Q}_{V_2|W_0})\in\\ \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})}}\mathbb{E}\Big[c_e(U,V_1,V_2)\Big], \tag{12}$$

where the expectation in (12) *is evaluated with respect to the distribution* $\mathcal{P}_U\mathcal{Q}_{W_0W_1|U}\mathcal{Q}_{V_1|W_0W_1}\mathcal{Q}_{V_2|W_0}$.

Theorem 1. *Let* $(R_0,R_1)\in\mathbb{R}_+^2$*, we have*

$$\forall\varepsilon>0,\exists\hat{n}\in\mathbb{N},\forall n\geq\hat{n},\qquad \Gamma_e^n(R_0,R_1)\leq\Gamma_e^\star(R_0,R_1)+\varepsilon, \tag{13}$$

$$\forall n\in\mathbb{N},\qquad \Gamma_e^n(R_0,R_1)\geq\Gamma_e^\star(R_0,R_1). \tag{14}$$

The proof of Theorem 1 is stated in Sec. V and IV.

Lemma 1. *The sequence* $\big(n\Gamma_e^n(R_0,R_1)\big)_{n\in\mathbb{N}^\star}$ *is sub-additive.*

The proof of Lemma 1 is stated in Appendix A . Using Fekete's Lemma for the sub-additive sequence $\big(n\Gamma_e^n(R_0,R_1)\big)_{n\in\mathbb{N}^\star}$, we get

$$\lim_{n\to\infty}\Gamma_e^n(R_0,R_1)=\inf_{n\in\mathbb{N}^\star}\Gamma_e^n(R_0,R_1)=\Gamma_e^\star(R_0,R_1). \tag{15}$$

This result describes two features of the asymptotic behavior of the encoder's long run cost function $\Gamma_e^n(R_0,R_1)$:

1) $\Gamma_e^n(R_0,R_1)$ converges to the single-letter cost $\Gamma_e^\star(R_0,R_1)$,
2) $\Gamma_e^\star(R_0,R_1)$ is a lower bound to $\Gamma_e^n(R_0,R_1)$.

It's important to note that just like the case of Shannon's lossy source coding, the single-letter solution $\Gamma_e^\star(R_0,R_1)$ in the strategic setting is only achievable asymptotically by block coding and cannot be achieved in the one-shot scenario with the transmission of a single symbol. That is, if $n=1$, $\Gamma_e^1(R_0,R_1)\neq\Gamma_e^\star(R_0,R_1)$.

B. Sketch of the Converse Proof of Theorem 1

The goal is to show that for any block length $n\in\mathbb{N}$, the optimal long-run cost $\Gamma_e^n(R_0,R_1)$ of the encoder, resulting from block coding and the Bayesian beliefs induced by the coding strategies, admits a lower bound $\Gamma_e^\star(R_0,R_1)$, formulated using auxiliary random variables and single-letter Bayesian beliefs.

The game played by blocks of n-sequences can be viewed as a sequence of single-letter games played at each stage $t \in \{1, \dots, n\}$. The proof consists of establishing the relations between the beliefs as well as the equilibria for both single-letter and block games. We begin by identifying the auxiliary random variables and the single-letter distribution induced by the coding such that the Markov chain properties of our model are satisfied. Then we show that the marginal distribution $\mathcal{P}^{\sigma}_{W_0W_1|U}$ induced by the encoding σ satisfies the information constraints imposed by successive refinement coding. Then we characterize the set of single-letter Bayes-Nash equilibria induced by $\mathcal{P}^{\sigma}_{W_0W_1|U}$ using the marginals $\mathcal{P}^{\tau_1}_{V_1|W_0W_1}$ and $\mathcal{P}^{\tau_2}_{V_2|W_0}$ such that τ_1 and τ_2 are best responses to σ of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. This allows switching from optimizing over coding strategies to optimizing over single-letter distributions and hence showing the desired inequality.

C. Sketch of the Achievability Proof of Theorem 1

Our proof consists of three main parts:

1) We introduce an alternative formulation of the problem which allows the restriction of the optimization to a dense subset of $\mathbb{Q}_0(R_0, R_1)$ composed of distributions $\mathcal{Q}_{W_0W_1|U}$ inducing Bayes-Nash equilibria that are essential, i.e for which small perturbations of $\mathcal{Q}_{W_0W_1|U}$ induce small perturbations in the equilibrium strategies $(\mathcal{Q}_{V_1|W_1W_0}, \mathcal{Q}_{V_2|W_0}) \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})$. This reformulation eliminates irregular jumps of the encoder cost function resulting from perturbing the equilibrium strategies.
2) We generate the codebook for the successive refinement setup as in [6, Section 13.5] and we show that the probability of error over the codebook is small.
3) We show that the Bayesian posterior beliefs of each decoder induced by the coding functions are close, under the Kullback-Leibler (*KL*)-divergence, to the single-letter posterior beliefs described using auxiliary random variables.

D. Special Case with Independent Decoders Cost Functions

Consider the problem of strategic communication via a successive refinement network in which the cost function of one decoder does not depend on the action of the other decoder, i.e. suppose for all u, v_1, v_1', v_2, v_2' we have $c_1(u, v_1, v_2) = c_1(u, v_1, v_2')$ and $c_2(u, v_1, v_2) = c_2(u, v_1', v_2)$. Therefore, for every encoding σ, and for all $\tau_1, \tau_2, \tau_1', \tau_2'$, we have $BR_1(\sigma, \tau_2) = BR_1(\sigma, \tau_2')$ and $BR_2(\sigma, \tau_1') = BR_2(\sigma, \tau_1')$.

Lemma 2. *For all σ, τ_1, τ_2, we have*

$$BNE(\sigma) = BR_1(\sigma, \tau_2) \times BR_2(\sigma, \tau_1). \tag{16}$$

Therefore, the encoding problem to be solved by the encoder, writes as follows

$$\Gamma_e^n(R_0, R_1) = \inf_{\sigma} \max_{\substack{\tau_1 \in BR_1(\sigma), \\ \tau_2 \in BR_2(\sigma)}} c_e^n(\sigma, \tau_1, \tau_2). \tag{17}$$

Let $(R_0, R_1) \in \mathbb{R}_+^2$. Consider the optimal single-letter cost function $\tilde{\Gamma}_e(R_0, R_1)$ given by

$$\tilde{\Gamma}_e(R_0, R_1) = \inf_{\substack{\mathcal{Q}_{W_0W_1|U} \\ \in \mathbb{Q}_0(R_0, R_1)}} \max_{\substack{\mathcal{Q}_{V_1|W_0W_1} \in \mathbb{Q}_1(\mathcal{Q}_{W_0W_1|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})}} \mathbb{E}\Big[c_e(U, V_1, V_2)\Big]. \tag{18}$$

where the infimum is taken over the set $\mathbb{Q}_0(R_0, R_1)$ defined in (9), and the decoders' target distributions are defined as follows

$$\mathbb{Q}_1(\mathcal{Q}_{W_1W_0|U}) = \underset{\mathcal{Q}_{V_1|W_0W_1}}{\arg\min}\, \mathbb{E}_{\substack{\mathcal{Q}_{W_0W_1|U} \\ \mathcal{Q}_{V_1|W_0W_1}}} \Big[c_1(U, V_1)\Big], \tag{19}$$

$$\mathbb{Q}_2(\mathcal{Q}_{W_0|U}) = \underset{\mathcal{Q}_{V_2|W_0}}{\arg\min}\, \mathbb{E}_{\substack{\mathcal{Q}_{W_0|U} \\ \mathcal{Q}_{V_2|W_0}}} \Big[c_2(U, V_2)\Big]. \tag{20}$$

Corollary 1. *Let $(R_0, R_1) \in \mathbb{R}_+^2$. We have*

$$\Gamma_e^\star(R_0, R_1) = \tilde{\Gamma}_e(R_0, R_1). \tag{21}$$

In the following, we proceed with the proof of the converse statement (14) of our main result.

IV. Converse proof of Theorem 1

Let $(R_0, R_1) \in \mathbb{R}_+^2$ and $n \in \mathbb{N}^\star$. Fix (σ, τ_1, τ_2), and consider a random variable T uniformly distributed over $\{1, 2, ..., n\}$ and independent of $(U^n, M_0, M_1, V_1^n, V_2^n)$. We identify the auxiliary random variables $W_0 = (M_0, T)$, $W_1 = M_1$, $(U, V_1, V_2) = (U_T, V_{1,T}, V_{2,T})$, distributed according to $\mathcal{P}_{UW_0W_1V_1V_2}^{\sigma\tau_1\tau_2}$ defined for all $(u, w_0, w_1, v_1, v_2) = (u_t, m_0, m_1, t, v_{1,t}, v_{2,t})$ by

$$\begin{aligned}
&\mathcal{P}_{UW_0W_1V_1V_2}^{\sigma\tau_1\tau_2}(u, w_0, w_1, v_1, v_2) = \mathcal{P}_{U_TM_0M_1,tV_{1,t}V_{2,t}}^{\sigma\tau_1\tau_2}(u_t, m_0, m_1, t, v_{1,t}, v_{2,t}) \\
=&\frac{1}{n} \sum_{\substack{u_{t+1}^n \\ u^{t-1}}} \sum_{\substack{v_2^{t-1}, v_{2,t+1}^n \\ v_1^{t-1}, v_{1,t+1}^n}} \left(\prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_0M_1|U^n}^{\sigma}(m_0, m_1|u^n) \mathcal{P}_{V_1^n|M_0M_1}^{\tau_1}(v_1^n|m_0, m_1) \mathcal{P}_{V_2^n|M_0}^{\tau_2}(v_2^n|m_0).
\end{aligned}$$

Remark 3. *Notations $U_T, V_{1,T}$ and $V_{2,T}$ refer to components $U_t, V_{1,t}$ and $V_{2,t}$ of the sequences U^n, V_1^n and V_2^n respectively selected uniformly at random for $t \in \{1, 2, .., n\}$.*

Lemma 3. *The distribution $\mathcal{P}^{\sigma\tau_1\tau_2}_{UW_0W_1V_1V_2}$ has marginal on $\Delta(\mathcal{U})$ given by $\mathcal{P}_U$ and satisfies the Markov chain properties*

$$(U, V_2) \multimap (W_0, W_1) \multimap V_1,$$
$$(U, W_1, V_1) \multimap W_0 \multimap V_2.$$

Proof. [Lemma 3] The i.i.d. property of the source ensures that the marginal distribution is $\mathcal{P}_U$. By the definition of the decoding functions τ_1 and τ_2 we have

$$(U_T, V_{2,T}) \multimap (M_1, M_0, T) \multimap V_{1,T},$$
$$(U_T, M_1, V_{1,T}) \multimap (M_2, M_0, T) \multimap V_{2,T}.$$

□

Therefore, the distribution of auxiliary random variables decomposes as $\mathcal{P}^{\sigma\tau_1\tau_2}_{UW_0W_1V_1V_2} = \mathcal{P}_U\mathcal{P}^{\sigma}_{W_0W_1|U}\mathcal{P}^{\tau_1}_{V_1|W_0W_1}\mathcal{P}^{\tau_2}_{V_2|W_0}$.

Lemma 4. *For all σ, the distribution $\mathcal{P}^{\sigma}_{W_0W_1|U} \in \mathbb{Q}_0(R_0, R_1)$.*

Proof. [Lemma 4] We consider an encoding strategy σ, then

$$nR_0 \geq H(M_0) \geq I(U^n; M_0) = \sum_{t=1}^{n} I(U_t; M_0|U^{t-1}) \tag{22}$$
$$= nI(U_T; M_0|U^{T-1}, T) \tag{23}$$
$$= nI(U_T; M_0, U^{T-1}, T) \tag{24}$$
$$\geq nI(U_T; M_0, T) = nI(U; W_0). \tag{25}$$

In fact, (23) follows from the introduction of the uniform random variable $T \in \{1, \ldots, n\}$, (24) comes from the i.i.d. property of the source and (25) follows from the identification of the auxiliary random variables (U, W_0). Similarly,

$$n(R_1 + R_0) \geq H(M_1, M_0) \geq I(U^n; M_1, M_0)$$
$$= \sum_{t=1}^{n} I(U_t; M_1, M_0|U^{t-1}) = nI(U_T; M_1, M_0|U^{T-1}, T)$$

$$\geq nI(U_T; M_1, M_0, T) = nI(U; W_1, W_0).$$

□

Lemma 5. *For all* (σ, τ_1, τ_2) *and* $i \in \{1, 2\}$*, we have*

$$c_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}\big[c_e(U, V_1, V_2)\big]. \tag{26}$$

where the expected values are evaluated with respect to $\mathcal{P}_U \mathcal{P}^{\sigma}_{W_0W_1|U} \mathcal{P}^{\tau_1}_{V_1|W_0W_1} \mathcal{P}^{\tau_2}_{V_2|W_0}$*. Moreover, for each* (m_0, m_1, v_1^n, v_2^n) *and* $i \in \{1, 2\}$ *we have*

$$C_i^{\sigma}(m_0, m_1, v_1^n, v_2^n) = \mathbb{E}\Big[C_i^{\star}(w_0, w_1, v_1, v_2)\Big], \tag{27}$$

$$\Psi_1^{\sigma}(\tau_1, \tau_2, m_0, m_1) = \mathbb{E}\Big[\Psi_1^{\star}(\mathcal{P}^{\tau_1}_{V_1|W_0W_1}, \mathcal{P}^{\tau_2}_{V_2|W_0}, w_0, w_1)\Big], \tag{28}$$

$$\Psi_2^{\sigma}(\tau_1, \tau_2, m_0) = \mathbb{E}\Big[\Psi_2^{\star}(\mathcal{P}^{\tau_1}_{V_1|W_0W_1}, \mathcal{P}^{\tau_2}_{V_2|W_0}, w_0)\Big]. \tag{29}$$

Proof. [Lemma 5] By Definition 2, we have

$$\begin{aligned} c_e^n(\sigma, \tau_1, \tau_2) = &\sum_{\substack{u^n, m_0, m_1, \\ v_1^n, v_2^n}} \Big(\prod_{t=1}^n \mathcal{P}_U(u_t)\Big) \mathcal{P}^{\sigma}_{M_0M_1|U^n}(m_0, m_1|u^n) \mathcal{P}^{\tau_1}_{V_1^n|M_0M_1}(v_1^n|m_0, m_1) \\ &\times \mathcal{P}^{\tau_2}_{V_2^n|M_0}(v_2^n|m_0) \cdot \left[\frac{1}{n}\sum_{t=1}^n c_e(u_t, v_{1,t}, v_{2,t})\right] \\ = &\sum_{t=1}^n \sum_{\substack{u_t, m_0, m_1, \\ v_{1,t}, v_{2,t}}} \mathcal{P}^{\sigma, \tau_1, \tau_2}(u_t, m_0, m_1, t, v_{1,t}, v_{2,t}) c_e(u_t, v_{1,t}, v_{2,t}) = \mathbb{E}\big[c_e(U, V_1, V_2)\big]. \end{aligned}$$

For each (m_0, m_1, v_1^n, v_2^n) and $i \in \{1, 2\}$ we have

$$\begin{aligned} C_i^{\sigma}(m_0, m_1, v_1^n, v_2^n) &= \sum_{u^n} \mathcal{P}^{\sigma}_{U^n|M_0M_1}(u^n|m_0, m_1) \left[\frac{1}{n}\sum_{t=1}^n c_i(u_t, v_{1,t}, v_{2,t})\right] \\ &= \frac{1}{n}\sum_{u^n} \prod_{t=1}^n \mathcal{P}^{\sigma}_{U_t|M_0M_1}(u_t|m_0, m_1) \sum_{t=1}^n c_i(u_t, v_{1,t}, v_{2,t}) \\ &= \sum_{t=1}^n \sum_{u_t} \mathcal{P}^{\sigma}(u_t|m_0, m_1, t) c_i(u_t, v_{1,t}, v_{2,t}) \\ &= \mathbb{E}\Big[C_i^{\star}(v_1, v_2, w_0, w_1)\Big]. \end{aligned}$$

$\Psi_1^{\sigma}(\tau_1, \tau_2, m_0, m_1)$

$$
\begin{aligned}
&= \sum_{u^n} \Big(\prod_{t=1}^{n} \mathcal{P}_U(u_t) \Big) \mathcal{P}^1(m_0|m_0,m_1) \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}(v_1^n|m_0,m_1) \mathcal{P}^{\tau_2}(v_2^n|m_0) \left[\frac{1}{n} \sum_{t=1}^{n} c_1(u_t, v_{1,t}, v_{2,t}) \right] \\
&= \sum_{t=1}^{n} \sum_{u_t} \mathcal{P}_U(u_t) \mathcal{P}^1(m_0, t|m_0, m_1, t) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|m_0, m_1, t) \mathcal{P}^{\tau_{2,t}}(v_{2,t}|m_0, t) \big[c_1(u_t, v_{1,t}, v_{2,t}) \big] \\
&= \sum_{u} \mathcal{P}_U(u) \mathcal{P}^1(w_0|w_0, w_1) \sum_{v_1, v_2} \mathcal{P}^{\tau_{1,t}}(v_1|w_0, w_1) \mathcal{P}^{\tau_{2,t}}(v_2|w_0) C_1^\star(v_1, v_2, w_0, w_1) \\
&= \mathbb{E}\Big[\Psi_1^\star(\mathcal{P}^{\tau_{1,t}}_{V_1|W_0W_1}, \mathcal{P}^{\tau_{2,t}}_{V_2|W_0}, w_0, w_1) \Big].
\end{aligned}
$$

Similarly, equation (29) can be shown. □

In the following, we show that for each encoding strategy σ, each equilibrium (τ_1, τ_2) of the game of block n, induces an equilibrium for the single-letter game.

Lemma 6. *For all σ, we have*

$$
\begin{aligned}
\mathbb{BNE}(\mathcal{P}^\sigma_{W_0W_1|U}) = \Big\{ (\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}), \quad \exists (\tau_1, \tau_2),\ \tau_1 \in BR_1(\sigma, \tau_2), \tau_2 \in BR_2(\sigma, \tau_1), \\
\mathcal{Q}_{V_1|W_0W_1} = \mathcal{P}^{\tau_1}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0} = \mathcal{P}^{\tau_2}_{V_2|W_0} \Big\}. \qquad (30)
\end{aligned}
$$

Proof. [Lemma 6] Fix σ and let $(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}) \in \mathbb{BNE}(\mathcal{P}^\sigma_{W_0W_1|U})$. We consider (τ_1, τ_2) such that

$$
\begin{aligned}
\mathcal{P}^{\tau_1}_{V_1^n|M_0M_1}(v_1^n|m_0, m_1) &= \prod_{t=1}^{n} \mathcal{Q}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t), \\
\mathcal{P}^{\tau_2}_{V_2^n|M_0}(v_2^n|m_0) &= \prod_{t=1}^{n} \mathcal{Q}_{V_2|W_0}(v_{2,t}|m_0, t).
\end{aligned}
$$

Then $\forall (w_0, w_1, v_1) = (m_0, m_1, t, v_{1,t})$,

$$
\begin{aligned}
\mathcal{P}^{\tau_1}_{V_1|W_0W_1}(v_1|w_0, w_1) =& \mathcal{P}^{\tau_1}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t) = \sum_{v_1^{t-1}, v_{1,t+1}^n} \prod_{s=1}^{n} \mathcal{Q}_{V_1|W_0W_1}(v_{1,s}|m_0, m_1, s) \\
=& \mathcal{Q}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t) \sum_{v_1^{t-1}, v_{1,t+1}^n} \prod_{s \neq t} \mathcal{Q}_{V_1|W_0W_1}(v_{1,s}|m_0, m_1, s) \\
=& \mathcal{Q}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t) = \mathcal{Q}_{V_1|W_0W_1}(v_1|w_0, w_1). \qquad (31)
\end{aligned}
$$

Suppose that $\tau_1 \notin BR_1(\sigma, \tau_2)$. Then, there exists $\bar{\tau}_1 \neq \tau_1$ such that

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{P}^\sigma_{U|W_0W_1}} \big[\Psi_1^\star(\mathcal{P}^{\bar{\tau}_1}_{V_1|W_0W_1}, \mathcal{P}^{\tau_2}_{V_2|W_0W_2}, w_0, w_1) \big] = \Psi_1^\sigma(\bar{\tau}_1, \tau_2, m_0, m_1) \\
&< \Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) = \mathbb{E}_{\mathcal{P}^\sigma_{U|W_0W_1}} \big[\Psi_1^\star(\mathcal{P}^{\tau_1}_{V_1|W_0W_1}, \mathcal{P}^{\tau_2}_{V_2|W_0W_2}, w_0, w_1) \big], \qquad (32)
\end{aligned}
$$

which contradicts $(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{BNE}(\mathcal{P}^{\sigma}_{W_0W_1|U})$. Therefore, $\tau_1 \in BR_1(\sigma, \tau_2)$ and thus $\mathcal{Q}_{V_1|W_0W_1}$ belongs to the right-hand side of (30). Similarly, one can show that $\tau_2 \in BR_2(\sigma, \tau_1)$. The other inclusion is direct. □

For any strategy σ, we have

$$\max_{\tau_1,\tau_2} c_e^n(\sigma, \tau_1, \tau_2) = \max_{\tau_1,\tau_2} \mathbb{E}_{\substack{\mathcal{P}^{\sigma}_{W_0W_1|U} \\ \mathcal{P}^{\tau_1}_{V_1|W_0W_1}\mathcal{P}^{\tau_2}_{V_2|W_0}}} \Big[c_e(U, V_1, V_2)\Big] \tag{33}$$

$$= \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}) \in \\ \mathbb{BNE}(\mathcal{P}^{\sigma}_{W_0W_1|U})}} \mathbb{E}_{\substack{\mathcal{P}^{\sigma}_{W_0W_1|U} \\ \mathcal{Q}_{V_1|W_0W_1}\mathcal{Q}_{V_2|W_0}}} \Big[c_e(U, V_1, V_2)\Big] \tag{34}$$

$$\geq \inf_{\substack{\mathcal{Q}_{W_0W_1|U} \\ \in \mathbb{Q}_0(R_0,R_1)}} \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}) \in \\ \mathbb{BNE}(\mathcal{P}^{\sigma}_{W_0W_1|U})}} \mathbb{E}\Big[c_e(U, V_1, V_2)\Big] \tag{35}$$

$$= \Gamma_e^{\star}(R_0, R_1). \tag{36}$$

Equations (33) and (34) follow from Lemma 5 and Lemma 6, whereas (35) comes from Lemma 4. This concludes the converse proof of Theorem 1.

In the following, we prove the achievability statement (13) of our main result.

V. Achievability Proof of Theorem 1

We denote by $\mathcal{T}_{\delta}^n(P_X)$ the set of all $\delta-$typical sequences corresponding to P_X. The set of all such $k-$tuples is denoted by $\mathcal{T}_{\delta}^n(\mathcal{P}_{X_1\ldots X_k})$. By a slight abuse of notation, the δ-typical set can be denoted by $\mathcal{T}_{\delta}^n(X)$ or $\mathcal{T}_{\delta}^n$ when the context is not ambiguous.

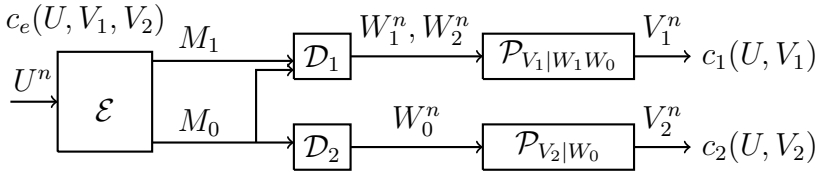

Fig. 3: Achievability proof via successive refinement source coding.

A. Essential Equilibria

The random coding scheme may induce some perturbations in the probability distribution $\mathcal{Q}_{W_0W_1|U}$ of the Bayesian game of Definition 7. A Bayesian game is essential [22, Definition 4.1] if small perturbations of the probability distributions may induce small perturbations of the set of Bayes-Nash equilibria. According to [22, Theorem 4.2], the set of essential Bayesian games is a dense subset of the set of Bayesian games.

Without loss of generality, we consider the target probability distribution $\mathcal{Q}_{W_0W_1|U}$ induces an essential Bayesian games.

Definition 10. *Given $\mathcal{Q}_{W_0W_1|U} \in \mathbb{Q}_0(R_0, R_1)$. An equilibrium $(\mathcal{Q}_{V_1|W_1W_0}, \mathcal{Q}_{V_2|W_0}) \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})$ is* essential *if for all $\varepsilon > 0$, there exists an open neighborhood Ω including $\mathcal{Q}_{W_0W_1|U}$ such that for all $\tilde{\mathcal{Q}}_{W_0W_1|U} \in \Omega$, we have*

$$\begin{aligned} & (\tilde{\mathcal{Q}}_{V_1|W_1W_0}, \tilde{\mathcal{Q}}_{V_2|W_0}) \in \mathbb{BNE}(\tilde{\mathcal{Q}}_{W_0W_1|U}) \\ \implies \quad & \max_{w_0,w_1} ||\mathcal{Q}_{V_1|W_0W_1}(\cdot|w_0,w_1) - \tilde{\mathcal{Q}}_{V_1|W_0W_1}(\cdot|w_0,w_1)|| + ||\mathcal{Q}_{V_2|W_0}(\cdot|w_0) - \tilde{\mathcal{Q}}_{V_2|W_0}(\cdot|w_0)|| \leq \epsilon, \quad (37) \end{aligned}$$

where $||\cdot||$ denote the L_1-norm. We denote by $\mathbb{EBNE}(\mathcal{Q}_{W_0W_1|U})$ the set of essential equilibria.

Equation (37) ensures that the Bayes-Nash equilibrium of the perturbed game $(\tilde{\mathcal{Q}}_{V_1|W_1W_0}, \tilde{\mathcal{Q}}_{V_2|W_0}) \in \mathbb{BNE}(\tilde{\mathcal{Q}}_{W_0W_1|U})$ is close under the L_1-norm, to the original Bayes-Nash equilibrium $(\mathcal{Q}_{V_1|W_1W_0}, \mathcal{Q}_{V_2|W_0}) \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})$.

Definition 11. *For $(R_0, R_1) \in \mathbb{R}_+^2$, we define the set*

$$\begin{aligned} \tilde{\mathbb{Q}}_0(R_0,R_1) = \Big\{ & \mathcal{Q}_{W_0W_1|U}, \quad R_0 > I(U;W_0),\ R_0 + R_1 > I(U;W_1,W_0), \\ & \min_{u,w_0,w_1} \mathcal{Q}_{W_0W_1|U}(w_0,w_1|u) > 0,\ \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U}) = \mathbb{EBNE}(\mathcal{Q}_{W_0W_1|U}) \Big\}. \quad (38) \end{aligned}$$

In the following, we show that optimizing over the full set of target distributions results in the same cost as when the optimization is taken over the set of target distributions with full support that induce essential Bayes-Nash Equilibria.

Definition 12. *We replace the set $\mathbb{Q}_0(R_0, R_1)$ by the set $\tilde{\mathbb{Q}}_0(R_0, R_1)$ and we define the following program:*

$$\tilde{\Gamma}_e(R_0,R_1) = \inf_{\substack{\mathcal{Q}_{W_0W_1|U} \in \\ \tilde{\mathbb{Q}}_0(R_0,R_1)}} \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}) \\ \in \mathbb{EBNE}(\mathcal{Q}_{W_0W_1|U})}} \mathbb{E}\Big[c_e(U,V_1,V_2)\Big].$$

Lemma 7. *For $(R_0, R_1) \in \mathbb{R}_+^2$, we have*

$$\Gamma_e^\star(R_0,R_1) = \tilde{\Gamma}_e(R_0,R_1). \quad (39)$$

The proof of Lemma 7 follows from [22, Theorem 4.2] and the Lemmas 8 and 9, below.

Lemma 8. *Given $\mathcal{Q}_{W_0W_1|U} \in \mathbb{Q}_0(R_0, R_1)$, the set $\mathbb{EBNE}(\mathcal{Q}_{W_0W_1|U})$ is dense in*

$\mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})$.

Proof. [Lemma 8] Let $(R_0, R_1) \in \mathbb{R}_+^2$ and consider the Bayes-Nash equilibrium correspondence $\mathbb{Q}_0(R_0, R_1) \rightrightarrows \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})$. According to [22, Theorem 3.3], for finite games, this correspondance is non-empty, compact-valued and upper semi-continuous. From [25, Theorem 2], see also [22, Lemma 2.1], there exists dense subset of probability distributions $\mathcal{Q}_{W_0W_1|U}$ such that this correspondence is lower semi-continuous, hence continuous. In conclusion, [22, Theorem 4.2] shows that there exists dense subset of probability distributions $\mathcal{Q}_{W_0W_1|U}$ such that $\mathbb{BNE}(\mathcal{Q}_{W_0W_1|U}) = \mathbb{EBNE}(\mathcal{Q}_{W_0W_1|U})$, i.e. the Bayes-Nash equilibria are essential Bayes-Nash equilibria. □

Lemma 9. *Let $(R_0, R_1) \in \mathbb{R}_+^2$, the set $\tilde{\mathbb{Q}}_0(R_0, R_1)$ is a dense subset of $\mathbb{Q}_0(R_0, R_1)$.*

Proof. [Lemma 9] The set of distributions $\mathcal{Q}_{W_0W_1|U}$ with full support is dense in $\Delta(\mathcal{W}_0 \times \mathcal{W}_1)^{|\mathcal{U}|}$. The finite intersection of dense subsets is dense. By using Lemma 8, $\tilde{\mathbb{Q}}_0(R_0, R_1)$ is a dense subset of $\mathbb{Q}_0(R_0, R_1)$. □

The last part of our achievability proof consists of showing that for sufficiently large n, the expected cost of the encoder evaluated with respect to the long-run beliefs of the decoders is close under the L_1-norm to the expected cost evaluated with respect to the single-letter beliefs. This requires showing that Bayesian beliefs of the decoders induced by the coding strategies in the game of block n, are close under the KL-divergence to the single-letter beliefs defined with respect to the auxiliary random variables W_0 and W_1. Decoder $\mathcal{D}_1$ has belief about the state U whereas decoder $\mathcal{D}_2$ has belief about the state U and the observation M_1 of decoder $\mathcal{D}_1$.

B. Codebook Generation

Fix a conditional probability distribution $\mathcal{Q}_{W_0W_1|U} \in \tilde{\mathbb{Q}}_0(R_0, R_1)$. Without loss of generality, we assume that there exists $\eta > 0$ such that

$$R_0 = I(U; W_0) + \eta, \tag{40}$$

$$R_1 = I(U; W_1|W_0) + \eta, \tag{41}$$

- *Codebook generation*: Randomly and independently generate $2^{\lfloor nR_0 \rfloor}$ sequences $w_0^n(m_0)$ for $m_0 \in [1 : 2^{\lfloor nR_0 \rfloor}]$, according to the i.i.d distribution $\prod_{t=1}^n \mathcal{Q}_{W_0}(w_{0,t})$. For each $(m_1, m_0) \in [1 : 2^{\lfloor nR_1 \rfloor}] \times [1 : 2^{\lfloor nR_0 \rfloor}]$ generate a sequence $w_1^n(m_1, m_0)$ randomly and conditionally independently according to the i.i.d conditional distribution $\prod_{t=1}^n \mathcal{Q}_{W_1|W_0}(w_{1,t}|w_{0,t}(m_0))$.

- *Encoding function*: Encoder $\mathcal{E}$ observes u^n and selects m_0 such that $(u^n, w_0^n(m_0)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0|U})$, and m_1 such that $(u^n, w_1^n(m_1, m_0), w_0^n(m_0)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0W_1|U})$ and sends m_0 to both decoders and m_1 to the decoder $\mathcal{D}_1$. If such a jointly typical tuple doesn't exist, the source encoder sets (m_0, m_1) to $(1, 1)$.
- *Decoding function*: Decoder $\mathcal{D}_1$ observes the pair (M_0, M_1) and decoder $\mathcal{D}_2$ observes M_0

C. Analysis of error probability

We define the following error events

$$\mathcal{F}_0 = \{(U^n, W_0^n(m_0)) \notin \mathcal{T}_\delta^n, \ \forall m_0\}, \tag{42}$$

$$\forall m_0, \quad \mathcal{F}_1(m_0) = \{(U^n, W_0^n(m_0), W_1^n(m_1, m_0)) \notin \mathcal{T}_\delta^n \ \forall m_1\}. \tag{43}$$

We denote the codebook by $\mathcal{C}$. For each $m_0 \in \{1, ..2^{\lfloor nR_0 \rfloor}\}$, we denote the inner codebook by $\mathcal{C}(m_0) = \{w_1^n(m_0, m_1) \in \mathcal{C}, m_1 \in \{1, ..2^{\lfloor nR_1 \rfloor}\}\}$. For each codeword $m_0 \in \mathcal{C}$, for each symbol $w_1 \in \mathcal{W}_1$, for each $t \in \{1, \dots, n\}$, for $\delta > 0$, we denote

$$\mathcal{A}^t(w_1|m_0) = \big\{w_1^n \in \mathcal{C}(m_0), w_{1,t} = w_1\big\}, \ \forall w_1, m_0 \tag{44}$$

$$\mathcal{J}(m_0) = \left\{ t \in \{1, .., n\}, \left|\left| \mathcal{Q}_{W_1|W_0}(\cdot|w_{0,t}) - \frac{|\mathcal{A}^t(\cdot|m_0)|}{|\mathcal{C}(m_0)|} \right|\right| > \delta \right\}, \ \forall m_0, \tag{45}$$

where $||\cdot||$ denote the L_1-norm.

Let $\mathcal{J} = \bigcup_{m_0} \mathcal{J}(m_0)$. We have by the union of events bound $\mathcal{P}(|\mathcal{J}| > \delta) \leq \sum_{m_0} \mathcal{P}(|\mathcal{J}(m_0)| > \delta)$.

By the covering lemma [6, Lemma 3.3], $\mathcal{P}(\mathcal{F}_0)$ tends to zero as $n \longrightarrow \infty$ if $R_0 \geq \ I(U; W_0) + \eta$, and $\mathcal{P}(\mathcal{F}_1(M_0)|\mathcal{F}_0^c)$ goes to zero by the covering lemma if $R_1 \geq \ I(U; W_1|W_0) + \eta$. Moreover, for all (w_1, t, m_0), we have

$$\sum_{w_1^n \in \mathcal{C}(m_0)} \mathcal{P}\big[W_{1,t} = w_1 | w_0^n(m_0)\big] = \sum_{w_1^n \in \mathcal{C}(m_0)} \mathbb{E}\big[\mathbb{1}_{\{W_{1,t}=w_1\}} \big| w_0^n(m_0)\big] = \mathbb{E}\left[|\mathcal{A}^t(w_1|m_0)| \middle| w_0^n(m_0) \right] \tag{46}$$

$$= \sum_{w_1^n \in \mathcal{C}(m_0)} \mathcal{Q}(w_{1,t}|w_{0,t}) = \big|\mathcal{C}(m_0)\big| \cdot \mathcal{Q}_{W_1|W_0}(\cdot|W_{0,t}). \tag{47}$$

Therefore, as $n \longrightarrow \infty$, $\mathcal{P}(\mathcal{J}(m_0))$ goes to zero for all $m_0 \in \{1, ..2^{\lfloor nR_0 \rfloor}\}$.

The expected probability of error over the codebook being small means that for all $\varepsilon_2 > 0$, for all

$\eta > 0$, there exists $\bar{\delta} > 0$, for all $\delta \leq \bar{\delta}$, there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$ we have:

$$\mathbb{E}\big[\mathcal{P}(\mathcal{F}_0)\big] \leq \varepsilon_2, \tag{48}$$

$$\mathbb{E}\big[\mathcal{P}(\mathcal{F}_1(m_0)|\mathcal{F}_0^c)\big] \leq \varepsilon_2, \tag{49}$$

$$\mathbb{E}\big[\mathcal{P}(|\mathcal{J}(m_0)| > \delta)\big] \leq \varepsilon_2. \tag{50}$$

D. Control of the Bayesian belief

We introduce the indicator of error events $E_\delta^1 \in \{0, 1\}$ defined as follows

$$E_\delta^1 = \begin{cases} 1, & \text{if } (u^n, w_1^n, w_0^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1 W_0|U}). \\ 0, & \text{otherwise.} \end{cases} \tag{51}$$

We control the Bayesian belief of decoder $\mathcal{D}_2$ about the type of $\mathcal{D}_1$.

Lemma 10. *For all $(w_0^n, w_1^n, w_0, w_1, t)$, we have*

$$\lim_{n \mapsto \infty} \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} D(\mathcal{P}_{W_{1,t}|W_0^n}(.|W_0^n)||\mathcal{Q}_{W_{1,t}|W_{0,t}}(.|W_{0,t}))\middle| E_\delta^1 = 0\right] = 0. \tag{52}$$

The proof of Lemma 10 is stated in App. B. Given the types (m_0, m_1) of the decoders, the Bayesian posterior belief about the source U at stage $t \in \{1, \ldots, n\}$ is close to the single-letter Bayesian belief.

Lemma 11. *For all $\delta > 0$, we have*

$$\lim_{n \mapsto \infty} \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} D(\mathcal{P}_{U_t|M_0 M_1}||\mathcal{Q}_{U|W_1 W_0}(\cdot|W_{1,t}, W_{0,t}))\middle| E_\delta^1 = 0\right] \leq \delta. \tag{53}$$

The proof of Lemma 11 is stated in App. C.

In the following, we define the Bayesian game $G^{\sigma,t}$ at stage t for $t \in \{1, 2, ...n\}$, and we show that for each σ, (τ_1, τ_2) is an equilibrium of G^σ if and only if $(\tau_{1,t}, \tau_{2,t})$ is an equilibrium of $G^{\sigma,t}$ for all $t \in \{1, 2, ...n\}$.

E. Bayesian game of stage $t \in \{1, 2, ...n\}$

For each σ and for all $t \in \{1, 2, ...n\}$, we define the Bayesian game at stage $t \in \{1, 2, ...n\}$. Then, we relate the Bayesian game of the coding problem $\Gamma_e^n(R_0, R_1)$, to the Bayesian game at stage $t \in \{1, 2, ...n\}$, and finally, to the Bayesian game of the single-letter solution $\Gamma_e^\star(R_0, R_1)$.

Definition 13. *For each encoding σ, and $t \in \{1,2,...n\}$, the Bayesian game $G^{\sigma,t}$ of stage $t \in \{1,2,...n\}$ consists of:*

- *The decoders $\mathcal{D}_i, i \in \{1,2\}$ as the players of the game,*
- *$\mathcal{V}_{i,t}$ is the set of action of $\mathcal{D}_i$,*
- *(M_0, M_1) is the type of decoder $\mathcal{D}_1$, and M_0 is the type of decoder $\mathcal{D}_2$.*
- *$\tau_{1,t} : \{1,2,..2^{\lfloor nR_0 \rfloor}\} \times \{1,2,..2^{\lfloor nR_1 \rfloor}\} \mapsto \Delta(\mathcal{V}_{1,t})$ is a behavior strategy of decoder $\mathcal{D}_1$, and $\tau_{2,t} : \{1,2,..2^{\lfloor nR_0 \rfloor}\} \mapsto \Delta(\mathcal{V}_{2,t})$ is a behavior strategy of $\mathcal{D}_2$.*
- *The decoder $\mathcal{D}_1$ knows the type of the decoder $\mathcal{D}_2$. The decoder $\mathcal{D}_2$ forms a Bayesian belief $\mathcal{P}^{\sigma}_{M_1|M_0}$ about the type of the decoder $\mathcal{D}_1$.*
- *The cost function of decoder $\mathcal{D}_i$ at stage $t \in \{1,2,...n\}$ is defined by*

$$C_i^{\sigma,t}(m_0, m_1, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{P}^{\sigma}(u_t|m_0, m_1) c_i(u_t, v_{1,t}, v_{2,t}), \quad \forall v_{1,t}, v_{2,t}, m_0, m_1.$$

- *Given the strategies $(\tau_{1,t}, \tau_{2,t})$ of the decoders at stage $t \in \{1,2,...n\}$, the expected costs are given by*

$$\Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) = \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|m_0, m_1) \mathcal{P}^{\tau_{2,t}}(v_{2,t}|m_0) C_1^{\sigma,t}(v_{1,t}, v_{2,t}, m_0, m_1).$$

Similarly, $\Psi_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0)$ can be defined.

Definition 14. *For each encoding σ, and $t \in \{1,2,...n\}$, the essential Bayesian game $\tilde{G}^{\sigma,t}(M_0, M_1, V_{1,t}, V_{2,t})$ of stage $t \in \{1,2,...n\}$ consists of:*

- *The decoders $\mathcal{D}_i, i \in \{1,2\}$ as the players of the game,*
- *$\mathcal{V}_{i,t}$ is the set of action of $\mathcal{D}_i$,*
- *M_0, M_1 is the type of decoder $\mathcal{D}_1$, and M_0 is the type of decoder $\mathcal{D}_2$.*
- *$\tau_{1,t} : \{1,2,..2^{\lfloor nR_0 \rfloor}\} \times \{1,2,..2^{\lfloor nR_1 \rfloor}\} \mapsto \Delta(\mathcal{V}_{1,t})$ is a behavior strategy of decoder $\mathcal{D}_1$, and $\tau_{2,t} : \{1,2,..2^{\lfloor nR_0 \rfloor}\} \mapsto \Delta(\mathcal{V}_{2,t})$ is a behavior strategy of $\mathcal{D}_2$.*
- *The decoder $\mathcal{D}_1$ knows the type of the decoder $\mathcal{D}_2$. The decoder $\mathcal{D}_2$ forms a Bayesian belief $\mathcal{P}^{\sigma}_{W_{1,t}|W_{0,t}}$ about the type of the decoder $\mathcal{D}_1$.*
- *The cost function of decoder $\mathcal{D}_i$ at stage $t \in \{1,2,...,n\}$ is defined by*

$$\tilde{C}_i^{\sigma,t}(m_0, m_1, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}(m_0), w_{1,t}(m_0, m_1)) c_i(u_t, v_{1,t}, v_{2,t}), \quad \forall v_{1,t}, v_{2,t}, m_0, m_1.$$

- *Given the strategies* $(\tau_{1,t}, \tau_{2,t})$ *of the decoders at stage* $t \in \{1, 2, ...n\}$*, the expected costs are given by*

$$\tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) = \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}(m_0), w_{1,t}(m_0, m_1))\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}(m_0))\tilde{C}_1^{\sigma,t}(v_{1,t}, v_{2,t}, m_0, m_1).$$

Similarly, $\tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0)$ *can be defined.*

For each encoding strategy σ and stage $t \in \{1, 2, ...n\}$, the set $BNE(\sigma, t)$ of Bayes-Nash equilibria $(\tau_{1,t}, \tau_{2,t})$ of $\tilde{G}^{\sigma,t}(M_0, M_1, V_{1,t}, V_{2,t})$ is given by

$$B\tilde{N}E^t(\sigma) = \Big\{(\tau_{1,t}, \tau_{2,t}), \quad \tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \leq \tilde{\Psi}_1^{\sigma,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, m_0, m_1), \forall\ \tilde{\tau}_{1,t}, m_0, m_1,$$
$$\tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0) \leq \tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tilde{\tau}_{2,t}, m_0)\ \forall\ \tilde{\tau}_{2,t}, m_0\Big\}. \tag{54}$$

Definition 15. *For each encoding* σ*, and* $t \in \{1, 2, ...n\}$*, the Bayesian game* $\hat{G}^{\sigma,t}(W_0^n, W_1^n, V_{1,t}, V_{2,t})$ *of stage* $t \in \{1, 2, ...n\}$ *consists of:*

- *The decoders* $\mathcal{D}_i, i \in \{1, 2\}$ *as the players of the game,*
- $\mathcal{V}_{i,t}$ *is the set of action of* $\mathcal{D}_i$*,*
- $(W_0^n, W_1^n) = (W_{0,t}, W_{1,t}, W_0^{t-1}, W_{0,t+1}^n, W_1^{t-1}, W_{1,t+1}^n)$ *is the type of decoder* $\mathcal{D}_1$*, and* W_0^n *is the type of decoder* $\mathcal{D}_2$*.*
- $\tau_{1,t} : \ \{1, 2, ..2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, ..2^{\lfloor nR_1 \rfloor}\} \mapsto \Delta(\mathcal{V}_{1,t})$ *is a behavior strategy of decoder* $\mathcal{D}_1$*, and* $\tau_{2,t} : \{1, 2, ..2^{\lfloor nR_0 \rfloor}\} \mapsto \Delta(\mathcal{V}_{2,t})$ *is a behavior strategy of* $\mathcal{D}_2$*.*
- *The decoder* $\mathcal{D}_1$ *knows the type of the decoder* $\mathcal{D}_2$*. The decoder* $\mathcal{D}_2$ *forms a Bayesian belief* $\mathcal{P}^{\sigma}_{W_1^n|W_0^n}$ *about the type of the decoder* $\mathcal{D}_1$*.*
- *The cost function of decoder* $\mathcal{D}_i$ *at stage* $t \in \{1, 2, ...n\}$ *is defined by*

$$\hat{C}_i^{\sigma,t}(w_{0,t}, w_{1,t}, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{P}^{\sigma}(u_t|w_{0,t}, w_{1,t})c_i(u_t, v_{1,t}, v_{2,t}), \quad \forall v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}.$$

- *Given the strategies* $(\tau_{1,t}, \tau_{2,t})$ *of the decoders at stage* $t \in \{1, 2, ...n\}$*, the expected costs are given by*

$$\hat{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) = \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t})\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t})\hat{C}_1^{\sigma,t}(v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}).$$

Similarly, $\hat{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t})$ *can be defined.*

Lemma 12. *For every σ, $(\tau_1, \tau_2) \in BNE(\sigma)$ if and only if $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$ for all $t \in \{1, 2, ..., n\}$.*

Proof. Given σ, let $(\tau_1, \tau_2) \in BNE(\sigma)$. Assume that there exists t such that $(\tau_{1,t}, \tau_{2,t}) \notin BNE(\sigma, t)$. This means that at t, at least one of the decoders is better off if it deviates from its strategy. Without loss of generality, assume $\mathcal{D}_1$ deviates to $\tilde{\tau}_{1,t}$ and selects $\tilde{v}_{1,t}$ accordingly. This shifts the action sequence v_1^n that corresponds to τ_1, to $\tilde{v}_1^n = (v_{1,1}, v_{1,2}, ..., \tilde{v}_{1,t}, ..., v_{1,n})$. Thus $\tau_1 \notin BR_1^{\sigma}(\tau_2)$, and $(\tau_1, \tau_2) \notin BNE(\sigma)$. Conversely, if $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$ for all $t \in \{1, 2, ..., n\}$, we define (τ_1, τ_2) such that

$$\mathcal{P}^{\tau_1}(v_1^n|m_0, m_1) = \prod_{t=1}^{n} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|m_0, m_1), \quad \forall v_1^n, m_0, m_1 \tag{55}$$

$$\mathcal{P}^{\tau_2}(v_2^n|m_0) = \prod_{t=1}^{n} \mathcal{P}^{\tau_{2,t}}(v_{2,t}|m_0), \quad \forall v_2^n, m_0. \tag{56}$$

Suppose that $(\tau_1, \tau_2) \notin BNE(\sigma)$. Without loss of generality, assume $\tau_1 \notin BR_1(\sigma, \tau_2)$, i.e there exists $\tilde{\tau}_1 \in BR_1(\sigma)$ such that $\Psi_1^{\sigma}(\tau_1, \tau_2, m_0, m_1) \geq \Psi_1^{\sigma}(\tilde{\tau}_1, \tau_2, m_0, m_1)\ \forall m_0, m_1$. Therefore, there exists $t \in \{1, ..., n\}$ such that $\Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \geq \Psi_1^{\sigma,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, m_0, m_1)\ \forall m_0, m_1$. Thus, $(\tau_{1,t}, \tau_{2,t}) \notin BNE(\sigma, t)$ which leads to the desired contradiction. □

We use Lemmas 10, 11, and 12, in order to evaluate the expected cost of the decoder $\mathcal{D}_1$ for strategies (τ_1, τ_2) of types (w_0^n, w_1^n).

$$\Psi_1^{\sigma}(\tau_1, \tau_2, w_0^n, w_1^n) = \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}(v_1^n|w_0^n, w_1^n)\mathcal{P}^{\tau_2}(v_2^n|m_0) \sum_{u^n} \mathcal{P}(u^n|w_0^n, w_1^n) \left[\frac{1}{n}\sum_{t=1}^{n} c_1(u_t, v_{1,t}, v_{2,t})\right] \tag{57}$$

$$= \frac{1}{n}\sum_{t=1}^{n} \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_0^n, w_1^n)\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_0^n) \sum_{u_t} \mathcal{P}(u_t|w_0^n, w_1^n)c_1(u_t, v_{1,t}, v_{2,t}) \tag{58}$$

$$\simeq \frac{1}{n}\sum_{t=1}^{n} \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_0^n, w_1^n)\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_0^n) \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t})c_1(u_t, v_{1,t}, v_{2,t}) \tag{59}$$

$$= \frac{1}{n}\sum_{t=1}^{n} \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t})\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}) \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t})c_1(u_t, v_{1,t}, v_{2,t}) \tag{60}$$

$$= \sum_{w_0, w_1} \pi_{W_0W_1}(w_0, w_1) \sum_{v_1, v_2} \mathcal{P}(v_1|w_0, w_1)\mathcal{P}(v_2|w_0) \sum_{u} \mathcal{Q}(u|w_0, w_1)c_1(u, v_1, v_2), \tag{61}$$

where $\pi_{W_0W_1}(\cdot)$ denotes the empirical distribution induced by the sequences (w_0^n, w_1^n) and (59) follows

from Lemma 11. Similarly, we evaluate the cost of decoder $\mathcal{D}_2$ for strategies (τ_1, τ_2) of type w_0^n.

$$\Psi_2^{\sigma}(\tau_1, \tau_2, w_0^n) \tag{62}$$

$$= \sum_{w_1^n} \mathcal{P}(w_1^n|w_0^n) \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}(v_1^n|w_0^n, w_1^n)\mathcal{P}^{\tau_2}(v_2^n|w_0^n) \sum_{u^n} \mathcal{P}(u^n|w_0^n, w_1^n) \left[\frac{1}{n}\sum_{t=1}^{n} c_2(u_t, v_{1,t}, v_{2,t})\right] \tag{63}$$

$$= \frac{1}{n}\sum_{t=1}^{n} \sum_{w_1^n} \mathcal{P}(w_1^n|w_0^n) \sum_{\substack{v_{1,t},\\ v_{2,t}}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_0^n, w_1^n)\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_0^n) \sum_{u_t} \mathcal{P}(u_t|w_0^n, w_1^n) c_2(u_t, v_{1,t}, v_{2,t}) \tag{64}$$

$$\simeq \frac{1}{n}\sum_{t=1}^{n} \sum_{w_1^n} \mathcal{P}(w_1^n|w_0^n) \sum_{\substack{v_{1,t},\\ v_{2,t}}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_0^n, w_1^n)\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_0^n) \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t}) c_2(u_t, v_{1,t}, v_{2,t}) \tag{65}$$

$$= \frac{1}{n}\sum_{t=1}^{n} \sum_{w_1^n} \mathcal{P}(w_1^n|w_0^n) \sum_{\substack{v_{1,t},\\ v_{2,t}}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t})\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}) \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t}) c_2(u_t, v_{1,t}, v_{2,t}) \tag{66}$$

$$= \frac{1}{n}\sum_{t=1}^{n} \sum_{w_{1,t}} \mathcal{P}(w_{1,t}|w_0^n) \sum_{\substack{v_{1,t},\\ v_{2,t}}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t})\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}) \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t}) c_2(u_t, v_{1,t}, v_{2,t}) \tag{67}$$

$$\simeq \frac{1}{n}\sum_{t=1}^{n} \sum_{w_{1,t}} \mathcal{Q}(w_{1,t}|w_{0,t}) \sum_{\substack{v_{1,t},\\ v_{2,t}}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t})\mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}) \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t}) c_2(u_t, v_{1,t}, v_{2,t}) \tag{68}$$

$$= \sum_{w_0} \pi_{W_0}(w_0) \sum_{w_1} \mathcal{Q}(w_1|w_0) \sum_{v_1, v_2} \mathcal{Q}(v_1|w_0, w_1)\mathcal{Q}(v_2|w_0) \sum_{u} \mathcal{Q}(u|w_0, w_1) c_2(u, v_1, v_2), \tag{69}$$

where $\pi_{W_0}(\cdot)$ denotes the empirical distribution of the sequence w_0^n, (65) comes from Lemma 11 and (68) comes from Lemma 10. This concludes the achievability proof of Theorem 1. □

APPENDIX A
PROOF OF LEMMA 1

Let $n, m \in \mathbb{Z}$. We denote by σ_c^{n+m}, the concatenation of the strategies σ^n, σ^m where σ^n is implemented over the first n stages and σ^m is implemented over the last m stages. For decoder $i \in \{1, 2\}$, consider the best responses ${\tau_i}^n \in BR_i(\sigma^n)$ and ${\tau_i}^m \in BR_i(\sigma^m)$. Then, the concatenation $\tau_{i,c}^{n+m}$ of ${\tau_i}^n$ and ${\tau_i}^m$ is also a best response $\tau_{i,c}^{n+m} \in BR_i(\sigma_c^{n+m})$. Therefore, we have the inequality

$$\begin{aligned} & n\Gamma_e^n(R_1, R_2) + m\Gamma_e^m(R_1, R_2) \\ =& \inf_{\sigma^n} \max_{\substack{{\tau_1}^n \in BR_1(\sigma^n),\\ {\tau_2}^n \in BR_2(\sigma^n)}} \mathbb{E}\Big[\sum_{t=1}^{n} c_e(U_t, V_{1,t}, V_{2,t})\Big] + \inf_{\sigma^m} \max_{\substack{{\tau_1}^m \in BR_1(\sigma^m),\\ {\tau_2}^m \in BR_2(\sigma^m)}} \mathbb{E}\Big[\sum_{t=1}^{m} c_e(U_t, V_{1,t}, V_{2,t})\Big] \qquad (70) \\ =& \inf_{\sigma_c^{n+m}} \max_{\substack{\tau_1^{n+m} \in BR_1(\sigma_c^{n+m}),\\ \tau_2^{n+m} \in BR_2(\sigma_c^{n+m})}} \mathbb{E}\Big[\sum_{t=1}^{n+m} c_e(U_t, V_{1,t}, V_{2,t})\Big] \qquad (71) \end{aligned}$$

$$\geq \inf_{\sigma^{n+m}} \max_{\substack{\tau_1{}^{n+m}\in BR_1(\sigma^{n+m}),\\ \tau_2{}^{n+m}\in BR_2(\sigma^{n+m})}} \mathbb{E}\Big[\sum_{t=1}^{n+m} c_e(U_t, V_{1,t}, V_{2,t})\Big] \tag{72}$$

$$=(n+m)\Gamma_e^{n+m}(R_1,R_2), \tag{73}$$

where the notation σ_c^{n+m}stands for the encoding strategies obtained by concatenation.

APPENDIX B
BELIEF ABOUT THE TYPE OF $\mathcal{D}_1$: PROOF OF LEMMA 10

We introduce the indicator of error events $E_\delta^1 \in \{0,1\}$ defined as follows

$$E_\delta^1 = \begin{cases} 1, & \text{if } (u^n, w_1^n, w_0^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1W_0|U}). \\ 0, & \text{otherwise.} \end{cases} \tag{74}$$

Also recall the following sets: For all m_0, $\mathcal{C}(m_0) = \{w_1^n(m_0,m_1) \in \mathcal{C}, m_1 \in \{1,..2^{\lfloor nR_1 \rfloor}\}\}$, and for all m_0, $w_1 \in \mathcal{W}_1$, $t \in \{1,\dots,n\}$, $\mathcal{A}^t(w_1|m_0) = \big\{w_1^n \in \mathcal{C}(m_0), w_{1,t} = w_1\big\}$.

Lemma 13. *For all $(w_0^n, w_1^n, w_0, w_1, t)$, we have*

$$\lim_{n\mapsto\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{t=1}^{n} D(\mathcal{P}_{W_{1,t}|W_0^n}(.|W_0^n)||\mathcal{Q}_{W_{1,t}|W_{0,t}}(.|W_{0,t}))\Big| E_\delta^1 = 0\Big] = 0. \tag{75}$$

Proof. For any (w_1^n, w_0^n), and sufficiently small $\delta > 0$, and sufficiently large n we have

$$\mathcal{P}(w_1^n|w_0^n) = \sum_{u^n \in \mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)} \mathcal{P}(w_1^n, u^n|w_0^n) \tag{76}$$

$$= \sum_{u^n \in \mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)} \mathcal{P}(u^n|w_0^n)\mathcal{P}(w_1^n|u^n, w_0^n) \tag{77}$$

$$= \sum_{u^n \in \mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)} 2^{-nH(U|W_0)} \mathbb{1}(E_\delta^1 = 0) \tag{78}$$

$$=|\mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)|2^{-nH(U|W_0)} \tag{79}$$

$$=2^{nH(U|W_0W_1)}2^{-nH(U|W_0)} \tag{80}$$

$$=2^{-nI(U;W_1|W_0)} \tag{81}$$

$$=2^{-n(R_1-\eta)}. \tag{82}$$

For any $(t, w_{1,t}, w_0^n)$, and sufficiently small $\delta > 0$, we have

$$\mathcal{P}_{W_{1,t}|W_0^n}(w_{1,t}|w_0^n) = \sum_{w_1^n \in \mathcal{T}_\delta^n(P_{W_1|W_0}, w_0^n)} \mathcal{P}(w_1^n|w_0^n) \tag{83}$$

$$= \sum_{w_1^n \in \mathcal{A}^t(w_{1,t}|m_0)} 2^{-n(R_1-\eta)} \tag{84}$$

$$= |\mathcal{A}^t(w_{1,t}|m_0)| 2^{-n(R_1-\eta)} \tag{85}$$

$$= \frac{|\mathcal{A}^t(w_{1,t}|m_0)|}{|\mathcal{C}(m_0)|} \tag{86}$$

$$= \mathcal{Q}_{W_{1,t}|W_{0,t}}(w_{1,t}|w_{0,t}) \tag{87}$$

where (84) follows from (82) and (44), and (87) follows from (50). Therefore, we get

$$\lim_{n\mapsto\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{t=1}^n D(\mathcal{P}_{W_{1,t}|W_0^n}(.|W_0^n)||\mathcal{Q}_{W_{1,t}|W_{0,t}}(.|W_{0,t}))\Big| E_\delta^1 = 0\Big] \tag{88}$$

$$= \lim_{n\mapsto\infty} \sum_{w_1^n, w_0^n} \mathcal{P}^\sigma(w_1^n, w_0^n \Big| E_\delta^1 = 0) \frac{1}{n}\sum_{t=1}^n \sum_{w_1} \mathcal{P}(w_1|w_0^n) \log_2 \frac{\mathcal{P}(w_1|w_0^n)}{\mathcal{Q}(w_{1,t}|w_{0,t})} \tag{89}$$

$$= 0. \tag{90}$$

where (90) follows since $\lim_{n\mapsto\infty} \log \frac{\mathcal{P}_{W_{1,t}|W_0^n}(w_{1,t}|w_0^n)}{\mathcal{Q}_{W_{1,t}|W_{0,t}}(w_{1,t}|w_{0,t})} = 0$. □

APPENDIX C
CONTROL OF BELIEFS ABOUT THE STATE U: PROOF OF LEMMA 11

We denote the Bayesian posterior belief about the state $\mathcal{P}^\sigma_{U_t|M_1M_0}(\cdot|m_1, m_0) \in \Delta(\mathcal{U})$ by $\mathcal{P}_t^{m_1,m_0}$. We show that on average, the Bayesian belief is close in KL distance to the target belief $\mathcal{Q}_{U|W_1W_0}$ induced by the single-letter distribution $\mathcal{Q}_{W_1W_0|U}$. The indicator of error event $E_\delta^1 \in \{0, 1\}$ is as given in (74). Assuming the distribution $\mathcal{Q}_{U|W_1W_0}$ is fully supported, the beliefs about the state are controlled as follows

$$\mathbb{E}\Big[\frac{1}{n}\sum_{t=1}^n D(\mathcal{P}_t^{m_1,m_0}||\mathcal{P}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t}))\Big| E_\delta^1 = 0\Big] \tag{91}$$

$$= \sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1, m_0, w_1^n, w_0^n \Big| E_\delta^1 = 0) \cdot \frac{1}{n}\sum_{t=1}^n D(\mathcal{P}_t^{m_1,m_0}||\mathcal{P}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})) \tag{92}$$

$$= \sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1, m_0, w_1^n, w_0^n \Big| E_\delta^1 = 0) \cdot \frac{1}{n}\sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1m_0}(u) \log_2 \frac{\mathcal{P}_t^{m_1m_0}(u)}{\mathcal{P}_{U|W_1W_0}(u|w_{1,t}, w_{0,t})} \tag{93}$$

$$= \sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1, m_0, w_1^n, w_0^n \Big| E_\delta^1 = 0) \cdot \frac{1}{n}\sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1m_0}(u) \log_2 \frac{1}{\mathcal{P}_{U|W_1W_0}(u|w_{1,t}, w_{0,t})}$$

$$- \sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\frac{1}{n}\sum_{t=1}^{n}\sum_{u}\mathcal{P}_t^{m_1m_0}(u)\log_2\frac{1}{\mathcal{P}_t^{m_1m_0}(u)} \tag{94}$$

$$=\frac{1}{n}\sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\sum_{t=1}^{n}\sum_{u}\mathcal{P}_t^{m_1m_0}(u)\log_2\frac{1}{\mathcal{P}_{U|W_1W_0}(u|w_{1,t},w_{0,t})}$$
$$- \frac{1}{n}\sum_{t=1}^{n}H(U_t|M_1,M_0,E_\delta^1=0) \tag{95}$$

$$=\frac{1}{n}\sum_{u^n,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\log_2\frac{1}{\prod_{t=1}^{n}\mathcal{P}_{U|W_1W_0}(u_t|w_{1,t},w_{0,t})}-\frac{1}{n}\sum_{t=1}^{n}H(U_t|M_1,M_0,E_\delta^1=0) \tag{96}$$

$$=\frac{1}{n}\sum_{u^n,w_1^n,w_0^n\in\mathcal{T}_\delta^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\log_2\frac{1}{\prod_{t=1}^{n}\mathcal{P}_{U|W_1W_0}(u_t|w_{1,t},w_{0,t})}-\frac{1}{n}\sum_{t=1}^{n}H(U_t|M_1,M_0,E_\delta^1=0) \tag{97}$$

$$\leq\frac{1}{n}\sum_{u^n,w_1^n,w_0^n\in\mathcal{T}_\delta^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot n\cdot\big(H(U|W_1,W_0)+\delta\big)-\frac{1}{n}H(U^n|M_1,M_0,E_\delta^1=0) \tag{98}$$

$$\leq\frac{1}{n}I(U^n;M_1,M_0\Big|E_\delta^1=0)-I(U;W_1,W_0)+\delta+\frac{1}{n}+\log_2|\mathcal{U}|\cdot\mathcal{P}^{\sigma,\tau_1,\tau_2}(E_\delta^1=1) \tag{99}$$

$$\leq\eta+\delta+\frac{1}{n}+\log_2|\mathcal{U}|\cdot\mathcal{P}^{\sigma,\tau_1,\tau_2}(E_\delta^1=1). \tag{100}$$

- Equation (92) comes from the definition of expected K-L divergence.
- Equation (93) comes from the definition of K-L divergence.
- Equation (94) comes from splitting the logarithm.
- Equation (95) follows since:

$$\sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\frac{1}{n}\sum_{t=1}^{n}\sum_{u}\mathcal{P}_t^{m_1m_0}(u)\log_2\frac{1}{\mathcal{P}_t^{m_1m_0}(u)} \tag{101}$$

$$=\sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\frac{1}{n}\sum_{t=1}^{n}H(U_t|M_1=m_1,M_0=m_0) \tag{102}$$

$$=\frac{1}{n}\sum_{t=1}^{n}\sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\ H(U_t|M_1=m_1,M_0=m_0) \tag{103}$$

$$=\frac{1}{n}\sum_{t=1}^{n}\sum_{m_1,m_0} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0\Big|E_\delta^1=0)\cdot\ H(U_t|M_1=m_1,M_0=m_0) \tag{104}$$

$$=\frac{1}{n}\sum_{t=1}^{n}H(U_t|M_1,M_0,E_\delta^1=0). \tag{105}$$

- Equation (96) follows since:

$$\sum_{m_1,m_0,w_1^n,w_0^n} \mathcal{P}^{\sigma,\tau_1,\tau_2}(m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\frac{1}{n}\sum_{t=1}^{n}\sum_{u}\mathcal{P}_t^{m_1m_0}(u)\log_2\frac{1}{\mathcal{P}_{U|W_1W_0}(u|w_{1,t},w_{0,t})} \tag{106}$$

$$=\frac{1}{n}\sum_{t=1}^{n}\sum_{u_t,m_1,m_0,w_1^n,w_0^n}\mathcal{P}^{\sigma,\tau_1,\tau_2}(u_t,m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\log_2\frac{1}{\mathcal{P}_{U|W_1W_0}(u_t|w_{1,t},w_{0,t})} \tag{107}$$

$$=\frac{1}{n}\sum_{t=1}^{n}\sum_{u^n,m_1,m_0,w_1^n,w_0^n}\mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\log_2\frac{1}{\mathcal{P}_{U|W_1W_0}(u_t|w_{1,t},w_{0,t})} \tag{108}$$

$$=\frac{1}{n}\sum_{u^n,m_1,m_0,w_1^n,w_0^n}\mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,m_1,m_0,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\log_2\frac{1}{\prod_{t=1}^{n}\mathcal{P}_{U|W_1W_0}(u_t|w_{1,t},w_{0,t})} \tag{109}$$

$$=\frac{1}{n}\sum_{u^n,w_1^n,w_0^n}\mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,w_1^n,w_0^n\Big|E_\delta^1=0)\cdot\log_2\frac{1}{\prod_{t=1}^{n}\mathcal{P}_{U|W_1W_0}(u_t|w_{1,t},w_{0,t})}. \tag{110}$$

- Equation (97) follows since the support of $\mathcal{P}^{\sigma,\tau_1,\tau_2}(u^n,w_1^n,w_0^n|E_\delta^1)=\mathbb{P}\{(u^n,w_1^n,w_0^n)\in\mathcal{T}_\delta^n\}$ is included in $\mathcal{T}_\delta^n$.
- Equation (98) follows from the typical average lemma property (Property 1 pp.26 in [6]) given in lemma 16, and the chain rule of entropy: $H(U^n|M_1,M_0,W_1^n,W_2^n)\leq\sum_{t=1}^{n}H(U_t|M_1,M_0,W_1,W_0)$.
- Equation (99) comes from the conditional entropy property and the fact that $H(U^n)=nH(U)$ for an i.i.d random variable U and lemma 17.
- Equation (100) follows since $I(U^n;M_1,M_0)\leq H(M_1,M_0)\leq\log_2|J|=n\cdot(R_1+R_0)=n\cdot(I(U;W_1,W_0)+\eta)$ and lemma 17.

If the expected probability of error is small over the codebooks, then it has to be small over at least one codebook. Therefore, equations (48) and (49) imply that:

$$\forall\epsilon_2>0,\forall\eta>0,\exists\bar{\delta}>0,\forall\delta\leq\bar{\delta},\exists\bar{n}\in\mathbb{N},\forall n\geq\bar{n},\exists b^\star,\ \text{s.t. }\mathcal{P}_{b^\star}(E_\delta^2=1)\leq\varepsilon_2. \tag{111}$$

The strategy σ of the encoder consists of using $b^\star$ in order to transmit the pair (m_1,m_0) such that $(U^n,W_0^n(m_0),W_1^n(m_0,m_1))$ is a jointly typical sequence. By construction, this satisfies equation (111).

Lemma 14. *Let $\mathcal{Q}_{W_1W_0|U}\in\tilde{Q}_0(R_1,R_0)$, then $\forall\varepsilon>0$, $\forall\alpha>0,\gamma>0$, there exists $\bar{\delta}$, $\forall\delta\leq\bar{\delta}$, $\exists\bar{n}$, $\forall n\geq\bar{n}$, $\exists\sigma$, such that $1-\mathcal{P}^\sigma(B_{\alpha,\gamma,\delta})\leq\varepsilon$.*

Proof. of lemma 14 We have:

$$1-\mathcal{P}_\sigma(B_{\alpha,\gamma,\delta}):=\mathcal{P}_\sigma(B_{\alpha,\gamma,\delta}^c) \tag{112}$$

$$= \mathcal{P}_\sigma(E_\delta^1 = 0)\mathcal{P}_\sigma(B^c_{\alpha,\gamma,\delta}|E_\delta^1 = 0) \tag{113}$$

$$\leq \mathcal{P}_\sigma(B^c_{\alpha,\gamma,\delta}|E_\delta^1 = 0) \tag{114}$$

$$\leq \varepsilon_2 + \mathcal{P}_\sigma(B^c_{\alpha,\gamma,\delta}|E_\delta^1 = 1). \tag{115}$$

Moreover,

$$\mathcal{P}_\sigma(B^c_{\alpha,\gamma,\delta}|E_\delta^1 = 0) = \sum_{w_1^n, w_0^n, m_1, m_0} \mathcal{P}_\sigma\Bigg((w_1^n, w_0^n, m_1, m_0) \in B^c_{\alpha,\gamma,\delta}\Bigg| E_\delta^1 = 0\Bigg) \tag{116}$$

$$= \sum_{w_1^n, w_0^n, m_1, m_0} \mathcal{P}_\sigma\Bigg((w_1^n, w_0^n, m_1, m_0) \text{ s.t. } \frac{|T_\alpha(w_1^n, w_0^n, m_1, m_0)|}{n} \leq 1 - \gamma\Bigg| E_\delta^1 = 0\Bigg) \tag{117}$$

$$= \mathcal{P}_\sigma\Bigg(\frac{\#}{n}\Bigg\{t, D\Bigg(\mathcal{P}_t^{m_1,m_0}\Bigg|\Bigg|\mathcal{Q}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})\Bigg) \leq \frac{\alpha^2}{2\ln 2} < 1 - \gamma\Bigg| E_\delta^1 = 0\Bigg\} \tag{118}$$

$$= \mathcal{P}_\sigma\Bigg(\frac{\#}{n}\Bigg\{t, D\Bigg(\mathcal{P}_t^{m_1,m_0}\Bigg|\Bigg|\mathcal{Q}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})\Bigg) > \frac{\alpha^2}{2\ln 2} \geq \gamma\Bigg| E_\delta^1 = 0\Bigg\} \tag{119}$$

$$\leq \frac{2\ln 2}{\alpha^2\gamma} \cdot \mathbb{E}_\sigma\Bigg[\frac{1}{n}\sum_{t=1}^n D\Bigg(\mathcal{P}_t^{m_1,m_0}\Bigg|\Bigg|\mathcal{Q}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})\Bigg)\Bigg] \tag{120}$$

$$\leq \frac{2\ln 2}{\alpha^2\gamma} \cdot \Bigg(\eta + \delta + \frac{2}{n} + 2\log_2|\mathcal{U}| \cdot \mathcal{P}_\sigma(E_\delta^2 = 1),\Bigg) \tag{121}$$

- *Equations* (116) *to* (119) *are simple reformulations.*
- *Equation* (120) *comes from using Markov's inequality given in lemma 15.*
- *Equation* (121) *comes from equation* (100)*.*

Lemma 15. *(Markov's Inequality). For all* $\varepsilon_1 > 0$ *,* $\varepsilon_2 > 0$ *we have:*

$$\mathbb{E}_\sigma\Bigg[\frac{1}{n}\sum_{t=1}^n D\Bigg(\mathcal{P}_t^{m_1,m_0}\Bigg|\Bigg|\mathcal{Q}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})\Bigg)\Bigg] \leq \varepsilon_0 \tag{122}$$

$$\implies \mathcal{P}_{w_1^n, w_0^n, m_1, m_0}\Bigg(\frac{\#}{n}\Bigg\{t, D\Bigg(\mathcal{P}_t^{m_1,m_0}\Bigg|\Bigg|\mathcal{Q}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})\Bigg) > \varepsilon_1\Bigg\} > \varepsilon_2\Bigg) \leq \frac{\varepsilon_0}{\varepsilon_1 \cdot \varepsilon_2}. \tag{123}$$

Proof. of lemma 15 We denote by $D_t = D(\mathcal{P}_t^{m_1,m_0}||\mathcal{Q}_{U|W_1W_0}(\cdot|W_{1,t}, W_{0,t})$ *and* $D^n = \{D_t\}_t$ *the K-L divergence. We have that:*

$$\mathcal{P}\Bigg(\frac{\#}{n}\Bigg\{t, \text{s.t.} D_t > \varepsilon_1\Bigg\} > \varepsilon_2\Bigg) = \mathcal{P}\Bigg(\frac{1}{n} \cdot \sum_{t=1}^n \mathbb{1}\Bigg\{D_t > \varepsilon_1\Bigg\} > \varepsilon_2\Bigg) \tag{124}$$

$$\leq \frac{\mathbb{E}\left[\frac{1}{n}\cdot\sum_{t=1}^{n}\mathbb{1}\left\{D_t>\varepsilon_1\right\}\right]}{\varepsilon_2} \tag{125}$$

$$= \frac{\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left[\mathbb{1}\left\{D_t>\varepsilon_1\right\}\right]}{\varepsilon_2} \tag{126}$$

$$= \frac{\frac{1}{n}\sum_{t=1}^{n}\mathcal{P}\left(D_t>\varepsilon_1\right)}{\varepsilon_2} \tag{127}$$

$$\leq \frac{\frac{1}{n}\sum_{t=1}^{n}\frac{\mathbb{E}\left[D_t\right]}{\varepsilon_1}}{\varepsilon_2} \tag{128}$$

$$= \frac{1}{\varepsilon_1\cdot\varepsilon_2}\cdot\mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n}D_t\right] \leq \frac{\varepsilon_0}{\varepsilon_1\cdot\varepsilon_2}. \tag{129}$$

- *Equations* (124)*,* (126)*,* (127) *and* (129) *are reformulations of probabilities and expectations.*
- *Equations* (125) *and* (128)*, come from Markov's inequality* $\mathcal{P}(X\geq\alpha)\leq\frac{\mathbb{E}[X]}{\alpha},\ \forall\alpha>0$*.*

APPENDIX D
ADDITIONAL LEMMAS

Lemma 16. *(Typical Sequences Property 1, pp.26 in [6]). The typical sequences* $(u^n,w_1^n,w_2^n)\in\mathcal{T}_\delta^n$ *satisfy:*

$$\forall\varepsilon>0,\ \exists\bar{\delta}>0,\ \forall\delta\leq\bar{\delta},\ \forall n,\ \forall(u^n,w_1^n,w_2^n)\in\mathcal{T}_\delta^n,$$
$$\left|\frac{1}{n}\cdot\log_2\frac{1}{\prod_{t=1}^{n}\mathcal{P}(u|w_{1,t},w_{2,t})}-H(U|W_1,W_2)\right|\leq\varepsilon, \tag{130}$$

where $\bar{\delta}=\varepsilon\cdot H(U|W_1,W_2)$*.*

Lemma 17. *Let* U^n *an i.i.d random variable and* M *a random variable. For all* $\varepsilon>0$*, there exists* $\bar{n}\in\mathbb{N}$*, such that for all* $n\geq\bar{n}$*, we have*

$$H(U^n|E_\delta=0)\geq n\cdot\Big(H(U)-\varepsilon\Big). \tag{131}$$

Proof.

$$H(U^n|E_\delta=0) = \frac{1}{\mathcal{P}(E_\delta=0)}\cdot\Big(H(U^n|E_\delta=1)-\mathcal{P}(E_\delta=1)\cdot H(U^n|E_\delta=1)\Big) \tag{132}$$

$$\geq H(U^n|E_\delta)-\mathcal{P}(E_\delta=1)\cdot H(U^n|E_\delta=1)\Big) \tag{133}$$

$$\geq H(U^n) - H(E_\delta) - \mathcal{P}(E_\delta = 1) \cdot H(U^n | E_\delta = 1)\Big) \tag{134}$$

$$\geq H(U^n) - n \cdot \varepsilon. \tag{135}$$

□

- *Equation* (132) *follows from the conditional entropy definition.*
- *Equation* (133) *follows since* $\mathcal{P}(E_\delta = 0) \leq 1$.
- *Equation* (134) *comes from the property* $H(U^n|M, E_\delta) = H(U^n, M, E_\delta) - H(M) - H(E_\delta) \geq H(U^n) - H(M) - H(E_\delta)$.
- *Equation* (135) *follows since* U *is i.i.d and the definition of* $E_\delta = 1$. *Hence, for all* ε, *there exists an* $\bar{n} \in \mathbb{N}$ *such that for all* $n \geq \bar{n}$ *we have* $H(\mathcal{P}(E_\delta = 1)) + H(M) + \mathcal{P}(E_\delta = 1) \cdot \log_2 |\mathcal{U}| \leq \varepsilon$.

REFERENCES

[1] *C. E. Shannon, "A mathematical theory of communication,"* Bell system technical journal*, vol. 27, no. 3, pp. 379–423, 1948.*

[2] *E. Kamenica and M. Gentzkow, "Bayesian persuasion,"* American Economic Review*, vol. 101, pp. 2590 – 2615, 2011.*

[3] *M. Brückner, C. Kanzow, and T. Scheffer, "Static prediction games for adversarial learning problems,"* Journal of Machine Learning Research*, vol. 13, no. 85, pp. 2617–2654, 2012.*

[4] *V. Koshelev, "Hierarchical coding of discrete sources,"* Problemy Peredachi Informatsii*, vol. 16, pp. 31–49, 1980.*

[5] *B. Rimoldi, "Successive refinement of information: characterization of the achievable rates,"* IEEE Transactions on Information Theory*, vol. 40, no. 1, pp. 253–259, 1994.*

[6] *A. El Gamal and Y.-H. Kim,* Network information theory. *Cambridge University Press, 2011.*

[7] *F. Koessler, M. Laclau, and T. Tomala, "Interactive information design,"* Mathematics of Operations Research*, vol. 47, no. 1, pp. 153–175, 2021.*

[8] *M. Le Treust and T. Tomala, "Persuasion with limited communication capacity,"* Journal of Economic Theory*, vol. 184, p. 104940, 2019.*

[9] ——, *"Strategic communication with decoder side information,"* Information Symposium on Information Theory (ISIT)*, 2021.*

[10] *R. Bou Rouphael and M. Le Treust, "Strategic successive refinement coding for bayesian persuasion with two decoders," in* IEEE Information Theory Workshop (ITW), Kanazawa Japan*, Oct. 2021.*

[11] *V. Crawford and J. Sobel, "Strategic information transmission,"* Econometrica*, vol. 50, no. 6, pp. 1431–51, 1982.*

[12] *S. Sarıtaş, S. Yüksel, and S. Gezici, "Quadratic multi-dimensional signaling games and affine equilibria,"* IEEE Transactions on Automatic Control*, vol. 62, no. 2, p. 605–619, Feb 2017.*

[13] *S. Sarıtaş, P. Furrer, S. Gezici, T. Linder, and S. Yüksel, "On the number of bins in equilibria for signaling games," in* 2019 IEEE International Symposium on Information Theory (ISIT)*, 2019, pp. 972–976.*

[14] *S. Sarıtaş, S. Yüksel, and S. Gezici, "Dynamic signaling games with quadratic criteria under Nash and Stackelberg equilibria,"* Automatica*, vol. 115, no. C, May 2020.*

[15] *E. Akyol, C. Langbort, and T. Başar, "Information-theoretic approach to strategic communication as a hierarchical game,"* Proceedings of the IEEE*, vol. 105, no. 2, pp. 205–218, 2017.*

[16] A. D. Wyner and J. Ziv, "The rate-distortion function for source coding with side information at the decoder," IEEE Transactions on Information Theory*, vol. 22, no. 1, pp. 1–11, 1976.*

[17] R. Bou Rouphael and M. Le Treust, "Impact of private observation in bayesian persuasion," International Conference on NETwork Games COntrol and OPtimization NetGCoop*, Mar. 2020.*

[18] A. S. Vora and A. A. Kulkarni, "Achievable rates for strategic communication," in 2020 IEEE International Symposium on Information Theory (ISIT)*, 2020, pp. 1379–1384.*

[19] ——, "Zero error strategic communication," in 2020 International Conference on Signal Processing and Communications (SPCOM)*, 2020, pp. 1–5.*

[20] R. Deori and A. A. Kulkarni, "Information revelation through signalling," Systems & Control Letters*, vol. 169, p. 105378, 2022.*

[21] S. Dughmi, D. Kempe, and R. Qiang, "Persuasion with limited communication," in Proceedings of the 2016 ACM Conference on Economics and Computation*, ser. EC '16. New York, NY, USA: Association for Computing Machinery, 2016, p. 663–680.*

[22] J. Yu, "Essential equilibria of n-person noncooperative games," Journal of Mathematical Economics*, vol. 31, pp. 361–372, 1999.*

[23] J. Nash, "Non-cooperative games," Annals of Mathematics*, pp. 286–295, 1951.*

[24] D. Fudenberg and J. Tirole, Game Theory. *MIT Press, Cambrige, MA, 1991.*

[25] M. J. Fort, "Points of continuity of semicontinuous functions," Publ. Math. Debrecen*, no. 2, pp. 100–102, 1951.*